LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 4

Problem 1 Let $\Omega \neq \emptyset$ be a set.

a): Let and $C_1, \dots, C_n$ a partition of Ω into non-empty pairwise disjoint subsets, i.e. $\Omega = \dot{\bigcup}_{k=1}^{n} C_k$. Let furthermore $\mathcal{C} := \left\{ \bigcup_{i \in I} C_i \mid I \subseteq \{1, \dots, n\} \right\}$, with the convention $\bigcup_{k \in \emptyset} C_k = \emptyset$.

(1): Check that $\mathcal{C}$ is a $\sigma-$algebra.

(2): Let $f : \Omega \to \mathbb{R}$ be a map. Show that f is measurable with respect to $\mathcal{C}$ (i.e. $\forall t \in \mathbb{R} : \{\omega \in \Omega \mid f(\omega) \geq t\} =: \{f \geq t\} \in \mathcal{C}$), if and only if f is constant on each C_i

b): Let $g : \Omega \to \mathbb{R}$ be a map that takes only finitely many values , i.e. $g(\Omega) = \{y_1, \dots, y_p\}$. Define $C_k := g^{-1}(\{y_k\})$ and $\mathcal{C}$ as in a):

1): Check that g is $\mathcal{C}-$measureable.

2): Let $\mathcal{F}$ be any $\sigma-$algebra such that g is $\mathcal{F}-$measureable. Show that $\mathcal{C} \subseteq \mathcal{F}$

3): Let $f : \Omega \to \mathbb{R}$ be a nother map. Show that f is $\mathcal{C}$-measureable if and only if there is a function $h : \mathbb{R} \to \mathbb{R}$ such that $f = h \circ g$.

Problem 2 (Probability and random variables) We make an experiment where secants of the unit circle $S^1 \subset \mathbb{R}$ are randomly chosen in the following ways:

a) Choose two (different) points on the circle at random (with uniform distribution), connect them with a straight line.

b) Choose one point $p \neq (0, 0)$ inside the circle at random (with uniform distribution) and draw the secant which goes through p and is orthogonal to the line from $(0, 0)$ to p.

Define suitable probability spaces $(\Omega_a, \mathcal{F}_a, \mathbb{P}_a)$ and $(\Omega_b, \mathcal{F}_b, \mathbb{P}_b)$ for both cases. Define random variables $X_j : \Omega_j \to \mathbb{R}(j = a, b)$ that assign to a choice of points the length of the respective secant. Calculate the probability distribution of X_a and X_b and the respective expectation values.

Problem 3 Let Ω be a set and $\mathcal{F}$ a corresponding $\sigma-$algebra. Let furthermore $\forall n \in \mathbb{N}$, $f_n : \Omega \to \mathbb{R}$ such that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measureable functions. Show that the following functions are measureable:

(1): $\Psi(\omega) = \sup_{n\in\mathbb{N}} f_n(\omega)$

(2): $\overline{f}(\omega) := \limsup_{n\to\infty} f_n(\omega)$ (limit superior of the sequence $f_n(\omega)$ for every ω).

(3): $f(\omega) := \begin{cases} \lim_{n\to\infty} f_n(\omega) & \text{if the limit exists and is in } \mathbb{R} \cup \{-\infty, \infty\} \\ 0 & \text{else} \end{cases}$

You are allowed to use that

- products and linear combinations of measureable functions are measureable
- if a function $h : \Omega \to \mathbb{R}$ is measureable, then $\forall a \leq b \in \mathbb{R}$:

  A) $f^{-1}([a,b]) \in \mathcal{F}$ (this includes the case $f^{-1}(\{a\}) \in \mathcal{F}$)

  B) $f^{-1}([a,b)) \in \mathcal{F}$

  C) $f^{-1}((a,b]) \in \mathcal{F}$

  D) $f^{-1}((a,b)) \in \mathcal{F}$ hold.

The solutions to these exercises will be discussed on Monday, 14.05.