3. Models With Continuous Symmetries

3.1 (A version of) The Mermin-Wagner theorem.

Setting & Statement of the theorem

3.2 Spin wave. Energy cost of a spin flip

ω^{SW} in $N=2$, $d=2$.

enters later proof ideas,

3.3 Comparing measures: relative entropy

absolute continuity, Radon-Nikodym derivative

relative entropy, total variation distance, Pinsker's inequality

Application: small change in energy $\to$ small change in Gibbs meas.

3.4 Proof of Theorem 3.1.

for $N=2$. Via additional Prop. on finite-vol. meas.

3.5 Corollaries

decay of correlations / absence of long-range order

absence of spontaneous magnetization.

3. Models With Continuous Symmetries

(→ Friedli-Velenik Chapter 9)

<u>Remember 2D Ising</u>: had found 2 distinct inf. vol. Gibbs measures, constructed with + or − b.c.
Global spin flip $\theta : (\sigma_x)_{x\in\mathbb{Z}^d} \mapsto (-\sigma_x)_{x\in\mathbb{Z}^d}$
transforms one meas. P into the other
↪ measures not invariant under spin flip $P\circ\theta^{-1} \neq P$
↪ <u>discrete spin flip symmetry is broken</u>

<u>This chapter</u>: situation very different for <u>continuous symmetries</u> – in dimension 1 and 2, continuous symm. typically not broken: <u>Mermin-Wagner theorem</u>

⚠ Thm needs assumptions – there are systems 1D for which a continuous symmetry is broken! (e.g. 1D jellium, maybe in exercise)

3.1 (A version of the) Mermin-Wagner theorem

<u>Setting</u>: $N \in \mathbb{N}$, single-spin space

$$\Omega_0 = \mathbb{S}^{N-1} = \{ \nu \in \mathbb{R}^N : \underbrace{\|\nu\|}_{(\sum \nu_i^2)^{1/2}} = 1 \}$$

e.g. $N=1$ $\quad \mathbb{S}^0 = \{+1, -1\}$

$N=2$ $\quad \mathbb{S}^1 = \{ (\cos\theta, \sin\theta) : \theta \in \mathbb{R} \}$.

Focus on $N=2$, results extend to $N \geq 3$.

⚠ do not confuse: dim. for single spin and di dim. of underlying lattice

- Config space for infinite system:

  $$\Omega = \Omega_0^{\mathbb{Z}^d}$$

  σ-algebra: product of Borel-σ-algebra

  i.e. smallest σ-alg. that contains all sets of the form $\bigtimes_{x\in\mathbb{Z}^d} O_x$ where
  - only finitely many $O_x \neq \mathbb{S}^{N-1}$
  - each $O_x \subset \mathbb{S}^{N-1}$ is "open"

  $(\forall \nu \in O_x \ \exists \varepsilon > 0$ s.t. $\nu' \in \mathbb{S}^{N-1}$, $\|\nu' - \nu\| < \varepsilon \Rightarrow \nu' \in O_x)$

  equivalently: smallest σ-alg. s.t. all functions of the form $\omega = (\omega_x)_{x\in\mathbb{Z}^d} \mapsto f_\Delta((\omega_x)_{x\in\Delta})$,

  $f_\Delta : (\mathbb{S}^{N-1})^\Delta \to \mathbb{R}$ continuous

  are measurable.

- Reference measure on single-spin space:

  Ω_0, Borel-σ-algebra

  μ_0 = unique rotationally invariant meas. with total mass 1

  Eg $N=2$ $\quad \int_{\mathbb{S}^1} f(\omega_0)\, d\mu_0(\omega_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

  In order to avoid clash with letter μ for Gibbs measures, will write $\int_{\mathbb{S}^{N-1}} f(\omega_0)\, d\omega_0$.

- Reference meas. on Ω_0^Λ, Ω : product measure

- Useful notation: $i \in \mathbb{Z}^d$, $S_i : \Omega \to \mathbb{S}^{N-1}$

$$\omega = (\omega_x)_{x \in \mathbb{Z}^d} \longmapsto S_i(\omega) = \omega_i$$

- Energy: will look at energies of the form

invariant w.r.t. global rotations!

$$\mathcal{H}_\Lambda = -\frac{1}{2} \sum_{\substack{i,j \in \Lambda: \\ \|i-j\|=1}} W(S_i \cdot S_j) \qquad \Lambda \Subset \mathbb{Z}^d$$

$W : [-1;1] \to \mathbb{R}$ twice continuously differentiable

Ex $W(S_i \cdot S_j) = -S_i \cdot S_j$ — $N=2$: XY model; $N=3$: classical Heisenberg model.

