06.07.2018

Proof of Theorem 4.1 (breaking of rotational symmetry):
based on Theorem 4.10
(orientational long-range order,
$\liminf_{L\to\infty} \langle \|m_L\|^2 \rangle_{\mu_L} \geq 1 - \frac{\beta_0}{\beta} > 0 \quad \beta > \beta_0$)

The proof is a little involved, we will not do it fully in class. However the strategy follows a pattern that is relevant in many situations (you can ask yourself if you could do something similar for the Ising model...).

$$m_L(\omega) = \frac{1}{|\mathbb{T}_L|} \sum_j \underbrace{S_j(\omega)}_{=\omega_j}$$

Step 1: Go from info on norm $\|m_L\|^2$ to info on projection $m_L \cdot e_j \quad j = 1, \dots, N$
wlog $j = 1$.

Uses equiv. of norms & prob. estimates:

(1a) $\underbrace{\|m_L\|^2}_{\in \mathbb{R}^N} = \sum_{k=1}^{N} |m_L \cdot e_k|^2 \leq N \max_{k=1,\dots,N} |m_L \cdot e_k|^2$

Therefore if $\|m_L\|^2 \geq \delta\sqrt{N}$, then there exists $k \in \{1, \dots, N\}$ s.t. $|m_L \cdot e_k| \geq \delta$.

So

$$\mu_L(\{\omega : \|m_L(\omega)\|^2 \geq \delta\sqrt{N}\}) \leq \sum_{k=1}^{N} \mu_L(\{\omega : |m_L(\omega) \cdot e_k| \geq \delta\})$$

$$\underset{\substack{\uparrow \\ \text{rotational symm. of } \mu_L \to k}}{\leq} N \mu_L(\{\omega : |m_L(\omega) \cdot e_1| \geq \delta\})$$

also $\mu_L(\{\omega : |m_L(\omega)\cdot e_1| \geq \delta\}) = \mu_L(\{\omega : m_L(\omega)\cdot e_1 \geq \delta\})$
$+ \mu_L(\{\omega : m_L(\omega)\cdot e_1 \leq -\delta\})$

$= 2\mu_L(\{\omega : m_L(\omega)\cdot e_1 \geq \delta\})$
again by symm. of μ_L

so $\mu_L(\{\omega : m_L(\omega)\cdot e_1 \geq \delta\})$ $\geq \frac{1}{2N}\mu_L(\{\omega : \|m_L\| \geq \delta\sqrt{N}\})$.

(1b) However Theorem 4.10 only tells us something about $\langle \|m_L\|^2 \rangle_{\mu_L}$, so an additional step is needed to go from info on $\langle \|m_L\|^2\rangle_{\mu_L}$ to $\mu_L(\{\omega : \|m_L\| \geq \delta\sqrt{N}\})$.

It seems plausible enough that if the average of $\|m_L\|^2$ is bounded away from zero, then $\|m_L\|$ cannot be too small with high probability.

Intuition implemented as follows:

- given $\varepsilon > 0$, estimate

$$\langle \|m_L\|^2 \rangle_{\mu_L} = \langle \underbrace{\|m_L\|^2 \mathbb{1}_{\{\|m_L\| \leq \varepsilon\}}}_{\leq \varepsilon^2} \rangle_{\mu_L} + \langle \underbrace{\|m_L\|^2}_{\leq 1} \mathbb{1}_{\{\|m_L\| \geq \varepsilon\}} \rangle_{\mu_L}$$

$$\leq \varepsilon^2 + \mu_L(\{\omega : \|m_L(\omega)\| \geq \varepsilon\})$$

$$\Rightarrow \mu_L(\{\omega : \|m_L(\omega)\| \geq \varepsilon\}) \geq \langle \|m_L\|^2 \rangle_{\mu_L} - \varepsilon^2.$$

- From Theorem 4.10, we know that

$$\liminf_{L\to\infty} \langle \|m_L\|^2 \rangle_{\mu_L} \cancel{> 0} \geq 1 - \frac{\beta_0}{\beta} > 0.$$

