CONNECTIONS WITH BOHMIAN MECHANICS

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ABSTRACT OF THE DISSERTATION

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We begin by formulating non-relativistic Bohmian mechanics for theories in which the wave functions are sections of a Hermitian vector bundle over a Riemannian manifold. We give geometric interpretations of the Pauli equation and the Dirac equation. We then explain the influence of non-trivial topology in formulating Bohmian theories and use that to explain the quantum story of identical particles. We take a break from Bohmian mechanics briefly to investigate when the covariant derivatives of a mapping between manifolds may be used in a Taylor polynomial fashion in order to find the best polynomial approximation to the map. We finish by first formulating and then giving some results about a new algorithm for computing solutions to Schrödinger's equation.

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Dedication

I dedicate this to Tilla Weinstein.

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Chapter 1

A quick orientation

1.1 A brief word about Bohmian Mechanics

Briefly, Bohmian mechanics is a quantum theory which considers the state of an N-particle system to be the wave function and the locations of the N particles. The dynamics is specified by Schrödinger's equation and a first-order equation for the evolution of the positions. Simply put, the particles are guided by the wave which is why wave behavior is seen. The particle behavior occurs because there are particles. Although not an obvious fact, it is true that the entire quantum measurement formalism can be derived from an analysis of Bohmian mechanics [25, 27]. One of the advantages of Bohmian mechanics is that it is easy to understand the starting point. There are no axioms about measurements. The whole theory is the dynamical system mentioned above. Although Bohmian mechanics is different from classical mechanics as its laws of motion are different, it is not a philosophically radical departure from the starting point of mechanics. That is to say, it is still based on a particle ontology.

1.2 Organization of thesis

Chapter 2 gives a rough sketch of the Bohmian story. It describes the simplest versions of Bohmian mechanics and states a few important results. It finishes by relating the historical story of how Bohmian mechanics has been viewed by and has influenced standard quantum theory as well as describing some of the more recent results involving quantum field theory and relativistic Bohmian mechanics.

Chapter 3 contains generalizations of non-relativistic Bohmian mechanics. We formulate Bohmian dynamics for wave functions as sections of a complex Hermitian vector bundle over a Riemannian manifold. This is rather straightforward. From this perspective, we then explain how the Pauli equation can be roughly viewed as using a generalization of the Laplacian. We finish by explaining how the Dirac equation and the gamma matrices arise directly from a Bohmian point of view.

Chapter 4 explains the influence of topology on theory formation from a Bohmian perspective. Our primary example in that chapter is the Aharonov-Bohm effect. We explain from three different perspectives how the famous phase arises as "one goes around a non-contractible loop". Quite generally, to specify a Bohmian theory on a multiply connected space, one needs to choose a character of the fundamental group of the configuration space. We shall explain how a character arises in three distinct, but ultimately equivalent, ways. Having to choose a character is what we call the Abelian Quantization Principle. We conclude the chapter by quantizing the classical system of a particle moving along a circle immersed in a constant electric field.

Chapter 5 discusses Bohmian mechanics for many particles. Most of the chapter describes the setup for identical particles. The setup starts with realizing that the multiply connected space of N-element subsets of physical space is the appropriate configuration space for identical particles. As the fundamental group has two characters, we have the choice of bosons or fermions. Our discussion also covers the case of identical particles with spin. In this case, the wave function is a section of a vector bundle over the configuration space; the fiber is the tensor product of the 1-particle spin space with itself N times using the physical positions as the index set for the tensoring. We apply the Abelian Quantization Principle to conclude the usual Bose-Fermi alternative. The context of identical particles is also an excellent source of examples for better understanding the generality of the Abelian Quantization Principle. The spin bundle that we define is also rather mathematically interesting. We give a rough sketch of some triviality and non-triviality results about these bundles. The full story is given in an appendix. We then give an alternative explanation of the Bose-Fermi alternative, one which does not rely upon the Abelian Quantization Principle. Unlike the Abelian

Quantization Principle approaches, the argument we give cannot be made in standard quantum mechanics; it needs Bohmian mechanics in an essential way. Somewhat ironically, this story is closest to what the textbooks would seem to want to use. We finish the chapter with a speculative idea about a theory of distinguished particles in which the particles are, fundamentally, identical. More to the point, the differences in the particles are in the wave function and arise in a fashion similar to why energy eigenstates arise. In other words, for a particle to behave as an electron, the wave function needs to be special. We shall only give the setup and leave a more detailed explanation for other work.

Chapter 6 is a chapter with no Bohmian mechanics in it whatsoever. Nevertheless, the work in that chapter was inspired by Bohmian mechanics and, in particular, some of the material is required for a full understanding of some of the work in chapter 7. Briefly, the chapter starts with explaining how to find higher covariant derivatives of a mapping between manifolds. After establishing the higher order chain rules and some key lemmas, the chapter pursues the question of Taylor polynomial approximations of these mappings. The idea is that a connection gives a local linear structure to the manifold. Using these linear structures, one can view any mapping between manifolds with connections as a mapping between vector spaces, at least between open subsets of the vector spaces. Given a degree, we can then try to find the best polynomial approximation to the mapping; this is a Taylor polynomial as in ordinary multi-variable calculus. We then can ask the question whether the Taylor polynomials are the same as a Taylor-type polynomial formed out of the covariant derivatives. Generically, this is not the case. We investigate when it is. The short answer is that the range needs to be flat although this statement needs to be understood properly. Examples of maps whose ranges are sufficiently flat are sections of vector bundles, maps into flat spaces, and maps which map geodesics to geodesics.

The final chapter describes two new algorithms for solving Schrödinger's equation. The algorithms are based upon Bohmian ideas, but Bohmian mechanics is irrelevant to their implementation. Essentially, we write the Schrödinger equation as two coupled, real equations. The algorithms decouple the equations in order to solve one first and

then the other. We use trajectories as a way of approximating the solutions of these PDEs. The results are rather limited. We do have a small set of examples in which we have excellent, robust convergence. We also use the algorithm to derive the propagator for quadratic potentials. We finish the chapter with deriving a PDE for the Bohmian family of trajectories, one which we can find solutions for before solving Schrödinger's equation. The PDE involves derivatives up to the fourth order and is very much a nonlinear PDE. Nevertheless, we do solve the equation in a special situation. The formulations of the PDE and the algorithms are for Bohmian systems on Riemannian manifolds, but only for complex-valued wave functions.

The appendices largely contain known material presented in what appears to be new ways. Appendix A discusses the set-based tensor product with a remark about its uses in connection with contractions. Appendix B discusses a useful basis frame for a bundle and has a short discussion on curvature. Appendix C discusses a technical point about what to do when the Hamiltonian is not essentially self-adjoint and explains what a Bohmian viewpoint can add to the discussion. Appendix D is actually new material. It gives an in-depth discussion of the triviality and non-triviality of the bundles arising in the identical particle discussion. Appendix E gives a heuristic answer to a question that the Bohmian PDE raised; generically, a full set of Bohmian trajectories arises from exactly one initial wave function. Appendix F is a supplement to Chapter 7. It discusses the derivative of the determinant on manifolds as well as establishing the usefulness of characteristics in solving the continuity equation and the Hamilton-Jacobi equation on Riemannian manifolds.

Chapter 2

The Bohmian story

2.1 Bohmian mechanics

Bohmian mechanics is a theory about particles with definite locations. The fundamental objects to specify are the trajectories in physical space of these particles. The object which defines the trajectories is the wave function, familiar from quantum mechanics. More precisely, the state of the system in Bohmian mechanics is given by the pair (Q, ψ); $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathbb{R}^{3N}$ is the configuration of the N particles in our system and ψ is a wave function on the configuration space $\mathbb{R}^{3N}$, as usual in a quantum theory. The state of the system changes according to Bohm's equation and Schrödinger's equation:

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k}\mathrm{Im}\frac{(\psi, \nabla_k\psi)}{(\psi,\psi)}(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) =: v_k^{\psi}(Q) \quad k = 1, \ldots, N \tag{2.1}$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\Delta_k\psi + V\psi = H\psi, \tag{2.2}$$

where V is the potential function and (ϕ, ψ) is the inner product on the value space, which we call the local inner product, in distinction from the inner product on the Hilbert space of wave functions. For complex-valued wave functions, the potential is a real-valued function on configuration space and the inner product is $\bar{\phi}\psi$, where the bar denotes complex conjugation.

2.2 Spinor Bohmian mechanics

To incorporate spin, the story barely changes. The value space for the wave function changes from $\mathbb{C}$ to the spinor space, W. By assumption, for one particle, W is a complex, irreducible representation space of $SU(2)$, the covering group of the rotation group $SO(3)$. If it is the spin s representation then $W = \mathbb{C}^{2s+1}$. The representation of

the group $SU(2)$ leads to a representation of the group's Lie algebra which is the same Lie algebra as that of $SO(3)$. The operators S_x, S_y, S_z are the three generators of the representation that correspond to the infinitesimal rotations about the x-axis, y-axis, and z-axis, respectively. They form the spin vector $\boldsymbol{S}_s = (S_x, S_y, S_z)_s$. For N particles, the value space becomes $\mathbb{C}^{2s_1+1} \otimes \cdots \otimes \mathbb{C}^{2s_N+1}$, where the i^{th} particle has spin s_i. The spin vector for each particle, $\boldsymbol{S}^{(i)} := \mathrm{Id} \otimes \ldots \otimes \mathrm{Id} \otimes S_{s_i} \otimes \mathrm{Id} \otimes \ldots \otimes \mathrm{Id}$, is the 1-particle spin vector for spin s_i acting only on the i^{th} factor:

$$\boldsymbol{S}^{(i)}(w_1 \otimes \cdots \otimes w_N) := w_1 \otimes \cdots \otimes w_{i-1} \otimes \boldsymbol{S}_{s_i}(w_i) \otimes w_{i+1} \otimes \cdots \otimes w_N.$$

The potential V takes the form

$$U(q) + \sum_{i=1}^{N} \mu_i \boldsymbol{S}^{(i)} \cdot \boldsymbol{B}(\boldsymbol{q}_i)$$

where U is a real-valued function and $\boldsymbol{B}$ is the magnetic field. Bohm's equation is unchanged, except we now interpret (ψ, ϕ) as a spinor inner product. The elements of the spin spaces are spinors, a terminology we use even in the case of the many particle spin space.

2.3 The emergence of the quantum formalism

In this section, we shall give a rough explanation for how the quantum formalism emerges from Bohmian mechanics. The full explanation may be found in [25, 27]. The collapse of the wave function is the simplest part of the story. The hardest part, and the one which we shall not say very much about, is the emergence of the correct empirical statistics in measurements. Somewhere in between in difficulty is the understanding of why self-adjoint operators are related to measurements. The discussion in this section is only for spinless particles.

We start with the notion of the conditional wave function. The wave function in Bohmian mechanics is the wave function for the entire universe. What about wave functions for a subsystem? A subsystem is some collection of particles. We shall use X for the system's actual configuration and use x as a variable on the configuration space

of those particles. The particles not involved in the subsystem will be called the environment and we shall use Y for its actual configuration and y as the variable. The wave function of the universe, Ψ, may then be written in terms of those two variables: $\Psi(x, y)$. The Bohmian velocity for the universal configuration is $v^{\Psi}(X, Y)$. To obtain a wave function for the subsystem, we define the conditional wave function $\psi(x) := \Psi(x, Y)$.[1] Immediately, one sees that the Bohmian velocity of the conditional wave function at X agrees with the actual Bohmian velocity. The conditional wave function's evolution is governed by the evolution of Ψ as well as the evolution of the environment Y. In certain situations, the conditional wave function will evolve according to a Schrödinger equation defined on the subsystem; we then say that the conditional wave function is an effective wave function. In other situations, the conditional wave function will not evolve according to Schrödinger's equation.

In a measurement situation, the system under consideration is assumed to have an effective wave function. During a measurement, the environment's evolution is influenced by the evolution of the system. This generally will lead to the system's wave function no longer evolving according to a Schrödinger equation. Instead, it evolves in a completely different manner. This is the process of collapse. After the measurement is over, the conditional wave function often returns to being an effective wave function. This is how collapse occurs in Bohmian mechanics. In terms of the actual Bohmian evolution of the universe, nothing special happens when collapse occurs. Furthermore, collapse is not a precise statement; although the conditional wave function exists and changes as it changes, the length of collapse is presumably the period of time in which conditional wave function is not considered to be an effective wave function. This, of course, depends on the level of approximation desired in the evolution of the conditional wave function.

In quantum mechanics, self-adjoint operators are related to measurements. What do these very abstract mathematical objects have to do with measurements? Mathematically, self-adjoint operators bijectively correspond to projection-valued measures

[1]For particles with spin, R. Tumulka has recently pointed out that density matrices replace the conditional wave function.

(PVM); this is the spectral theorem of functional analysis. The measure is a measure on the results of the experiments. They encapsualte the relevant behavior of the environment. The projection implements the collapse of the wave function. This is true for repeatable measurements. In more general measurements, such as when the measurement cannot be repeated, the projections are replaced with positive operators.

There remains the question of the probabilities. That is to say, we need to explain why $|\psi|^2$ gives the distribution of the results of measurements. In short, the empirical agreement between Bohmian mechanics and standard quantum mechanics is grounded in equivariance. In Bohmian mechanics, if the configuration is initially randomly distributed according to $|\psi_0|^2$, then the evolution is such that the configuration will be distributed according to $|\psi_t|^2$ at time t. This is equivariance. More precisely, the measure $\rho_0 = |\psi_0|^2$ can evolve either according to the evolution of ψ under the Schrödinger evolution or as a density evolving according to the Bohmian flow. Equivariance is the statement that those two evolutions agree. The continuity equation, a consequence of Schrödinger's equation, is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v^{\psi}) = 0 \tag{2.3}$$

and it guarantees equivariance.

This answer is rather lacking in usefulness, even though it may seem satisfying. Bohmian mechanics is a deterministic dynamics. In particular, the initial configuration of the universe is whatever it is. A satisfactory statement, and a true statement, would be that for a typical initial configuration of the universe, the universe evolves in such a way that the experimental results of quantum mechanics emerges. Typicality is with respect to $|\Psi_0|^2$. This is what is explained in [25] and I cannot find a short way of explaining it appropriately.

2.4 Particular experiments

Any theory, no matter how beautiful its mathematics or sound its explanations in abstract terms, must give a convincing explanation of important experiments. We collect here some of the most prominent experiments in quantum mechanics and explain the

picture as presented by Bohmian mechanics. We emphasize that this is technically unnecessary as the experimental information is encoded in $|\psi|^2$. Thus the quantum analysis of the system is sufficient even from a Bohmian perspective. Indeed, the point of the quantum formalism, as seen from the Bohmian perspective, is as a thermodynamical-type formalism which allows one to predict the outcome of experiments without having to do a detailed analysis of the, rather complicated, Bohmian motions. Nevertheless, it is important to appreciate what the measurements are or, more appropriately, are not measuring.

All of the experiments below are described in very rough terms; our intention is to only give a rough story of the situation and not a detailed analysis beyond the need for an informative and plausible argument.

2.4.1 A measurement of "momentum"

Let us consider a free particle. The experiment is the following. We start with a wave function essentially supported around a position $\boldsymbol{q}_0$; this can be created by a nondestructive measurement of position. We then let the system evolve and measure the position at some later time t. A wave packet, with wabe vector $\boldsymbol{k}$, will evolve so that its center is approximately $\boldsymbol{q}_0 + \frac{\hbar}{m}\boldsymbol{k}t$, i.e. if $\psi_0 \approx A(\boldsymbol{q})e^{i\boldsymbol{k}\cdot\boldsymbol{q}}$ with $A(\boldsymbol{q})$ slowly varying for most of its support and the support being relatively small centered around $\boldsymbol{q}_0$, then it will approximately evolve to $\psi_t \approx A_t(\boldsymbol{q})e^{i\boldsymbol{k}\cdot\boldsymbol{q}}$ where A_t has a somewhat larger support with center $\boldsymbol{q}_t := (\boldsymbol{q}_0 + \frac{\hbar}{m}\boldsymbol{k}t)$. If we do a position measurement at time t, we shall find the particle in the packet i.e. near q_t. Thus, it might seem natural to regard as the particle's momemtum $m\frac{\boldsymbol{q}_t - \boldsymbol{q}_0}{t} = \hbar\boldsymbol{k}$. For a general wave function we then consider a superposition of such initial wave packets. Given enough time, they will separate into different wave packets whose centers are quite distant. The particle will be distributed according to the norm squared of the coefficients in the original superposition. That is to say, we shall see the distribution of the Fourier transform when we do repeated measurements. Notice that all we needed to do in order to measure momentum in this sense is to be able to measure the approximate position of the particle.

Is this a measurement of the initial momentum of the actual Bohmian particle?

Absolutely not. There is no reason to believe that the particle which ends up in the wave packet for momentum $\hbar\boldsymbol{k}$ actually has that as an initial momentum. Additionally, we do not have conservation of the quantity $m\boldsymbol{v}$.

2.4.2 Scattering experiments

The above momentum measurements lead to a nice explanation of scattering experiments. Shoot at a system of interest a particle in a wave packet, one with a prescribed momentum $\hbar\boldsymbol{k}_0$. It will interact with the system and then separate from the system and again proceed freely. By setting up detectors, one can find the position of the particle and use the momentum measurement idea to deduce $\hbar\boldsymbol{k}_t$. By repeating this, one can find the distribution of the Fourier transform of the resultant wave function after interacting with the system.

What does this tell us about the scatterer and what does it do to the scatterer? To answer this, we should write down the Hamiltonian of the combined system. We have the Hamiltonian of the target, H_o, and of the projectile, H_p. There also needs to be an interaction term H_{int} which should be localized so that it is significant only when the projectile is near the target. The projectile's Hamiltonian shall be assumed to be free. We wish to determine what V_o is. Let us assume that H_o has eigenstates and that ψ_0 is a product of an eigenstate of H_o with a wave packet of momentum $\hbar\boldsymbol{k}$ directed towards the interaction region. That is to say, assume $\psi_0 = |E_M\rangle \otimes |\phi_{\boldsymbol{k}}\rangle$ where $H_o|E_M\rangle = E_M|E_M\rangle$. Let H be the Hamiltonian of the system, i.e. $H := H_o + H_p + H_{int}$. We then have that $H\psi_0 \approx E_M\psi_0 - \frac{\hbar^2\boldsymbol{k}\cdot\boldsymbol{k}}{2m}\psi_0$ where the approximation is coming from assuming that the wave packet is essentially a plane wave on its support and that the initial wave function is such that H_{int} is irrelevant. The evolution is such that the packet travels into the target, interacts with it, and then leaves the interaction region. Let U_t be the unitary evolution operator; thus $HU_t\psi_0 = U_tH\psi_0$. When the wave function is supported away from the interaction region, then $H \approx H_o + H_p$. Thus, if ψ_t is such that this is a good approximation, then $(H_o + H_p)U_t\psi_0 = U_t(H_o + H_p)\psi_0$ and we see that ψ_t is an approximate eigenstate of the system with the same eigenvalue. Thus, $H\psi_t \approx (E_M - \frac{\hbar^2\boldsymbol{k}\cdot\boldsymbol{k}}{2m})\psi_t$. After the interaction with the object, the state can be represented

approximately as $\psi_t \approx \sum_{n,\boldsymbol{l}} C_{n,\boldsymbol{l}} |E_n\rangle \otimes |\phi_{\boldsymbol{l},t}\rangle$ where $\phi_{\boldsymbol{l}}$ is a wave packet has momentum $\hbar\boldsymbol{l}$, evolves under the free evolution, and originates at the target's location. Applying H to this state, we find $H\psi_t \approx \sum_{n,\boldsymbol{l}} (E_n - \frac{\boldsymbol{l}\cdot\boldsymbol{l}}{2}) C_{n,\boldsymbol{l}} |E_n\rangle \otimes |\phi_{\boldsymbol{l},t}\rangle$. Since the terms in the sum are approximately orthogonal (approximately disjoint supports in configuration space because of the wave packets), we find that $E_M - \frac{\hbar^2 \boldsymbol{k}\cdot\boldsymbol{k}}{2m} = E_n - \frac{\hbar^2 \boldsymbol{l}\cdot\boldsymbol{l}}{2m}$ for every term in the sum such that $C_{n,\boldsymbol{l}} \neq 0$. That is to say, the eigenvalue differences in the initial and final state of the target are equal to the differences in the energy of the projectile, as provided by a momentum measurement. By using the superposition principle, one sees that this measurement procedure will only yield the eigenvalue differences; the differences are the only physically relevant information since the ground state energy corresponds to an irrelevant choice of a constant in the potential.

This explains the discrete values obtained when we measure energy for systems with bound states. In particular, thinking of photons as particles, if we have various atoms and have photons interacting with them, then the energy differences in the photons lie in a prescribed set of values which depends on the particular type of atom that they are interacting with.

As a final note about scattering experiments, observe that in Bohmian mechanics we may also analyze time-of-arrival experiments. Such experiments do not quite fit the quantum formalism, but physicists, of course, do have methods for obtaining the correct predictions. An analysis of such experiments may be found in [22].

2.4.3 The double-slit experiment

The double slit experiment is one of the most famous experiments in all of quantum mechanics. It is often taken to be the example of all the quantum mysteries. Bohmian mechanics accounts for the phenomena in a trivial manner.

The setup is that of particles being directed towards a system with two slits and a position detector, e.g. a screen, placed a distance away from the slits. The system is arranged so that the wave function is approximately a plane wave when it hits the two slits. After it passes through the slits, there will be interference from the two emerging waves. The interference pattern can be seen on the screen, built up out of many dots.

If the particles are sent one at a time, this pattern is built up over time. This is often considered to be rather mysterious. Bohmian mechanics explains that the dots on the screen occur because there are particles hitting the screen; the pattern develops as a result of the particles being guided by a wave.

If one of the slits is closed, then the interference pattern disappears. This follows since there is now only one wave and thus there is no interference, ignoring diffraction. The particle, being guided by the wave, will also not reflect any interference.

There is a final modification of this experiment and this is supposed to be the most troubling aspect of the experiment. Place a detector in such a way that the slit which the particle goes through is identified, but arrange it so that, classically at least, one would expect no disruption in the motion of the particle. It happens that there is a disruption in the motion. The reason is that the wave function is not a wave in physical space, but rather it is a wave in configuration space; detection changes the relevant configuration space. In this experiment, as in all experiments, we are using the effective wave function of the system. This is the actual wave function of the entire universe with the irrelevant degrees of freedom conditioned on. Furthermore, we are assuming that this conditioned wave function is evolving according to Schrödinger's equation for the system. When we detect the position of the particle, more degrees of freedom become relevant; in particular, the positions of the particles in the detector. This separates the two parts of the projectile's wave function. That is, the conditional wave function ceases to be an effective wave function. We can no longer use the Schrödinger evolution of the system to evolve the original wave function. But, once we take into account that the two parts of the wave function have separated and that only one portion is relevant, the one with the actual particle as reflected in the environment, then we can return to evolving that portion according to Schrödinger's equation for the system. The conditional wave function automatically takes into account the separation and collapse; all that one needs to check is that the conditional wave function is effective.

2.4.4 Schrödinger's cat

Picking up from the end of the double slit experiment discussion, we can ask about Schrödinger's cat. The experiment is to hook up a quantum system to the fate of a cat. That is, do some measurement of some quantum mechanical quantity, such as a spin measurement, and decide the fate of a cat based on the outcome of that experiment. The dilemma comes from the axioms of quantum mechanics. When is the cat's fate decided? Does the collapse of the system occur only when someone with a PhD comes along [9]? Or can a cat collapse itself because it is alive or because it is a big system?

Bohmian mechanics resolves it by stating that the cat has a definite state because there are actual particles with definite positions. But predicated upon this is that the state is recognizable. This really means that the wave function's support decomposes into two, essentially disjoint, supports. This is where size matters. Indeed, imagine a cat which is alive versus one which is dead. Just by breathing, having the blood flowing, wagging the tail, etc., many particles are in a different position than they would be if the cat was dead. Let us say that the average separation of each particle in the live cat versus the dead cat is about 1 cm. Making the assumption that there about 10^{23} particles in the cat, we find that the separation distance in configuration space is roughly $\sqrt{10^{23}}$ cm; this is a rather large distance. One would therefore expect that the configurations corresponding to the live or dead cat remain separated. That is, we have an effective collapse.

The cat is illustrative of a different side of collapse. For a microscopic system, it is the entanglement of the environment which causes the collapse. But if one enlarges the system to include the relevant environmental degrees of freedom, then the collapse does not happen. Instead, one has a very large separation between the different possibilties. It is the location of the particle which then determines the relevant packet. But it is only an approximation to discard the other parts of the wave function. For the conditional wave function, there is no approximation except for when to consider it an effective wave function. For macroscopic systems, the system itself evolves in such a way that portions of the wave function will have negligible influence on the configuration's

evolution.

2.4.5 Position measurements

We now describe a measurement of the position operator which does not accurately measure the position of the particle. By a measurement of the operator, we mean an experiment whose statistical information is encapsulated in the associated PVM. In the case of position, given a wave function ψ, a measurement of position is one in which the result R is distributed according to $|\psi|^2$ as the measurement is repeated on identically prepared systems.

The measurement described in this section will satisfy that criterion, but whether physicists would accept the measurement or not as a measurement of position is not clear to this author. But, from a Bohmian perspective, it is definitely not a measurement of position. Bohmian mechanics provides a way of actually determining whether the experiment did measure what it was supposed to have measured.

Before discussing the measurement, we need to mention some important facts about Bohmian mechanics. It is a first-order theory. This implies that trajectories do not cross. We have already mentioned that it is equivariant. In one dimension, this means that if q_0 is the initial starting point, then q_t is, more or less, uniquely defined by

$$\int_{-\infty}^{q_0} |\psi_0|^2(x)dx = \int_{-\infty}^{q_t} |\psi_t|^2(x)dx.$$

As a particular application, if $|\psi_t|^2$ is periodic in time, then the Bohmian velocities are periodic in time with the same period. We emphasize that this is special to one dimension.