- Energy with boundary conditions:

$$\mathcal{H}_{\Lambda|\tau} = \mathcal{H}_\Lambda + \sum_{\substack{i \in \Lambda, j \in \Lambda^c \\ \|i-j\|=1}} W(S_i \cdot \tau_j)$$

Gibbs $\mu^{(\beta)}_{\Lambda|\tau} = \mu_{\Lambda|\tau}$

- Formalism of infinite-vol. Gibbs measures, DLR conditions, existence: extends to this set-up.

$\mathcal{G}(\beta)$:= infinite-vol. Gibbs meas. at inverse temp. β.

Think $P \in \mathcal{G}(\beta)$

$$\Leftrightarrow \forall \Delta \subset\subset \mathbb{Z}^d : \quad A_\Delta \subset \Omega_0^\Delta$$

$$P\left(S_\Delta(\omega) \in A_\Delta \mid S_{\Delta^c}(\omega) = \tau_{\Delta^c} \right)$$

$$= \mu_{\Delta|\tau_{\Delta^c}}(A_\Delta)$$

$$= \frac{1}{\text{Norm.}} \int_{\Omega_0^\Delta} \mathbb{1}_A \left((\omega_x)_{x \in \Delta} \right) e^{-\beta \mathcal{H}_{\Lambda|\tau}(\omega)} \prod_{x \in \Omega_0} d\omega_x .$$

Rigorously: $\int_\Omega f \, dP = \int_\Omega \left(\int_{\Omega_0^\Lambda} f(\sigma_\Lambda \tau_{\Lambda^c}) \, d\mu_{\Lambda|\tau}(\sigma) \right) dP(\tau)$ $\forall$ local bounded meas. f.

Theorem 3.1 (Mermin-Wagner theorem)

Assume $N \geq 2$, $W \in C^2([-1;1])$, $d = 1$ or 2.
Then all infinite-volume Gibbs measures are invariant under the action of $SO(N)$:

$$\forall \mu \in \mathcal{G}(\beta), \quad \forall r \in SO(N): \quad r(\mu) = \mu.$$

⚠ $r(\mu) = \mu$ - precisely:
let $r \in SO(N)$ ($N \times N$ orthogonal matrix, determinant 1)

r acts on Ω via: $\omega = (\omega_x)_{x \in \mathbb{Z}^d} \mapsto r\omega := (r\omega_x)_{x \in \mathbb{Z}^d}$

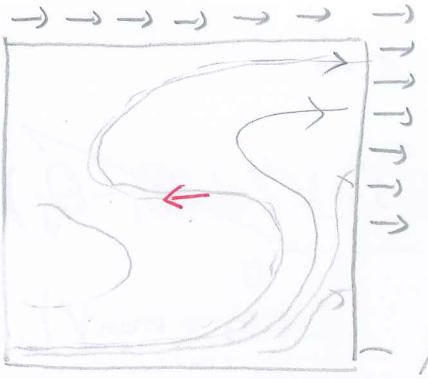
r acts on measures via:

$$(r\mu)(A) := \mu(r^{-1}(A)) = \mu(\{\omega \in \Omega : r\omega \in A\}).$$

3.2 Spin wave. Energy cost of a spin flip

- Ising model, Peierls arg^t: key input was probability of seeing a -1-spin at some fixed point eg. $x = 0$ wrt Gibbs meas with $+$-b.c.
  $\rightsquigarrow$ was small because of energy penalty.

- What about energy penalty for continuous spin?
  Situation changes! Supp $N=2$, $d=2$, $W : [-1,1] \to \mathbb{R}$ decreasing
  $\rightsquigarrow$ minimal at 1.
  $\Lambda_n = \{-n, \dots, n\}^2$ $\qquad S_i \cdot S_j = 1$
  b.c. $\tau_x = (1, 0)$ $\forall x \in \Lambda_n^c$
  $\rightarrow$



← spin wave / $\omega^{SW}_{\Lambda_n}$
Friedli-Velenik
fig. 9.3

= "field lines"
tangential to spin

- ground state / minimizer of $\mathcal{H}_{\Lambda|\tilde{\omega}}$:
  $\omega_x = (1,0) \quad \forall x \in \Lambda_n$ $\quad \to \to \to \to$

  (unique if W is strictly monotone on $[-1;1]$, otherwise just a minimizer).

- Energy cost of spin flip $\omega_0 = (1,0) \longrightarrow \omega_0 = (-1,0)$?

  i.e. $\min\{\mathcal{H}_{\Lambda|\tilde{\omega}}(\omega_\Lambda) : \omega_\Lambda \in \Omega_0^\Lambda,\ \omega_0 = (-1,0)\}$?