Choose $\varepsilon^2 := \frac{1}{2}\left(1 - \frac{\beta_0}{\beta}\right) > 0$, then previous estimate yields

$$\liminf_{L\to\infty} \mu_L(\{\omega : \|m_L(\omega)\| \geq \varepsilon\}) \geq \frac{1}{2}\left(1 - \frac{\beta_0}{\beta}\right) > 0.$$

and then going back to (1a), choosing $\delta = \frac{\varepsilon}{\sqrt{N}}$

$$\liminf_{L\to\infty} \mu_L(\{\omega : \cancel{\|m_L(\omega)\|}\, m_L(\omega)\cdot e_1 \geq \delta\}) \geq \underbrace{\frac{1}{2N}\,\frac{1}{2}\left(1 - \frac{\beta_0}{\beta}\right)}_{> 0}$$

(1) $$\boxed{\liminf_{L\to\infty} \mu_L(\{\omega : m_L(\omega)\cdot e_1 \geq \delta\}) > 0}$$, $\delta > 0$ suitabley chosen (depends on $N, \beta \ldots$)

(End of step1)

Note: analogous property of Ising model would be: take Gibbs measure with periodic b.c./on torus, zero external field, temperature below Curie temp. Then the claim would be: the probablity that the average magnetization is $\geq \delta$, for some suitabley chosen $\delta > 0$, stays bounded away from zero as $L \to \infty$.

You should convince yourself that there is no contradiction with spin flip symmetry - ~~the me~~. perhaps $\mu_L = \frac{1}{2}(\mu_L^+ + \mu_L^-)$ with μ_L^+ having positive average magn. , μ_L^- negative ...

In a similar vein, there is also no contradiction between the boxed eqn. above and the rotational symmetry of μ_L for the classical Heisenberg model!

Next, we need to figure out how to go from Eq. (1) to existence of infinite-volume Gibbs measures with certain properties.

Idea Again the analogy with Ising (at $h=0$) may help us: if, for the Ising model, we know that the average magnetization (wrt a Gibbs meas. on torus or on cube with some b.c., or free bc) is $\geq \delta$ with a probability bounded bounded away from zero as $L \to \infty$

then the relation between magnetization and pressure $\left(\langle m_L \rangle \propto \frac{1}{L} \frac{\partial}{\partial h} \log Z_L(\beta, h) \right)$ together with spin flip symmetry suggests that there should be phase transition in the sense of non-analyticity of thermodynamic potentials, and the non-analyticity should be a jump discontinuity of $\frac{\partial p}{\partial h}$ at $h = 0$.

Such phase transitions are actually special because there is a general theorem that links jump discontinuity of $\frac{\partial p}{\partial h}$ and non-uniqueness of Gibbs measure:

<u>Theorem</u> (Friedli-Velenik Prop. 6.91):

Ising model, h external field, $\beta p(\beta, h) := \lim \frac{1}{|\Lambda|} \log Z_\Lambda(\beta, h)$

(a) $\forall \mu \in \mathcal{G}_\theta(\beta, 0)$: $\mathcal{G}_\theta(\beta, h)$: translationally inv. Gibbs measures

$$\frac{\partial p}{\partial h}(\beta, 0-) \le \langle \cancel{\mathbb{1}}\ \sigma_0 \rangle_\mu \le \frac{\partial p}{\partial h}(\beta, 0+)$$

(b) $\exists \mu^\pm \in \mathcal{G}_\theta(\beta, 0)$ s.t:

$$\frac{\partial p}{\partial h}(\beta, 0+) = \langle \sigma_0 \rangle_{\mu^+}, \quad \frac{\partial p}{\partial h}(\beta, 0-) = \langle \sigma_0 \rangle_{\mu^-}$$

In particular, if $\frac{\partial p}{\partial h}(\beta, \cdot)$ has a jump discontinuity at $h = 0$, then μ^+ and μ^- from (b) are different: jump discontinuity of $\frac{\partial p}{\partial h}$ implies non-uniqueness of Gibbs meas.!