Our example is in one dimension with the harmonic oscillator potential $V(x) = x^2$, ignoring constants. The evolution of any wave function is fully periodic in time with period 4π and periodic up to a sign in 2π units of time. In π units of time, the wave function ψ will be $i\psi(-x)$. Let x_m be the median of the probability distribution $|\psi|^2$. If a particle starts to the left of the median, then it remains to the left of the median. Thus, if it begins to the left of x_m, then at time $t = \pi$, it is to the left of the median at that time which is $-x_m$.

If we look at the probability distribution, we will see that it has been reflected across the origin. The position operator at $t = \pi$, $\hat{Q}_\pi$, is therefore the same as minus the position operator at $t = 0$, $-\hat{Q}_0$. This means that we can measure the initial position operator by measuring the position operator at time $t = \pi$ and then taking minus that result as the result for measuring the initial position operator. If we assume that the measurement of the operator at time $t = \pi$ accurately measures the position of the particle at that time, then the above measurement for $\hat{Q}_0$ disagrees with the actual initial position except if the particle initially started at the median. We thus have a "measurement" of position which disagrees with the Bohmian position.[2]

2.4.6 Measurements of "spin"

Measurements of spin are the canonical measurements to discuss in foundational matters. The basic experiment involves the Stern-Gerlach device. Essentially, a neutral spin-$\frac{1}{2}$ particle is shot into a region with an inhomogeneous magnetic field. This magnetic field interacts with the spin of the particle and forces the particle to move up or down depending on whether the spin is up or down in the appropriate direction. This is a repeatable experiment. And, as is well known, a particle that has been measured spin up in the x direction and then spin up in the z direction and is then measured in the spin x direction, will have spin up only 50% of the time.

Bohmian mechanics adds just a little to the picture. The Stern-Gerlach device splits the wave function into two pieces. The particle travels in the support of just one piece. If a measurement of the position of this particle is made, then the conditional wave function changes; the "empty" packet is discarded. But if the system is kept isolated, then the conditional wave function remains effective. In particular, one might be able to arrange it so that the two pieces recombine. This is why collapse can be a tricky concept and best discussed in the context of Bohmian mechanics, a theory in which collapse can be analyzed.

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[2] Another example, in a certain sense simpler, is the two-dimensional oscillator. For that example, the wave function evolution is still periodic in time, but the different trajectories do not have a common period. See [27] for details.

The most interesting aspect of the Bohmian story of spin measurements is the following, as suggested in [4]. Prepare a system with sufficient symmetry and the particle will go up or down based on whether it is above or below the line of symmetry. This is due to the fact that Bohmian mechanics is a first-order theory. One can setup the Stern-Gerlach device so that going up registers as the particle being spin up in the z direction. One can also arrange the magnets, e.g. reverse the polarities, so that going up represents being in the spin down state of the z direction. Thus, depending on the setup of the magnets, the very same Bohmian initial conditions for the system will give different results for the two experiments. This implies that the spin observable is not an intrinsic property of the system; rather, the spin observable intimately depends on the details of the experiment being done. That is essentially why the theorems proving Bohmian mechanics impossible do not work. The theorems assume a naive realist view towards the operators. But the operators are just compact descriptions of the probability measure for the outcomes of experiments. There does not have to be an underlying quantity common to experiments measuring the same "observable".

2.5 Global existence and uniqueness

There are global existence results for complex-valued wave functions where the configuration space is an open subset of $\mathbb{R}^n$. Although the dynamics is undefined at the nodes, it has been shown that typically the particle evolution does not reach the nodes. Indeed, it was shown in [10] that there is global existence in time for typical initial conditions where typicality is with respect to the measure $|\psi|^2$. This includes showing that the particles do not reach, in finite time, the nodes, the singularities of the potential, or spatial infinity.

The proof is rooted in equivariance although equivariance only holds once the existence results are established. Nevertheless, the basic idea is that the density vanishes at the places where the particle dynamics ceases to exist. Since particles flow with the density, this suggests that only a set of measure zero will reach the nodes, singularities or infinity. The proof requires a series of estimates involving the flux around these points. That typicality is all that can be expected is demonstrated with a variety of

examples. These results do need to be extended to a more general setting, but the expectation is that this is just a straightforward extension of the arguments.

2.6 The historical story, hidden variables, and nonlocality

Here is a rough sketch of the history. As I am not a qualified historian, nor did I do any personal research into the story, please take it only as my personal viewpoint.

Bohmian mechanics has its roots at the very start of quantum theory. Indeed, the folklore is that Einstein originally envisioned the electromagnetic wave as guiding the particulate photons. Although this came to nothing, it is not that distant from the Bohmian viewpoint. Bohmian mechanics was presented by Louis de Broglie to the distinguished physicists of the Solovay Congress. The lore has it that de Broglie's model was a bit overly complicated and that he did not answer a question from Pauli in a satisfactory manner. Thus, they dismissed the theory. Also around this time, Madelung came up with a hydrodynamic picture which chemists enjoy citing. It was also pretty close to Bohmian mechanics. Nevertheless, the idea did not take hold.

Instead, the quantum orthodoxy decided that such attempts were impossible. They called such models hidden variable models. The idea is that the state of the system is the wave function, and that any attempt to add additional variables, i.e. the hidden variables, in order to deduce the measurement axioms is doomed to failure. The first "proofs" were given by von Neumann in the early 1930's. As usual, the mathematics was fine, but the assumptions of what a hidden variable theory had to satisfy were too restrictive.

In the early 1950's, Bohm wrote a book on quantum theory, a book which is still cited today as a good quantum book. In his book, he presented a "no hidden variables" argument. Rumor has it that Bohm said that Einstein had convinced Bohm that Bohm's argument did not work. His faith shaken in the standard dogma, Bohm came up with a counterexample to the "no hidden variables" theorem of von Neumann which we now call Bohmian mechanics. Excitedly, he published his work. The physics community responded negatively, but not convincingly.

One mistake that Bohm made was this emphasis on the quantum potential. This was an attempt to make the theory appeal to classical intuitions. But as Bohmian mechanics is not at all a classical theory, this made the theory rather complicated and it was deemed to be ad hoc. Nevertheless, the physics community now had a counterexample to the "no hidden variables" theorems. Instead of resolving the contradictions, the physicists not only ignored it, but continued to teach students that the type of theory that Bohm had formulated was necessarily incompatible with quantum predictions.

Indeed, it seems that even Bohm abandoned his theory for a long while. It lay dormant until J. S. Bell came across Bohm's theory. This was a revelation to Bell, resulting in one of the most shocking theoretical and experimental results of the latter half of the 20^{th} century. When Bell saw Bohm's theory, he realized two things. One was the simple fact that it worked. Rather than focussing on the quantum potential, he focussed on the density's being equivariant. This allowed him to easily formulate the theory for spin. But, more importantly, he realized that the theory was nonlocal. This is obvious as the wave function is defined on the configuration space, not on physical space. In fact, when evaluating the Bohmian velocity of a particle, one must, in general, take into account the positions of all the particles which seems inherently nonlocal. Bell asked whether one could remove the nonlocality. He also asked what went wrong with the hidden variables arguments.

The result of these considerations was Bell's famous inequalities. They pertain to Bohm's version of the EPR experiment spins are used rather than position and momentum. The essential idea is that particles can be setup in such a way that the measurement on one particle tells us the value of a relevant observable for the other particle; the correlation is verifiable. The particles are to be widely separated so that no local influences can intervene. By measuring an incompatible observable on the second particle, one has results for two incompatible observables. With only local influences propagating, one would conclude that the values must already be predetermined. Bell's inequalities then showed that it is impossible for such values to be predetermined assuming the quantum mechanical predictions. The conclusion is not that hidden variables are impossible. Rather, quantum mechanics is, fundamentally, nonlocal. To be sure,

the nonlocality seems benign enough to avoid signalling paradoxes, but it is still true that the fundamental theory seems to be nonlocal. One of the advantages of Bohmian mechanics is that it forces one to deal with nonlocality in an honest way.

Sadly, Bell's work was not a clarion call to embrace Bohmian mechanics. Although Bell seems to have taken Bohmian mechanics as the sensible version of quantum theory, he was invariably misunderstood by the physics community. They would cite his work as proof that hidden variables could not explain the results of quantum measurements. His work actually shows that no local theory can explain it. But the point of particle motion is neither to have locality nor to eliminate probabilities. Rather the point of Bohmian mechanics is that it avoids the mysticism of quantum mechanics. For example, instead of postulating collapse, one derives collapse of the wave function from the theory.

2.7 Objections to Bohmian mechanics

It seems that the objections to Bohmian mechanics come in four generic varieties. The first is a misunderstanding of the facts of the situation. To put it straight, Bohmian mechanics is a consistent theory whose predictions agree with those of quantum mechanics. There can be no disagreement. Thus, any evidence in support of orthodox quantum theory is evidence for support of Bohmian mechanics.

This naturally leads to "Who cares?" There is not much to say about this. We have already explained the clarity that Bohmian mechanics gives to the measurement problem. That itself should be sufficient.

I would add that standard quantum mechanics appears as pure magic while Bohmian mechanics is the mechanism behind the show. The relevance of this statement is in the teaching of quantum mechanics. The standard approach mystifies students beyond the common mystification associated with abstract mathematics. Fundamentally, many thoughtful students are not comfortable with the measurement axioms. Unlike other areas of physics in which explanations are offered, quantum mechanics lends no explanations to collapse or the special role of measurement. Adopting a Bohmian point of view leads to a clear picture and, at the very least, one can cite that the axioms

mathematically follow from the evolution equations.

There are many objections to standard quantum theory that one can raise, but the problem is that there is no actual standard quantum theory when measurements are discussed. And it is exactly at the point of measurements, at the contact with reality, that Bohmian mechanics is relevant. It explains, in an obvious manner, what this mathematical theory has to do with reality. More fundamentally, one should start, when developing a theory, with an idea of what the primitive variables are for the formulation of this theory. Primitive variables should not be a deduction of a theory, although experiments may suggest suitable variables to choose. Presumably the primitive variables should not be results of measurements which are rather complicated objects centered upon human existence. One does not want a human-centered theory; rather, one wants a clear theory in which certain behavior of the variables would give a reasonable explanation of why the world behaves as it does.

Bohmian mechanics postulates the existence of particles with definite positions which evolve according to a prescribed dynamics. It happens that this dynamics involves a wave function evolving according to Schrödinger's equation. As we shall now explain, demanding a particle ontology almost immediately leads to Bohmian mechanics.

2.7.1 Where does Bohmian mechanics come from?

The third objection to Bohmian mechanics is that it is cooked up. We shall state five different derivations of Bohmian mechanics.

1. The starting place for deriving Bohm's equation is the same as for deriving Schrödinger's equation. We start with the idea that a wave is involved. In particular, we assume we have a generic plane wave, i.e.

$$\psi(\boldsymbol{q}, t) := e^{i(\boldsymbol{k}\cdot\boldsymbol{q}-\omega t)}. \tag{2.4}$$

The empirical data somehow suggested that the momentum of the particle was given by the de Broglie relation

$$\boldsymbol{p} = \hbar\boldsymbol{k}. \tag{2.5}$$

The energy of the particle was related to the frequency via the Planck formula

$$E = \hbar\omega. \tag{2.6}$$

We now look for an equation of motion for the particle as well as for the wave equation.

To obtain $\boldsymbol{k}$ from ψ, we take the spatial derivative and divide by i and ψ. Thus, (2.5) and (2.4) suggests that

$$m\boldsymbol{v} = \boldsymbol{p} = \hbar\boldsymbol{k} = \frac{\hbar}{i}\frac{\nabla\psi}{\psi}.$$

Considering, instead of (2.4), a general complex-valued wave function, we find that we would end up with a complex velocity vector. The simplest choice is to take the real part of the expression above which, since we divided by i, translates into taking the imaginary part. We have thus derived Bohm's equation:

$$\boldsymbol{v}^{\psi} := \frac{\hbar}{m}\mathrm{Im}\frac{\nabla\psi}{\psi}.$$

What about Schrödinger's equation? Well, classically, the energy of a particle is

$$E = \frac{\boldsymbol{p}\cdot\boldsymbol{p}}{2m} + V.$$

We look for the simplest wave equation whose plane wave solutions (2.4) are consistent with (2.5), (2.6), and (1). We find that (2.6) implies for plane waves, that

$$E = -\frac{\hbar}{i\psi}\frac{\partial\psi}{\partial t}.$$

As for the $\boldsymbol{p}$ term, we have two possibilities:

$$\boldsymbol{p}\cdot\boldsymbol{p} = -\hbar^2\frac{\nabla\psi}{\psi}\cdot\frac{\nabla\psi}{\psi} \tag{2.7}$$

or

$$\boldsymbol{p}\cdot\boldsymbol{p} = -\hbar^2\frac{\Delta\psi}{\psi}. \tag{2.8}$$

Choosing (2.8) leads to

$$-\frac{\hbar}{i\psi}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\Delta\psi}{\psi} + V,$$

which becomes Schrödinger's wave equation after multiplying through by ψ. As for (2.7), this is not a linear equation. For a linear equation, knowing the solutions for any initial plane wave actually specifies the evolution for any function. This is not true for nonlinear equations.

We have arrived at Bohmian mechanics. We shall discuss in Chapter 3 how to arrive at the Pauli equation.

2. Briefly, if we start with Schrödinger's equation, then we can derive a continuity equation for $\rho = |\psi|^2$. By dividing the probability current by ρ, we obtain a velocity field, the Bohm velocity field. Equivariance is then immediate, and with it the empirical agreement with quantum mechanics. In terms of finding Bohmian laws of motion, this has been a successful and easy method for determining appropriate laws. But as a derivation, it has a feeling of fitting theory to given data. Thus, it is best if we have other methods. But we do note that one of the guiding principles in forming quantum theories is to establish that there is an appropriate probability current associated with the equation. This suggests that we should perhaps formulate the Bohmian law of motion and then find a wave equation based on demanding equivariance.

3. Schrödinger's equation also implies a Hamilton-Jacobi equation. This is Bohm's method of derivation. It describes particles moving under a force consisting of the classical force and a new quantum force. But we have to supplement the force with the constraint on the initial velocity field given by the wave function according to Bohm's equation. The strange feature about this is that the constraint is actually the law of motion one wants.

4. In [25], the authors give a rather convincing derivation of Bohm's equation based on symmetry considerations. The quantity $\nabla\psi$ is a natural quantity for a rotation invariant way of obtaining a vector. Dividing by ψ comes from noting that it is only the ray in Hilbert space which is physically relevant. The imaginary part arises from time reversal invariance. Under time reversal, the wave function is sent to its complex conjugate and velocity fields change sign. To obtain the factor of

$\frac{\hbar}{m}$, we must consider Galilean boosts. Under a boost, we have that $\psi \mapsto e^{-i\frac{m}{\hbar}\boldsymbol{w}\cdot\boldsymbol{q}}\psi$ where $\boldsymbol{w}$ is the velocity of the new frame with respect to the old frame. A velocity $\boldsymbol{v}$ should tansform to $\boldsymbol{v}-\boldsymbol{w}$ in the new frame and the Bohm velocity will become $\boldsymbol{v}^{\psi}_{old}-\boldsymbol{w}$ if and only if the factor in front is $\frac{\hbar}{m}$.

5. A new derivation is in [23]. It seems to apply to a wide variety of cases, such as the non-relativistic case, quantum field theory, and the Dirac equation. In some situations, it must be interpreted correctly. We shall only show here that it agrees in the non-relativistic case. The idea is that a tangent vector is a first-order differential operator acting on functions and obeying the Leibniz rule. Thus, v is defined if we know what $v(f)$ is for all f. In particular, if we have coordinates, then the tangent vector v has coordinate components $v^j = v(x^j)$. We also note that if $q(t)$ is a trajectory whose tangent vector is v at time 0, then defining $g := f \circ q$, we have $\dot{g}(0) = v(f)$.

   From a quantum perspective, functions can be thought of as multiplication operators acting on the Hilbert space of wave functions. In general, an operator A's time evolution, in the Heisenberg picture, is given by[3]

$$\dot{A}_t = \frac{i}{\hbar}[H, A_t].$$

   Thus, transform f into a multiplication operator: $(\hat{f}\psi)(q) := f(q)\psi(q)$. We shall assume that f is smooth and is compactly supported, as local information is all that we care about. Then the Bohmian velocity field v with a given ψ can be defined as the differential operator whose action satisfies

$$(v(f))(q) = \dot{f}(q) = \mathrm{Re}\frac{(\psi, \dot{\hat{f}}\psi)}{(\psi,\psi)}(q) = \mathrm{Re}\frac{(\psi, \frac{i}{\hbar}[H, \hat{f}]\psi)}{(\psi,\psi)}(q).$$

[3]This follows by shifting from the Schrödinger picture, in which A does not evolve and the expected value of the operator at time t is given by $(\psi_t, A\psi_t)$, to the Heisenberg picture in which only the operator evolves and the expected value at time t is $(\psi_0, A_t\psi_0)$. Indeed, let $U_t = e^{-\frac{i}{\hbar}Ht}$ be the (unitary) Schrödinger evolution operator. Then to obtain the same answer in both cases, we need $A_t = U_t^{-1}AU_t$. By differentiating with respect to time, we find that

$$\dot{A}_t = \frac{i}{\hbar}HA_t + A_t(-\frac{i}{\hbar}H) = \frac{i}{\hbar}[H, A_t].$$

We shall now investigate this for Schrödinger's equation. First, note that any potentials in H commute with f and thus they contribute nothing to the velocity field. Let us therefore take H to be $-\frac{\hbar^2}{2m}\Delta$. The commutator is easy to compute; we find

$$[\Delta, \hat{f}]\psi = \Delta f\psi + 2\nabla\psi \cdot \nabla f.$$

Thus, we have that

$$v(f) = v \cdot \nabla f$$

where v is the Bohm velocity field.

2.7.2 Relativistic extensions of Bohmian mechanics

The fourth objection is that it is irreconcilable with quantum field theory and with relativity. These claims are false as recent work has demonstrated.

Nonlocality is often assumed to be problematical for a hidden variable perspective. There are two responses to this. As stated previously, it is a problem for standard quantum theory as well, but due to the shiftiness of what is being described, the problem can be hidden for a long time. The second point is that there is no inherent inconsistency with nonlocal influences in a relativistic theory, as discussed at length in [42]. More to the point, there are relativistic, nonlocal Bohmian models [46, 34].[4] The configuration space for N particles moving in the space-time manifold M might be taken to be M^N. Then the wave function in a relativistic setting would seem to be a function on M^N. That is to say, it is a multi-time wave function. When evaluating the Bohmian evolution, one must evaluate the wave function by using the positions of the other particles. The problem is that we no longer have simultaneity to tell us what the full configuration is. One model involves a foliation of space-time. That is to say, simultaneity surfaces are built into the structure. Although perhaps distasteful, if a compelling Lorentz-invariant law for the foliation can be found, then this should be acceptable. With the foliation method, the empirical predictions are transparent and they are equivalent to those of

[4] One might argue as to what Bohmian means; in this section, we take it to mean a theory in which there is a clear ontology with prescribed dynamics and no reference to measurements in the formulation of the theory.

quantum theory. In particular, the foliation is unobservable. Another idea involves no additional structure. It uses the light cones to accomplish what the simultaneity surfaces do. To have a chance at agreeing with experiments, it is necessary to use the forward light cones. Thus, one might say there are two different arrows of time involved. But the probabilistic import of the model is still unclear. Existence of the dynamics is also rather unclear.

Quantum field theory is perhaps the object most pointed towards when decrying the Bohmian point of view. Bohm took the approach of giving ontological status to the fields. A recent approach [24] is to retain particles as the ontology with the wave function retaining its role as guiding the particles. Describing the evolution of the wave function is the primary concern of quantum field theory and it is difficult. Understanding the role of particles is quite easy. The wave function is a function on the disjoint union of all of the configuration spaces for the various number of particles. That is, there is no fixed number of particles. The Bohmian motion consists of two parts. One is the usual part in which the particles evolve deterministically guided the wave function. The other part is stochastic. In a random fashion, particles are created or annihilated. The motion proposed is equivariant implying that this model gives the same empirical predictions as standard quantum field theory. But it is a theory in which no discussion about measurements is necessary in order to make this a theory about reality. It is true that this is not deterministic, but that was never the issue driving Bohmian mechanics. The real issue was the demand for a simple, well-defined theory of reality.

Chapter 3

Geometrically formulating Bohmian theories

In this chapter, we shall formulate a Bohmian dynamics defined on a general Riemannian manifold $\mathcal{Q}$ with the wave function being a section of a cc-Hermitian vector bundle E over $\mathcal{Q}$. We begin with scalar wave functions over $\mathcal{Q}$ and then generalize to vector bundles. This sets up the general non-relativistic Bohmian framework. The Pauli equation may be viewed as both a special case as well as a generalization of this setup. We explain how to view the equation as expressing a generalized version of the Laplacian. After making a few standard observations, we finish this chapter with a discussion of how a desire for an ultralocal Bohmian theory leads to the Dirac equation. It is rooted in a structure which we call the square-root of the tangent space. It is intimately related to the role of Clifford algebra in the Dirac equation.

3.1 A quick review of tensor analysis

In this section, we shall give a review of the uses and abuses of tensor analysis. An appreciation of these methods is essential for an understanding of this chapter. For most of this section and the rest of the chapter, the statements can either be thought of as in terms of the whole bundle or just at a point. Even though it is best to think of all the structures as just at a point, we shall write our notations without denoting the point. For example, we shall write $T\mathcal{Q}$ instead of $T_q\mathcal{Q}$ even though we will actually be discussing the tangent space at the point q.

We start with a metric g on the manifold $\mathcal{Q}$. It is a $(0,2)$-tensor meaning that it is an element of $(T^*\mathcal{Q})^{\otimes 2}$. It acts on $(T\mathcal{Q})^{\otimes 2}$; alternatively one could say that it is a bilinear form acting on tangent vectors. In coordinates, $g = g_{jk}dx^j \otimes dx^k$ where the summation convention on repeated indices is implied. An important use of g is to lower

an index, i.e. g may be viewed as a linear map from $T\mathcal{Q}$ to $T^*\mathcal{Q}$. Indeed, given a tangent vector v, $g(v,\cdot)$ is a 1-form. If it is necessary, we will denote the role of g as a loweing operator by writing $g_\flat$. In other words, $g_\flat(v) := g(v,\cdot)$. Even more compactly, one may write $v_\flat$ indicating that v has been converted to a 1-form using g.

An assumption on a metric is that it is nondegenerate which means that given a vector v, there is a w such that $g(v,w) \neq 0$. This assumption is equivalent to demanding that g as a lowering operator has an inverse. We denote the raising operator as $G^\sharp$ and it is a map from $T^*\mathcal{Q}$ to $T\mathcal{Q}$. One can then define G as a bilinear form acting on $T^*\mathcal{Q}$ by defining $G(\omega,\nu) := \omega(G^\sharp(\nu))$ where ω and ν are 1-forms. Such a bilinear form is the same as a $(2,0)$-tensor i.e. it is an elment of $(T\mathcal{Q})^{\otimes 2}$. In coordinates, $G = G^{jk}\partial_j \otimes \partial_k$ where (G^{jk}) is the inverse matrix to (g_{jk}).

Most of our use of G is to contract it with $(0,2)$-tensors. On product elements, $G(\omega \otimes \nu) := G(\omega,\nu)$; it is then extended by linearity to all of $T^*\mathcal{Q}^{\otimes 2}$. Let β be a $(0,2)$-tensor. Then the notations that may be used are $G(\beta) = G \circ \beta = \beta(G) = G^{jk}\beta_{jk}$.

We introduce some more notation. The symmetric subspace of a i-fold tensor product of a vector space W with itself is denoted by $\bigodot^i W$. The antisymmetric part of such a tensor product is $\bigwedge^i W$. The direct sum over i of the antisymmetric parts is the exterior algebra where the product is the wedge product which may be defined as the antisymmetrization of the tensor product of the terms.

A metric is a symmetric tensor, i.e. $g \in \bigodot^2 T^*\mathcal{Q}$. Its dual, G, is also symmetric. This means that G, when acting on a $(0,2)$ tensor, will only notice the symmetric part of the tensor. In particular, if $\beta \in \bigwedge^2 T^*\mathcal{Q}$, then $G(\beta) = 0$.

The final element of a metric on a manifold is that it should be parallel. This requires choosing a connection on the tangent bundle. A connection can be viewed in various ways. In this chapter, we shall be using two viewpoints. The first is that a connection provides a means for parallel transporting vectors along paths. To say it in another way, it provides an isomorphism of the local structures at different points on the manifold although the isomorphism may depend on the path. The second viewpoint is that it provides a method for differentiating vectors. We shall use D to denote the covariant differentiation operator. This discussion works equally for any vector bundle.

As part of the definition of a connection, the connection must satisfy the Leibiniz rule for tensor products, i.e. $D(v \otimes w) := Dv \otimes w + v \otimes Dw$ where v and w are sections of two vector bundles.[1] It should also commute with contraction, e.g. if ω is a section of the dual bundle in which v is a section of, then $D[\omega(v)] = D[\omega](v) + \omega(D[v])$. In particular, once you have defined a connection on a vector bundle, then there is an induced connection on its dual space and all of the various tensor bundles that one can form from these spaces.

For a metric, the connection which is chosen is the Levi-Civita connection. It is uniquely specified by demanding that the metric is parallel and that the torsion of the connection vanishes. The fact that the metric vanishes leads to the commutation of differentiation and contraction with either g or G. For example, if v and w are vector fields, then $D[g(v, w)] = g(D[v], w) + G(v, D[w])$.

For more general vector bundles, one can also have an inner product on the fibers. One almost always has a connection on the bundle and demands that the inner product on the fibers is parallel. This does not define the connection uniquely. Therefore, we shall call such a bundle a cc-Hermitian vector bundle where "cc" stands for connection-carrying indicating that a connection has been chosen and the inner product will always be assumed to parallel under that connection. We shall generally use $(\cdot, \cdot)$ to denote the inner product. We will always use D for the covariant derivative regardless of which bundles are being discussed.