For $x \in \Lambda_n$, define

$$\begin{cases} \theta_x^{SW} := \left(1 - \dfrac{\log(1+\|x\|_\infty)}{\log(1+n)}\right)\pi, & \omega_x^{SW} := (\cos\theta_x^{SW}, \sin\theta_x^{SW}) \\ \|x\|_\infty := \max(|x_1|, |x_2|). \end{cases}$$

Then: $\theta_0^{SW} = \left(1 - \dfrac{\log 1}{\log(1+n)}\right)\pi = \pi$, $\omega_0^{SW} = (-1,0)$.

- $\|x\|_\infty = n \Rightarrow \theta_x^{SW} = 0$, $\omega_x^{SW} = (1,0)$

$\Rightarrow \min\{\mathcal{H}_{\Lambda|\tilde{\omega}}(\omega_\Lambda) : \omega_0 = (-1,0)\} \le \mathcal{H}_{\Lambda|\tilde{\omega}}(\omega_\Lambda^{SW})$

- For $S_i = (\cos\theta_i; \sin\theta_i)$:

  write $W(S_i \cdot S_j) = V(\theta_j - \theta_i)$, V 2π-periodic, C^2, $V(0) = \min V$.
  $\qquad\qquad\qquad \| \qquad\qquad\qquad \|$
  $\qquad W(\cos(\theta_i - \theta_j))$

$$C := \sup_{\theta \in [-\pi,\pi]} V''(\theta) < \infty$$

$$W(S_i \cdot S_j) = V(\theta_i - \theta_j) \leq V(0) + \underbrace{\cancel{V'(0)(\theta_i - \theta_j)}}_{0} + \frac{1}{2} C (\theta_i - \theta_j)^2$$

$(V(0) = \min V)$.

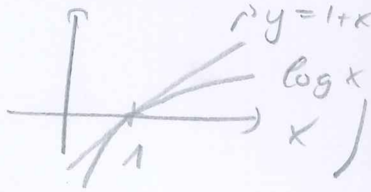
- $i, j \in \Lambda_n \subset \mathbb{Z}^2$ nearest neighbors

$\hookrightarrow$ w.l.o.g. $\|i\|_\infty = \|j\|_\infty - 1$

$$\theta_i^{SW} - \theta_j^{SW} = \frac{\log(1 + \|j\|_\infty) - \log(1 + \|i\|_\infty)}{\log n} \underset{\pi}{\sim}$$

$$= \frac{\log\left(1 + \frac{1}{\|j\|_\infty}\right)}{\log n} \underset{\pi}{\sim}$$

$$\leq \frac{\pi}{\log(1+n)} \frac{1}{\|j\|_\infty} \qquad \left(\log(1+u) \leq u \right)$$



Altogether:

$$0 \leq \mathcal{H}_{\Lambda|\tau}(\omega^{SW}) - \min \mathcal{H}_{\Lambda|\tau}$$

$$\leq \frac{1}{2} C \frac{\pi^2}{(\log(1+n))^2} \sum_{r=1}^{n+1} 4(2r-1) \frac{1}{r^2}$$

r: possible values of $\|j\|_\infty$

$4 \cdot (2r-1) \geq$ no. of relevant (i,j)-bonds

$$\leq \frac{8 C \pi^2}{\log(1+n)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

$\Rightarrow$ $\boxed{\text{The energy cost of a spin flip goes to zero as the size of the box goes to infinity!}}$

Proof idea for Mermin-Wagner: would like to exploit that spins in finite region can be rotated at little energy cost.

Wanted: translate somehow "small energy change" into "small change in probability"

tool: $\rightarrow$ Relative entropy.

3.3 Comparing measures: relative entropy.

- $p_1, \dots, p_n \geq 0$, $q_1, \dots, q_n \geq 0$, $\sum_{i=1}^n p_i = 1 = \sum_{i=1}^n q_i$

p, q prob. meas. on $\{1, \dots, n\}$:

Relative entropy defined as

Convention: $0 \log 0 := 0$

$$h(p|q) := \begin{cases} \sum_{i=1}^n p_i \log \frac{p_i}{q_i} & \text{if } q_i = 0 \Rightarrow p_i = 0 \\ \infty & \text{if for some } i: q_i = 0 \text{ but } p_i > 0 \end{cases}$$

$= \sum_{i=1}^n q_i \frac{p_i}{q_i} \log \frac{p_i}{q_i}$

(Note: for $q_i = 1/n$ uniform distribution,

get $h(p|q) = \sum_{i=1}^n p_i \log n + \sum_{i=1}^n p_i \log p_i$

$= \log n - S(p)$, $S(p) := \sum_{i=1}^n (-p_i \log p_i)$

Shannon entropy

⚠ Shannon entropy comes with a minus, relative entropy does not!)

