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Exercises on Mathematical Statistical Physics
Sheet 11

Problem 1 (Green's functions and random walks) Let E be a finite or countably infinite set and $P = (P(x,y))_{x,y\in E}$ a transition matrix (i.e., $P(x,y) \geq 0$ and $\sum_y P(x,y) = 1$). Define

$$G(x,y) := \delta_{x,y} + P(x,y) + \sum_{n=2}^{\infty} \sum_{x_1,\dots,x_{n-1}\in E} P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y) \qquad (x,y\in E).$$

a) Explain why $G(x,y)$ is the expected number of visits to site y of a random walker in E started at site x and evolving according to the transition matrix P.

b) Define $((I-P)G)(x,y)$ by the usual matrix product formulas. Show that if $G < \infty$ on $E\times E$, then $\big((I-P)G\big)(x,y) = \delta_{x,y}$ for all $x,y\in E$. (In this sense, $G = (I-P)^{-1}$.)

c) Suppose that E is finite. Is it possible that $G<\infty$ on $E\times E$?

d) Let $E = \mathbb{Z}^d$. Assume that $G<\infty$ on $E\times E$, that G is symmetric and positive semi-definite: $G(x,y) = G(y,x)$ for all x,y and $\sum_{x,y\in\mathbb{Z}^d} f_x G(x,y) f_y \geq 0$ for all $f = (f_x)_{x\in\mathbb{Z}^d}$ for which the sum converges. Suppose that there exists a measure μ on $\Omega := \mathbb{R}^{\mathbb{Z}^d}$ (equipped with the product of the Borel-σ-algebras) such that

$$\int_\Omega \mathrm{e}^{\mathrm{i}\sum_{x\in\mathbb{Z}^d} f_x\varphi_x}\mathrm{d}\mu(\varphi) = \mathrm{e}^{-\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} f_x G(x,y) f_y}.$$

for all $f = (f_x)_{x\in\mathbb{Z}^d}\in\mathbb{R}^{\mathbb{Z}^d}$ that have only finitely many non-zero entries. Show that for all $x,y\in\mathbb{Z}^d$,

$$\int_\Omega \varphi_x\varphi_y \mathrm{d}\mu(\varphi) = G(x,y).$$

e) Let $E = \mathbb{Z}^d$ and

$$P(x,y) := \begin{cases} \frac{1}{2d}, & \text{if } x,y \text{ are nearest neighbors,}\\ 0, & \text{else.}\end{cases}$$

Show that for all $\varphi\in\mathbb{R}^{\mathbb{Z}^d}$ that have only finitely many non-zero entries,

$$\sum_{x,y\in\mathbb{Z}^d} \varphi_x\big(\delta_{x,y} - P(x,y)\big)\varphi_y$$

is proportional to $\sum_{x,y\in\mathbb{Z}^d:||x-y||=1}(\varphi_y-\varphi_x)^2$. Compute the proportionality constant.

Problem 2 (Markov chains and reflection positivity) Let E be a finite, non-empty set, $\Omega = E^{\mathbb{Z}}$, and $\mathcal{F}$ the product σ-algebra. Let $P = (P(x,y))_{x,y\in E}$ be a transition matrix and μ a measure on E that satisfies the detailed balance $\mu(x)P(x,y) = \mu(y)P(y,x)$. Let $\mathbb{P}$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}\big(\{\omega \in \Omega : \omega_{-n} = x_{-n}, \dots, \omega_n = x_n\}\big) = \mu(x_{-n})P(x_{-n}, x_{-n+1}) \cdots P(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$ and $(x_{-n}, \dots, x_n) \in E^{2n+1}$. For the purpose of this exercise, the axis $\mathbb{Z}$ is best thought of as discrete time and Ω as a space of paths in E; $\mathbb{P}$ is a probability measure on paths.

a) Let $\theta : \Omega \to \Omega$ be the reflection defined by $(\theta\omega)_k := \omega_{-k}$. Further let $\mathcal{A}_+$ the collection of complex-valued maps $F \in L^2(\Omega, \mathbb{P})$ that depend $(\omega_k)_{k\geq 0}$ only, and $\mathcal{A}_+^{\mathrm{loc}}$ those elements $F \in \mathcal{A}_+$ that are also local. Define the bilinear form B by

$$B(F,G) := \int_\Omega \overline{F(\theta\omega)} F(\omega) \mathrm{d}\mathbb{P}(\omega) \quad (F, G \in \mathcal{A}_+).$$

Show that $B(F,F) \geq 0$ for all $F \in \mathcal{A}_+^{\mathrm{loc}}$.

b) Let $s : \Omega \to \Omega$ be the reflection around $n = 1/2$, i.e., $(s\omega)_k := \omega_{1-k}$. Show that (i) if $\int_\Omega \overline{G \circ s}\, G \mathrm{d}P \geq 0$ for all local $G : \omega \to \mathbb{C}$ that depend on $(\omega_k)_{k\geq 1}$ alone, then necessarily the matrix $(\mu(x)P(x,y))_{x,y\in E}$ is positive semi-definite. (ii) Consider

$$E = \{1,2\}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu = (1/2, 1/2).$$

Does the criterion from b) (i) hold true?

c) Define an equivalence relation on $\mathcal{A}_+$ by $F \sim G$ if and only if $B(F-G, F-G) = 0$. The space of equivalence classes is equipped with the scalar product $\langle [F]_\sim, [G]_\sim \rangle := B(F,G)$. Let $\mathcal{H}$ be the completion of the space of equivalence classes with respect to this norm.

   (i) Show that for every $F \in \mathcal{A}_+^{\mathrm{loc}}$, there is a function $g : E \to \mathbb{C}$ such that $F \sim G$ with $G(\omega) := g(\omega_0)$.

   (ii) Let $L^2(E, \mu_0)$ be the Hilbert space of functions $g : E \to \mathbb{C}$ with scalar product $\langle g, h \rangle = \sum_{x_0 \in E} \overline{g(x_0)} h(x_0) \mu(x_0)$. Show that there is a norm-preserving, linear bijection $U : L^2(E, \mu_0) \to \mathcal{H}$ (thus $\mathcal{H}$ and $L^2(E, \mu_0)$ are isomorphic). You are allowed to use that $\mathcal{A}_+^{\mathrm{loc}}$ is dense in $\mathcal{A}$.

d) Let $\tau : \Omega \to \Omega$ be the shift operator $(\tau\omega)_k := \omega_{k+1}$. For $F \in \mathcal{A}_+^{\mathrm{loc}}$, let $g := U^{-1}[F]_\sim$. Show that

$$[F \circ \tau]_\sim = U(Pg).$$

Remark: The exercise hints at how, in principle, the "time-zero" Hilbert space $L^2(E, \mu_0)$ and the transition matrix P can be recovered from a reflection positive measure on path-space. This was the initial motivation for the notion of reflection positivity (also called *Osterwalder-Schrader positivity*), in the context of constructive quantum field theory; when time is continuous rather than discrete, one looks for a semi-group $(P_t)_{t\geq 0}$ rather than just one matrix P and associates it with a Hamilton operator H by $P_t = \exp(-tH)$ (the full theory is quite involved, however!).