We shall now get into the heart of what we need. Given a section ψ of a cc-Hermitian vector bundle E over the Riemannian manifold $\mathcal{Q}$, we have that $D\psi$ is an element of $E \otimes T^*\mathcal{Q}$ which is often called an E-valued 1-form. We might also call this a 1-form valued section of E which, although literally incorrect, is often useful terminolgy. The gradient of ψ is $\nabla\psi := G^\sharp(D\psi)$. It is an element of $E \otimes T\mathcal{Q}$ and we might call it a tangent vector valued section of E. For Bohmian mechanics, the velocity field is a multiple of $\mathrm{Im}(\psi, \nabla\psi)$ which is indeed a section of the tangent bundle.

The Laplacian of ψ is defined to be the metric trace of the second covariant derivative

[1] A section of E is a map ψ from $\mathcal{Q}$ into E such that $\psi(q) \in E_q$, i.e. it maps a point q of $\mathcal{Q}$ into elements of the vector fiber over q.

of ψ. In symbols, we have $\Delta\psi := G(D^2\psi)$. Note that the second covariant derivative requires the use of the connection on the cotangent bundle induced by the metric. This is the divergence of the gradient. Indeed, given a tangent vector valued section, one can take the contraction of the derivative with the tanget vector value. In symbolds, $D \cdot \phi := D_j\phi^j$ which one could also view as $\nabla \cdot \phi := g(\nabla\phi)$. Thus, the divergence of the gradient is

$$\nabla \cdot (\nabla\psi) = g(G^\sharp D(G^\sharp(D\psi))) = g(G^\sharp(G^\sharp(D^2\psi))) = G(D^2\psi) = \Delta\psi.$$

It is a fact that this is equivalent to the Laplace-Beltrami operator acting on sections of a vector bundle [28]. The Laplace-Beltrami operator is defined on E-valued forms by using Hodge-$*$ duality and the exterior derivative. In the case of a section, this agrees with the Laplacian described above. For other forms, it does not. They differ by some curvature terms.

As a final note, many of the very useful computational tools of Euclidean analysis do carry over into Riemannian manifolds and bundles. An important difference is in the coordinate expressions although one can choose coordinates so that at one particular point, the expressions look Euclidean. In general, if $\{e_i\}$ is a local basis for E and $\{dx^l\}$ is a local basis for $T^*\mathcal{Q}$, then the Laplacian of $\psi := \psi^i e_i$ is

$$\Delta\psi = G^{jk}(\frac{\partial^2\psi^i}{\partial x^j \partial x^k}e_i + \psi^i D^2[e_i](\partial_j \otimes \partial_k) + \frac{\partial\psi^i}{\partial x^j}D_{\partial_k}[e_i] + \frac{\partial\psi^i}{\partial x^k}D_{\partial_j}[e_i] + \frac{\partial\psi^i}{\partial x^j}e_i D_{\partial_k}[dx^j].$$

3.2 Bohmian dynamics on manifolds

A Bohmian dynamics exists in more general situations than the one described in Chapter 2. In this section, the wave functions will still be complex-valued functions over the configuration space. In fact, we can let the configuration space, $\mathcal{Q}$, be an arbitrary Riemannian manifold. Given a metric g on a manifold, the Laplacian may be defined to be the metric trace of the second derivative as explained above. The gradient $\nabla\psi$ is the (complex) tangent vector metrically equivalent to the 1-form $D\psi$. Then v^ψ, the Bohmian velocity field associated to a wave function ψ, is

$$v^\psi := \hbar \operatorname{Im}\frac{(\psi, \nabla\psi)}{(\psi, \psi)} \tag{3.1}$$

where we have absorbed the mass into the metric. The time evolution of the state (Q_t, ψ_t) is given by

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) \tag{3.2a}$$

$$i\hbar\frac{\partial\psi_t}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi_t + V\psi_t =: H\psi_t. \tag{3.2b}$$

Thus, given $(\mathcal{Q}, g, V)$, we have formally specified a Bohmian theory.

The above is suitable for any Riemannian manifold. A primary example is when the configuration space describes multiple particles moving in physical space. Let the Riemannian manifold M, with metric g, be the physical space. Then the configuration space for N distinguished particles moving in M is $\mathcal{Q} := M^N$. Let the masses of the particles be m_i. Then the relevant metric on M^N, acting on the tangent space $T_{(q_1,\ldots q_N)}M^N = \bigoplus_{i=1}^N T_{q_i}M$, is

$$g^N(v_1 \oplus \cdots \oplus v_N, w_1 \oplus \cdots \oplus w_N) := \sum_{i=1}^N m_i g(v_i, w_i).$$

Using g^N for the metric, the Bohmian velocity field is given by (3.1) and the system evolves according to (3.2). In Chapter 5, we give a more interesting discussion of many particle systems.

3.3 The Hamilton-Jacobi equation on manifolds

As there is no analogue of this for theories whose wave functions are sections of a bundle, we pause here to establish the standard fact that the modified Hamilton-Jacobi equation is a consequence of the Schrödinger evolution. Let $\psi =: Re^{i\frac{S}{\hbar}}$ where R is positive and S is real. Then, the Schrödinger equation becomes, after dividing by $e^{i\frac{S}{\hbar}}$,

$$i\hbar\frac{\partial R}{\partial t} - R\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2}(\Delta R + \frac{2i}{\hbar}\nabla R\cdot\nabla S + \frac{i}{\hbar}\Delta S - \frac{1}{\hbar^2}\nabla S\cdot\nabla S) + VR.$$

Taking the real part of that equation and dividing by $-R$, we have

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\nabla S\cdot\nabla S - V + \frac{\hbar^2\Delta R}{2R}$$

which is the Hamilton-Jacobi equation for the potential $V - \frac{\hbar^2\Delta R}{2R}$ on the Riemannian manifold $\mathcal{Q}$ with metric g.

3.4 Bohmian dynamics involving bundles

More generally, we can consider a Bohmian dynamics for wave functions taking values in some complex vector bundle E over the Riemannian configuration space $\mathcal{Q}$. Our bundles will always be cc-Hermitian vector bundles. The "wave functions" then become smooth sections of the vector bundle; $C^\infty(\mathcal{Q}, E)$ will denote the smooth sections while $C_0^\infty(\mathcal{Q}, E)$ will denote the smooth sections with compact support. The local inner product acting on the vectors v_q, w_q will be denoted by (v_q, w_q). The global inner product on the Hilbert space of wave functions is the local inner product integrated against the Riemannian volume element associated with the metric g; we denote the Hilbert space by $L^2(\mathcal{Q}, E)$. The potential V is now a self-adjoint section of the matrix, or more properly endomorphism, bundle acting on the vector bundle's fibers.[2] In the context of a theory involving just spin, V is restricted to being a pointwise linear combination of the spin matrices and the identity; this does exclude a variety of potentials in the case of many particles and generally in the case of 1 particle where spin-$\frac{1}{2}$ happens to be an exception.

The equations are the same. Indeed, v^ψ, the Bohmian velocity field associated to a wave function ψ, is

$$v^\psi := \hbar \operatorname{Im} \frac{(\psi, \nabla\psi)}{(\psi, \psi)}. \tag{3.3}$$

The time evolution of the state (Q_t, ψ_t) is given by

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) \tag{3.4a}$$

$$i\hbar \frac{\partial \psi_t}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi_t + V\psi_t := H\psi_t. \tag{3.4b}$$

The wave function ψ will always be assumed to be in L^2 and smooth, i.e. $\psi_t \in C^\infty(\mathcal{Q}, E) \cap L^2(\mathcal{Q}, E)$. We shall generally assume that the formal Hamiltonian, H, is essentially self-adjoint on the domain of compactly-supported, smooth wave functions.[3]

[2] For us, the terms matrix and endomorphism will be synonymous; the use of the term matrix does not imply that we are using a particular basis. We avoid the term linear operator as that may be confused with an operator acting on the Hilbert space of wave functions.

[3] Appendix C gives an extremely brief discussion about situations in which the Hamiltonian is not essentially self-adjoint on the domain $C_0^\infty(\mathcal{Q}, E)$. As a particular example, when $\mathcal{Q}$ is $\mathbb{R}^3$ with the z-axis removed, the Laplacian is not essentially self-adjoint on $C_0^\infty(\mathcal{Q}, \mathbb{C})$ [2]. A choice of a self-adjoint extension is required corresponding to boundary conditions on the z-axis.

We shall often use $-\Delta+V$ to indicate that we are discussing the formal Schrödinger-type Hamiltonians. It has been shown that $-\Delta + V$ is essentially self-adjoint on the domain $C_0^\infty(\mathcal{Q}, E)$ for cc-Hermitian vector bundles whose base space is a complete Riemannian manifold with suitable restrictions on the potential V; see [53, 40] for details.

3.5 The continuity equations for bundle-valued Bohmian dynamics

We shall now establish the continuity equation for the Bohmian dynamics descrubed by (3.4). This is just the standard argument. The density ρ is actually $|\psi|^2\nu$ where ν is the Riemannian volume element. The probability current ρv^ψ may then be considered as a $(n-1)$-form and the divergence is the exterior derivative of the current. In terms of establishing the continuity equation, one can also view ρ as a function and then ρv^ψ would be a vector field. The divergence is then the contraction of the derivative of the vector field. In any event, we want to show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v^\psi) = 0.$$

As a starting point, we can compute from Schrödinger's equation that

$$\begin{aligned}\frac{\partial \rho}{\partial t} =& (\frac{\partial \psi}{\partial t}, \psi) + (\psi, \frac{\partial \psi}{\partial t}) \\ =& (\frac{1}{i\hbar}(-\frac{\hbar^2}{2}\Delta\psi + V\psi), \psi) + (\psi, \frac{1}{i\hbar}(-\frac{\hbar^2}{2}\Delta\psi + V\psi)) \\ =& \frac{1}{i\hbar}\{\frac{\hbar^2}{2}((\Delta\psi, \psi) - (\psi, \Delta\psi)) + (-(V\psi, \psi) + (\psi, V\psi))\}.\end{aligned}$$

The potential term automatically vanishes since V is a self-adjoint matrix-valued function. The Laplacian terms may be compactly written as $-\hbar\mathrm{Im}(\psi, \Delta\psi) = -\hbar\mathrm{Im}\{\nabla \cdot (\psi, \nabla\psi) - g((\nabla\psi, \nabla\psi))$. Since the last term does not contribute to the imaginary part, we are left with the divergence of the probability current.

3.6 The Pauli equation

Let $\mathcal{Q}$ be a Riemannian manifold, with metric g, and let E be a cc-Hermitian vector bundle. Physics often requires an additional structure. This strucutre may be viewed as an entanglement of the spatial structure with the bundle structure; it comes in the form of a representation.

3.6.1 A bit of representation theory

Recall that a Lie algebra representation is a linear map from a Lie algebra into the set of matrices acting on a vector space satisfying that the image of a Lie bracket is the commutator of the images. For a Riemannian manifold, the relevant Lie group is $SO(g)$, the matrices that preserve the inner product; its Lie algebra is $\mathfrak{so}(g)$. We also note that $Spin(g)$ is the double cover of $SO(g)$ and has the same Lie algebra since the Lie algebra is the tangent space to the identiy.

We will assume that our bundle and manifold come equipped with a Lie algebra representation of $\mathfrak{so}(g)$ induced by a representation of either $SO(g)$ or $Spin(g)$. We shall denote the Lie algebra representation as $S : \mathfrak{so}(g) \to E \otimes E^*$. A representation of a Lie algebra is a linear map which preserves Lie brackets. We shall assume that the Lie group representation is a unitary representation; this does imply that S takes values in the anti-self-adjoint valued matrices.

We now explain the very standard statement that $\bigwedge^2 T^*\mathcal{Q}$ is canonically isomorphic, up to a choice of sign, to the Lie algebra $\mathfrak{so}(g)$. Indeed, the Lie group $SO(g)$ is the set of matrices $A \in T^*\mathcal{Q} \otimes T\mathcal{Q}$ which preserve the metric g. That is to say, if $A \in SO(g)$, then for every $v, w \in T\mathcal{Q}$, we have the defining property

$$g(Av, Aw) = g(v, w). \tag{3.5}$$

This group of matrices is a manifold and the Lie algebra is the tangent space to the identity operator. More to the point, if we take a path $A(t) \in SO(g)$ such that $A(0) = \mathrm{Id}$, then $\frac{\partial A}{\partial t}$ defines an element of the Lie algebra; every such element of the Lie algebra arises in this way. By computing the derivative of (3.5), we find $g(\dot{A}v, w) + g(v, \dot{A}w) = 0$ which, using the symmetry of g, leads to

$$g(\dot{A}v, w) = -g(\dot{A}w, v).$$

The isomorphism between the 2-forms and the Lie algebra is given by the map $\dot{A} \mapsto \omega$ defined by $\omega(v, w) := g(\dot{A}v, w)$. By the above equation, $\omega \in \bigwedge^2 T^*\mathcal{Q}$. By using the non-degeneracy of g, we can make a bijective correspondence to the set of antisymmetric matrices. Thus, given a 2-form ω, one can map it to an antisymmetric matrix A and then exponentiate tA to obtain a path α in the Lie group such that $\dot{\alpha}(0) = A$.

We may therefore view an element of the Lie algebra $\mathfrak{so}(g)$ as an element of $\bigwedge^2 T^*\mathcal{Q}$. The dual of this space is $\bigwedge^2 T\mathcal{Q}$. We conclude that a representation may be viewed as a special element of $\bigwedge^2 T\mathcal{Q} \otimes E^* \otimes E$.

3.6.2 A generalization of the Laplacian

We start with the already stated observation that

$$\Delta\psi := G \circ (D^2\psi).$$

It is critical to appreciate this statement. Accepting it almost immediately leads to the Pauli equation as an obvious extension. Just to be clear, $D^2\psi \in (T^*M)^{\otimes 2} \otimes E$ is the second covariant derivative, i.e. it is a $(0,2)$ E-valued tensor. The symmetric $(2,0)$ tensor $G \in T\mathcal{Q}^{\otimes 2}$ is the dual of the metric $g \in (T^*M)^{\otimes 2}$. We may therefore plug the tensor G into the tensor $D^2\psi$ to obtain a section of E. Note that in the definition of the Laplacian, we actually should say that G means $G \otimes \mathrm{Id}$.

There is an obvious way to modify this definition. We can contract the second covariant derivative with any matrix-valued $(2,0)$ tensor. That is to say, let $T \in T\mathcal{Q}^{\otimes 2} \otimes E \otimes E^*$. Then $T \circ D^2\psi$ makes sense and defines the T-Laplacian which shall be denoted by Δ_T.

What should T be? Since G is part of the given local information, we may assume that the symmetric part of T is $G \otimes \mathrm{Id} \in \bigodot^2 T\mathcal{Q} \otimes E \otimes E^*$. This choice is simple, does not lead to any new objects, and is certainly invariant under transformations preserving the metric. This leaves us with the task of specifying the antisymmetric part of T. The easiest possibility, as originally done, is to take the antisymmetric part to be zero. Assume that we do not do this. Since the antisymmetric part is an element of $\bigwedge^2 T\mathcal{Q} \otimes E \otimes E^*$, we see that the representation S is the same kind of tensor as the antisymmetric part of T. The same reasoning that leads to choosing G leads us to choosing S.

To understand what happens with T, we need to understand the second covariant derivative of a section ψ. The second derivative of ψ decomposes into a symmetric part and an antisymmetric part. The symmetric part is what it is; this is the term

that contributes to the usual Laplacian and is the only part that G notices. The antisymmetric part is the curvature of the connection applied to ψ; this is what S will contract with. It is a tensor in ψ meaning that it depends only on ψ through its value at that point. It turns out that if a connection preserves a Hermitian inner product, then its derivative is anti-self-adjoint in its action on the bundle. See Appendix B for an explanation of this fact. We shall use F to denote the curvature and F is an element of $\bigwedge^2 T^*\mathcal{Q} \otimes E \otimes E^*$.

Let us look at what we have. Taking $\Delta\psi$ to mean the metric trace of the second covariant derivative, we see that

$$\Delta_T\psi = \Delta\psi + S(F(\psi))$$

where $S(F)$ is a contraction which leaves a self-adjoint matrix. Thus, we can write

$$\Delta_T\psi = \Delta\psi + V\psi$$

for some self-adjoitn V depending on the representation and the curvature. We do end up with a Pauli-type equation. We also see that the Bohmian motion is still equivariant with respect to Δ_T for this choice of T. We also mention that one can easily multiply both G and S, independently, in the definition of T.

As an illuminating example, let $\mathcal{Q} = \mathbb{R}^3$ and $E = \mathbb{R}^3 \times \mathbb{C}^2$. We have as a background connection, the trivial connection denoted by D. We are considering the case of a spin-$\frac{1}{2}$ particle. Let S be the representation and $\boldsymbol{S}$ its vector version using the duality between $(1,0)$ tensors and antisymmetric $(2,0)$ tensors special to three dimensions. In particular, $\boldsymbol{S}_x = \frac{\hbar}{2}\sigma$ is what comes from $S(y \wedge z)$. The curvature is the magnetic field. It is most naturally a 2-form, but it becomes a vector under duality. The usual Paulie equation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}(\nabla + i\boldsymbol{A})\cdot(\nabla + i\boldsymbol{A})\psi + \mu\boldsymbol{S}\cdot\boldsymbol{B}\psi$$

for some constant μ and vector potential $\boldsymbol{A}$. The way it would be written above is the following. Let ω be the 1-form equivalent to A under the metric and β the 2-form equivalent to B; the 2-form is equal to the exterior derivative of ω. Then the T-based Laplacian equation is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}(G+S)(D+i\omega)^2\psi$$

which is the same as the Pauli equation after converting the objects and expanding.

3.7 Other equivariant motions?

We shall now ask whether T leads to a different, yet equivariant, motion for the particle. The answer is yes. Finding other equivariant and invariant motions is not new; see [20, 13]. This section is a bit technical and can be skipped without loss of information.

Let $T \in T\mathcal{Q} \otimes E^* \otimes T\mathcal{Q} \otimes E$. Then, viewing $D\psi \in T^*\mathcal{Q} \otimes E$, we can contract on the first two tensor factors of T with $D\psi$ to obtain $T \circ D\psi \in T\mathcal{Q} \otimes E$. We then use the inner product to obtain a complex tangent vector. Taking the imaginary part leaves us with a real tangent vector. Multiplying by $\hbar$ and dividing by $\rho := (\psi, \psi)$ leads to a velocity field, $v^{T,\psi}$. When do ρ and $v^{T,\psi}$ lead to an equivariant motion? In symbols, we are defining

$$v^{T,\psi} := \hbar \mathrm{Im} \frac{(\psi, T \circ D\psi)}{(\psi, \psi)}$$

and trying to find out what conditions guarantee that the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v^{T,\psi}) = 0$$

is satisfied?

As a starting point, we can compute from Schrödinger's equation that

$$\begin{aligned}
\frac{\partial \rho}{\partial t} =& (\frac{\partial \psi}{\partial t}, \psi) + (\psi, \frac{\partial \psi}{\partial t}) \\
=& (\frac{1}{i\hbar}(-\frac{\hbar^2}{2}\Delta\psi + V\psi), \psi) + (\psi, \frac{1}{i\hbar}(-\frac{\hbar^2}{2}\Delta\psi + V\psi)) \\
=& \frac{1}{i\hbar}(-\frac{\hbar^2}{2}(-(\Delta\psi, \psi) + (\psi, \Delta\psi)) + (-(V\psi, \psi) + (V\psi, \psi)));
\end{aligned}$$

we include the spin terms into the potential V. The potential term automatically vanishes since V is a self-adjoint matrix-valued function. The Laplacian terms may be compactly written as $-\hbar\mathrm{Im}(\psi, \Delta\psi)$. Explicitly, we need

$$\nabla \cdot (\rho v^{T,\psi}) = \hbar \mathrm{Im}(\psi, G \circ D^2\psi).$$

Computing the divergence term, we find

$$\begin{aligned}
\nabla \cdot (\rho v^{T,\psi}) &= \hbar \mathrm{Im}(\nabla \cdot (\psi, T \circ D\psi)) \\
&= \hbar \mathrm{Im}((D_k\psi, T^{jk} D_j\psi) + (\psi, (D_k T^{jk}) D_j\psi) + (\psi, T \circ D^2\psi))
\end{aligned}$$

where we use $A_j B^j$ to indicate a contraction over the indices when there may be ambiguity; the last term has essentially an unambiguous contraction. We can think of ψ, $D\psi$ and the symmetric part of $D^2\psi$ as independent of each other. In particular, we may choose some to be zero and the others non-zero. We therefore find that we need

$$\mathrm{Im}(D_k\psi, T^{jk}D_j\psi) = 0 \tag{3.6a}$$

$$\mathrm{Im}(\psi, (D_k T^{jk})D_j\psi) = 0 \tag{3.6b}$$

$$\mathrm{Im}(\psi, T \circ D^2\psi) = \mathrm{Im}(\psi, G \circ D^2\psi). \tag{3.6c}$$

Whenever these equations are satisfied, we have an equivariant motion.

These equations rule out many possibilities, but we shall find that we do have more possibilities than just the usual Bohmian velocity. The main equation is (3.6a). The essential trick is that we have complete freedom over the covariant derivatives of ψ. Indeed, as discussed in Chapter 6, we can create a ψ whose symmetric covariant derivatives are specified at a single point. In particular, we can choose the first derivatives anyway we like. Pick a basis of the tangent space at a point q; although this discussion only holds for q, since q is arbitrary, this is sufficient and we shall suppress the dependency on q. For a basis element x, we choose $D[\psi](x) = \phi$ and $D[\psi](y) = 0$ for all the other basis elements. With this choice, (3.6a) becomes

$$\mathrm{Im}(\phi, T^{xx}\phi) = 0$$

which evidently implies that T^{xx} is self-adjoint as this hold for all ϕ. We now choose $D[\psi](x) = \phi$, $D[\psi](y) = \tau$ and all of the other basis directional derivatives to vanish. Then (3.6a) becomes

$$\begin{aligned} 0 =& \mathrm{Im}\{(\phi, T^{xx}\phi) + (\tau, T^{xy}\phi) + (\phi, T^{yx}\tau) + (\tau, T^{yy}\tau)\} \\ =& \mathrm{Im}\{(\tau, T^{xy}\phi) + (\phi, T^{yx}\tau)\}. \end{aligned}$$

If we now take $\tau = \phi$, we complete the argument that the symmetric part of T is self-adjoint. If we take $\tau = i\phi$, then we end up with

$$0 = \mathrm{Im}\{-i(\phi, T^{xy}\phi) + i(\phi, T^{yx}\phi) = Re(\phi, T^{xy} - T^{yx}\phi)$$

which indicates that the antisymmetric part of T is anti-self-adjoint. We now move on to (3.6c). The symmetric part of the second derivative is an object that we can specify

arbitrarily at a point. Thus, repeating an argument similar to the above, we have that the symmetric part of T is equal to $G \otimes \mathrm{Id}$. The antisymmetric part of the second derivative is an object that we do not have control over. Indeed, it is the curvature operator. Therefore, if T's antisymmetric part is anti-self-adjoint, then it couples with the curvature operator in a way to ensure self-adjointness. That is to say, if T_A is the antisymmetric part of T and F is the curvature operator, then

$$(\psi, T \circ D^2\psi) = (\psi, G \circ D^2\psi) + (\psi, T_A \circ F\psi).$$

As $T_A \circ F$ is self-adjoint, its imaginary part vanishes.

We conclude that the only choice for T is $G \otimes \mathrm{Id} + S$ where T_A is a linear map of the Lie algebra of $\mathfrak{so}(g)$ taking values in the Lie algebra of anti-self adjoint matrices. It is therefore natural to take $T_A := S$ to be the representation map. Equation (3.6b) will be satisfied if T is parallel which it is if it is composed of G and S.

3.7.1 The addition of a curl

For equivariance, we could always add on a divergence-free vector field to the probability current. This will not change the empirical equivalence with standard quantum mechanics. One can then ask the question as to whether there is a divergence free velocity field that arises from the structures at hand. Indeed there is.

As we have said, S is a anti-self-adjoint matrix-valued antisymmetric (2,0) tensor. If we define

$$\nu := \hbar \mathrm{Im}(\psi, S\psi) = -\hbar \mathrm{Im}(S\psi, \psi),$$

then we have an antisymmetric (2,0) tensor. Such an object can always be used to define a divergence free vector field. Indeed, take the derivative of ν and contract; one of the two vector slots must be chosen which is really a choice of a sign. This leaves a vector field. More explicitly, we note that

$$(D[\nu])^{j\cdot}_{j} = \frac{\hbar}{i}(D[\mathrm{Im}(\psi, S^{j\cdot}\psi)])_j = \hbar \mathrm{Im}(\psi, S \circ D\psi)$$

which is essentially what we had in the last section. Thus, adding this to the probability current will not change the equivariance.

But we wish to demonstrate that it is equivariant just because $\nu \in \bigwedge^2 T\mathcal{Q}$. Let v be the vector field obtained by contracting the first derivative of ν with one of its tangent vector's components. To compute the divergence, take the derivative of the vector field and contract again. We claim that zero is the answer.

There are at least two ways to understand this. The first is to view an element of $\bigwedge^2(T\mathcal{Q})$, which ν is, as an $(n-2)$-form using the volume form. Then compute the exterior derivative to obtain a $(n-1)$-form and this is equivalent to a tangent vector field. The divergence is computed by taking the exterior derivative of the $(n-1)$-form and comparing the resultant n-form to the volume form. Since $d^2 = 0$, we have that the divergence vanishes.