More generally:

μ, ν 2 prob. meas. on $(\Omega, \mathcal{F})$

- Say μ <u>absolutely</u> <u>continuous</u> wrt ν if:

$$\forall A \in \mathcal{F} \quad \nu(A) \Rightarrow \mu(A).$$, written: $\mu \ll \nu$

- <u>Theorem:</u>

μ ac wrt ν $\Rightarrow \exists \rho: \Omega \to \mathbb{R}_+$ meas. s.t.

$$\forall A \in \mathcal{F}: \mu(A) = \int_A \rho \, d\nu$$

written as: $\mu = \rho \nu$.

- Fct ρ is called <u>Radon-Nikodym derivative</u> of μ wrt ν, written $\rho = \frac{d\mu}{d\nu}$
- ρ is unique up to ν-null sets.

$(\mu = \rho \nu = \tilde{\rho} \nu \Rightarrow \exists N \in \mathcal{F};\ \nu(N) = 0,\ \rho = \tilde{\rho}$ on $N)$

<u>Def. 3.2</u>

Let μ, ν be 2 prob. meas. on some space $(\Omega, \mathcal{F})$
The <u>relative entropy</u> $h(\mu|\nu)$ is defined as

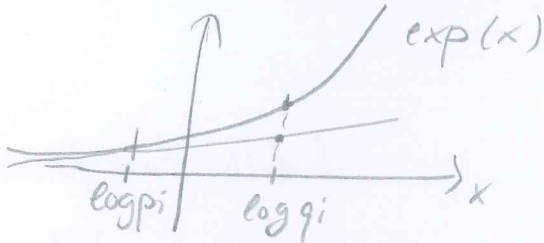
$$h(\mu|\nu) := \begin{cases} \int (\rho \log \rho) \, d\nu = \int_\Omega \left(\log \frac{d\mu}{d\nu}\right) d\mu & \text{if } \mu \ll \nu,\ \rho = \frac{d\mu}{d\nu} \\ \infty & \text{otherwise} \end{cases}$$

Also called: <u>Kullback-Leibler divergence</u>.

<u>Prop. 3.3</u> $h(\mu|\nu) \geq 0$ and "$= 0$" iff $\mu = \nu$.

<u>Proof in the discrete case:</u> p, q $\{1, .., n\}$ (or $\mathbb{N}$):

wlog $q_k > 0$ everywhere (otherwise, make space on which measures live smaller)



from strict convexity of exp:

$$\forall i: \quad e^{\log q_i} \geq e^{\log p_i} + e^{\log p_i}(\log p_i - \log q_i),$$

"$=$" iff $p_i = q_i$

$$q_i \geq p_i + p_i \log\frac{q_i}{p_i}$$

$$q_i + p_i \log\frac{p_i}{q_i} \geq p_i$$

$$\Rightarrow \cancel{\underbrace{\sum_i q_i}_{=1}} + h(p|q) \geq \cancel{\underbrace{\sum p_i}_{=1}}, \quad \text{"$=$" iff } \forall i: p_i = q_i$$

$\square$

general case: Friedli-Velenik Appendix B12

<u>Def. 3.4</u> μ, ν prob meas on some space $(\Omega, \mathcal{F})$

The <u>total variation distance</u> between μ and ν is

$$\|\mu - \nu\|_{TV} := 2 \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|$$

⚠ In probability, often defined without factor 2.

<u>Advantage of including factor:</u>

have $\mu(A^c) - \nu(A^c) = (\cancel{1} - \mu(A)) - (\cancel{1} - \nu(A)) = \nu(A) - \mu(A)$

$$\Rightarrow \|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} \left(|\mu(A) - \nu(A)| + |\mu(A^c) - \nu(A^c)| \right)$$

<u>Ex</u> p, q measures on discrete space, then

$$\|p - q\|_{TV} = \sum_k |p_k - q_k| \quad \ell^1\text{-distance}.$$

<u>Indeed:</u> "$\leq$" $|p(A) - q(A)| + |p(A^c) - q(A^c)|$

$$= \Big|\sum_{k \in A} (p_k - q_k)\Big| + \Big|\sum_{k \in A^c} (p_k - q_k)\Big|$$

$$\leq \sum_k |p_k - q_k|$$

holds true for all $A \Rightarrow \quad \|p-q\|_{TV} \leq \sum_k |p_k - q_k|.$

"$\geq$" $\quad A := \{k : p_k > q_k\}, \quad A^c = \{k : p_k \leq q_k\}$

$$|p(A) - q(A)| + |p(A^c) - q(A^c)|$$

$$= \sum_{k\in A} \underbrace{(p_k - q_k)}_{\geq 0} + \sum_{k\in A^c} \underbrace{(q_k - p_k)}_{\geq 0}$$

$$= \sum_k |p_k - q_k|$$

$\rightsquigarrow \quad \|p-q\|_{TV} \geq \sum_k |p_k - q_k|. \qquad \square$

2 useful facts (not proven in class).

