LMU Munich
Summer term 2018

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Exercises on Mathematical Statistical Physics
Math Sheet 1

Problem 1 (Entropy density of a lattice gas). For $\Lambda = \{1, \ldots, L\}^2 \subset \mathbb{Z}^2$ and $\boldsymbol{n} = (n_x)_{x\in\Lambda} \in \{0,1\}^\Lambda$, define

$$H_\Lambda(\boldsymbol{n}) := -J \sum_{\substack{(x,y)\in\Lambda^2 \\ ||x-y||=1}} n_x n_y$$

where $J \in \mathbb{R}$ is some fixed interaction parameter. We are looking for a function $s(u,\rho)$ such that, on a heuristic level,

$$\left|\left\{\boldsymbol{n} \in \{0,1\}^\Lambda \;\middle|\; H_\Lambda(\boldsymbol{n}) \approx u|\Lambda|,\ \sum_{x\in\Lambda} n_x \approx \rho|\Lambda|\right\}\right| \approx \exp\big(|\Lambda| s(u,\rho)\big).$$

This exercise presents some elements of the required mathematical steps towards that goal. For $N \in \mathbb{N}_0$ and $E \in \mathbb{R}$, define

$$\Omega(E,\Lambda,N) := \{\boldsymbol{n} \in \{0,1\}^\Lambda \mid H_\Lambda(\boldsymbol{n}) \leq E,\ \sum_{x\in\Lambda} n_x = N\}$$

and $S(E,\Lambda,N) := \log|\Omega(E,\Lambda,N)|$ with the convention $\log 0 = -\infty$.

(a) Show that for all $\Lambda, \boldsymbol{n}, E, N$, we have $|H_\Lambda(\boldsymbol{n})| \leq 2|J| \sum_{x\in\Lambda} n_x$ and $S(E,\Lambda,N) \in [0, \log 2] \cup \{-\infty\}$.

(b) *Monotonicity*: Show that if $E_1 \leq E_2$, then $S(E_1,\Lambda,N) \leq S(E_2,\Lambda,N)$, and that if $\Lambda_1 \subset \Lambda_2$, then $S(E,\Lambda_1,N) \leq S(E,\Lambda_2,N)$.

(c) *Superadditivity*: Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^2$ be two sets with distance $\operatorname{dist}(\Lambda_1,\Lambda_2) > 1$. Show that for all $E_1, E_2 \in \mathbb{R}$ and $N_1, N_2 \in \mathbb{N}_0$, we have

$$S(E_1+E_2, \Lambda_1 \cup \Lambda_2, N_1+N_2) \geq S(E_1,\Lambda_1,N_1) + S(E_2,\Lambda_2,N_2).$$

(d) Let $\mathcal{Q} := \{(u,\rho) \in \mathbb{R} \times (0,1) \mid \exists k \in \mathbb{N} :\ 4^k\rho \in \mathbb{N}\}$. Consider the sequence of squares $C_k = \{1, \ldots, 2^k - 1\}^2$. Notice $\lim_{k\to\infty} |C_k|/4^k = 1$. Show that for all $(u,\rho) \in \mathcal{Q}$, the limit

$$s(u,\rho) := \lim_{k\to\infty} \frac{1}{4^k} S(4^k u, C_k, 4^k\rho) \in \mathbb{R} \cup \{-\infty\}$$

exists.

(e) Show that s is *mid-point concave*, i.e.,

$$s\left(\tfrac{u_1+u_2}{2}, \tfrac{\rho_1+\rho_2}{2}\right) \geq \tfrac{1}{2}\big(s(u_1, \rho_1) + s(u_2, \rho_2)\big)$$

for all (u_1, ρ_1) and $(u_2, \rho_2) \in \mathcal{Q}$, and monotone increasing in u, i.e., $s(u_1, \rho) \leq s(u_2, \rho)$ for all $(u_1, \rho), (u_2, \rho) \in \mathcal{Q}$ with $u_1 \leq u_2$.

(f) Fix $(u, \rho) \in \mathcal{Q}$. Let $N_k := 4^k \rho_0$ and $E_k := 4^k u$ so that

$$\lim_{k\to\infty} \frac{N_k}{|C_k|} = \rho, \quad \lim_{k\to\infty} \frac{E_k}{|C_k|} = u.$$

Show that if $u' \mapsto s(u', \rho)$ is strictly increasing in $(-\infty, u]$, then for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{|C_k|} \log\left|\left\{\boldsymbol{n} \in \{0,1\}^\Lambda \;\middle|\; \sum_{x\in\Lambda} n_x = N_k,\ E_k - N_k\varepsilon < H_{C_k}(\boldsymbol{n}) \leq E_k\right\}\right| = s(u, \rho).$$

Problem 2 (Laplace integrals in $\mathbb{R}^n$). Let B be a compact domain in $\mathbb{R}^n$, and $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ a C^∞ function on the interior of B and continuous on B. The function ϕ has a minimum at, and only at the point $x_0 \in B$ (an interior point) and the Hessian matrix

$$(H(x_0))_{ij} := \left.\frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}\right|_{x=x_0}$$

is *positive definite*. Show that the following so-called Laplace's integral

$$I(\lambda) := \int_B e^{-\lambda\phi(x)} d^n x \sim \sqrt{\frac{(2\pi/\lambda)^n}{\det(H(x_0))}}\, e^{-\lambda\phi(x_0)}, \quad \lambda \to \infty.$$

Recall: $f(x) \sim g(x),\ x \to \infty \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = \text{const.}$ [Hint: Use Taylor's polynomial approximation to ϕ around point x_0, and try to bound the integral appropriately from both sides.]

The solutions to these exercises will be discussed on Monday, 23.04.