The other way is to use the covariant derivative. The only reason to consider this method is that we are using covariant derivatives and sometimes this use conflicts with a differential form approach. But in this case, we end up with the same result. It is important to recall that covariant differentiation preserves symmetry types, i.e. $D^2[\nu]$ is an element of $\bigwedge^2(T\mathcal{Q}) \otimes T^*\mathcal{Q} \otimes T^*\mathcal{Q}$. By computation, we have

$$(D^2[f\nu])^{jk}_{jk} = (D^2[f])_{jk}\nu^{jk} + (D[f])_j(D[\nu])^{jk}_k + (D[f])_k(D[\nu])^{jk}_j + f(D^k[\nu])^{jk}_{jk}.$$

The second derivative of a function for a symmetric connection is symmetric which implies, in conjunction with the antisymmetry of ν, that the first term vanishes. The mixed terms vanish since we clearly have

$$(D[f])_k(D[\nu])^{jk}_j = -(D[f])_k(D[\nu])^{kj}_j = -(D[f])_j(D[\nu])^{jk}_k.$$

Thus, multiplication by a function commutes with the divergence operator on antisymmetric 2-tensors.

Let e and f be two vector fields forming part of a local basis at a point q. Then

$$\begin{aligned}(D^2[e \otimes f - f \otimes e])^{jk}_{jk} =& D^2[e]^j_{jf} - D^2[f]^j_{je} + D[e]^j_j D[f]^k_k - D[f]^j_j D[e]^k_k \\ &+ D[e]^j_k D[f]^k_j - D[f]^j_k D[e]^k_j + D^2[f]^k_{ek} - D^2[e]^k_{fk}.\end{aligned}$$

Clearly, the middle terms, involving the first derivatives, vanish since multiplication is commutative. The other terms vanish because essentially we are left with Ricci

curvature terms but in such a way that they cancel. Specifically,

$$(D^2[e]^j_{jf} - D^2[e]^k_{fk}) - (D^2[f]^j_{je} - D^k[f]^k_{ek}) = Ric(e,f) - Ric(f,e).$$

As Ricci is a symmetric tensor, we have that the right hand side vanishes. Thus, the divergence vanishes.

Therefore, one can add this spin-based velocity field to the usual Bohmian velocity and still maintain equivariance. There would seem to be no way to distinguish between these theories. As described in [13], the non-relativistic limit of the Bohm motion for the Dirac equation seems to include this term. We conclude with remarking that if we chose $G \otimes \mathrm{Id} + S$ for computing the velocity field, then we are using the same tensor to compute the Laplacian as well as the velocity field. In other words, instead of choosing G and S, we can view the whole specification of the theory as choosing the connection (curvature), the object T (G and S), and the inner product.

3.8 The Dirac equation

In this section, we will argue that the structure of the Dirac equation is based upon what may be called a square-root of the tangent bundle. It is this structure which is suggested by the Bohmian viewpoint. In order to minimize additional structures on space-time, we actually require our square-root of the tangent bundle to be a square-root of the metric. We shall see that this puts a severe constraint on the choices and leads to the Dirac equation being a square-root of the Klein-Gordon equation. We shall assume we have a Lorentz metric on space-time and shall denote it by g and its dual by G. As throughout this chapter, almost all statements are about the local structure and one should have in mind the tangent space at a point and the fiber at a point rather than the bundle although we shall omit the reference to the point.

3.8.1 Brief background material

Let W be a complex vector space. Its dual space, W^*, is the space of complex-linear functionals acting on W. The conjugate dual of W is the space of conjugate-linear functionals acting on W; it is denoted by $\overline{W}^*$and it is also the conjugate space of W^*.

The dual of W^* is canonically isomorphic to W under the mapping $\psi \mapsto \psi(\omega) := \omega(\psi)$ where $\psi \in W$ and $\omega \in W^*$. The dual of $\overline{W}^*$ is the conjugate space $\overline{W}$ of W. It is the same as the conjugate dual of W^*. It is canonically isomorphic to W under the mapping $\psi \mapsto \overline{\psi}(\omega) := \overline{\omega(\psi)}$ and under the multiplication that takes $c \odot \psi \mapsto \overline{c}\psi$. This multiplication allows the appropriate conjugate-linearity to be expressed in the tensor products. Heuristically, the conjugate space corresponds to the other choice of what i could mean.

3.8.2 Square-roots of tangent spaces and metrics

From a Bohmian perspective, we want to have a theory of particles moving on space-time. We restrict our attention to one particle. The idea is to try to define a covariant vector field on space-time in terms of ψ. In the non-relativistic theory, one uses the first derivative of ψ to do this. But what if the mapping between wave functions and velocity fields were ultralocal, i.e. did not depend on derivatives, and yet it is still covariant?

Such maps do exist; we call such a map a square-root of the tangent space. As a starting point, the map should be a sesquilinear-quadratic form taking values in the tangent space; it should be conjugate linear in the first slot and linear in the second. Abstractly, $\nu \in \overline{W}^* \otimes W^* \otimes \mathbb{C}T\mathcal{Q}$ which we few as a linear mapping $\nu : \overline{W} \otimes W \to \mathbb{C}T\mathcal{Q}$. Given a $\psi \in W$, we immediately have that $\nu(\overline{\psi}, \psi) \in \mathbb{C}T\mathcal{Q}$. We want a real velocity field and therefore demand that $\nu(\overline{\psi}, \psi)$ is a real tangent vector. Then the Bohmian trajectories are the integral curves of the velocity field. Note that the parametrization is irrelevant; all we need are the 1-dimensional submanifolds whose tangent spaces correspond to the span of $\nu(\overline{\psi}, \psi)$.[4] From this we can deduce that both ν and ψ should be thought of as projective objects. We also demand that ν be non-degenerate in the sense that $\nu(\overline{\psi}, \psi) = 0$ if and only if $\psi = 0$. In summary, given ν acting on the bundle and a section ψ, we have a velocity field for the law of motion for the particle.

Before we call ν a square-root of the tangent space, we also demand that it be covariant. As part of this, we shall need a representation R of the Lorentz group acting

[4] As we are working in space-time rather than just space, we know that the position in the manifold is all that should be relevant.

on the bundle. To be a square root of the tangent space, ν must satisfy, for every Lorentz transformation Λ acting on the tangent space and every ψ, $\phi \in W$.

$$\Lambda(\nu(\psi, \phi)) = \nu(R(\Lambda)\psi, R(\Lambda)\phi).$$

Given a covariant ν, we then have a covariant law of motion for the particle.

To form a law that the wave functions must satisfy, we shall need one more structure, which we call μ, an appropriately linear mapping $\overline{W}^* \to W$. Thinking of ν as a mapping from $W \to \overline{W}^* \otimes T\mathcal{Q}$, we see that $\mu \circ \nu$ leads to a mapping $\gamma : W \to W \otimes \mathbb{C}T\mathcal{Q}$. Assuming that μ is invertible, which we do, we can define a, not necessarily positive definite, sesquilinear product $\langle \psi, \psi \rangle := \{\mu^{-1}(\psi)\}(\overline{\psi})$. Thus, we have that $\nu(\overline{\psi}, \psi) = \langle \overline{\psi}, \gamma\psi \rangle$. As this should be real for all ψ we have that γ should be appropriately self-adjoint with respect to $\langle \cdot, \cdot \rangle$. We also demand that μ be covariant or, equivalently, that $\langle R(\Lambda)\psi, R(\Lambda)\phi \rangle = \langle \psi, \phi \rangle$.

What else can we generate with these structures? With $\langle \cdot, \cdot \rangle$ and γ, we can construct more tensor objects. For example, $T := \langle \overline{\psi}, \gamma^2\psi \rangle$ is a $(2, 0)$ tensor. If we wish to minimize new structures on space-time, then it would seem reasonable to demand that the symmetric part of T is actually $G\langle \overline{\psi}, \psi \rangle$ which immediately means that the symmetric part of γ^2 is $G \otimes \mathrm{Id}$. Making this one choice leads to very few possibilities, after taking into account equivalences, of what the pair (ν, μ) represent.[5] Indeed, γ is a representation map for a Clifford algebra as we shall now explain.

The refrence for this material is [36]. A Clifford algebra for the vector space V, endowed with the metric β, is a an associative algebra with unit 1 equipped with a linear mapping $\alpha : V \to \mathrm{Cliff}(V, \beta)$ such that $\alpha(V)$ generates the algebra and the anticommutator of the images is a multiple of the identity, i.e. $\{\alpha(x), \alpha(y)\} = \alpha(x)\alpha(y) + \alpha(y)\alpha(x) = \beta(x, y)1$ for all $x, y \in V$.[6] It is a fact that a Clifford algebra is unique up to isomorphism. It is constructed from the tensor algebra of V by essentially viewing $x \otimes y + y \otimes x$ as $\beta(x, y)1$, i.e. mod out by the ideal generated by that equation. If we allow β to be identically zero, then the Clifford algebra generated is the exterior

[5] The author does not know whether covariance of ν and μ are sufficient to deduce this.

[6] One also requires a universality property which we shall not state.

algebra. For us, the vector space W is $\mathbb{C}T^*\mathcal{Q}$ and the inner product β is G.

A space of spinors for (V, β) is a vector space W equipped with a linear map $\gamma : V \to W \otimes W^*$ such that the only invariant subspaces of W under $\gamma(V)$ is W and 0 and that $\{\gamma(x), \gamma(y)\} = \beta(x, y)\mathrm{Id}$. The map γ then extends to an irreducible representation of the Clifford algebra and all irreducible representations arise in this way. For us, γ is the same γ we were discussing before once we convert the spinor γ to a map of the form $W \to V^* \otimes W$. It is a fact that for even dimensions, there is exactly one space of spinors while for odd dimensions, there is two inequivalent spaces of spinors. We therefore have a complete specification of the additional structures in four dimensions if the Clifford algebra representation is irreducible. Furthermore, $\gamma^2 = G \otimes \mathrm{Id} + S$, where S is a particular representation, up to equivalence, of the Lorentz Lie algebra, $\mathfrak{so}(\mathrm{g})$. For even dimensions, the Lie algebra representation decomposes into the two, inequivalent, half-spin representations of $\mathfrak{so}(g)$. For odd dimensions, the Lie algebra representation is irreducible. Notice that the covariance plays no role in determining the uniqueness of these structures; covariance even seems to be implied from demanding that the symmetric part of γ^2 is the metric.

The pair (ν, μ), or equivalently $(\gamma, \langle \cdot, \cdot \rangle)$, is what we call a square-root of the metric since, when we square γ, the symmetric part is the metric.

The usual Gamma matrices also give rise to an irreducible representation of the Clifford algebra for G. By the uniqueness cited above, we find that a square-root of the Lorentz metric on four-dimensional space corresponds to an equivalent representation as the usual Gamma matrices.

3.8.3 Formulating the Dirac equation

We shall continue to use D for the covariant derivative and it shall be used for differentiating any object such as vector fields or sections of the bundle E.

We start by noting that ν and μ should be parallel. Equivalently, we demand that,

for any section ψ, we have

$$D\nu(\overline{\psi},\psi) = \nu(D\overline{\psi},\psi) + \nu(\overline{\psi},D\psi) \tag{3.7}$$

$$D\langle\overline{\psi},\psi\rangle = \langle D\overline{\psi},\psi\rangle + \langle\overline{\psi},D\psi\rangle. \tag{3.8}$$

Using the irreducibility of the Clifford algebra representation, it can be shown that the set of possible connections is characterized by the class of 1-forms rather than general matrix-valued 1-forms.

To obtain the Dirac equation, one could try to formulate the simplest equation with the structures at hand. What do we have? We have a mapping γ taking wave functions to elements of $\mathbb{C}T\mathcal{Q}\otimes W$. The covariant derivative takes wave functions to elements of $\mathbb{C}T^*\mathcal{Q}\otimes W$. Both are covariant operations. If we contract with these two operators, then we have an invariant operator taking sections of E to sections of E. This is the Dirac operator which is defined to be $\not{\partial}\psi := \gamma^j D_j\psi$. Seeking an invariant equation for the wave function using the Dirac operator, we see that the Dirac equation is an obvious candidate:

$$\frac{\hbar}{i}\not{\partial}\psi = a\psi. \tag{3.9}$$

There is another way to argue for this equation. As before, we want to minimize the structures placed on space-time. Given a vector field, one can take its derivative. Demanding that this vanishes leads to a rather uninteresting vector field. If we demand, on the other hand, that only the divergence vanishes, then a natural candidate is the Dirac equation. This is the space-time version of equivariance. The velocity field being divergence-free is the equation

$$0 = D_j\langle\overline{\psi},\gamma^j\psi\rangle = \langle D_j\overline{\psi},\gamma^j\psi\rangle + \langle\overline{\psi},\not{\partial}\psi\rangle = \overline{\langle\overline{\psi},\not{\partial}\psi\rangle} + \langle\overline{\psi},\not{\partial}\psi\rangle$$

where we have used the previously mentioned fact that $\langle\overline{\tau},\gamma\phi\rangle = \langle\overline{\gamma\tau},\phi\rangle$ for all $\tau,\phi\in W$. Thus, we need the real part of $\langle\overline{\psi},\not{\partial}\psi\rangle$ to vanish. This certainly follows if $ib\psi = \not{\partial}\psi$ for some real constant b which is the Dirac equation.

A final fact is that the Dirac operator is the square root of a Pauli-type Laplacian. Indeed,

$$a^2\psi = (\frac{\hbar}{i}\not{\partial})(\frac{\hbar}{i}\not{\partial})\psi = -\hbar^2(\gamma^2)^{jk}D^2_{jk}\psi = -\hbar^2(\Delta\psi + S(F)\psi)$$

where F is the curvature of the connection. Thus, a solution to the Dirac equation also satisfies the Klein-Gordon-Pauli equation.

Chapter 4

Theory formation and the influence of the topology of configuration space

4.1 Introduction

In this chapter, we analyze the influence of the topology of the configuration space in formulating Bohmian theories. We formulate the Abelian Quantization Principle which states that, in addition to the usual data such as the potential, one needs to choose a character of the fundamental group of the configuration space when forming a Bohmian theory. To conclude the principle, we consider three different approaches based on: the covering space; connections; and self-adjoint extensions. We demonstrate that each approach generates equivalent theories. This will set us up for the next chapter. In particular, we will conclude that, in the theory of identical particles moving in 3-space, one has the choice between bosons and fermions. This is the usual Bose-Fermi alternative.

The current goal is to explain the relevance of a space being multiply connected. The prime example to have in mind is the Aharonov-Bohm effect. Briefly, the setup is a particle moving along a circle embedded in 3-space; there is also a magnetic field whose support is disjoint from the circle but whose integral over the disk does not vanish. We would like to model this situation in terms of a Bohmian system involving just the circle with no need to embed it into an ambient space.

We start by considering what role the covering space might have. The idea is to consider wave functions on the universal covering space such that they still define a Bohmian motion on the base space. For the circle, the covering space is a line; the useful wave functions are those that are 2π-periodic up to a phase. In general, we find a periodicity condition characterized by a character of the fundamental group. Another

approach is to consider different connections. This is very familiar in the context of the Aharonov-Bohm effect. The important point is that the characters of the fundamental group are in bijective correspondence with the flat connections; the correspondence is that of the holonomy operator. For the circle, this is the integral of the vector potential. The final approach is to consider different self-adjoint extensions of the Hamiltonian. On the circle, if we remove a point, then we may view the space as an interval. As is well-known, there are a variety of possible self-adjoint Laplacians on the interval; they are characterized by different boundary conditions. The extensions that are permissible are those for which, given any other point, there is a corresponding Hamiltonian which provides the same Bohmian evolution of the configuration.

The notion of multiply connected spaces giving rise to different quantum theories is not new. A sampling of the literature is the following. The use of a covering space was done at least as early as 1950 as in [14]; it was more fully done in [21]. The use of vector potentials was done in [3]. Path integrals on multiply connected spaces began largely with the work of [50] and [37]; see [51] for details. Most of these works are dedicated to scalar-valued wave functions.

This chapter is organized as follows. In Section 4.2, we discuss the circle and its relation to the Aharonov-Bohm effect. In that section, we give the principle for Bohmian theories on manifolds for which the wave functions are complex-valued functions over the configuration space. We give the general formulation and argument for the principle in Section 4.3; the generalization refers to taking wave functions to be sections of a given complex vector bundle over the configuration space. The entire discussion, centered upon deriving how characters arise, is based upon appealing to the covering space of the configuration space. Section 4.4 and Section 4.5 discuss approaches that do not use the covering space. We contrast the various equivalent approaches in Section 4.6 and discuss in Section 4.7 how the principle is generating locally equivalent theories. We conclude with a discussion of quantization. In particular, quantization of systems on the configuration space whose potential can only be formulated on the covering space is described in Section 4.8.4. Appendix A reviews the necessary notion of a set-indexed tensor product. Appendix C deals with a technical point involving the self-adjointness

of the formal Hamiltonian.

4.2 The circle and the Aharonov-Bohm effect

In this section, all wave functions will be complex-valued functions on the configuration space.

Consider the system of a single particle constrained to travel along a circle. Thinking of the circle as sitting in $\mathbb{R}^3$, we assume that going through the center of the circle is a magnetic field. We demand that the field vanishes in a neighborhood around the circle, but that its integral over a disk, with the circle as the boundary, does not vanish. Our goal is a Bohmian theory that accurately describes this system in which the configuration space is explicitly the circle.

This is a simplified version of the usual Aharonov-Bohm effect. The standard answer is that the vector potential which generates the magnetic field should be included in the covariant derivative. This does work as explained in Section 4.4. But we want to describe this without getting into bundles. Instead, we shall focus on the topology of the circle and see that the possibilities naturally arising from being multiply-connected already provides the setting for such a model.

We are looking for other Bohm-type theories governing an evolution of a particle on the circle. One important constraint is that the Bohmian velocity field needs to be well-defined on the circle. The inspiration in this approach comes from noting that if we write ψ in the polar form $Re^{iS/\hbar}$, then $v^\psi = \nabla S$. Thus, the Bohmian velocity field is locally a gradient. From classical experiences, this immediately brings to mind the universal covering space which, for the circle, is just $\mathbb{R}$. We can therefore ask which wave functions on $\mathbb{R}$ could provide an evolution on the circle. The tentative answer are those wave functions for which S is periodic up to a constant, i.e. $S(\theta + 2k\pi) = S(\theta) + k\beta$. This translates into a multiplicative periodicity condition on ψ:

$$\psi(\theta + 2k\pi) = \gamma^k \psi(\theta). \tag{4.1}$$

As a side note, we could say that the periodicity condition is represented by a homomorphism between the additive group of integers into the multiplicative group of the

unit circle in the complex plane. More abstractly, we could say that we have a unitary, 1-dimensional representation of the covering group, and therefore of the fundamental group, for the circle.

Given this periodicity condition, it is clear to see that the associated Bohmian velocity field on $\mathbb{R}$ is 2π-periodic. Hence, one can project the velocity field down to the circle in order to define a Bohm-type evolution of the particle. As for this condition being preserved under the Schrödinger evolution on the covering space, linearity of Schrödinger's equation suggests it; locally, the wave function differs only by a constant. We shall say more about this in the next section.

We may therefore define a γ-based theory in the following way. The position of the particle at time t, Q_t, is a point on the circle. The wave function is a smooth function on $\mathbb{R}$ satisfying (4.1). The evolution of the wave function is governed by Schrödinger's equation on $\mathbb{R}$. The generated Bohmian velocity field projects, in a single-valued way, to the circle. The position then evolves according to that velocity field.

As for the relation to the Aharonov-Bohm setup, we make the following claim, which will be established in more generality in Section 4.4.3. Let $\beta := \int_{D^2} \boldsymbol{B}\boldsymbol{\omega}$. This integral requires a choice of orientation and the choice induces an orientation on the circle. If increasing θ on $\mathbb{R}$ agrees with this orientation, then $\gamma := e^{-i\beta}$ will give the appropriate theory.

The above story holds much more generally. We first state the following principle for scalar-valued wave functions. We then explain the terminology and notation in a following subsection. An even more general statement is given in the next section.

Principle 1 (Abelian Quantization Principle, covering formulation, scalar). *To specify a Bohmian theory involving complex-valued wave functions, one must choose a Riemannian configuration space $\mathcal{Q}$, a potential V, and a character γ of the fundamental group $\pi_1(\mathcal{Q})$ of $\mathcal{Q}$. The Bohmian theory corresponding to the character γ is defined by* (3.2a)*, where ψ_t is a smooth wave function on $\widehat{\mathcal{Q}}$ satisfying the character's associated periodicity condition and locally satisfying* (3.2b)*, i.e. ψ_t evolves according to the Schrödinger dynamics generated by the lift of H from $\mathcal{Q}$ to $\widehat{\mathcal{Q}}$.*

4.2.1 Notations and relevant facts

The notations are summarized by the table:

Configuration space	$\mathcal{Q}$	S^1
Universal covering space	$\widehat{\mathcal{Q}}$	$\mathbb{R}$
Points in $\mathcal{Q}$, $\widehat{\mathcal{Q}}$	q, $\hat{q}$	$e^{i\theta}$, θ
Projection map	$\rho : \widehat{\mathcal{Q}} \to \mathcal{Q}$	$e^{i\cdot} : \mathbb{R} \to S^1$
Covering fiber over q, $e^{i\theta}$	$\rho^{-1}(q)$	$\{\theta + 2k\pi\}_{k\in\mathbb{Z}}$
Fundamental group of $\mathcal{Q}$	$\pi_1(\mathcal{Q})$	$\mathbb{Z}$
Covering transformation	σ	$\sigma_k : \theta \to \theta + 2\pi k$
Covering group	$Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$	σ_k
Character of fundamental group	γ_σ	$\gamma_k := \gamma_1^k$
Bundle, lifted bundle	E, $\widehat{E}$	$S^1 \times \mathbb{C}$, $\mathbb{R} \times \mathbb{C}$

Throughout this chapter, $\mathcal{Q}$ will be a Riemannian manifold with metric g. Its universal covering space will be $\widehat{\mathcal{Q}}$. We shall always take $\widehat{\mathcal{Q}}$ to be endowed with the lifted metric $\hat{g}$; this assures that it is canonically locally isometric to $\mathcal{Q}$. More to the point, the differential operators are locally identical.

The universal covering space $\widehat{\mathcal{Q}}$ of $\mathcal{Q}$ is, by construction, a simply connected space. The *covering fiber* for $q \in \mathcal{Q}$ is the set of points in $\widehat{\mathcal{Q}}$ that project to q under the projection, or covering, map, $\rho : \widehat{\mathcal{Q}} \to \mathcal{Q}$, associated with the covering space. A function $f : \widehat{\mathcal{Q}} \to \mathbb{C}$ is projectable iff $f(\hat{q}) = f(\hat{r})$ whenever $\rho(\hat{q}) = \rho(\hat{r})$. The projection $\tilde{f} : \mathcal{Q} \to \mathbb{C}$ is defined by $\tilde{f}(q) = f(\rho^{-1}(q))$. Projectability of a vector field can be translated into demanding that $((D\rho)v^\psi)(\hat{q}) = ((D\rho)v^\psi)(\hat{r})$ whenever $\rho(\hat{q}) = \rho(\hat{r})$; $D\rho : T\widehat{\mathcal{Q}} \to T\mathcal{Q}$ is the differential of the map ρ. The lift of a function or a vector field is always well-defined and it is the inverse of projection.

If E is a bundle over $\mathcal{Q}$, then the lift of E, denoted $\widehat{E}$, is a bundle over $\widehat{\mathcal{Q}}$; the fiber at $\hat{q}$ is defined to be the fiber of E at q, i.e. $\widehat{E}_{\hat{q}} := E_q$ where $q := \rho(\hat{q})$. When we do this, we can then lift sections of the vector bundle as well as matrix-valued functions acting on the bundle. It is important to realize that with this construction, it makes sense to ask if $v \in E_{\hat{q}}$ is equal to $w \in E_{\hat{r}}$ whenever $\hat{q}$ and $\hat{r}$ are elements of the same covering

fiber. As a particular example, the lift of the tangent bundle of $\mathcal{Q}$ to $\widehat{\mathcal{Q}}$ is canonically isomorphic to the tangent bundle of $\widehat{\mathcal{Q}}$.

A *covering transformation* is an isometry which maps the covering space to itself and which preserves the covering fibers. The group of such transformations is the covering group and is denoted by $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. The fundamental group at a point q, denoted by $\pi_1(\mathcal{Q}, q)$, is the set of equivalence classes of closed loops, through q, where the equivalence relation is that of homotopy, or smoothly deforming one curve into the other. The product in the group is concatenation; in particular, given $\sigma\tau$, we first follow τ and then follow σ. This is in contrast to the common definition of the product in the opposite order. We do it this way as it seems more natural in terms of parallel transport, which is of fundamental concern to us. The fundamental groups at different points are isomorphic to each other as well as to the covering group, but the isomorphisms are not canonical. By the fundamental group, written $\pi_1(\mathcal{Q})$, we shall mean the abstract group which the covering group and all of the fundamental groups are isomorphic to. In particular, a representation of the covering group is a representation of the fundamental group. A character of the fundamental group is any unitary, 1-dimensional representation of the fundamental group, i.e., it is a homomorphism between the fundamental group and the multiplicative group of the complex numbers of modulus one.

4.3 Bohmian mechanics and topology: the Abelian Quantization Principle

Since it is the same argument, we shall argue for the Abelian Quantization Principle in the context of a Bohmian theory in which the wave functions are sections of a vector bundle. We start with a Riemannian configuration space $\mathcal{Q}$ and a vector bundle E endowed with a connection and a parallel inner product. We are also given a self-adjoint matrix-valued potential V acting on E. We claim that, in addition to this information, we also need to choose a character of the fundamental group to complete the theory formation. That this is an *a priori* necessity is debatable since the theory corresponding to the trivial character requires no need to appreciate the other possibilities. But the

physics certainly seems to require an appreciation of these options.