Fact 1: $\quad \|\mu - \nu\|_{TV} = \sup\{|\int f d\mu - \int f d\nu| : \|f\|_\infty \leq 1\}$

(this is nice from a functional-analytic pt of view. goes hand in hand with interpretation measures $\hat{=}$ elements of the dual of space of fcts)

(would not be true without the factor 2 in the def. of $\|.\|_{TV}$) !)

Fact 2 if $\mu \ll \nu$, then

$$\|\mu - \nu\|_{TV} = \int \left|\frac{d\mu}{d\nu} - 1\right| d\nu.$$

Prop. 3.5 (Pinsker's inequality):

$$\|\mu - \nu\|_{TV} \leq \sqrt{2h(\mu|\nu)}$$

trivial if μ is not ac wrt ν. If $\mu \ll \nu$: $\left(\int\left|\frac{d\mu}{d\nu} - 1\right| d\nu\right)^2 \leq 2\int \left(\log\frac{d\mu}{d\nu}\right) d\mu.$

Heuristics: $\rho = \frac{d\mu}{d\nu}$

$$\frac{d}{dx} x \log x \Big|_{x=1} = (\log x + 1)\Big|_{x=1} = 1$$

$$\frac{d^2}{dx^2} x \log x \Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$$

$$\curvearrowright \; x \log x = \cancel{1 \log 1}_{0} + 1(x-1) + \frac{1}{2}(x-1)^2 + \text{remainder}$$

$$h(\mu|\nu) = \int \rho \log \rho \, d\nu = \int (\rho - 1)\, d\nu + \frac{1}{2}\int (\rho-1)^2 d\nu$$
$$+ \text{remainder}$$

now $\int (\rho - 1)\, d\nu = \int d\mu - \int d\nu = 1 - 1 = 0$

and $\left(\int |\rho-1|\, d\nu\right)^2 \leq \left(\int |\rho - 1|^2 d\nu\right) \underbrace{\cancel{\left(\int 1^2 d\nu\right)}}_{=1}$ (Cauchy-Schwarz!)

$$\Rightarrow \; h(\mu|\nu) \geq \frac{1}{2}\Big(\underbrace{\int |\rho - 1|\, d\nu}_{= \|\mu-\nu\|_{TV}^2}\Big)^2 + \text{remainder}$$
$$= \frac{1}{2} \|\mu-\nu\|_{TV}^2 + \text{remainder}.$$

If there was no remainder, we would be done!

<u>Full proof</u>: Friedli-Velenik, p. 512.

$\rightsquigarrow$ <u>Up to now</u>: Relative entropy – total variation distance – Pinsker's inequality.

Application in stat mech? Remember: our goal was: small change in energy $\longleftrightarrow$ small change in Gibbs measure?

In finite spaces: supp. H, $\tilde{H}$ are 2 energy fcts

$$\mu(\sigma) = \frac{1}{Z(\beta)} e^{-\beta H(\sigma)}, \qquad \nu(\sigma) = \tilde{\mu}(\sigma) = \frac{1}{\tilde{Z}(\beta)} e^{-\beta \tilde{H}(\sigma)}$$

associated Gibbs measures.

Then

$$h(\mu|\nu) = \sum_\sigma \mu(\sigma) \log \frac{\mu(\sigma)}{\nu(\sigma)}$$

$$= \sum_\sigma \frac{1}{Z} e^{-\beta H(\sigma)} \log \frac{e^{-\beta H(\sigma)}/Z}{e^{-\beta \tilde{H}(\sigma)}/\tilde{Z}}$$

$$= \sum_\sigma \frac{1}{Z} e^{-\beta H(\sigma)} \left\{ \beta\left(\tilde{H}(\sigma) - H(\sigma)\right) + \log \frac{\tilde{Z}}{Z} \right\}$$

$$= \Big(\underbrace{\sum_\sigma \frac{1}{Z} e^{-\beta H(\sigma)} \left\{ \beta\left(\tilde{H}(\sigma) - H(\sigma)\right)\right\}}_{\le \beta \sup_\sigma |\tilde{H}(\sigma) - H(\sigma)|} \Big) + \underbrace{\log \frac{\tilde{Z}}{Z}}_{\le \dots}$$

$$\le 2\beta \sup_\sigma |\tilde{H}(\sigma) - H(\sigma)|$$

Pinsker

$$\sum_\sigma |\mu(\sigma) - \nu(\sigma)| = \|\mu - \nu\|_{TV} \le \sqrt{4\beta \|\tilde{H} - H\|_\infty}$$

This is exactly the type of bound we were after!
Works just the same for finitely many continuous spins (replace sums over σ with integrals).