Called <u>first-order phase transition</u>

See Friedli-Velenik Prop. 6.91 for generalization to other models, and proof.

> ⚠ Remember long-term:
>
> - 2 definitions of phase-transitions in math: (at least...)
>   - non-uniqueness of infinite vol. Gibbs meas.
>   - non-analyticity of thermodynamic potentials
> - in general, these 2 defns are not equivalent
> - however if the non-analyticity of the thermodynamic potential is in fact a jump discontinuity, then there is a general theorem that links the two definitions of phase transition.

~ STRATEGY for the classical Heisenberg:

<u>Step ~~1~~ 2</u>: define an analogue of the pressure at some external field

<u>Step ~~1~~ 3</u>: show that Eq. (1) implies non-differentiability of the pressure at $h=0$

<u>Step ~~1~~ 4</u>: show with the help of ~~II~~ FV-Prop. 6.91 that there exists an infinite-vol. Gibbs meas μ s.t.

$$\mu(\{\omega : m_L(\omega)\cdot e_1 \geq \delta\}) > 0.$$

<u>Step ~~0~~ 5</u>: Conclude.

Step 2: define for $h \in \mathbb{R}$ (!)

Note: $\psi(0) = 0$!

$$\psi(h) := \lim_{L\to\infty} \frac{1}{|\mathbb{T}_L|} \log \left\langle \exp\left(h \sum_{j\in\mathbb{T}_L} S_j \cdot e_1\right)\right\rangle_{\mu_L}$$

suppress β-dependence to lighten notation.

Existence of the limit: similar to existence of the thermodynamic limit of the pressure of the Ising model.

Step 3: note
$$\sum_{j\in\mathbb{T}_L} S_j \cdot e_1 = |\mathbb{T}_L| \left(\frac{1}{|\mathbb{T}_L|}\sum_{j\in\mathbb{T}_L} S_j\right)\cdot e_1 = |\mathbb{T}_L|\, m_L \cdot e_1,$$

so
$$\psi(h) = \lim_{L\to\infty}\frac{1}{|\mathbb{T}_L|}\log\langle \exp(h|\mathbb{T}_L|\, m_L\cdot e_1)\rangle_{\mu_L}.$$

For $\underline{h \geq 0}$ and $\delta > 0$ from Eq. (1):

$$\langle e^{h|\mathbb{T}_L| m_L \cdot e_1}\rangle_{\mu_L} \geq \langle e^{h|\mathbb{T}_L| m_L\cdot e_1} \mathbb{1}_{\{m_L^{(\omega)}\cdot e_1 \geq \delta\}}\rangle_{\mu_L}$$
$$\geq e^{h\delta|\mathbb{T}_L|}\mu_L(\{\omega : m_L(\omega)\cdot e_1 \geq \delta\})$$
$$\geq e^{h\delta|\mathbb{T}_L|}\varepsilon_\delta, \quad \varepsilon_\delta > 0 \text{ s.t. } \liminf_{L\to\infty} \mu_L(\cdots) \geq \varepsilon_\delta > 0!$$

Therefore
$$\frac{1}{|\mathbb{T}_L|}\log\langle e^{h|\mathbb{T}_L| m_L\cdot e_1}\rangle_{\mu_L} \geq \underbrace{\frac{1}{|\mathbb{T}_L|}\log\varepsilon_\delta}_{\to 0 \text{ as } L\to\infty!} + h\delta$$

Take limit $L\to\infty$

$$\leadsto \boxed{\psi(h) \geq h\delta. \quad \text{for all } h \geq 0}$$

Consequence: $$\psi'(0+) = \lim_{h\searrow 0}\frac{\psi(h) - \cancel{\psi(0)}^{=0}}{h} \geq \delta > 0.$$

Because of the symmetries of the underlying meas μ_L, have $\psi(h) = \psi(-h)$, deduce

$$\psi'(0-) = \lim_{h \searrow 0} \frac{\cancel{\psi(0)} - \psi(-h)}{h} = \lim_{h \searrow 0} \frac{-\psi(h)}{h} \leq -\delta < 0$$

$\rightsquigarrow$ Have $\psi'(0-) \leq -\delta < \delta \leq \psi'(0+)$

$$\psi'(0-) < \psi'(0+)$$

$\rightsquigarrow$ jump discontinuity of $\psi'(h)$ at $h=0$.