As in the case of the circle, the idea is to appeal to the universal covering space $\widehat{\mathcal{Q}}$ in order to trivialize the topology. We lift all of the objects of our theory to $\widehat{\mathcal{Q}}$. This includes lifting E to $\widehat{E}$ by defining $E_{\hat{q}} := E_q$. By lifting the local objects, we can write down a Schrödinger's equation for sections of $\widehat{E}$; we refer to the Hamiltonian on $\widehat{\mathcal{Q}}$ as the lifted Hamiltonian. As before, we are only interested in wave functions which generate a Bohmian evolution on $\mathcal{Q}$. In particular, the velocity field defined in (3.3) needs to be projectable.

Although we no longer have an S, as we are dealing with vector bundles, it is quite clear that projectability is guaranteed if we have a scalar periodicity condition as in the case of the circle. More precisely, if ψ is a section of $\widehat{E}$ satisfying

$$\psi(\sigma\hat{q}) = \gamma_\sigma \psi(\hat{q}) \tag{4.2}$$

for every $\hat{q} \in \widehat{\mathcal{Q}}$ and $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, and where γ_σ is a constant of modulus one, then it is clear that

$$v^\psi(\hat{q}) = v^\psi(\sigma\hat{q}).$$

In order to say that we have a Bohmian dynamics on $\mathcal{Q}$, we must demonstrate that if the initial wave function satisfies (4.2), then it remains projectable. In fact, we shall demonstrate that it continues to satisfy the same periodicity condition. It follows from the linearity of Schrödinger's equation and the fact that the lifted Hamiltonian is locally the same when compared at different points in the same covering fiber. The idea is that the wave function at different "levels" is the same up to an irrelevant constant. Its evolution should therefore be the same on the different levels and the periodicity condition will thus be preserved.

A more precise argument is the following. If ψ_t is any solution to Schrödinger's equation on $\widehat{E}$, then, for any covering transformation σ, we have that $\psi_t^\sigma := \psi_t \circ \sigma$ is another solution to Schrödinger's equation.[1] On the other hand, if we multiply ψ_t by a

[1]Since the potential is identical at each point of the covering fiber, we only have to worry about the Laplacian. But since σ is an isometry we have $\Delta\psi_t^\sigma = (\Delta\psi_t) \circ \sigma$.

constant scalar, we still have a solution. Using the periodicity condition and uniqueness of initial conditions, we have that the periodicity condition is preserved.[2]

Given the periodicity condition (4.2), it is a fact that the map $\gamma : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \mathbb{C}$, must be a representation of the covering group, as the following argument demonstrates. Let σ_1, $\sigma_2 \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. Then we have the following equalities

$$\gamma_{\sigma_1\sigma_2}\psi(\hat{q}) = \psi(\sigma_1\sigma_2\hat{q}) = \gamma_{\sigma_1}\psi(\sigma_2\hat{q}) = \gamma_{\sigma_1}\gamma_{\sigma_2}\psi(\hat{q}).$$

We therefore conclude the fundamental relation

$$\gamma_{\sigma_1\sigma_2} = \gamma_{\sigma_1}\gamma_{\sigma_2}. \tag{4.3}$$

We thus have that each periodicity condition corresponds to a character of the covering group. The periodicity condition also forms a character of the fundamental group since the fundamental group and covering group are isomorphic.

We have the following principle, which shall be denoted by $\mathrm{AQP}_{\widehat{\mathcal{Q}}}$:

Principle 2 (Abelian Quantization Principle, $\widehat{\mathcal{Q}}$). *To specify a Bohmian theory, one needs to pick a Riemannian manifold $\mathcal{Q}$, a cc-Hermitian vector bundle E over $\mathcal{Q}$, a potential V, and a character γ of the fundamental group $\pi_1(\mathcal{Q})$. The Bohmian theory $(\mathcal{Q}, E, V, \gamma)_{\widehat{\mathcal{Q}}}$ is defined by* (3.4) *where v^{ψ_t} is the projection to $\mathcal{Q}$ of the Bohmian velocity field defined on $\widehat{\mathcal{Q}}$ and the wave function ψ_t is a section of the lifted bundle $\widehat{E}$ satisfying* (4.2) *and locally satisfying* (3.4b)*.*

4.3.1 Remarks

1. The trivial periodicity condition, $\sigma \mapsto 1$ for all σ, corresponds to lifted wave functions. Indeed, if we lift a wave function from $\mathcal{Q}$ to $\widehat{\mathcal{Q}}$, then, for every $\hat{q} \in \widehat{\mathcal{Q}}$ and $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, the wave function satisfies the periodicity condition

$$\psi(\sigma\hat{q}) = \psi(\hat{q}).$$

[2] Assume that ψ_0 satisfies the periodicity condition. Let ψ_t be the solution of Schrödinger's equation with that initial condition. Define $\phi_t^\sigma := \gamma_\sigma\psi_t$. By linearity, this is a solution to Schrödinger's equation. As stated above, ψ_t^σ is also a solution. We claim that $\psi_0^\sigma = \phi_0^\sigma$. This is clear since $\psi_0^\sigma(\hat{q}) := \psi_0(\sigma\hat{q}) = \gamma_\sigma\psi_0(\hat{q}) =: \phi_0^\sigma(\hat{q})$. Thus, by uniqueness of initial conditions, $\psi_t^\sigma = \phi_t^\sigma$ which implies that the periodicity condition is preserved.

2. Having a projectable Bohmian velocity field places no constraint on the amplitude. One might then wonder whether the amplitude is allowed to vary based on $\hat{q}$. Generically, such a condition will not be preserved under the Schrödinger evolution. In other words, if $r_\sigma(\hat{q})$ is the proposed amplitude, then in order to argue that it is preserved, we should have $\Delta r_\sigma \psi = r_\sigma \Delta\psi$. That is to say, we need $(\Delta r_\sigma)\psi + \nabla r_\sigma \cdot \nabla\psi = 0$. If r_σ is independent of ψ, we see that ∇r_σ must vanish and hence r_σ is a constant.

3. What if r_σ is a constant? In standard quantum mechanics, the condition on wave functions on the covering space is that their densities should project down unambiguously [45]. As this is not our motivation, one might wonder why γ_σ should be of modulus one. If the fundamental group is finite, or even just idempotent, then the constant must be a root of unity as a result of γ being a representation. If the group is not idempotent, then the representation properties do not demand that γ_σ be of modulus one. But if γ_σ has a modulus not equal to one, then $|\psi(\sigma^m \hat{q})| = r_\sigma^m \psi(\hat{q})$, for any integer m; our amplitude is growing exponentially in space as we proceed from level to level. With such a growth, there is reason to believe that the evolution of the system will be ill-defined.

4. For vector bundles, multiplying by a scalar is not the only possibility. We could imagine replacing γ with a matrix, Γ. Again, the problem is preservation of the condition under the Schrödinger evolution. In particular, it is necessary that $[\Gamma, V] = 0$. This could happen, e.g. V is a scalar function times the identity, but it should not be expected. In section 5.4.1, we discuss this possibility in the context of spin and demonstrate that it is not possible.

5. Another question is whether the periodicity condition characterizes all projectable wave functions. This is clearly false since any real wave function has a zero Bohmian velocity field and hence it projects unambiguously. If we make the more important restriction that the wave function should be projectable for all time, it still fails. Indeed, it is possible to have on the covering space a wave function which is essentially a real eigenstate of the potential. It is therefore invariant

and its velocity field vanishes implying projectability. This is not a problem as our goal is to formulate a theory. We are not interested in having just one wave function, but we are rather demanding a robust class of wave functions that our theory will have as initial conditions.[3]

6. Different characters may give rise to equivalent theories. In the examples we discuss, the theories are not equivalent, and this is generically the case. If our bundles are 1-dimensional, then they are distinct. We shall say more about this in Section 5.5 in a context for which this can be more readily understood.

4.4 Different connections

We have described a class of Bohmian theories that arise by considering those wave functions, defined on the covering space of the configuration space, which provide an unambiguous evolution of the configuration. We motivated that line of pursuit with the circle in the situation of the Aharonov-Bohm effect. In this section, we shall pursue theories that arise by considering different connections, i.e. different possibilities for what the derivative is. This approach has the advantage that it does not depend on Bohmian considerations. Instead of changing the base space, we change the bundle structures of the value space of the wave function. We automatically have a well-defined Bohmian velocity.

[3]It can be argued that a wave function which is projectable for all time, but does not satisfy the periodicity condition, must have the property that the nodal set is of codimension 1. Since wave functions generically have a nodal set of codimension 2, as both the real and imaginary parts must vanish, we can expect that the condition is only preserved under very special conditions such as the wave function being an eigenstate and, therefore, not evolving. This question was pursued in the context of identical particles in [7]. As a particular example in that setup, and one which should perhaps be read after the Chapter 5, consider the hydrogen atom with two electrons and no interaction between the electrons. Tensor the ground state with a real-valued excited state. The state is then essentially invariant under the evolution and has no particular symmetry. Since it is essentially real, the Bohmian velocities are zero and, hence, consistent. We would expect, however, that under a small perturbation of the potential (or including interactions) the states would evolve in such a way as to lose their consistency condition.

4.4.1 The circle

We start with the circle. The usual wave functions over the circle are sections of the trivial bundle $S^1 \times \mathbb{C}$. For trivial bundles, there is a natural choice for a derivative; namely, form the difference quotient. Since it is a trivial bundle, there is no issue in comparing values in different fibers. But it is a fact that this trivial connection is not the only flat connection on the circle. Indeed, any real-valued 1-form defines a new connection and if the integral of the 1-form around a loop is not zero, then the new connection is fundamentally different from the trivial connection.

We claim that the set of complex numbers of modulus one are in bijective correspondence with the distinct classes of connections. Define the real-valued 1-form $d\theta$ on the circle such that its integral around the circle is 2π.[4] Let ∇ be the trivial connection associated with $S^1 \times \mathbb{C}$. Then $\nabla + i\frac{\beta}{2\pi}d\theta$ defines a flat connection; the parallel transport of v along the path α is given by

$$P_\alpha v = e^{\int_\alpha i\frac{\beta}{2\pi}d\theta} v.$$

Thus, around a closed loop of multiplicity n, we have that multiplication by $e^{in\beta}$ is the parallel transport operator. Note that although it is no longer the trivial flat bundle, the bundle is still trivial as a vector bundle.

As for the relation to the Aharonov-Bohm setup, the fact of the matter is that β should be the integral of the magnetic field over the disk. Furthermore, there is a vector potential which generates the magnetic field and, restricted to the circle, it is $\frac{\beta}{2\pi}d\theta$. This is essentially the original explanation of the Aharonov-Bohm effect.

4.4.2 General discussion

The circle is a nice example since it implements the idea of changing the connection on the bundle. In more general situations, this changing of the connection actually requires a change of the underlying bundle structure. We therefore formulate the general discussion in terms of constructing a new bundle out of an old bundle. This method

[4] Parametrize the circle with the multi-valued angle function θ and differentiate.

provides the appropriate connection and it may or may not change the underlying vector bundle.

Inspired by the case of fermions, as discussed in the next chapter, we shall formulate the new principle slightly differently than one might expect: tensor the original bundle with a flat cc-Hermitian line bundle. It is a fact that the flat cc-Hermitian line bundles are in bijective correspondence, up to equivalences, with the characters of the fundamental group. Indeed, the representation associated with a flat bundle is its holonomy representation; the *holonomy representation*, at q, is the map taking each element $\sigma \in \pi_1(\mathcal{Q}, q)$ to the matrix associated with parallel transporting vectors around a representative curve of σ. Flatness implies this map is well-defined, i.e. parallel transport on flat bundles only depends upon the homotopy class of the curve. For a line bundle, a matrix is just a complex number; to preserve the inner product, it must be an element of the unit circle. Therefore, every flat line bundle has an associated character.

The connection for the tensor product $F \otimes E$ of two bundles is defined using the Leibniz rule:

$$\nabla^{F\otimes E}(f \otimes e) = (\nabla^F f) \otimes e + f \otimes (\nabla^E e).$$

It can also be realized by tensoring the holonomy operators together. Computing the Laplacian, we find

$$\Delta^{F\otimes E}(f \otimes e) = (\Delta^F f) \otimes e + f \otimes (\Delta^E e) + 2(\nabla^F f) \cdot \otimes (\nabla^E e).$$

Furthermore, it is an easy fact that in computing the curvature, the cross terms cancel rather than add.[5] Thus, if E has curvature and F is a flat line bundle, then $F \otimes E$ will have the same curvature as E. Any potential V, defined on E, goes to $\mathrm{Id} \otimes V$, which may reasonably be viewed as V since F is one dimensional, i.e $\mathrm{Id} = 1$. Thus, we have

[5]The curvature, ω, is the antisymmetric part of the second derivative. In particular, for a tensor product, we have

$$\begin{aligned} D^2(f \otimes e)(x,y) - D^2(f \otimes e)(y,x) =& D^2 f(x,y) \otimes e + Df(x) \otimes De(y) + f \otimes D^2 e(x,y) + \\ & Df(y) \otimes De(x) - (D^2 f(y,x) \otimes e + Dfy \otimes De(x) + \\ & f \otimes D^2 e(y,x) + Df(x) \otimes De(y)). \end{aligned}$$

Thus, we have $\omega^{F\otimes E} = \omega^F \otimes \mathrm{Id} + \mathrm{Id} \otimes \omega^E$. In particular, if F is a flat line bundle, we can appropriately say that the curvature is unchanged under tensoring with F.

the Abelian Quantization Principle based on changing the connection; we shall denote it by AQP_∇.

Principle 3 (Abelian Quantization Principle, ∇). *To specify a Bohmian theory, one needs to pick a Riemannian manifold $\mathcal{Q}$, a cc-Hermitian vector bundle E over $\mathcal{Q}$, a potential V, and a character γ of the fundamental group $\pi_1(\mathcal{Q})$. Let F be the flat cc-Hermitian line bundle corresponding to γ. The Bohmian theory $(\mathcal{Q}, E, V, \gamma)_\nabla$ is defined by* (3.4) *where the wave function ψ_t is a section of the bundle $F \otimes E$.*

4.4.3 Equivalence of the two approaches

The bundle approach and the covering space approach are completely equivalent as we shall now explain. We start with the bundle E and denote its holonomy operator for the path α by P^E_α.[6] We now have two approaches for forming theories. One is to consider various periodicity conditions, involving scalars, on the covering space; the other is to tensor E with the various flat line bundles. Both are characterized by a character of the fundamental group. We contend that either way produces equivalent theories for equivalent characters.

Before we produce the equivalence, we first discuss the canonical isomorphism between the covering group and the fundamental group based at a point. There are two natural bijections and it is the less obvious one which is relevant, as we now explain. Let ρ be the projection map from $\widehat{\mathcal{Q}}$ to $\mathcal{Q}$. Then $\rho(\alpha) = \rho(\sigma\alpha)$ for all paths α on $\widehat{\mathcal{Q}}$ and for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. This is trivially true since the covering transformations map each covering fiber to itself and ρ projects all points of the fiber to the same base point. Given $\hat{q} \in \widehat{\mathcal{Q}}$, and $q := \rho(\hat{q})$, we can define an isomorphism between $\pi_1(\mathcal{Q}, q)$ and $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. The obvious bijection from $Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \pi_1(\mathcal{Q}, q)$ is to map the covering transformation σ into the projection of any path taking $(\hat{q} \to \sigma\hat{q})$. This is fine as a bijection, but it fails to be an isomorphism.[7] To find an isomorphism, we map it to the

[6] We should emphasize that we are not assuming that the connection on E is flat. Thus, the holonomy operator depends on the path itself and not just the homotopy class of the path.

[7] To be an isomorphism, we would need $\rho(\hat{q} \to \sigma\tau\hat{q}) \sim \rho(\hat{q} \to \sigma\hat{q}) \circ \rho(\hat{q} \to \tau\hat{q})$. We certainly have $\rho(\hat{q} \to \sigma\tau\hat{q}) \sim \rho(\tau\hat{q} \to \sigma\tau\hat{q}) \circ \rho(\hat{q} \to \tau\hat{q})$ and therefore only need $\rho(\hat{q} \to \sigma\hat{q}) \sim \rho(\tau\hat{q} \to \sigma\tau\hat{q})$. But, in actual fact, we can only conclude $\rho(\hat{q} \to \sigma\hat{q}) \sim \rho(\tau\hat{q} \to \tau\sigma\hat{q})$ by applying τ to the left hand side. If the

inverse of the above map. More precisely, map σ into $\rho(\sigma\hat{q} \to \hat{q}) \sim \rho(\hat{q} \to \sigma^{-1}\hat{q})$. This is an isomorphism as is easy to check.[8] Hence, we have a representation of the fundamental group if and only if the image under this isomorphism is a representation of the covering group. Abstractly, they will be the same representation. This isomorphism is exactly what comes out of the following equivalence.

Theorem 1. *The principles generate equivalent Bohmian theories, i.e.*

$$(\mathcal{Q}, E, V, \gamma)_{\widehat{\mathcal{Q}}} \simeq (\mathcal{Q}, E, V, \gamma)_{\nabla}.$$

Proof. Let F be the flat line bundle associated with γ. The proof will go as follows. We start with $(\mathcal{Q}, E, V, \gamma)_{\nabla}$. We lift the bundle $G := F \otimes E$ and show that the lifted sections are essentially the sections of the lift of E which satisfy the periodicity condition γ. We also show that this mapping commutes with the evolutions.

We start by lifting the bundle F to the covering space; as usual, we denote its lift by $\widehat{F}$. Since $\widehat{F}$ is flat and $\widehat{\mathcal{Q}}$ is simply connected, it is trivializable. Let ϕ be a trivialization map of $\widehat{F}$. That is to say, $\phi : \widehat{F} \to \widehat{\mathcal{Q}} \times \mathbb{C}$ should be a bundle isomorphism preserving the connection, i.e. $\nabla(\phi(\psi)) = \phi(\nabla\psi)$ for any section ψ of $\widehat{F}$. We now lift G to $\widehat{G} = \widehat{F} \otimes \widehat{E}$. The Bohmian theory on $\widehat{G}$ with the trivial periodicity condition is completely equivalent to the Bohmian theory on G. Apply $\Phi := \phi \otimes \mathrm{Id}$ to $\widehat{G}$; this is clearly an isomorphism between $\widehat{G}$ and $\widehat{E}$. Furthermore, it commutes with the covariant derivative due to ϕ being connection-preserving. We also know that it commutes with all potentials since it is essentially a scalar. More to the point, if ψ_t is a section of $\widehat{G}$ and it is a solution to the appropriate Schrödinger's equation, then we have

$$i\frac{\partial \psi_t}{\partial t} = H\psi_t \Leftrightarrow i\frac{\partial \Phi(\psi)_t}{\partial t} = H\Phi(\psi)_t \tag{4.4}$$

$$v^{\psi_t} = v^{\Phi(\psi)_t}. \tag{4.5}$$

Thus, once we establish the claim below, we will have shown that these theories are equivalent.

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covering group is not abelian, then this bijection is not an isomorphism, but an anti-isomorphism.

[8] To be an isomorphism, we would need $\rho(\hat{q} \to (\sigma\tau)^{-1}\hat{q}) \sim \rho(\hat{q} \to \sigma^{-1}\hat{q}) \circ \rho(\hat{q} \to \tau^{-1}\hat{q})$. Since $\rho(\hat{q} \to (\sigma\tau)^{-1}\hat{q}) \sim \rho(\tau^{-1}\hat{q} \to \tau^{-1}\sigma^{-1}\hat{q}) \circ \rho(\hat{q} \to \tau^{-1}\hat{q})$, we need $\rho(\hat{q} \to \sigma^{-1}\hat{q}) \sim \rho(\tau^{-1}\hat{q} \to \tau^{-1}\sigma^{-1}\hat{q})$ which is what we have by applying τ^{-1} to the left hand side.

Our claim is that ψ_G is a lifted section of G if and only if $\psi_E := \Phi(\psi_G)$ satisfies $\psi_E(\sigma\hat{q}) = \gamma_\sigma \psi_E(\hat{q})$. To see this, consider what the partial trivialization is. Given points $\hat{q}, \widehat{r} \in \widehat{\mathcal{Q}}$, then the trivialization ϕ satisfies

$$\phi_{\widehat{r}} = P^{\widehat{\mathcal{Q}}\times\mathbb{C}}_{\hat{q}\to\widehat{r}} \phi_{\hat{q}} P^{\widehat{F}}_{\widehat{r}\to\hat{q}} = \phi_{\hat{q}} P^{\widehat{F}}_{\widehat{r}\to\hat{q}};$$

this follows from the fact that the trivialization is parallel and that parallel transport on the trivial flat bundle is trivial. Thus, if we map ψ_G, a lifted section of G, using this partial trivialization to a section of $\widehat{E}$, ψ_E, we find the following:

$$\begin{aligned}\psi_E(\sigma\hat{q}) =& \Phi_{\sigma\hat{q}}\psi_G(\sigma\hat{q}) = \Phi_{\sigma\hat{q}}\psi_G(\hat{q}) \\ =& \Phi_{\hat{q}}(P^{\widehat{F}}_{\sigma\hat{q}\to\hat{q}} \otimes \mathrm{Id})\psi_G(\hat{q}) = \Phi_{\hat{q}}(P^{\widehat{F}}_{\sigma\hat{q}\to\hat{q}} \otimes \mathrm{Id})\Phi^{-1}_{\hat{q}}\psi_E(\hat{q}) = \gamma_{\rho(\sigma\hat{q}\to\hat{q})}\psi_E(\hat{q});\end{aligned}$$

it is important to note that $P^{\widehat{F}}_{\sigma\hat{q}\to\hat{q}} = \gamma_{\rho(\sigma\hat{q}\to\hat{q})}$ which is the defining property of F. This demonstrates that the mapping injects into the set of wave functions in $\widehat{E}$ satisfying the periodicity condition. We must show that it is onto. But this is just a rewriting of the above series of equations.

□

As an immediate application of this theorem, we have that the covering space description of the Aharonov-Bohm effect is equivalent to the vector potential description; it is important to appreciate the correct isomorphism in order to make the correct correspondence.

4.5 Self-adjoint extensions from cutting

We now pursue the functional analytic approach. This approach is concerned with different self-adjoint extensions of the formal Hamiltonian on the same L^2 space. We will formulate a principle based on this approach and it will correspond to the previous approaches. In particular, for each character of the fundamental group, we shall have a corresponding Bohmian theory. We first introduce the rough picture and then provide bite-sized pieces of a precise treatment.

The idea is to cut $\mathcal{Q}$ in order to make it simply connected. That is to say, we shall specify a set $\kappa \subset \mathcal{Q}$ such that $\bar{\mathcal{Q}}_\kappa := \mathcal{Q} \setminus \kappa$ is a simply connected open set. We shall also

demand that κ be appropriately smooth and minimal. If E is a bundle over $\mathcal{Q}$, then $\bar{E}_\kappa$ denotes the restriction of E to $\bar{\mathcal{Q}}_\kappa$. We also have an induced Hamiltonian acting on the sections with compact support in $\bar{\mathcal{Q}}_\kappa$. But such wave functions no longer form a domain of essential self-adjointness for H in the Hilbert space $L^2(\mathcal{Q}, E)$. We therefore need to choose a self-adjoint extension of H. Generally, there are many different extensions. For us, a self-adjoint extension will be acceptable only if the resulting Bohmian evolution cannot be used to detect which cut was used to generate the theory.

4.5.1 The circle

Let us consider our canonical example, $\mathcal{Q} = S^1$, $E = \mathcal{Q} \times \mathbb{C}$. We parametrize the circle with the angle θ. A cut κ corresponds to choosing a point; say $\kappa = 0$ in the parametrization. Once we cut it, then we may view the space as the interval $[0, 2\pi]$. We wish to consider various self-adjoint extensions of the Laplacian defined on $C_0^\infty([0, 2\pi])$. The domains of essential self-adjointness which correspond to the ones we have found before consist of functions in $C^\infty([0, 2\pi])$ satisfying the boundary condition $D^n\psi(2\pi) = \gamma D^n\psi(0)$.[9] This corresponds to the periodicity condition $\psi(\theta + 2\pi) = \gamma\psi(\theta)$ on $\widehat{\mathcal{Q}}$ and it also corresponds to having the connection $\nabla - \frac{\ln(\gamma)}{2\pi} d\theta$ on $\mathcal{Q} \times \mathbb{C}$.

It is a fact that the full set of self-adjoint extensions is a family parameterized by four parameters. The extensions above are characterized by just one parameter. How do we discard the other extensions? The idea is that the Bohmian evolution should not depend upon which cut was chosen; in this regard, a cut should be thought of as a choice of coordinates. For example, if we choose the extension with boundary conditions that ψ vanishes at both endpoints, then we know that typically, the particle's trajectory will not cross the cut point. This is the kind of dependence that we are trying to avoid. A variety of extensions have the property that the velocity field is ill-defined at the cut point; we certainly would exclude those extensions. There is one family of extensions for which the velocity field is defined and continuous at the cut point, but it is not smooth at the cut point. We exclude them as well. This leaves us with only the ones

[9] Here γ must be of modulus one so that the Hamiltonian is symmetric on this domain as an integration by parts argument demonstrates.

we had previously.

We have roughly described why the other cuts are unsuitable from our perspective. The question is now whether the boundary conditions we consider good actually satisfy being cut-independent. Choose γ_1 and γ_2 such that $\gamma = \frac{\gamma_1}{\gamma_2}$. Another cut, $\kappa' \in (0, 2\pi)$, would correspond to $[\kappa', \kappa' + 2\pi]$. A unitary equivalence map $U^{\kappa'\kappa}$ from κ' to κ, which preserves the Bohmian velocities except at the cut points, is

$$U^{\kappa'\kappa}(\theta)\psi(\theta) := \begin{cases} \gamma_1\psi(\theta) & \theta \in (0, \kappa') \\ \gamma_2\psi(\theta) & \theta \in (\kappa', 2\pi) \end{cases}.$$

As this map is constant away from the cuts, it is rather self-evident that the Bohmian velocity field is unchanged under this mapping. Furthermore, this is a bijection between the domains; choosing the γ_i so that the ratio is γ is the key fact. This ensures that the discontinuity at κ' disappears while the smoothness at κ is replaced with a discontinuity ensuring that the wave function satisfies the boundary condition. As the wave function evolution will commute with this mapping, we can see that the Bohmian evolution is independent of which cut is chosen in the formation of the Hamiltonian.

