Definition $(\mathfrak{A}, \|\cdot\|, *)$ is called a **C*-algebra** if $\mathfrak{A}$ is a complex algebra, $\|\cdot\| : \mathfrak{A} \to \mathbb{R}$ is a norm, $\mathfrak{A}$ is complete in the $\|\cdot\|$-topology (which makes it a Banach algebra) and $* : \mathfrak{A} \to \mathfrak{A}$ is an involution with $\forall\, a, b \in \mathfrak{A}$, $\lambda \in \mathbb{C}$

$$a^{**} = a$$
$$(a+b)^* = a^* + b^*$$
$$(ab)^* = b^* a^*$$
$$(\lambda a)^* = \bar{\lambda} a^*$$
$$\|a^* a\| = \|a^*\| \|a\| = \|a\|^2 \quad (C^*\text{-property})$$

NB: We did not assume commutativity or the existence of a $1 \in \mathfrak{A}$.

The latter can be adjoined (see homework) and thus in many practical cases we will silently use never the less.

Examples:

1) The main example comes from a Hilbert space $\mathcal{H}$ where $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ is the algebra of bounded operators (w.r.t. the operator norm

$$\|a\| := \sup_{\|\psi\|=1} \|a\psi\| \qquad)$$

where $*$ is the adjoint.

2) Any closed subalgebra of $\mathcal{B}(\mathcal{H})$ is also a C^*-algebra (and it can be shown that every C^*-algebra arises this way although the corresponding $\mathcal{H}$ tends to be huge and is constructed using the Hahn-Banach-Thm)

3) Restricted to finite dimensions $\mathcal{H} = \mathbb{C}^n$

$\mathfrak{A} = \mathrm{Mat}(n \times n, \mathbb{C})$

All finite dimensional C^*-algebras can be decomposed into a direct sum of such full matrix algebras.

4) Let X be a locally compact Hausdorff space. Then the continuous functions vanishing at infinity

$$\mathcal{C}_0(X) := \{ f : X \to \mathbb{C} \mid f \text{ continuous}, \forall \varepsilon > 0 \ \exists K \subset X \text{ compact} : |f|_{X \setminus K} < \varepsilon \}$$

is a commutative C^*-algebra upon pointwise addition, multiplication and complex conjugation. The norm is given by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

(finite thanks to vanishing at ∞)

Every commutative C^*-algebra arises this way, i.e. for a commutative $\mathfrak{A}$ there is a locally cpt Hausdorff space $X(\mathfrak{A})$ s.th.

$$\mathcal{C}_0(X(\mathfrak{A})) \cong \mathfrak{A}$$

The following examples will be studied in this course

<u>Quantum spin chain:</u> For each point $x \in \mathbb{Z}^d$, assign a finite dimensional Hilbert space $\mathcal{H}_x \cong \mathbb{C}^{n_x}$ and define for $\Lambda \subset\subset \mathbb{Z}^d$

$$\mathfrak{A}_\Lambda := \mathcal{B}\Big(\bigotimes_{x \in \Lambda} \mathcal{H}_x\Big)$$

For $\Delta \subset \Lambda \subset\subset \mathbb{Z}^d$ we can imbed

$$\mathfrak{A}_\Delta \hookrightarrow \mathfrak{A}_\Lambda$$

by extending an operator by the identity on $\mathcal{H}_x$ for $x \in \Lambda \setminus \Delta$.

Furthermore for $\Delta_1 \cap \Delta_2 = \emptyset$ we have $\mathfrak{A}_{\Delta_1} \perp \mathfrak{A}_{\Delta_2}$ (as subalgebras for each $\mathfrak{A}_\Lambda$ where $\Delta_1 \cup \Delta_2 \subset \Lambda \subset\subset \mathbb{Z}^d$).

We can thus form the algebra of all local operators

$$\mathfrak{A}_{loc} := \bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathfrak{A}_\Lambda$$

This contains all operators that act non-trivially on finitely many $\mathcal{H}_x$. This is, however, not closed. So we have to go to the "quasi-local operators"

$$\mathfrak{A} := \overline{\mathfrak{A}_{loc}}^{\|\cdot\|}$$

to obtain a C*-algebra. Those contain the operators that can be approximated by local operators and thus have to approach (fast enough) the identity when one goes to infinity in $\mathbb{Z}^d$.

For example the Ising-model falls in this category.