3.4 Proof of Theorem 3.1

(For $N=2$ $\Omega_0 = S^1$ ~~, $d \le 2$~~
General $N \ge 3$ $d \le 2$: see Frieddi-Vel.)

$\mu \in \mathcal{G}(\beta)$ inf. vol. Gibbs meas.
for $\varphi \in (-\pi; \pi]$: $r_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in SO(2)$ rotation of angle φ

goal: $r_\varphi \mu = \mu$, equivalently:

XM

$$\int_\Omega f(r_\varphi \omega)\, d\mu(\omega) = \int_\Omega f(\omega)\, d\mu(\omega)$$

for all $\varphi \in (-\pi,\pi]$ and all <u>local</u> bounded meas. f.

Supp. f depends only on spins σ_x with $x \in \Lambda_\ell$, $= \{-\ell, \dots, \ell\}^2$

DLR cond. for $\Lambda_n \supset \Lambda_\ell$: $n > \ell$

$$\int_\Omega f \circ r_\varphi\, d\mu = \int_\Omega \langle f \circ r_\varphi \rangle_{\Lambda|\eta}\, d\mu(\eta)$$

$$\int f\, d\mu = \int \langle f \rangle_{\Lambda|\eta}\, d\mu(\eta)$$

Can choose Λ arbitrarily large!

Theorem follows from:

<u>Prop. 3.7</u> $d = 1$ or 2, $N = 2$, hyp. of Thm 3.1. Then:

$\exists\, c_1, c_2 > 0$ s.t. for all boundary conditions η, all $0 < \beta < \infty$, all $\varphi \in (-\pi,\pi]$, and all $\ell \in \mathbb{N}_0$:

$$\left| \langle f \rangle_{\Lambda_n|\eta} - \langle f \circ r_\varphi \rangle_{\Lambda_n|\eta} \right| \le \beta^{1/2} |\varphi|\, \|f\|_\infty \begin{cases} \dfrac{c_1}{\sqrt{n-\ell}} & d=1 \\ \dfrac{c_2\sqrt{\ell}}{\sqrt{\log(n-\ell)}} & d=2 \end{cases}$$

for all $n > \ell$ and all bounded meas. fcts f that depend on $(\omega_x)_{x\in\Lambda_\ell}$ alone.

<u>Proof of Theorem 3.1 for $N=2$, based on Prop. 3.7:</u> f local $f(\omega_{\Lambda_\ell})$

$$\left| \int f\, d\mu - \int f \circ r_\varphi\, d\mu \right| \underset{\text{DLR \& triangle}}{\le} \int_\Omega \left| \langle f \rangle_{\Lambda_n|\eta} - \langle f \circ r_\varphi \rangle_{\Lambda_n|\eta} \right| d\mu(\eta)$$

$$\le \sup_\eta \left| \langle f \rangle_{\Lambda_n|\eta} - \langle f \circ r_\varphi \rangle_{\Lambda_n|\eta} \right|$$

holds true for all $n \geq \ell_{11}$ $\leadsto$ can let $n \to \infty$

& By Prop 3.7: $\sup_{\eta} |\langle f \circ r_\varphi \rangle_{\Lambda_n|\eta} - \langle f \rangle_{\Lambda_n|\eta}| \xrightarrow[n\to\infty]{} 0$

$\to$ Therefore: $|\int f d\mu - \int f \circ r_\varphi d\mu| \leq 0$

$\int f d\mu = \int f \circ r_\varphi d\mu$

for all local bounded f

General measure-theoretic arg $\leadsto$ $r_\varphi \mu = \mu$. $\square$

(For general N: idea is to write $r \in SO(N)$
as product of rotations acting in 2D subspaces only.
$\leadsto$ Friedli-Velenik Section 9.2.3.)

So it remains to prove Prop. 9.7. Starting point:

$$\langle r_\varphi f \rangle_{\Lambda|\eta} = \langle f \circ r_{-\varphi} \rangle_{\Lambda|\eta}$$
$$= \frac{1}{Z_{\Lambda|\eta}} \int f(r_{-\varphi}\omega) e^{-\beta H_\Lambda(\omega_\Lambda|\eta_{\Lambda^c})} \prod_{i\in\Lambda} d\omega_i$$

<u>Idea</u>: • move the rotation from the variable in f
to variable of H_Λ using change of variables
• do it in such a way that "rotated" energy
is close to original energy
so that we can estimate distances with
relative entropy & Pinsker as in the end
of Section 3.3

Let $\Psi : \mathbb{Z}^d \to (-\pi;\pi]$ with $\Psi_i = \varphi$ for all $i \in \Lambda_e$,
$\Psi_i = 0$ for all $i \in \Lambda^c$.
(Leave choice of Ψ open for now - but think of spin wave from Section 3.2 !)