Let us make a couple of remarks about approaches that did not work. If we were to demand that the wave function's evolution is independent of the cut, then the only allowed boundary condition is the trivial periodic boundary condition; the only domain which is independent of the cut is the one in which ψ, and its derivatives, agree at the boundary. Another possibility, one which is too weak, is that of unitary equivalence. On the circle, given any extension based on a cut, we can implement a rotation on the circle and obtain a unitarily equivalent evolution based on another cut. That is to say, unitary equivalence excludes nothing.

4.5.2 What is a cut?

We first need to define what a cut is. After doing that, we consider certain illuminating examples although the most illuminating is in the context of identical particles as discussed in the next chapter.

A cut, κ, is a closed subset of $\mathcal{Q}$ satisfying

1. Its complement, $\bar{\mathcal{Q}}_\kappa$, is a simply connected set.[10]

2. If $\bar{\mathcal{Q}}_\kappa \subset A$ and A is an open, simply connected subset of $\mathcal{Q}$, then $A = \bar{\mathcal{Q}}_\kappa$.

3. The cut κ should be smooth in the sense that about any point $p \in \kappa$, there is a neighborhood V such that $V \cap \kappa$ is the union of a finite number of regular, $(n-1)$-dimensional submanifolds.[11]

For S^1, an $(n-1)$-dimensional submanifold is a point. If we remove more than one point, then the resulting space ceases to be connected. Thus, a cut κ for the circle is the removal of a single point. For the torus, T^2, an example of a cut is the removal of a horizontal circle and a vertical circle. Their intersection is typically a single point. In the usual representation of the torus as a square in the plane, the boundary of the square is the cut κ and the interior is the fundamental domain, $\bar{\mathcal{Q}}_\kappa$. Analogously, if we take a cube in $\mathbb{R}^3$ and identify opposite sides to obtain $T^3 = S^1 \times S^1 \times S^1$, then the boundary of the cube is a cut. Notice that it consists of the union of three copies of T^2. The twelve edges of the cube are the three circles each of which corresponds to an intersection of two of these surfaces. The corners correspond to the single point of T^3 in which all three manifolds intersect. In contrast, a vertical cylinder's cut is a vertical line which is a single $(2-1)$-dimensional submanifold with no self-intersections.

4.5.3 BV-equivalence and cut-independent self-adjoint extensions

We shall now make a precise expression as to what it means for the Bohmian evolution to be independent of the cut. We define two self-adjoint operators H_1 and H_2 to be *BV-equivalent* iff there exists a unitary operator $U : L^2[\mathcal{Q}, E] \to L^2[\mathcal{Q}, E]$ such that

1. $v^\psi(q) = v^{U\psi}(q)$ almost everywhere (w.r.t. Lebesgue measure)

[10] We take simply connected to also imply that the space is connected.

[11] A regular submanifold is one in which every point of it has a coordinate chart in the manifold in which the submanifold looks like a subspace in that coordinate system. This assumption of regularity is to avoid having a boundary such as the following in the plane: $\{(x, \sin(\frac{1}{x}))|0 < x \leq 1\} \cup \{(0, y)|y \in [-1, 1]\}$. That is not the kind of boundary we want. Note that in that example, any small neighborhood about the origin has an infinite number of disconnected lines arising from the intersection of the submanifold with the neighborhood.

2. $UH_1\psi = H_2U\psi$ for every ψ in the domain of H_1.

This does define an equivalence relation on the set of self-adjoint operators.[12] We shall state what these conditions imply, but we shall only give a rough argument in establishing the implications.

Since the velocity fields must be defined at the same places (almost everywhere), the supports of ψ and $U\psi$ must be the same. This implies that U is a unitary matrix-valued function over $\mathcal{Q}$, i.e. $(U\psi)(q) = U(q)\psi(q)$, with $U(q)$ a unitary matrix acting on the value space of ψ.[13] Furthermore, U should be smooth except for a set of measure zero; the set of measure zero arises since we only demand agreement up to a set of measure zero. We can then use the preservation of the velocities to conclude that U has to be constant in any region of smoothness.[14] The second condition implies that U bijectively maps the domain of H_1 to the domain of H_2.[15]

Equipped with a satisfactory notion of equivalence, we shall state precisely what cut-independence of the extension means. Given a cut κ, we define H^0_κ to be the formal Hamiltonian $-\Delta + V$ with domain $C^\infty_0(\bar{E}_\kappa)$. By H_κ, we shall mean a self-adjoint extension of H^0_κ. Then H_κ is *cut-independent* iff for each cut κ', there is a corresponding $H_{\kappa'}$ which is BV-equivalent to H_κ. Although there are many choices, we shall simply assume that a BV-equivalence map taking $H_\kappa \to H_{\kappa'}$ has been chosen; it will be denoted by $U^{\kappa\kappa'}$.

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[12]Composition of two BV-equivalences is clear enough. As for the inverse of BV-equivalence being a BV-equivalence, the first property is still clear. For the second property, the domain is the issue. To resolve it, take the adjoint of the equation and use the facts that both H_1 and H_2 are self-adjoint and U is unitary. Thus, one can conclude $U^{-1}H_2 = H_1U^{-1}$.

[13]This follows from the fact that U commutes with all of the position spectral measures. If we were discussing only scalar valued wave functions, we could then say that U is diagonalizable in the position representation. The reason for the commutation is that if P_A is a projection operator onto the wave functions with support in A, then $UP_A(\psi_A + \psi_{A^c}) = U\psi_A = P_A(U\psi_A + U\psi_{A^c})$; all that is necessary is that U preserves supports.

[14]We really mean parallel, i.e. $\nabla U = 0$. We shall generally use the term constant especially since U will generally be a pointwise multiple of the identity.

[15]From an orthodox perspective, one would presumably demand that the local density is unaffected by the unitary transformation. This immediately leads to $U(q)$. To argue that it is locally constant, one would need to invoke the commutation with the Hamiltonian. In particular, one needs $[\Delta, U] = 0$. This immediately leads to, acting on a test function ψ, $(\Delta U)\psi + \nabla U \cdot \nabla\psi = 0$. Thus, U is locally constant.

4.5.4 The principle

We now establish a family of cut-independent self-adjoint extensions, one for each character. The construction is to appeal to the covering space to obtain the relevant evolution. To do this, we show that given a periodicity condition on the covering space, we do have a corresponding cut-independent self-adjoint extension.

In many ways, the cut approach is a non-smooth version of the covering space approach. Indeed, select a fundamental domain[16] R on the covering space whose boundary is contained in the lift of κ. Given a character, γ, define

$$\Lambda_\gamma := \{\phi \in \mathcal{D}(H) | \phi(\sigma\hat{q}) = \gamma_\sigma \phi(\hat{q}) \forall \hat{q} \in \widehat{\mathcal{Q}}\},$$

the set of smooth sections of the lifted bundle $\widehat{E}$ satisfying the periodicity condition defined by γ. The unitary evolution on the covering space preserves this domain and preserves the L^2 norm of each section restricted to the region. Define the dense domain of $L^2(\mathcal{Q}, E)$

$$E_{\kappa,\gamma} := \{\psi | \exists \phi \in \Lambda_\gamma \text{ s.t. } \forall q \in \bar{\mathcal{Q}}_\kappa, \psi(q) = \phi(\hat{q}) \text{ for the unique } \hat{q} \in (\rho^{-1}(q) \cap R)\}$$

and let $\rho_R(\phi)$ be the bijection making the correspondence between the elements of Λ_γ and $E_{\kappa,\gamma}$. Then the covering space evolution defines a unitary evolution on $E_{\kappa,\gamma}$ which takes the domain to itself. Thus, if we define H_κ to be the formal Hamiltonian defined on $E_{\kappa,\gamma}$, then H_κ is self-adjoint.

To establish that H_κ is cut-independent, we choose another cut κ'. We select a fundamental domain R' on $\widehat{\mathcal{Q}}$ with a boundary formed from the lift of κ'. We do the same process again to generate $H_{\kappa'}$, choosing the same periodicity condition as the one for H_κ. The BV-equivalence mapping is extremely simple. Let $f^{RR'}$ be the map taking R to R' which is defined away from the boundary. Then the BV-equivalence map is

$$U^{\kappa\kappa'}\psi(q) = U^{\kappa\kappa'}\rho_R(\phi)(q) := \phi(f^{RR'}(\rho_R^{-1}(q)) = \gamma_{\sigma(q,f^{RR'})}\rho_R(\phi)(q).$$

In words, we take our wave function in $E_{\kappa,\gamma}$, lift it to the appropriate wave function on the covering space, using the level R, then we project down the corresponding parts of

[16] A fundamental domain is a region of the covering space which bijectively corresponds, under the projection map, to $\bar{\mathcal{Q}}_\kappa$. In more general contexts, the boundary of a fundamental domain is allowed to be quite wild.

that wave function defined over R'. If $q \in R \cap R'$, then the BV-equivalence map at q is the identity. Elsewhere, it would be multiplication by the appropriate phase induced by the periodicity condition. Thus, given a character and a cut, we have a cut-independent self-adjoint extension of the formal Hamiltonian.

As a general principle, we can state the following.

Principle 4 (Abelian Quantization Principle, H). *To specify a Bohmian theory, one needs to pick a Riemannian manifold $\mathcal{Q}$, a cc-Hermitian vector bundle E over $\mathcal{Q}$, a potential V, and a character γ of the fundamental group $\pi_1(\mathcal{Q})$. The Bohmian theory $(\mathcal{Q}, E, V, \gamma)_H$ is defined by* (3.4) *where the wave function ψ_t is a section of the bundle E and H is the cut-independent self-adjoint extension defined by γ and the cut κ.*

4.5.5 Characterization of cut-independence

Given a cut-independent self-adjoint extension whose BV-equivalence map is of the form $\gamma_q \mathrm{Id}$, then there is an associated character of $\pi_1(\mathcal{Q})$. In general, the various unitary representations of the fundamental group characterize the possible cut-independent extensions; this can be argued in a similar vein as below. As shall be explained in the next chapter, there is good reason to ignore such possibilities. Therefore, for the rest of this section, BV-equivalence maps will be multiples of the identity.

Theorem 2. *Let E be a cc-Hermitian vector bundle over the Riemannian configuration space $\mathcal{Q}$ and V a potential acting on E. Let H_κ be a cut-independent self-adjoint extension generating the Bohmian theory $(\mathcal{Q}, E, V)_{H_\kappa}$ whose BV-equivalence maps are scalar multiples of the identity. Then there exists a character γ such that $(\mathcal{Q}, E, V)_{H_\kappa} \simeq (\mathcal{Q}, E, V, \gamma)_\nabla$.*

From the equivalence proved in Section 4.4, we can conclude that the set of cut-independent self-adjoint extensions is characterized by the cut and a character.

Proof. We start with the cut-independent H_κ. Each pair $(\bar{E}_{\kappa'}, U^{\kappa\kappa'})$ is to be a generalized chart for our bundle.[17] The transition map going from the chart of κ' to the chart

[17] By generalized, we mean that we are not actually going to write down coordinate charts. Instead

of κ'' is the (BV-equivalence) map

$$U^{\kappa'\kappa''} := U^{\kappa\kappa''}(U^{\kappa\kappa'})^{-1}.$$

These maps are smooth away from the cuts κ' and κ''; we shall establish this later. We then form the disjoint union of all of these charts and define our vector bundle to be the set of equivalence classes under the relation

$$(q,v)_{\kappa'} \sim (q,w)_{\kappa''} \text{ iff } U^{\kappa'\kappa''}v = w; \tag{4.6}$$

By $(q,v)_{\kappa'}$ we mean that $q \in \bar{\mathcal{Q}}_{\kappa'}$ and $v \in E_q$. Because BV-equivalence is an equivalence relation, the relation above is indeed an equivalence relation. Quite generally, it can be shown that this setup does define a bundle as is often done in the context of physics. We shall call this bundle G. Parallel transport is defined, in a chart, as the parallel transport in E. Furthermore, H_κ defines a Schrödinger evolution with the correct formal Hamiltonian on G; cut-independence is important to show that the evolution in any of the other charts is compatible. The first property for BV-equivalence establishes that the Bohmian evolution will be the same either using H_κ or the bundle G.

All that remains is to demonstrate that $G = F_\gamma \otimes E$, for some flat line bundle F_γ. For this, consider the parallel transport of the element $v \in E_q$, $q \in \bar{\mathcal{Q}}_\kappa$, around the closed loop α, parametrized in the interval $[0,1]$. For simplicity, assume that $\bar{\mathcal{Q}}_\kappa$ and $\bar{\mathcal{Q}}_{\kappa'}$ form a cover of α and that it is possible to find t_1, t_2 such that $\alpha_t \in \bar{\mathcal{Q}}_\kappa$ for $t \in [0,t_1] \cup [t_2,1]$ while $\alpha_t \in \bar{\mathcal{Q}}_{\kappa'}$ for $t \in [t_1,t_2]$. Then parallel transport along α is

$$P^G_{\alpha_0\to\alpha_1} v = P^{\bar{E}_\kappa}_{\alpha_{t_2}\to\alpha_1} U^{\kappa'\kappa}_{\alpha_{t_2}} P^{\bar{E}_{\kappa'}}_{\alpha_{t_1}\to\alpha_{t_2}} U^{\kappa\kappa'}_{\alpha_{t_1}} P^{\bar{E}_\kappa}_{\alpha_0\to\alpha_{t_1}} v = U^{\kappa'\kappa}_{\alpha_{t_2}} U^{\kappa\kappa'}_{\alpha_{t_1}} P^E_{\alpha_0\to\alpha_1} v$$

where we used the fact that the BV-equivalence maps commute with the parallel transport operators which is especially true as they are just scalars. Thus, we have that parallel transport is just parallel transport in E times a phase factor. To finish the argument, we need to argue that the phase factor depends only on the homotopy class of the curve. This follows from two facts. First, within a chart, we can smoothly vary

these are neighborhoods with a bundle structure already implied. Our construction does not require us to go to coordinates; it only requires us to have patches that we can put together differently.

the path α without changing the phase part since BV-equivalence maps are constant away from the boundaries. Second, parallel transport is independent of the charts.

Except for establishing smoothness of the BV-equivalence maps, we have arrived at our result. The flat line bundle that we tensor with E has the holonomy given by the phase holonomy constructed from the BV-equivalences as done above.

We now establish some smoothness results. We need this to demonstrate that the transition maps are smooth where they are supposed to be. This will also establish that the appropriate wave functions in the domain of H_κ, those that are smooth enough to provide a Bohmian evolution, actually become the smooth sections of G in the construction. Although this seems like a side remark, it is actually quite important as it demonstrates the restrictions placed on the BV-equivalence maps due to property 2 of the definition for being a BV-equivalent map.

Given an operator A, let $\mathcal{D}(A)$ be the domain of A. Define $C^\infty(A) := \bigcap_{n=1}^\infty \mathcal{D}(A^n)$. Let H_κ be a cut-independent self-adjoint extension and $H_{\kappa'}$ be a BV-equivalent extension. Then, from the definition of BV-equivalence, it is clear that $C^\infty(H_\kappa)$ is mapped into $C^\infty(H_{\kappa'})$. By Sobolev regularity theorems for the Laplacian, we actually have that $C^\infty(H_\kappa)$ consists of smooth functions away from the boundary and these functions, and their derivatives, have limits as one approaches the boundaries. Therefore, the image of a smooth section under a BV-equivalence map is smooth. Since a BV-equivalence map is multiplication by a function, we can deduce that the BV-equivalence map must be smooth away from the boundaries. Furthermore, the jumps along a given cut are prescribed by the domain of the extension corresponding to that cut.

Since BV-equivalence is an equivalence relation, the same statement holds for the composition of BV-equivalence maps. Thus, the map $U^{\kappa\kappa'}U^{\kappa''\kappa}$ is smooth along κ wherever κ does not intersect either of the other cuts. One can also then understand that under the equivalence map defining the bundles, sections in $C^\infty(H_\kappa)$ that might not look smooth along the boundary do get mapped to sections that are smooth along κ in the new chart although they will generally cease to be smooth along the cut of the new chart.

□

4.6 Comparisons

In this chapter, we have described three very different approaches to finding various theories. To summarize, we have shown that there is only one Abelian Quantization Principle, i.e.,

$$(\mathcal{Q}, E, V, \gamma) := (\mathcal{Q}, E, V, \gamma)_{\widehat{\mathcal{Q}}} \simeq (\mathcal{Q}, E, V, \gamma)_{\nabla} \simeq (\mathcal{Q}, E, V, \gamma)_{H}.$$

Although equivalent, each approach had a different feeling to it. Indeed, the covering space approach is to change the domain of the wave function into a simply connected space. The bundle approach is to change the notion of derivative; to a certain extent, we can view this as changing the range of the wave function. The self-adjoint extension approach directly changes the Hamiltonian. It is not at all obvious that all three approaches should have worked and it is quite remarkable that they produce the same possibilities. We shall now attempt to contrast the three approaches.

Our first approach was to use the covering space; this is a topological solution. Indeed, since topology is responsible for these extra possibilities, it seems quite natural to attempt to trivialize the topology. The usual topological mechanism for a trivialization is the use of the simply connected covering space particularly when gradients are involved. We used Bohmian mechanics to explain why the covering space is useful. The Bohmian constraint is the condition of projectability of the velocity field. The immediate solution is the scalar periodicity condition. We demonstrated that this works for any Bohmian theory. In the next chapter, we shall show that, generically, the scalar periodicity condition is the only periodicity condition that works. The true advantage of the covering space is its simplicity. Noticing that it is a character which is at work is immediate. It also provides a single framework for all of the various possible theories. A minor disadvantage is that the covering space causes certain worries to occur. One worry, easily dismissable, but very tempting, is to be bothered by the possibility of wave functions on the covering space whose Bohmian velocities project down to the base space for all time without satisfying any periodicity conditions. The appropriate response is that the goal is to find theories, not exceptions, but many may be unsatisfied with this. Another worry, of a very technical nature, is that for spaces with infinite

fundamental groups, the wave functions satisfying a periodicity condition are not in L^2. This technical problem can be dealt with in a variety of ways, such as appealing to the other approaches or perhaps to the theory of rigged Hilbert spaces, but it is still an annoyance.

The approach of considering different self-adjoint extensions is a functional analysis approach. The issue is, after all, about the evolution of a system. But in order to consider these possibilities, we had to the cut space in such a way to make it simply connected. Then our self-adjoint extension could be thought of as describing what happens along this virtual boundary. The picture to have is that as a wave packet passes through the boundary, it picks up a phase. But considering all self-adjoint extensions is to consider a great number of possible theories. The necessary demand for the extension to be physically reasonable was that the extension was cut-independent. This notion was grounded in the Bohmian evolution not detecting which cut was used. We immediately constructed a cut-independent extension for each character; one way was to appeal to the covering space. But in this context, it seemed necessary to rule out the vast number of other possibilities, which we did. Unlike the covering space, these possibilities did form their own theory. We dismissed them on the grounds that they introduced an element assumed to be not in the physics, in much the same way that physics demands an appropriate invariance under coordinate changes.

The bundle approach is a geometric solution. As wave functions are sections of a vector bundle, the idea is to consider various different connections potentially involving various bundles. The important fact is that having a non-trivial fundamental group implies the existence of inequivalent flat bundles. This allowed us to define different Bohmian theories by tensoring the original bundle with a flat line bundle. Interestingly, the bundle approach does not seem to offer any wrong directions. We had no isolated wave functions and we had no obvious alternative theories. In terms of tensoring with a line bundle, only the flat ones are allowed and they are easily classified. One could ask the question if it is possible to have locally equivalent bundles which are not constructed in this manner. This is the case for flat bundles, as all flat bundles of a given dimension are locally equivalent to each other, but from the experience with spin as detailed in

the next chapter, it seems that the only generically permitted construction is tensoring with a flat line bundle.

One strength of the bundle approach is in the following question: Given the existence of certain structures, such as spin, necessary for the kinds of theories we want, what are the possible bundles over the configuration space that can support these structures? Although interesting, this question is not what this work is addressing. Indeed, that question applies for simply connected spaces and requires the consideration of very specific theories such as those involving spin.

As to the role of Bohmian mechanics in these considerations, the main role is to solidify the notion that wave functions are not primary objects and to explain what is of primary concern, namely the particles and their evolution. Indeed, the wave function does not need to be well-defined on the base space, only the Bohmian velocity field needs to be. To put it more succinctly, wave functions should be thought of as more in the nature of potentials, which are certain artifacts of describing the dynamics, and less in the nature of electromagnetic fields, which are actually exact objects.

And yet we are not implying that these arguments cannot be made in standard quantum mechanics. If one accepts, as an axiom, that the local density is the central object in the theory, then the same results will follow in almost the same way. The cut approach would replace the preservation of the velocity fields with the preservation of the density. In the covering space approach, the projectability of the density would replace the projectability of the velocity field. The only actual change in the arguments would be the need to use the Laplacian to demonstrate local constancy of the periodicity or boundary conditions.

4.7 Local equivalence

A hidden principle that we have found to be a useful guide is that of local equivalence. Given the data $(\mathcal{Q}, E, V)$, the various theories generated by the Abelian Quantization Principle are all locally equivalent to each other. This means that only global considerations can distinguish the theories.

What does local equivalence mean? Roughly, the equations of motion look locally the same. The objects generating the theory such as g, ∇, and V are all local objects and, in a certain sense, locally they remain unchanged. The global difference comes in from the character. Another way of saying what local equivalence is, is to consider coordinates in a neighborhood and write down the equations, restricted to that neighborhood, in those coordinates. The theories are locally equivalent if, in the appropriate coordinates, the equations are the same. Appropriate coordinates means the following. Pick a point $q \in \mathcal{Q}$ and a simply connected neighborhood of q. Choose a basis of the tangent space at q and a basis of E at q. Then pick normal coordinates for $\mathcal{Q}$ and parallel transport the basis of the vector bundle along the geodesics in $\mathcal{Q}$. One can then show that the equations are the same.

Local equivalence helps us limit the possibilities. This chapter has shown that, as far as we can imagine, AQP describes all the possible locally equivalent theories. But if we allow the dimension of the bundle fiber to change, then we can find other theories for which the theories are locally equivalent in terms of individual components, but they are not, strictly speaking, locally equivalent as the number of components is different. More to the point, if G is the tensor product of E with some flat bundle F, then there is no local isomorphism between G and E if F is not a line bundle. Nevertheless, if the initial wave function is a product state, then its evolution will locally look like it is a section of E. The relevance of such possibilities is rather unclear.

4.8 Quantization of classical systems

We finish the chapter by indicating how to quantize a classical system from a Bohmian point of view. The procedure is little more than Schrödinger quantization, i.e. writing down Schrödinger's equation. One of the uses of Bohmian mechanics is that this is all that needs to be chosen; the Bohmian velocity law is always the same. Bohmian mechanics also provides the appropriate framework in order to recover the classical system in the classical limit of the quantum theory. Finally, we use the ideas of this chapter to quantize a classical system whose force is only locally a gradient, i.e. the potential is only well-defined on the universal covering space.

4.8.1 Newtonian dynamics on a manifold

We start with a Riemannian configuration space $\mathcal{Q}$, with metric g, and a real-valued potential function V defined on $\mathcal{Q}$. This is enough information to define a classical evolution on $\mathcal{Q}$. Indeed, the Levi-Civita connection, defined by the metric, allows the acceleration vector to be defined. The metric also provides a canonical isomorphism between 1-forms and tangent vectors. Thus, given V, we have Newton's equation: $\ddot{Q} = -\nabla V$. Equivalently, Newton's equation may be written, in terms of 1-forms, as $g(\ddot{Q}, \cdot) = -dV(\cdot)$. An immediate consequence of this equation is conservation of energy, $\frac{1}{2}g(\dot{Q}, \dot{Q}) + V(Q) = E$.

4.8.2 Bohmian Quantization

The question we address is what the corresponding quantum system is. The general prescription is that our wave functions are to be complex-valued functions and the Hamiltonian is $-\Delta + V$. The wave function evolves according to Schrödinger's equation (3.2b). This is Schrödinger quantization. To complete the theory, we take Bohm's equation (3.2a) as defining the evolution of the particles.

This should be contrasted with standard quantum mechanics in which the theory is only fully specified after the relevant observables are chosen. This can be tricky since generally the momentum operator does not exist, e.g. the half-line, or the position operator does not exist, e.g. the circle. The relevant data for quantum theories does exist, as can be deduced from Bohmian mechanics, but the story becomes much more complicated and requires the finesse of a master in order to do it correctly without the aid of Bohmian mechanics.

4.8.3 Classical limit

When one quantizes a system, the main purpose is to form a quantum theory which, under certain suitable conditions, will behave like the given system. From a Bohmian perspective, this is formally clear. Indeed, just as in $\mathbb{R}^n$, we can write Schrödinger's equation, using the polar form $\psi = Re^{iS/\hbar}$, as two coupled real equations. One of

the equations is the Hamilton-Jacobi equation for the above classical system with an additional term, the quantum potential ($\frac{-\hbar^2 \triangle R}{2R}$). Thus, in any region in which this potential may be considered negligible, the system will behave classically in that region. Giving a precise story of when it is negligible is subtle; see [5]. The main lesson to learn is that in Bohmian mechanics the classical limit only involves demonstrating that the trajectories of the actual particles are approximately classical. In contrast, standard quantum mechanics must explain what objects are supposed to behave in a way in which the classical world will emerge.

We conclude this subsection with what this chapter tells us about quantization. Although the above quantization procedure does define a unique theory given V, it is not necessarily the only Bohmian theory with that classical limit. If $\mathcal{Q}$ is multiply connected, then the Abelian Quantization Principle tells us that for each character of the fundamental group, we have a different Bohmian theory involving the potential V. Section 4.7 explained that these theories are locally equivalent. Since the classical limit is largely based on a local viewpoint, we can expect that these various theories all have the same classical limit. To distinguish between them, one must consider the quantum effects.