<u>Step 4</u> FV Prop. 6.91 translated to the present context says:

(a) For all $\mu \in \mathcal{G}_0(\beta)$ $\Leftarrow$ translationally invariant Gibbs meas (at $h=0$)

$$\psi'(0-) \leq \langle S_0 \cdot e_1 \rangle_\mu \leq \psi'(0+)$$

(b) We can always find $\mu_\pm \in \mathcal{G}_0(\beta)$ s.t.

$$\psi'(0+) = \langle S_0 \cdot e_1 \rangle_{\mu^+}, \quad \langle \psi'(0-) \rangle = \langle S_0 \cdot e_1 \rangle_{\mu^-}$$

In particular, from (b) and $\psi'(0+) \geq \delta > 0$:

(2) $\boxed{\exists \mu \in \mathcal{G}(\beta) \text{ s.t.: } \langle S_0 \cdot e_1 \rangle_\mu \geq \delta > 0}$

<u>Step 5</u> - Conclusion.

Let μ be as in (2). We would like to have

$\langle S_0 \rangle_\mu = m^* e_1$ with $m^* \geq \delta$. $\langle S_0 \rangle_\mu \cdot e_1$

But so far we only know that $m^* := \langle S_0 \cdot e_1 \rangle_\mu \geq \delta > 0$

Missing: other components of $\langle S_0 \rangle_\mu$ vanish i.e. $j = 2, \dots, N$:

$$\langle S_0 \rangle_\mu \cdot e_j = \langle S_0 \cdot e_j \rangle_\mu = 0$$

Case 1: the measure μ from (2) does the job – then we're done.

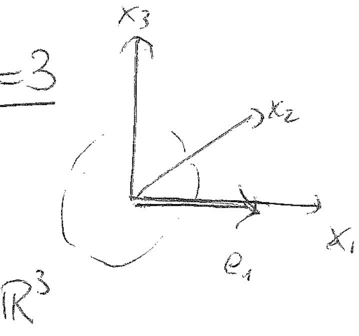
Case 2: $\langle S_0 \cdot e_j \rangle_\mu \neq 0$ for some $j = 2, \dots, N$:

define a new measure ν by averaging μ over all rotations that leave e_1 fixed.

N=2 $\quad \nu := \frac{1}{2}\left(\mu + r_{\vec{e}_1,\pi}\, \mu\right)$

$$r_{\vec{e}_1,\pi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ "rotation" of angle π around axis 1.

N=3



$$\nu = \frac{1}{2\pi} \int_0^{2\pi} r_{\vec{e}_1,\theta}\, \mu \, d\theta$$

$$r_{\vec{e}_1,\theta} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Check that 1) $\langle S_0 \cdot e_j \rangle_\nu = 0 \quad j = 2, \dots, N$

2) $\langle S_0 \cdot e_1 \rangle_\nu = \langle S_0 \cdot e_1 \rangle_\mu = m^* e_1$

$\rightarrow \langle S_0 \rangle_\nu = m^* e_1, \quad m^* > 0$

3) Convexity of Gibbs meas. & compactness wrt suitable topology $\leadsto$ ν is also an infinite-volume Gibbs measure.

$\leadsto$ have found $\nu \in \mathcal{G}(\beta)$ s.t. $\langle S_0 \rangle_\nu = m^* e_1$, $m^* > 0$.

Works just the same for other unit vectors $\vec{e} \in \mathbb{S}^{N-1}$. $\square$