Let $t_\Psi : \Omega \to \Omega$ be the trafo

$$\left(\begin{pmatrix}\cos\theta_i\\ \sin\theta_i\end{pmatrix}\right)_{i\in\mathbb{Z}^2} \longrightarrow \left(\begin{pmatrix}\cos(\theta_i+\Psi_i)\\ \sin(\theta_i+\Psi_i)\end{pmatrix}\right)_{i\in\mathbb{Z}^d}.$$

Because of $\Psi_i = \varphi$ on Λ_e:

$$\int f((r_{-\varphi}\omega)_{i\in\Lambda_e}) e^{-\beta H_\Lambda(\omega_\Lambda|\eta_{\Lambda^c})} \prod_{i\in\Lambda} d\omega_i$$

$$= \int_{[0;2\pi]^e} f\left(\begin{pmatrix}\cos(\theta_i-\Psi_i)\\ \sin(\theta_i-\Psi_i)\end{pmatrix}_{i\in\Lambda_e}\right) e^{-\beta H_\Lambda\left(\begin{pmatrix}\cos\theta_i\\ \sin\theta_i\end{pmatrix}_{i\in\Lambda}|\eta_{\Lambda^c}\right)} \prod_{i\in\Lambda}\frac{d\theta_i}{2\pi}$$

$\theta_i' = \theta_i - \Psi_i$

$$= \int_{[0;2\pi]^e} f\left(\begin{pmatrix}\cos\theta_i'\\ \sin\theta_i'\end{pmatrix}_{i\in\Lambda_e}\right) e^{-\beta H_\Lambda\left(\begin{pmatrix}\cos(\theta_i'+\Psi_i)\\ \sin(\theta_i'+\Psi_i)\end{pmatrix}_{i\in\Lambda}|\eta_{\Lambda^c}\right)} \prod_{i\in\Lambda}\frac{d\theta_i'}{2\pi}$$

$$= \int_{\Omega_0^\Lambda} f(\omega)\, e^{-\beta H_\Lambda(t_\Psi\omega_\Lambda | \underbrace{t_\Psi\eta_{\Lambda^c}}_{=\eta_{\Lambda^c}})} \prod_{i\in\Lambda} d\omega_i$$

$= \eta_{\Lambda^c}$ because $\Psi_i = 0$ on Λ^c.

Set $H_\Lambda^\Psi(\omega_\Lambda|\eta_{\Lambda^c}) := H_\Lambda(t_\Psi\omega_\Lambda | t_\Psi\eta_{\Lambda^c})$

Have checked

$$\int f(r_\varphi\omega_{\Lambda_e}) e^{-\beta H_\Lambda(\omega_\Lambda|\eta_{\Lambda^c})}\prod_{i\in\Lambda} d\omega_i = \int f(\omega_{\Lambda_e}) e^{-\beta H_\Lambda^\Psi(\omega_\Lambda|\eta_{\Lambda^c})} d\omega_\Lambda.$$

Special case: for $f=1$ (constant fct) see that partition functions are equal

$$Z_{\Lambda|\eta} = Z^{\Psi}_{\Lambda|\eta}.$$

Can write $\langle r_\varphi f\rangle_{\Lambda|\eta} = \langle f\rangle^{\Psi}_{\Lambda|\eta}$ hence

$$|\langle f\rangle_{\Lambda|\eta} - \langle r_\varphi f\rangle_{\Lambda|\eta}| = |\langle f\rangle_{\Lambda|\eta} - \langle f\rangle^{\Psi}_{\Lambda|\eta}|$$
$$\leq \|f\|_\infty \|\mu_{\Lambda|\eta} - \mu^{\Psi}_{\Lambda|\eta}\|_{TV}$$

defined as...