4.8.4 Potentials on the covering space

We have been assuming that we have a time-independent formal Hamiltonian well-defined on $\mathcal{Q}$. One can ask the question about quantizing systems with a potential which is only defined on the covering space; an example is having a constant electric field along a circle. Alternatively, we could also pursue time-dependent, topologically motivated connections, such as having a time-dependent magnetic field in the Aharonov-Bohm setup. On the surface, these seem to be areas where the approaches are inequivalent. On closer inspection, these are, in fact, the same phenomena.

We start with $\mathcal{Q}$ and a 1-form ω which is closed but not exact on $\mathcal{Q}$. The model spaces to think about are the circle and the torus. We take ω to be our classical force, and we have its potential V on the covering space, but not on the base space. We therefore consider the Bohmian system on the covering space which corresponds to

$-\Delta + V$, and the relevant set of wave functions will be those that consistently project down. We claim that the periodicity condition now becomes time-dependent. More to the point, our space of wave functions will be a disjoint union over the periodicity conditions and any particular wave function will evolve from one sector to the next. Note that the space of wave functions ceases to be a linear space.

To establish what we just claimed, we start with a wave function on the covering space satisfying

$$\psi(\sigma\hat{q}, 0) = \gamma_\sigma \psi(\hat{q}, 0). \tag{4.7}$$

Our claim is that if $V(\sigma\hat{q}) = V(\hat{q}) + \beta_\sigma$ for all $\hat{q}$ and σ, then

$$\psi(\sigma\hat{q}, t) = \gamma_\sigma e^{-i\beta_\sigma t}\psi(\hat{q}, t). \tag{4.8}$$

We can establish this by noting that if ϕ is any solution to Schrödinger's equation, then

$$\phi_\sigma(\hat{q}, t) := \gamma_\sigma^{-1} e^{i\beta_\sigma t}\phi(\sigma\hat{q}, t)$$

is also a solution to Schrödinger's equation. That is a simple computation. If ψ satisfies equation (4.7), then $\psi_\sigma(\hat{q}, 0) = \psi(\hat{q}, 0)$. Hence, by uniqueness of solutions,

$$\psi_\sigma(\hat{q}, t) = \psi(\hat{q}, t)$$

which is equation (4.8). That is to say, we have time-dependent periodicity conditions. Furthermore, as above, in the classical limit the motion on $\mathcal{Q}$ should look Newtonian with force ω.

We now wish to argue that the above Bohmian system is equivalent to a Hamiltonian with a time-dependent connection, but with a fixed periodicity condition. Quite generally, we claim that the Bohmian systems

$$(-\Delta + V, \nabla) \sim (-(\nabla - itdV)^2, \nabla - itdV) \tag{4.9}$$

are equivalent under the map $\psi \mapsto e^{iVt}\psi$.[18] Again, a simple computation establishes this. If we make this change, then the time-dependent mapping cancels the time-dependent change of phase in the periodicity condition. Hence we are left with the

[18] If V depends on t, then we would replace Vt with $W := \int V dt$ and replace tdV with dW.

initial periodicity condition as the condition for all t. We may thus define a flat bundle over $\mathcal{Q}$, by using the time-independent periodicity condition. As dV is well-defined on $\mathcal{Q}$, the time-dependent connection on the bundle over the covering space induces a time-dependent connection on the bundle over $\mathcal{Q}$.

As for the cuts, it seems that time-dependent boundary conditions would be necessary. Our scheme would be the following. Cut the space and define V up to the boundary. If we use time-independent conditions, then the potential we are modeling is the one corresponding to the lift of this discontinuous potential to the covering space. The cut would certainly matter and presumably no BV-equivalence exists between the different cuts. However, if we use a time-dependent boundary condition, as suggested by the time-dependent periodicity condition, then the extension will be cut-independent.

Let us say a few words about this in relation to a quantum system on the circle. On the circle, there are closed, but not exact, 1-forms. Indeed, a constant force along the circle is represented by the the non-exact 1-form $\frac{\beta}{2\pi}d\theta$. We shall suppress the $d\theta$ as is customarily done. On the covering space we have a potential function, $\frac{\beta}{2\pi}\theta$, and, therefore, a well-defined Schrödinger's equation. The periodicity condition is $\psi(\theta + 2\pi n, t) = \gamma^n e^{i\beta n t}\psi(\theta, t)$. Such wave functions obviously define a Bohmian evolution on S^1. The change of connection which eliminates this time-dependent periodicity condition is $\frac{\partial}{\partial\theta} - i\frac{\beta}{2\pi}t$. Forming the induced line bundle over the circle, we have the trivial bundle with connection

$$\nabla := \frac{\partial}{\partial\theta} - i(\frac{\alpha + t\beta}{2\pi})$$

where α satisfies $\gamma = e^{i\alpha}$. This connection incorporates both the periodicity condition and the time-dependent connection on the covering space.

Chapter 5

Many particle systems

This chapter demonstrates that Bohmian mechanics provides an argument for why the wave function for a system of identical particles must have a certain symmetry; this should be contrasted with the claim of some that the lack of precise trajectories in standard quantum mechanics is the reason for such a constraint. There are others who have discussed the Bohmian story of identical particles [54, 49, 18, 7]. Our particular approach has the appeal that it is founded upon the Abelian Quantization Principle.

The principle implies multiple possibilities only for configuration spaces with non-trivial topology. As we shall discuss, for N identical particles, the configuration space, ${}^N\mathbb{R}^3$, is the space of all N-element subsets of physical space. The space is a manifold which is not simply connected; its fundamental group has two characters. Applying the principle leads to the Bose-Fermi alternative. In order to apply our principle in the case of spin, we define the spin bundle over ${}^N\mathbb{R}^3$. It is an unusual definition; the fiber at the point $q \subset \mathbb{R}^3$, is the N-fold tensor product of the 1-particle spin space with itself using q as the index set for the tensor product. Once we have the spin bundle, then the principle immediately leads to the Bose-Fermi alternative.

The literature on identical particles is quite extensive.[1] We will only mention some of the key papers that we managed to come across. The literature about the symmetrization postulate in non-relativistic quantum mechanics seems to have begun with [43]. Both that paper and [31] focussed on observables. This approach is still used; in [29], there is an argument for the identical particle case based on observables and decomposition of the Hilbert space into permutation representations. A critique of such

[1] We found one compilation of references for the symmetrization postulate that contained over 350 references and yet it seemed far from complete.

approaches may be found in [44]. The topological approaches began with [37] and its use of Feynman path integrals on multiply connected spaces. An approach that changes the connection on the configuration space is that of [39]; they include a brief discussion for the case of spin. An orthodox approach, using the covering space and only discussing scalar wave functions, is discussed in [45]. The version for Nelson's stochastic mechanics may be found in Nelson [47] and its relevance is that it is that description upon which this chapter is based.

We do not address the connection between spin and statistics. Our accomplishment is to demonstrate that for each type of particle, there are two choices for theories that locally look identical; the spin-statistics is an additional statement that gives the relation between spin and statistics. It is usually derived in the context of relativistic quantum field theory. In [8], the authors give an approach that does establish the relationship between spin and statistics in the context of non-relativistic theories describing the creation and annihilation of particles with anti-particles. The approach is based on making a radical change to the configuration space. Indeed, it ceases to be a manifold and they devote many pages to arguing that the topology is reasonable and simply connected. For scalar wave functions, this implies that only bosonic wave functions are allowed. For spin, they specify how two particles can annihilate or create one another; this mimics a full rotation in physical space acting on a 1-particle spin space thereby providing the spin-statistics relation. In [11], the authors use the Schwinger representation of spin to give an argument for the spin-statistics relation in non-relativistic quantum mechanics. They postulated that a certain set of axioms, which their specific construction did satisfy, would force the spin-statistics relation to be true. The conjecture is false as they later found other constructions satisfying these axioms, but not the spin-statistics relation [12].

This chapter is organized as follows. We start by explaining ${}^N\mathbb{R}^d$. We follow this by describing a set of cuts for that space. We then apply the Abelian Quantization Principle to scalar-valued wave functions to conclude that the particles are either governed by bosonic or fermionic wave functions if $d \geq 3$. Anyons appear as possibilities if $d = 2$. We make full use of the notion of a set-indexed tensor product in Section 5.3 when we

discuss spin on the natural configuration space. Our principle immediately applies and we obtain the usual Bose-Fermi alternative in the case of spin. Section 5.4.1 considers, and then rules out, certain possibilities that go beyond the general principle. In the specific case of identical particles, we give, in Section 5.7, a different kind of argument for the Bose-Fermi alternative. Section 5.8 briefly highlights why classical mechanics, involving identical particles and formulated on ${}^N\mathbb{R}^d$, are not enriched by appealing to the covering space if $d \geq 3$. The final section describes a speculative idea as to how to describe a system of distinguished particles as arising from a theory of indistinguishable particles.

5.1 The configuration space of identical particles

This section explains why the space of N-element subsets of physical space is the correct configuration space. The usual way of dealing with identical particles is to label them and then demand that the labelling is irrelevant to the motion of the particles. The configuration space which we use, and which is certainly not novel, is a way of formulating the dynamics without introducing the irrelevant structure of labelling. In what follows, we shall take $\mathbb{R}^d$ to be the physical space and we require $d \geq 2$.[2]

As a starting point, we might say that a dynamical system of N identical particles is one in which the appropriate exchange of the initial conditions results in the new solutions being the exchange of the old solutions. Note that to exchange the particles, we have already distinguished them by more than just their locations. The statement is then asserting that the dynamics does not notice the distinguishing characteristics. A stronger statement, and one more to the point, is that exchanging two identical particles has no effect on the world. It would then be logical to make no distinction between the particles in forming the configuration space of the system. Fundamentally, we have N points in physical space. Stating this mathematically, our configuration space is the space of all N-element subsets of $\mathbb{R}^d$. Instead of $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)$, the configuration is

[2]For $d = 1$, the configuration space consists of a variety of simply connected components. The principle will not apply and the statistics, in as much as they are present, only arise as boundary conditions on the Hamiltonian.

$\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N\}$. Instead of $\mathbb{R}^{dN}$, the configuration space of N identical particles is given by

$$ {}^N\mathbb{R}^d \;:=\; \{S | S \subseteq \mathbb{R}^d, |S| = N\}. $$

A more familiar form is given by the following; it is the mathematical version of starting with a labelled system and then removing the labelling. We define Δ to be the set of configurations of $\mathbb{R}^{dN}$ having coincident points, that is to say,

$$ \Delta := \{(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) \in \mathbb{R}^{dN} | \exists\ i,\ j \text{ s.t. } \boldsymbol{q}_i = \boldsymbol{q}_j\}. $$

let $\mathbb{R}^{d,N}_{\neq}$ denote the set of noncoincident points in $\mathbb{R}^{dN}$, i.e.

$$ \mathbb{R}^{d,N}_{\neq} :=\; \mathbb{R}^{dN} \setminus \Delta. $$

Let S_N denote the group of permutations of the set $\{1, \ldots N\}$. This group acts freely on $\mathbb{R}^{d,N}_{\neq}$ by permuting the indices, i.e.

$$ \sigma(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) \mapsto (\boldsymbol{q}_{\sigma 1}, \ldots, \boldsymbol{q}_{\sigma N}) $$

and the exclusion of Δ implies that, for every nontrivial permutation, there are no fixed points. Then the natural configuration space ${}^N\mathbb{R}^d$ can be (canonically) identified with the Riemannian manifold of equivalence classes under that action; more compactly

$$ {}^N\mathbb{R}^d \cong \mathbb{R}^{d,N}_{\neq}/S_N. $$

From this identification, ${}^N\mathbb{R}^d$ is itself a Riemannian manifold. The identification also correctly suggests that, in analogy with the torus, ${}^N\mathbb{R}^d$ is not simply connected.

5.2 Applying the Abelian Quantization Principle

For $d \geq 3$, the universal covering space of ${}^N\mathbb{R}^d$ is $\mathbb{R}^{d,N}_{\neq}$. As said above, a permutation naturally defines a diffeomorphism on $\mathbb{R}^{d,N}_{\neq}$; the set of these diffeomorphisms forms the covering group. The fundamental group at a point $q \in {}^N\mathbb{R}^d$ is the set of bijections of the set to itself. Both groups are isomorphic to the permutation group S_N. The only characters of S_N are the trivial one and the alternating one. The trivial character defines the bosonic wave functions which satisfy the periodicity condition

$$ \psi(\sigma\hat{q}) = \psi(\hat{q}). \tag{5.1} $$

The alternating character defines the fermionic wave functions which satisfy the periodicity condition

$$\psi(\sigma\hat{q}) = (-1)^{\sigma}\psi(\hat{q}).^{3} \tag{5.2}$$

Applying the Abelian Quantization Principle, we conclude the Bose-Fermi alternative.

The case $d = 2$ is quite different since $\mathbb{R}^{2N}_{\neq}$ is not simply connected. The fundamental group of ${}^{N}\mathbb{R}^2$ is isomorphic to the braid group, not S_N. There is a one-parameter family of different characters for this group. Bosons and fermions are among the possibilities, but there are many more. The reason that $d = 2$ is different is that a set of paths that exchanges two particles twice may not be homotopic to the identity; more to the point, these paths are in $\mathbb{R}^2 \times [0, 1]$, a space in which knots can occur. The generators for the braid group may be chosen to be a certain subset of braids that exchange two particles;[4] they satisfy the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad j \neq i \pm 1,$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

The second relation implies that a character of the braid group assigns the same complex number to each generator. If that number is $e^{i\beta}$, then the periodicity condition is

$$\psi(\sigma(\hat{q})) = e^{i|\sigma|\beta}\psi(\hat{q}),$$

where $|\sigma|$ is the, appropriately counted, number of generators defining σ. Bosons correspond to $\beta = 0$ and fermions to $\beta = \pi$. The particles corresponding to the other possibilities are usually called *anyons*. They were first suggested in [39], and their investigation began in earnest with [32, 56]. See [45] for some more details and references.

5.2.1 Identical particles moving in M

More generally, let physical space be the Riemannian manifold M. Then the configuration space for identical particles would be the N-element subsets of M, denoted by

[3] The sign of the permutation, $(-1)^{\sigma}$, is 1 if σ can be written as a composition of an even number of transpositions, and -1 otherwise.

[4] More precisely, label the particles. Then σ_i is the path exchanging particles i and $i+1$. Furthermore, these paths, which may be thought of as parametrized circles in the plane, should all have the same orientations arising from the parametrization.

${}^N M$. The tangent space of the configuration space at the point q is $\bigoplus_{\boldsymbol{q}\in q} T_{\boldsymbol{q}} M$. Its metric, if the particles all have mass m, is

$$ {}^N g(\mathop{\oplus}_{\boldsymbol{q}\in q} v_{\boldsymbol{q}}, \mathop{\oplus}_{\boldsymbol{q}\in q} w_{\boldsymbol{q}}) := \sum_{\boldsymbol{q}\in q} m g(v_{\boldsymbol{q}}, w_{\boldsymbol{q}}). $$

Then, as above, we can apply the principle to ${}^N M$. We will always have bosons and fermions as possibilities, but there may be more possibilities depending on the dimension of M and M's fundamental group.

5.3 Identical particles with spin

The textbooks, when discussing identical particles with spin s, take the configuration space to be $\mathbb{R}^{d,N}_{\neq}$ and the value space of the wave function to be the N-fold tensor product of $W := \mathbb{C}^{2s+1}$, the 1-particle value space, with itself. We denote it in various ways: $W_1 \otimes \cdots \otimes W_N = \bigotimes_{i=1}^N W_i = \bigotimes_{i\in\{1,\ldots,N\}} W_i = W^{\otimes N}$. It is tempting to say that each factor of the tensor product corresponds to each particle. But if we do this, we are assigning a label $i \in \{1, \ldots, N\}$ to each particle. On ${}^N\mathbb{R}^3$, there is no consistent way of making such an assignment. Thus it may appear that the configuration space ${}^N\mathbb{R}^3$ is incompatible with spin, but it is, in fact, quite compatible. We associate a copy of W not with each particle, but rather with every point in physical space $\mathbb{R}^3$. Even in the standard story, labelling identical particles is not allowed. That is why, even for bosons, there is a permutation action on the value space. If we wish to work on ${}^N\mathbb{R}^3$, we must define a tensor product which only labels the particles by their locations. We do this by replacing the usual tensor product's index set $\{1, \ldots, N\}$ with the N-element set $\{\boldsymbol{q}_1, \ldots \boldsymbol{q}_N\} \in {}^N\mathbb{R}^3$. Instead of using $\bigotimes_{i\in\{1,\ldots,N\}} W_i = W^{\otimes N}$, we form $\bigotimes_{\boldsymbol{q}\in q} W_{\boldsymbol{q}} = W^{\otimes q}$.

5.3.1 The spin bundle and a Bohmian theory

The use of an arbitrary finite index set is a straightforward notion. We wish to replace $\{1, \ldots, N\}$ with the N-element set T. The ordering in the usual tensor product serves as a way to map the factors of the tensor product bijectively to the elements of $\{1, \ldots, N\}$.

Analogously, a product element in the T-indexed tensor product of W with itself N-times should be thought of as a function from T into W. The obvious analogues to the usual definitions of the tensor product hold and we state them in Appendix A; one can also find this definition in [15]. Here we shall content ourselves to make a notational correspondence. The tensor product space $W^{\otimes N}$ is replaced with $W^{\otimes T}$. The product element $\otimes_{i=1}^{N} w_i$ is replaced with $\otimes_{a\in T} w_a$. The usual tensor product rules still hold. For example, $\otimes_{a\in T} x_a + \otimes_{a\in T} y_a = \otimes_{a\in T} z_a$ if, for some $a' \in T$, we have both $x_{a'} + y_{a'} = z_{a'}$ and, for all the other $a \in T$, $x_a = y_a = z_a$. This trivial extension of the commonly used version of the tensor product is exactly what we need for spin.

We begin by defining the spin bundle,[5] which we shall write as $W^{\otimes \mathcal{Q}}$. Let W be the single-particle spin space for our N identical particles. Given $q \in \mathcal{Q}$, we define the spin space *at that point* to be $W^{\otimes q}$. This defines the fibers for our spin bundle; one can use the connection described below to define the bundle structure.[6]

We need to define the connection. We shall do this by specifying the equivalent notion of parallel transport. Given two points q, q' and a path α connecting them, then we have a set bijection σ_α between q and q' given by following the N trajectories in physical space that α represents. This bijection of the index sets provides a natural isomorphism between $W^{\otimes q}$ and $W^{\otimes q'}$; this isomorphism will be the parallel transport operator, or holonomy operator, associated to α. It is clear that it depends only on the homotopy class of α. This in turn implies that the connection is flat. We shall use ∇ to denote the gradient associated with this connection.

Section 3.4 has essentially already described the Bohmian mechanics on this bundle. Indeed, the wave function is a (smooth) section of the bundle. This means that $\psi(q) \in W^{\otimes q}$ for every $q \in \mathcal{Q}$. In particular, the value space of the wave function varies from point to point. The wave function defines a velocity field on $\mathcal{Q}$ as it did before. The

[5] For spin, we shall restrict our discussion to $d = 3$. We will use the notation $\mathcal{Q} := {}^N\mathbb{R}^3$ and use $\widehat{\mathcal{Q}}$ for the universal covering space, $\mathbb{R}^{d,N}_{\neq}$.

[6] One could always define the bundle structure first by locally labeling the particles in order to define a coordinate patch, but it is more appealing to use the identification between the points that the connection provides.

equations of motion are Bohm's equation and Pauli's equation:

$$\frac{d\boldsymbol{Q}_t}{dt} = \frac{\hbar}{m}\mathrm{Im}\frac{(\psi_t, \nabla_{\boldsymbol{Q}_t}\psi_t)}{(\psi_t,\psi_t)}(Q_t) \quad \text{for each } \boldsymbol{Q}_t \in Q_t, \tag{5.3a}$$

$$i\hbar\frac{\partial\psi_t}{\partial t} = (-\sum_{\boldsymbol{q}\in q}\frac{\hbar^2}{2m}\Delta_{\boldsymbol{q}} + V_q + \sum_{\boldsymbol{q}\in q}\mu(\boldsymbol{S}^{\boldsymbol{q}}\cdot\boldsymbol{B}_{\boldsymbol{q}}))\psi_t. \tag{5.3b}$$

By $\nabla_{\boldsymbol{Q}}\psi$, we mean that we fix all the elements of Q except for $\boldsymbol{Q}$.[7] The symbol $\frac{d\boldsymbol{Q}_t}{dt}$ represents the tangent vector of the trajectory in physical space going through $\boldsymbol{Q}_t$ at time t.

5.3.2 Applying AQP$_{\widehat{\mathcal{Q}}}$

The covering space version of the Abelian Quantization Principle tells us to lift the spin bundle to the covering space and look at wave functions satisfying a periodicity condition. When we pull-back the bundle, by definition elements in the same covering fiber have the same exact spin space. We shall use the notation $W^{\otimes\widehat{\mathcal{Q}}}$ for the lifted spin bundle since the lifted bundle is identical to the bundle formed using the index set $\{\boldsymbol{q}_1,\ldots\boldsymbol{q}_N\} = q$ at the point $(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N) = \hat{q}$. We shall also use the notation $\boldsymbol{q}\in\hat{q}$ with the idea of viewing $\hat{q}$ as a set instead of as an N-tuple.

The class of wave functions for each theory generated by the principle is characterized by the periodicity condition

$$\psi(\sigma(\hat{q})) = \gamma_\sigma\psi(\hat{q}).$$

Since our space is ${}^N\mathbb{R}^3$, we have two possibilities for different classes of wave functions: bosons ($\gamma_\sigma = 1$) and fermions ($\gamma_\sigma = (-1)^\sigma$). We have reached the main conclusion.

It is useful to relate our presentation to the usual story. In the usual story, one starts with $\mathbb{R}^{d,N}_{\neq}$ and one uses the trivial bundle $W^{\otimes N}$ with its trivial connection. By a bosonic wave function, one then means a wave function satisfying

$$\psi(\sigma\hat{q}) = R_\sigma\psi(\hat{q}). \tag{5.4}$$

[7] More precisely, $\nabla\psi$ is metrically equivalent to the 1-form $D\psi$ acting on $T_q\mathcal{Q} = \bigoplus_{\boldsymbol{q}\in q}T_{\boldsymbol{q}}\mathbb{R}^3$. As we want to transform a 1-form defined on $T_{\boldsymbol{Q}}\mathbb{R}^3$, we restrict $D\psi$ to the subspace in $T_q\mathcal{Q}$ that $T_{\boldsymbol{Q}}\mathbb{R}^3$ is. Equivalently, we are transforming the 1-form on $T\mathcal{Q}$ defined by $D\psi\{(\bigoplus_{\boldsymbol{Q}'\in Q,\boldsymbol{Q}'\neq\boldsymbol{Q}} w_{\boldsymbol{Q}'})\oplus v_{\boldsymbol{Q}}\} := D\psi\{0\oplus v_{\boldsymbol{Q}}\}$.

where R_σ is the permutation of the spin factors related to the permutation of the set $\{1,\ldots,N\}$ represented by σ. To relate our spin space, $W^{\otimes\hat{q}}$, to the trivial space, $W^{\otimes N}$, at the point $\hat{q}=(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N)$, we use the bijection between $\{1,\ldots,N\}$ and $\{\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N\}$ which the N-tuple provides.

We would like to point out that our general principle would not apply in the usual formulation. Indeed, if we used the trivial bundle, and its trivial connection, on the base space, we first run into the problem that the Hamiltonian is not particularly well-defined.[8] If we then lift the trivial bundle $\mathcal{Q}\times W^{\otimes N}$ to $\widehat{\mathcal{Q}}$, we get the trivial flat bundle, $\widehat{\mathcal{Q}}\times W^{\otimes N}$. As implied in Section 5.4.1, sections of $\widehat{\mathcal{Q}}\times W^{\otimes N}$ which satisfy the trivial periodicity condition will not be invariant under the evolution. Only the fermionic and bosonic periodicity conditions are invariant under the Pauli evolution.

5.3.3 Applying $\mathbf{AQP}_H$

As for the cut version, we have the same conclusions as before and for the same reason. For the cuts of ${}^N\mathbb{R}^3$ described in Section 5.3.4, we have the boundary condition of -1 for the fermions. The BV-equivalence maps consist of multiplication by 1 or -1 depending on the cut and the regions. This approach suggests that the difference between bosons and fermions is the Hamiltonian. In particular, a wave function, whose support is simply connected, has the potential to be either bosonic or fermionic depending on which evolution is chosen. Also note that, just as an example, the other fermionic wave functions will evolve under the bosonic evolution. But they should instantaneously cease to be fermionic as well as becoming unsuitable for a Bohmian evolution.

5.3.4 Cutting ${}^N\mathbb{R}^3$

A rather pleasant exercise is to describe how to cut for $\mathcal{Q}:={}^N\mathbb{R}^3$ so that it becomes simply connected. For each i, we define κ_i, $i=1,\ldots,3$, to be the set of points $q\in\mathcal{Q}$ such that at least two points in q have equal i^{th} components. In other words,

$$\kappa_i := \{q\in\mathcal{Q} | \exists \boldsymbol{q},\tilde{\boldsymbol{q}}\in q \text{ s.t. } \boldsymbol{q}^i=\tilde{\boldsymbol{q}}^i\}.$$

[8] Note that we do know the Hamiltonian on the covering space; the spin terms in the Hamiltonian at the points $\hat{q}$ and $\sigma\hat{q}$, are related by conjugation with R_σ.

