LMU Munich
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Exercises on Mathematical Statistical Physics
Math Sheet 2

Problem 1 (A theorem about matrices with positive entries.) Define the following relation for finite-dimensional matrices (and vectors as a special case):

$$(a_{ij}) \succ (\succeq) \quad :\Leftrightarrow \quad a_{ij} > (\geq)\, 0 \qquad \forall\, i,j$$

and, correspondingly,

$$(a_{ij}) \succ (\succeq)(b_{ij}) \quad :\Leftrightarrow \quad (a_{ij}) - (b_{ij}) > (\geq)\, 0.$$

Let $n \in \mathbb{N}$ and let $A \in \mathcal{M}(n \times n, \mathbb{C})$ be such that $A \succ 0$.

a) Prove that there exists $\lambda_0 > 0$ and $x_0 \in \mathbb{R}^n$, $x_0 \succ 0$, such that $Ax_0 = \lambda_0 x_0$.
(Hint: One route you may take is to apply a suitable fixed point argument. Alternatively, you may define the set $\Lambda := \{\mu > 0 |\, \exists\, x \succ 0$ such that $Ax \succeq \mu x\}$ and find its supremum.)

b) Prove that if $\lambda \neq \lambda_0$ is a (possibly complex) eigenvalue of A, then $|\lambda| < \lambda_0$.

c) Prove that the eigenvalue λ_0 is simple (i.e., its algebraic multiplicity is 1.)

Problem 2 (Irreducible, aperiodic Markov processes.) Consider a $n-$dimensional transition matrix P for a Markov process that is *irreducible* and *aperiodic* (according to the definitions given in class).

a) Prove that there exists $t \in \mathbb{N}$ such that $P^t \succ 0$.

b) Prove that the eigenvalue λ_0 of P^t determined by means of the theorem proved in Problem 1 is actually $\lambda_0 = 1$.

c) Prove that for every $x \in \mathbb{R}^n$ such that $x \succ 0$ and $\sum_{i=1}^{n} x_i = 1$ one has

$$||P^n x - x_0|| < C\mu^n$$

for some constant $C > 0$ and $\mu \in (0,1)$, where $\mathbb{R}^n \ni x_0 \succ 0$, $P^t x_0 = x_0$ (as in Problem 1), whence in particular

$$\lim_{n\to\infty} P^n x = x_0$$

in the vector-norm sense. (In fact this holds also in the matrix-norm sense, because of the finite dimensional setting.)

Problem 3 (Transfer matrices.) You are given the one dimensional spin chain with the energy function

$$H_\Lambda(\underline{\sigma}) = \sum_{J \subset \Lambda} \Phi(J) \prod_{x \in J} \sigma_x,$$

where the interaction is translationally invariant, i.e., for all $a \in Z$ and all $J \subset \mathbb{Z}$, we have $\Phi(J) = \Phi(J+a)$ where $J + a := \{j + a \mid j \in J, a \in \mathbb{Z}\}$, and has a finite range: $\Phi(J) = 0$, whenever $\operatorname{diam}(J) \geq R+1, R \in \mathbb{N}$ (e.g. for the nearest-neighbor interaction one would have $R = 1$); $\Lambda := \{1, 2, ..., m \cdot R\}$. The partition function is

$$Z_\Lambda(\beta) = \sum_{\sigma \in \{+1,-1\}^\Lambda} e^{-\beta H_\Lambda(\underline{\sigma})}$$

a) Let $B := \{+1, -1\}^R$ and $\Lambda_m := \{1, 2, ..., m \cdot R\}$. Find functions $f_\beta, g_\beta : B \to \mathbb{R}_+$ and $K_\beta : B \times B \to \mathbb{R}_+$, such that for all $m \in \mathbb{N}$

$$Z_{\Lambda_m}(\beta) = \sum_{b_1,...,b_m \in B} f_\beta(b_1) \cdot K_\beta(b_1, b_2) \cdot ... \cdot K_\beta(b_{m-1}, b_m) \cdot g_\beta(b_m)$$

Note that such f_β, g_β and K_β are not unique.

(Hint: split the Λ_m into the blocks of the length R and represent the energy function as the sum of two summands: one corresponding to the interaction within the blocks and one – between different blocks).

b) For $R = 1$ express the pressure

$$p(\beta) := \lim_{m \to \infty} \frac{1}{\beta m R} \ln(Z_{\Lambda_m}(\beta))$$

in terms of the principal (the biggest) eigenvalue of a suitably chosen matrix or linear map.

Steps:
(i) notice that K_β from the first part of this exercise can be seen as a matrix with positive entries, which can be chosen to be symmetric.
(ii) show that $Z_{\Lambda_m}(\beta)$ can be represented as some scalar product involving the n-th power of K_β.
(iii) Use the results of the Ex. 1 to express the pressure in terms of the principal eigenvalue of K_β.

c) Let $a, c, d : \mathbb{R} \to \mathbb{R}$ be C^∞ functions and

$$A := \begin{pmatrix} a(t) & c(t) \\ c(t) & d(t) \end{pmatrix}.$$

Show that if $A(t_0)$ has two distinct eigenvalues $|\lambda_1(t_0)| > |\lambda_2(t_0)|$, then there is some $\delta > 0$ such that $A(t)$ has two distinct eigenvalues for all $t \in (t_0 - \delta, t_0 + \delta)$, moreover $\lambda_1(t)$ is a C^∞ function in some neighborhood of t_0.

d) Using the results of the second and third parts of the exercise show that $p(\beta)$ is a C^∞ function.

The solutions to these exercises will be discussed on Monday, 30.04.