Pinsker:

$$\boxed{|\langle f\rangle_{\Lambda|\eta} - \langle r_\varphi f\rangle_{\Lambda|\eta}| \leq \|f\|_\infty \sqrt{2\, h(\mu_{\Lambda|\eta} | \mu^{\Psi}_{\Lambda;\eta})}}$$

- Evaluation of the relative entropy:

because of $Z_{\Lambda|\eta} = Z^{\Psi}_{\Lambda|\eta}$, have

$$\frac{d\mu_{\Lambda|\eta}}{d\mu^{\Psi}_{\Lambda|\eta}}(\omega_\Lambda) = e^{\beta[H^{\Psi}_{\Lambda|\eta}(\omega_\Lambda) - H_{\Lambda|\eta}(\omega_\Lambda)]}$$

$$h(\ldots|\ldots) = \beta \int_{\Omega_0^\Lambda} \left(H^{\Psi}_{\Lambda|\eta}(\omega_\Lambda) - H_{\Lambda|\eta}(\omega_\Lambda)\right) d\mu_{\Lambda|\eta}(\omega_\Lambda)$$

$$= \beta \left\langle H^{\Psi}_{\Lambda|\eta} - H_{\Lambda|\eta}\right\rangle_{\Lambda|\eta}$$

$V(\theta) = W(\cos\theta)$, $V\, C^2$, but assume no longer that $V'(0) = 0$: $\sup_{\mathbb{T}} |V''| \leq C < \infty$.

$$\beta \langle H^{\Psi}_{\Lambda|\eta} - H_{\Lambda|\eta} \rangle_{\Lambda|\eta}$$

$$= \beta \sum_{\{i,j\}} \langle V(\vartheta_i - \vartheta_j + \Psi_j - \Psi_i) - V(\vartheta_j - \vartheta_i) \rangle_{\Lambda|\eta}$$

sum over nearest-neighbor bonds

$\vartheta_i(\omega) = \dots$

$$\leq \beta \sum_{\{i,j\}} \left\{ \langle V'(\vartheta_i - \vartheta_j) \rangle_{\Lambda|\eta} (\Psi_j - \Psi_i) + \frac{C}{2} (\Psi_j - \Psi_i)^2 \right\}$$

Would like ~~to~~ to get rid of $V'(\vartheta_i - \vartheta_j)$.

<u>trick</u>: relative entropies are ≥ 0

$$\rightsquigarrow h(\mu_{\Lambda|\eta} | \mu^{\Psi}_{\Lambda|\eta}) \leq h(\mu_{\Lambda|\eta} | \mu^{\Psi}_{\Lambda|\eta}) + \underbrace{h(\mu_{\Lambda|\eta} | \mu^{-\Psi}_{\Lambda|\eta})}$$

gives rise to $\langle V'(\vartheta_i - \vartheta_j) \rangle_{\Lambda|\eta} (-\Psi_j + \Psi_i)$

$\rightsquigarrow$ cancels out the term we want to get rid of

Altogether:

$$\boxed{h(\mu_{\Lambda|\eta} | \mu^{\Psi}_{\Lambda|\eta}) \leq C\beta \sum_{\substack{\{i,j\} \\ \text{nearest} \\ \text{neighbors}}} (\Psi_j - \Psi_i)^2}$$

☒ Final step: choice of deformation Ψ:

can modify spin-wave construction from Section 3.2 (exercise! Ex. 9.2 in Fr. Vel.) $\rightsquigarrow$ in 2D: recognize $\frac{1}{\sqrt{\log(n-\ell)}}$

[alternative method: relation with random walkers $\hookrightarrow$ Fr. Vel.]

exercise: fill the dots.

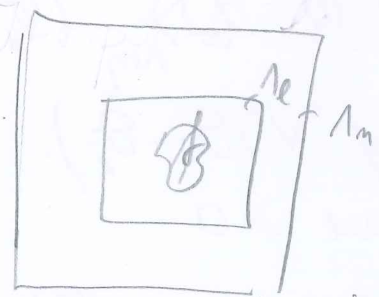
$\downarrow$

...

□

Main proof ideas Mermin-Wagner / summary:

1° Use DLR condition to transform question on infinite-vol. Gibbs meas. into question on finite-vol. Gibbs meas. with b.c.

2° 

in order to compare $\langle r_\varphi f \rangle_{\Lambda_n|\eta}$ and $\langle f \rangle_{\Lambda_n|\eta}$, $\varphi \in (-\pi;\pi]$:

(a) interpolate by angles Ψ_i

$$\Psi_i\big|_{i\in\Lambda_\ell} = \varphi, \quad \Psi_i\big|_{i\in\Lambda_\ell^c} = 0$$

(think spin wave),

(b) check $\langle r_\varphi f \rangle_{\Lambda_n|\eta} = \langle f \rangle^{\Psi}_{\Lambda_n|\eta}$

Gibbs for "rotated" energy.

3° Estimate $\langle f \rangle^{\Psi}_{\Lambda_n|\eta} - \langle f \rangle_{\Lambda_n|\eta}$

with Pinsker's inequality & Taylor expansion of $V(\theta_i - \theta_j + \Psi_j - \Psi_i)$,

use trick to make $V'(\theta_i - \theta_j)$ disappear

4° Make sure resulting bound is small: judicious choice of $(\Psi_i)_{i\in\mathbb{Z}^d}$.