Canonical Anti-Commutation relations:

Theorem and definition: Let h be a pre-Hilbert space (The "one particle HS"). Then there is a unique C*-algebra $CAR(h)$ generated by id and $a(f)$ for $f \in h$ satisfying

1) $f \mapsto a(f)$ is anti-linear
(i.e. $a(\lambda f) = \bar{\lambda}\, a(f)$)

2) $\{a(f), a(g)\} = 0 \quad \forall f, g \in h$

3) $\{a(f), a(g)^*\} = \langle f, g \rangle \, id$

Remarks:

- $a^*(f)$ is linear
- $\|a(f)\| = \|f\|$

(proof: $\|a^*(f)\, a(f)\|^2 =$

$\|a^*(f)\, a(f)\, a^*(f)\, a(f)\| =$

$\|-a^*(f)\, a^*(f) \underbrace{a(f)\, a(f)}_{=0} + a^*(f) \{a(f), a^*(f)\} a(f)\|$

$= \|f\|^2 \, \|a^*(f)\, a(f)\|$)

- $\dim h = n < \infty \Rightarrow CAR(h) \cong Mat(2^n \times 2^n, \mathbb{C})$
- For $U: h \to h$ bounded linear
  $V: h \to h$ bounded anti-linear

obeying

$$V^* U + U^* V = 0 = U V^* + V^* U$$
$$U^* U + V^* V = id = U U^* + V V^*$$

there exists a unique $\gamma \in \mathrm{Aut}\, CAR(h)$
with $\gamma(a(f)) = a(uf) + a^*(v(f))$

and $\gamma^{-1}(a(f)) = a(u^* f) + a^*(v^* f)$

This is called a Bogoliubov-transformation.
For $V \neq 0$, it mixes creation and annihilation operators.

- Instead of $a^*(f)$ and $a(f)$ we can also use
  "field operators" $B(f) := \frac{1}{\sqrt{2}}(a(f) + a^*(f))$
  (inverse: $a(f) = \frac{1}{\sqrt{2}}(B(f) + i B(if))$)

Then $\{B(f), B(g)\} = \mathrm{Re}(\langle f, g \rangle)$

This suggest, we could have started from a real Hilbert space h with a real positive symmetric bi-linear form $s : h \times h \to \mathbb{R}$
$\{B(f), B(g)\} = s(f,g)$
If $J : h \to h$ is a complex structure, i.e. $J^2 = -\mathrm{id}$ with $s(Jf, g) = -s(f, Jg)$

we could then obtain

$$a_J(f) := \frac{1}{\sqrt{2}} \left(B(f) + i\, B(Jf) \right)$$

such that

$$\{ a_J(f), a_J^*(g) \} = s(f,g) + i\, s(f, Jg)$$

The Bogoliubov transformation is then obtained from an isometry $T: h \to h$ with $s(Tf, Tg) = s(f,g)$ and $\gamma(B(f)) = B(Tf)$.

This then gives $U = \frac{1}{2}(T - JTJ)$, $V = \frac{1}{2}(T + JTJ)$

Canonical commutation relations (Bosons):

Def Let h be a symplectic space (i.e. real vector space with anti-symmetric bilinear form $\sigma : h \times h \to \mathbb{R}$ with

$\forall g \in h: \sigma(f,g) = 0 \implies f = 0$.

Then, there is a unique C^*-algebra $CCR(h)$ generated by $W(f)$ for $f \in h$

with

i) $W(f)^* = W(-f)$

ii) $W(f)\,W(g) = e^{i/2\,\sigma(f,g)}\,W(f+g)$

remarks: • If h is actually complex Hilbert space one can chose $\sigma(f,g) = \operatorname{Im}\langle f,g\rangle$

OTOH, in the real case, one can complexify using again a complex structure $J: h \to h$, $J^2 = -id$, $\sigma(Jf,g) = \sigma(Jg,f)$

$$\langle f,g\rangle := \sigma(f,Jg) + i\,\sigma(f,g)$$

- $W(0) = id$, $W(f)$ is unitary
- $\| W(f) - id \| = 2$ for $f \neq 0$ (homework)
- $CCR(h)$ is non-separable for $h \neq \{0\}$.
- Bogoliubov transformations are induced by symplectomorphisms $T: h \to h$ with $\sigma(Tf, Tg) = \sigma(f,g)$

$$\gamma(W(f)) = W(Tf)$$