Note that $q \in \bar{\mathcal{Q}}_{\kappa_i}$ iff q can be ordered by the i^{th} component of its elements. Indeed, a corresponding fundamental domain in $\widehat{\mathcal{Q}}$ is the set $\{\hat{q} \in \widehat{\mathcal{Q}} | \boldsymbol{q}_1^i < \boldsymbol{q}_2^i < \cdots < \boldsymbol{q}_N^i\}$. From the definition of a cut, it is reasonably clear that properties 1 and 2 are satisfied; the main observation is that the paths must preserve the ordering. Thus nontrivial exchange is forbidden. As for the third property, note that the set in which three points have the same i^{th} component is a $(n-2)$-dimensional submanifold. But if we restrict the submanifold to a small enough neighborhood of such a point, then we find that it becomes a union of 3 regular submanifolds, one for each pair. We could also have a $(n-2)$-dimensional part by having two pairs of points with the same i^{th} component for each pair, but different for the different pairs, e.g. $\boldsymbol{q}_1^i = \boldsymbol{q}_2^i \neq \boldsymbol{q}_3^i = \boldsymbol{q}_4^i$. Then, in small neighborhoods, this appears as the union of two submanifolds. If all of the i^{th} components are equal, then there are $\binom{N}{2}$ submanifolds in the local union. In analogous fashion, all of the self-intersections are locally the union of regular submanifolds.

5.3.5 Applying $\mathbf{AQP}_\nabla$

Instead of viewing fermionic wave functions as sections on the covering space satisfying a periodicity condition or as sections of the cut spin bundle satisfying certain virtual boundary conditions, we will view them as sections of a certain bundle over ${}^N\mathbb{R}^3$. For complex-valued functions, we will construct the Fermi line, a one dimensional bundle endowed with a flat connection such that parallel transporting a vector along an element σ of the fundamental group is equivalent to multiplying the vector by $(-1)^\sigma$. As for higher spins, we construct the appropriate bundle for fermions by tensoring the Fermi line with the spin bundle.

We start with some terminology. A Bose bundle is a bundle whose sections are bosonic wave functions; a Fermi bundle's sections are the fermionic wave functions. For spin 0, i.e. the scalar case, we shall refer to the Bose bundle as the Bose line and the Fermi bundle as the Fermi line. The number of particles for the discussion below will be, as usual, N.

The Bose line is the trivial flat line bundle. That is to say the Bose line is $\mathcal{Q} \times \mathbb{C}$. For higher spins, the Bose bundle is the spin bundle, a flat bundle which is inequivalent

to the trivial flat bundle. Indeed, the holonomy representation on the spin bundle is the permutation representation.[9] We shall describe the Fermi line in a moment; assume that we know what it is. The Fermi bundle for higher spins is then the tensor product of the Bose bundle with the Fermi line as the principle suggests. By construction, the holonomy representation of the fermion bundle is the alternating permutation representation, as it should be.

The question remains how to describe the Fermi line over ${}^N\mathbb{R}^3$. The Fermi line is a flat, 1-dimensional bundle such that parallel transport maps $w \mapsto -w$ along any path exchanging two particles and for any w. We shall first do this in the physically relevant case of three dimensions. For odd dimensions, the configuration space of identical particles is not orientable. That is to say, the bundle of pseudo-scalars, $\bigwedge^{Nd}(T^*({}^N\mathbb{R}^3))$, is nontrivial, i.e. the selection of a non-zero global volume form is impossible. The holonomy representation of this line bundle is the alternating character. Complexifying it preserves the holonomy and we end up with the Fermi line bundle.

We now do a different construction which works for all dimensions. Using the particle number N, form the N^N dimensional bundle $(\mathbb{C}^N)^{\otimes \mathcal{Q}}$. Parallel transport along a path is defined in the obvious way by using the bijection of the index set induced by the path. The 1-dimensional subbundle of totally antisymmetric elements is the Fermi line for N particles; symbolically, we could write the Fermi line as $\bigwedge^{\mathcal{Q}}(\mathbb{C}^N)$. To say it another way, the subspace of the fiber that we want is the top dimensional antisymmetric space obtained from $\mathbb{C}^N$ and the exterior algebra, i.e. the determinant line $\bigwedge^N(\mathbb{C}^N) \subset (\mathbb{C}^N)^{\otimes N}$. To start to understand this bundle, note that the operation of parallel transport does map this subbundle to itself. Furthermore, since parallel transport around a loop permutes the factors, it is easy to see that the total antisymmetry of the vector means the holonomy representation is equivalent to the alternating character. Explicitly,

$$\bigwedge_{\boldsymbol{q}\in q} w_{\boldsymbol{q}} \overset{\Gamma_\sigma}{\mapsto} \bigwedge_{\boldsymbol{q}\in q} w_{\sigma\boldsymbol{q}} = \det(\Gamma_\sigma) \bigwedge_{\boldsymbol{q}\in q} w_{\boldsymbol{q}} = (-1)^\sigma \bigwedge_{\boldsymbol{q}\in q} w_{\boldsymbol{q}}$$

[9] By *permutation representation*, we mean any representation equivalent to the one acting on $(\mathbb{C}^k)^{\otimes N}$ which takes any N-element permutation into the operator R_σ which does the same permutation on the tensor product index set N. The *alternating permutation representation* shall mean any representation equivalent to $\sigma \mapsto (-1)^\sigma R_\sigma$.

where Γ_σ is the holonomy operator associated to the path σ and each w_q is an element of $\mathbb{C}^N$.

5.3.6 Generalization

There is an obvious way to generalize the above story. Let M be a Riemannian manifold and let it be the space in which the particles move. Furthermore, let E be a cc-Hermitian bundle over M; E should be the 1-particle value space. Then to generate a theory of identical particle motion, we do the same as we did before. Let $\mathcal{Q} := {}^N M$. We define $E^{\otimes \mathcal{Q}}$ to have the fiber $E^{\otimes q} := \bigotimes_{\boldsymbol{q} \in q} E_{\hat{q}}$. We again define parallel transport by using the induced paths in M generated by a path in $\mathcal{Q}$. The parallel transport of a product element is the parallel transport of the factors in the product over the paths in physical space. In the spin bundle, parallel transport of a factor was trivial. We shall refer to $E^{\otimes \mathcal{Q}}$ as the position product bundle of E over M and $\bigoplus_{\mathcal{Q}} E$ as the position sum bundle of E over M.

Applying the Abelian Quantization Principle in any of its forms, we again have bosons and fermions as before. We may also have additional possibilities depending on the dimension of M and its fundamental group.

5.4 Beyond the principle?

This section describes what happens if we try to go beyond the Abelian Quantization Principle. We describe this in all three approaches as each has its own flavor to how the possibility is rejected. Of course, technically we only need to do this once. Although we are giving this argument in the context of identical particles with spin, this is really a discussion about whether the Abelian Quantization Principle is formulated in an appropriate level of generality.

5.4.1 Beyond AQP$_{\widehat{\mathcal{Q}}}$?

The covering space formulation suggests that the wave function on different levels should differ only by a multiplicative constant. Our motivation was the realization that this

is an invariant condition guaranteeing that the wave function provides a Bohmian evolution on the base space. Looking at Bohm's equation, it is clear that if we replace the constant scalar with a constant unitary matrix, then we again have a projective Bohmian velocity field. We thus formulate our periodicity condition as

$$\psi(\sigma\hat{q}) = \Gamma_\sigma(\hat{q})\psi(\hat{q}) \tag{5.5}$$

where Γ_σ is locally constant.[10] The condition of invariance under Pauli's equation is a very strong condition; whether Γ_σ is unitary or not, invariance demands that Γ_σ is a multiple of the identity. This is what we shall now explain.

Invariance means that if an initial ψ_0 satisfies (5.5), then its solution will satisfy (5.5) for all later times. Let ψ_t be the solution to Pauli's equation with initial condition ψ_0 and assume that (5.5) holds for all time. Using Pauli's equation, we compute

$$H\Gamma_\sigma\psi_t = H(\psi_t \circ \sigma) = i\hbar\frac{\partial(\psi_t \circ \sigma)}{\partial t} = i\hbar\frac{\partial(\Gamma_\sigma\psi_t)}{\partial t} = i\hbar\Gamma_\sigma\frac{\partial\psi_t}{\partial t} = \Gamma_\sigma H\psi_t;$$

of critical importance to this computation is the easy fact that if ψ_t is any solution of Pauli's equation, then $\psi_t \circ \sigma$ is a solution as well.[11] We thus have the fundamental relation

$$[H, \Gamma_\sigma]\psi_t = 0. \tag{5.6}$$

By local constancy, Γ_σ commutes with the Laplacian. We are left with

$$[\sum_{\boldsymbol{q}\in\hat{q}}(\boldsymbol{S^q}\cdot B_{\boldsymbol{q}}), \Gamma_\sigma]\psi(\hat{q}) = 0. \tag{5.7}$$

To use this equation effectively, we fix a point, $\hat{q}$, and parallel transport our objects from every $\hat{q}'$ to $\hat{q}$. Parallel transport preserves (5.7). Since Γ_σ is parallel, we now have

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[10] *Local constancy* means that we can obtain Γ_σ from parallel transporting it along a curve; equivalently, its covariant derivative satisfies

$$\nabla\Gamma_\sigma = 0.$$

Although it makes no sense to ask if it is globally constant since the vector fibers themselves depend on $\hat{q}$, it does make sense to ask if $\Gamma_{\sigma_1}(\hat{q}) = \Gamma_{\sigma_1}(\sigma_2\hat{q})$. This equation does not hold; instead we have to apply permutations: $\Gamma_{\sigma_1}(\sigma_2\hat{q}) = R_{\sigma_2}\Gamma_{\sigma_1}(\hat{q})R_{\sigma_2}^{-1}$ where R_{σ_2} is the appropriate permutation matrix representing the parallel transport operator for spinors.

[11] This follows from the fact that composing ψ with σ does not affect the Laplacian and that the spin terms for our bundle are equal at different points along the same covering fiber. In the standard story, a permutation in spin space would need to be invoked.

a family of equations constraining $\Gamma_\sigma(\hat{q})$. For a generic Hamiltonian, the $\boldsymbol{B}$ field will vary from point to point and we are led to the conclusion that the commutation relation holds for all spin terms. Furthermore, if ψ is sufficiently generic, then we would expect the commutation relations to hold for all spinors in the spin space at $\hat{q}$. We can then say that Γ_σ must commute with the $3N$ element set $\bigcup_{\boldsymbol{q}\in\hat{q}}\{S_x{}^{\boldsymbol{q}}, S_y{}^{\boldsymbol{q}}, S_z{}^{\boldsymbol{q}}\}$; this set is a set of generators for an irreducible representation.[12] Thus, by Schur's lemma, Γ_σ must be a multiple of the identity. Again, by local constancy, the scalar multiple is a global constant. We have the Bose-Fermi alternative and we find that we failed to go beyond the principle.

5.4.2 Beyond $\mathbf{AQP}_H$?

The cut approach goes in much the same way. The difference is that in the very formulation, we are looking for a domain of self-adjointness. Just being symmetric implies, roughly speaking, that the boundary conditions need to commute with the Hamiltonian. Thus, we immediately have the commutation relation between the representation of the fundamental group and the spin representation leading again to the fundamental group's representation being essentially one-dimensional.

5.4.3 Beyond $\mathbf{AQP}_\nabla$?

As for going beyond AQP_∇, we need to ask a different question. Our original question was modest; we only wanted to find a method for constructing different theories based on a given bundle. We formulated a principle as to how to do this and we established that it was equivalent to what we had done in the covering space approach. But we could also try to find all bundles that are locally equivalent but globally inequivalent. As an example, we might know that the theory we want has the property that the

[12] $W^{\otimes q}$ is an irreducible representation space of $\times_{\boldsymbol{q}\in q} SU(2)$, the q-indexed group product of $SU(2)$. This follows because the single-particle spin space, W, is an irreducible representation space of $SU(2)$; tensor products of representation spaces naturally represent the group product; and, by Proposition 4.14 on page 82 in [17], such product representations are irreducible if the factors themselves are irreducible. The proof in that text uses the theory of characters of a representation. To avoid confusion, please note that we are not discussing the natural representation of $SU(2)$ on $W^{\otimes q}$ which is a reducible representation.

associated bundle E is endowed with a flat connection. Then the question reduces to characterizing all flat bundles of a given dimension. We shall give the well-known answer to this question, and then explain how one might exclude these possibilities based on other constraints. In particular, the presence of the spin structures prevents us from considering all flat bundles over ${}^N\mathbb{R}^3$.

We wish to characterize all flat bundles over $\mathcal{Q}$. Each flat bundle over $\mathcal{Q}$ corresponds to a different Bohmian theory; locally each flat bundle is equivalent, but globally these are inequivalent flat bundles and distinguished by their holonomy representations. If the flat bundle has a parallel inner product, then the representation is unitary under that inner product. After taking into account the appropriate equivalence relations,[13] it is a fact that for each unitary representation of $\pi_1(\mathcal{Q})$ acting on W, there is exactly one flat bundle with a parallel inner product.[14] For each flat bundle, we have a Bohmian theory; it is a standard fact that equivalent flat bundles provide equivalent theories.[15] Thus, for flat bundles whose potentials are scalar multiples of the identity, we have a different theory for each unitary representation of $\pi_1(\mathcal{Q})$.

In the case of identical particles with spin, our potentials are not just scalar multiples of the identity. We want a bundle which can support the spin terms. Our demand is that that the spin operators should locally form a parallel set of operators and be appropriately linked to the physical positions in space. Specifically, we demand that

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[13]Two flat bundles, E and F over $\mathcal{Q}$, are equivalent if there is a smooth diffeomorphism f between the two bundles such that $f(q,\cdot)$ is an isomorphism from E_q onto F_q for each $q \in \mathcal{Q}$ and the pull-back of the connection on F agrees with the connection on E. Two representations are equivalent if there is an isomorphism between the vector spaces which induces an isomorphism of the representations; that is to say, if $f : V \to W$ is an isomorphism, $\sigma \mapsto A_\sigma$ is a representation of the group G on V, and $\sigma \mapsto B_\sigma$ is a representation of the group G on W, then f demonstrates the equivalence if and only if $fA_\sigma f^{-1} = B_\sigma$. It is an easy fact that if the representations formed by the parallel transport operators on E and F are equivalent, then E and F are equivalent as flat bundles. To see this, simply take the equivalence at one point, parallel transport it around, and use the representation properties to demonstrate that this is a consistent procedure, i.e. the parallel transported isomorphism is the same as the original one. The character of the representation, defined to be the mapping from the group to the trace of its representative, is in bijective correspondence with the equivalence classes of the representations.

[14]One way to construct them is to appeal to the covering space. Let $\sigma \mapsto \Gamma_\sigma$ be the corresponding representation of the covering group. Then define the equivalence relation $(\hat{q}, v) \sim (\sigma\hat{q}, \Gamma_\sigma v)$. The corresponding bundle and induced parallel transport is the correct flat bundle.

[15]Using a connection-equivalence bundle map, say $\Phi : E \to F$, we may map wave functions to wave functions and note that since $\nabla^E \Phi\psi = \Phi\nabla^F\psi$ and Φ_q is unitary, we have the same Bohmian velocities. As Φ also suitably commutes with the Hamiltonians, we have the same Bohmian evolution.

for each element $\boldsymbol{q} \in q \in \mathcal{Q}$, there is an associated spin vector, $\boldsymbol{S}_{\boldsymbol{q}}$. As we parallel transport these operators along a path α in $\mathcal{Q}$, the spin vector associated to $\boldsymbol{q}$ should be mapped into the spin vector associated with $\alpha_{\boldsymbol{q}}(t)$, the point in physical space at time t that $\boldsymbol{q}$ is mapped to via α.[16] Around a representative of a nontrivial element in $\pi_1(\mathcal{Q}, q)$, the spin vectors are permuted.[17] The holonomy operators are responsible for implementing this permutation. By using Schur's lemma appropriately, we can argue that the holonomy representation must either be the permutation representation or the alternating permutation representation.[18] Thus, the only flat bundles capable of supporting the spin terms are the Bose bundles and the Fermi bundles. The argument given above is the argument given in [39]. Thus, if we demand flatness and the presence of spin, we conclude that our principle is the most generic that we can expect.

What if we remove the constraint of flatness? Trying to characterize all bundles of a certain dimension with a prescribed curvature, is not, to our knowledge, an easy task. Our general principle allows us to construct new bundles from a given bundle, but to find all such bundles is an entirely different question. With the spin terms, we can at least substantially reduce the question. Indeed, we still demand the spin operators to be parallel and to be permuted upon transporting around any loop. Thus, along any path, Schur's lemma tells us that parallel transport along a closed curve is implemented by scalar multiples of permutation matrices. The permutation depends only on the homotopy class of the curve while the scalar will generally vary over the homotopy class. It then seems plausible to say that all bundles with the spin terms may be formed by tensoring the spin bundle with a line bundle. The question of classification then reduces to classifying all line bundles over $\mathcal{Q}$.[19]

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[16] Recall that a path $\alpha : [0, 1] \to \mathcal{Q}$ is a set of N trajectories in physical space. We denote the path starting at $\boldsymbol{q}$ by $\alpha_{\boldsymbol{q}} : [0, 1] \to \mathbb{R}^3$.

[17] This implies that the spin operators are not sections of the endomorphism bundle. Only the set of spin vectors is globally defined. We could define a local parallel section for each spin operator, but these cannot be extended globally on $\mathcal{Q}$.

[18] If we know the action of a matrix on all of the elements of an irreducible representation, then we know the matrix up to a scalar multiple. Since the action is that of a permutation, then the holonomy must be of the form of a character tensored with the permutation representation.

[19] From a discussion with R. Tumulka, it seemed that the key question is whether there are closed, but not exact 2-forms. This is because the curvature tensor for a line bundle is a closed 2-form. Thus, if all closed 2-forms are exact, then given a prescribed curvature, there is a 1-form which may be subtracted

5.5 Below the principle

Just as it is possible in special circumstances to formulate theories that go beyond the Abelian Quantization Principle, it is also possible to be in a situation in which the different characters give rise to equivalent theories. We have put this material here since the example is best understood at this point.

This bundle was suggested by Federico Bonetto in the context of a theory of mixed bosons and fermions; we discuss this here only in the scalar case. One approach to a system in which we have m bosons and n fermions is to to choose the configuration space to be the Cartesian product ${}^m\mathbb{R}^3 \times {}^n\mathbb{R}^3$. The bundle over that space would then be the ordinary tensor product of the the respective Bose and Fermi lines. Another possibility is to take the special $3k$-forms on ${}^N\mathbb{R}^3$ which respect the particle identity; in local coordinates, we have $d\boldsymbol{q}_{i_1} \wedge \cdots \wedge d\boldsymbol{q}_{i_k}$. Such a $3k$-form could be thought to represent k-fermions and $N-k$ bosons. What happens if we tensor this bundle with the Fermi line? As one would naively expect, this switches the fermions and bosons, i.e., it maps the $3k$-forms into $3(N-k)$-forms. Thus, the bundle of all special forms is equivalent to itself tensored with the Fermi line. In order for the two Bohmian theories to be equivalent, we would actually need the Hamiltonian to be the same under the equivalence; this seems to place some restrictions on the Hamiltonian. The interested reader can find more details about the interplay of different types of particles and ${}^N\mathbb{R}^3$ in [34] and Section 5.9.

For a simpler counterexample, take the bundle E to be the direct sum (Whitney sum) of the Fermi line, F, and the Bose line, B, over ${}^N\mathbb{R}^3$. Then the principle suggests that to find other theories, we should tensor the Fermi line, F, with E to obtain a new

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from the connection to achieve a flat connection; since flat connections have been characterized by π_1, this would conclude the argument. But closed 2-forms being exact is implied by the triviality of the second homotopy class, $\pi_2(M)$. In other words, if all continuous maps of S^2 into M are contractible in the space, then 2-forms are exact. Except for $d = 3$, $\pi_2(\mathbb{R}^{d,N}_{\neq})$ seems to be trivial. Indeed, pick two particles and place a sphere between two of them. This sphere can be contracted iff $d \geq 4$; this follows by viewing one particle as the origin and demanding that the contracting sphere never intersects the origin. We would suspect that, for $d \geq 4$, $\pi_2({}^N\mathbb{R}^3)$ is finite and therefore its cohomology is trivial. Whether or not the exceptional case $d = 3$ is an exception is not known to us. We conjecture that, just as anyons arise in two dimensions, in three dimensions we do have such 2-forms and quite possibly such bundles.

bundle G. But in fact, we have

$$G := F \otimes (F \oplus B) \cong F \otimes F \oplus F \otimes B \cong B \oplus F \cong E.$$

Assume that we had a potential $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acting on E. If one computes, one finds that $\mathrm{Id} \otimes V \mapsto \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$ under these isomorphisms. As derivatives are unaffected by these isomorphisms, we shall have equivalent theories iff $a = d$ and $b = c$.

Note that what happens in these case is the vanishing of the trace of the holonomy operators of E representing the paths whose holonomy operators in F are non-trivial. Since the character of a representation determines a representation up to isomorphism and the character of a tensor product is the product of the characters,[20] we have that G's holonomy representation is equivalent to E's holonomy representation. It seems that if we avoid this kind of degenerate situation, then we will obtain inequivalent theories.

5.6 Bundle triviality

A natural question to ask about the Bose bundles and Fermi bundles is whether these bundles are trivial or not. They are certainly not trivial flat bundles. But a bundle can have a non-trivial flat connection and still be trivial as a bundle, as is the case for the complex line over the circle. It turns out that some of the bundles are trivial and some are not. Given that information, it might then be reasonable to suggest that the spin-statistics relation would be correlated to the triviality of the bundle. More precisely, the conjecture would be that the Bose bundles for integral spin would be trivial while the Fermi bundles would be trivial for half-integral spin. This conjecture is false. In fact, there is no correlation. Here is the table of relations for $N = 2$, $d = 3$, spin s:

	$s = 2l$	$s = 2l + \frac{1}{2}$	$s = 2l + 1$	$s = 2l + \frac{3}{2}$
Bose	trivial	nontrivial	nontrivial	trivial
Fermi	nontrivial	nontrivial	trivial	trivial

[20] See [17]. Recall that the character of a representation is the mapping from the group into the complex numbers defined by taking the trace of the representatives.

for $l \in \mathbb{N}$. For $N > 2$, the Bose bundle is nontrivial for odd integral spin while the Fermi bundle is nontrivial for even integral spin. We do not know the triviality or nontriviality of the other bundles except for the Bose line which is clearly trivial. We give here a brief outline of the reason for these results. In Appendix D, we give a more complete account of these results.

We start with some facts about line bundles over ${}^N\mathbb{R}^d$. First, a flat line bundle with trivial holonomy is a trivial bundle. To trivialize, simply parallel transport a (1-element) basis to every point of the manifold. Since the holonomy is trivial, this is a well-defined operation. Second, if a flat line bundle has non-trivial holonomy, over ${}^N\mathbb{R}^d$, then it is non-trivial. As there are only two characters of the fundamental group, this is a statement directly about the Fermi line. The proof of this assertion lies in the following idea. The difference between two connections is a 1-form and for both of them to be flat, the 1-form must be closed. Furthermore, to have a nontrivial holonomy, the 1-form must be non-exact. Since all closed 1-forms on $\mathcal{Q}$ are exact, we are done. Thus the Fermi line is nontrivial. Furthermore, the configuration space's top form bundle has the holonomy of the alternating character iff d is odd. This follows by considering the wedge product of the basis elements and noting that the parallel transport along a loop transposing two particles involves the exchange of d-pairs of vectors. Thus, ${}^N\mathbb{R}^d$ is orientable iff d is even.

A useful tool in establishing the nontriviality of a vector bundle is its determinant line bundle. If E is a vector bundle of dimension l, then its determinant line is $\det(E) := \bigwedge^l(E) = (E^{\otimes l})_{Alt}$, the totally antisymmetric, 1-dimensional subbundle of the l-fold tensor product of E with itself. If E is a trivial bundle, then its determinant line is trivial. The converse fails to hold as the case of the nontrivial tangent bundle to the (orientable) 2-sphere demonstrates. We also have that the parallel transport operator on E induces a parallel transport operator on $\det(E)$. In fact, for a given path, the holonomy operator for that path in $\det(E)$ is multiplication by the determinant of the corresponding holonomy operator acting on E. This is related to why it is called the determinant line bundle. Furthermore, if E is flat, then $\det(E)$ is flat. Hence our plan

is to compute the determinant line bundle of our bundles. This is equivalent to computing the determinant of the matrices in the permutation and alternating permutation representation.

Given any of the bundles, our candidate permutation matrix will be the one corresponding to an exchange of two particles. As is easy to see,[21] we find that the determinant of the matrix for the Bose bundle is -1 raised to the power $\binom{2s+1}{2}(2s+1)^{N-2}$ while the determinant for the matrix for the Fermi bundle is -1 raised to the power $(2s+1+\binom{2s+1}{2})(2s+1)^{N-2}$. By computing when the exponents are odd, we arrive at our claims about the non-triviality.

Our final effort will be to demonstrate the triviality assertions. To do this, we need to argue that the direct sum of the Fermi line with itself, over ${}^2\mathbb{R}^3$, is trivial, i.e. $F \oplus F \simeq {}^2\mathbb{R}^3 \times \mathbb{C}^2$. The following proof was implied to exist in appendix D of [11]. The idea is to produce a trivialization. We start with writing the covering space as

$$\widehat{\mathcal{Q}} := \mathbb{R}^{2,3}_{\neq} = \mathbb{R}^3 \times \mathbb{R}^+ \times S^2,$$

i.e. center of mass, distance between the particles, and the direction from particle 1 to particle 2. To create the base space, we identify antipodal points on S^2. To find a basis for our bundle, it is sufficient to find a global basis of $S^2 \times \mathbb{C}^2$ such that both sections of the basis satisfy the alternating periodicity condition. Once we have such sections, we then extend them to $\widehat{\mathcal{Q}}$ by defining them to be constant in the other coordinates. To obtain these sections over S^2, we parameterize the sphere with spherical coordinates (θ, ϕ): $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. Then the following smooth, linearly independent sections

$$\psi_1(\theta,\phi) = \begin{pmatrix} \cos\phi \\ e^{i\theta}\sin\phi \end{pmatrix} \quad \psi_2(\theta,\phi) = \begin{pmatrix} e^{-i\theta}\sin\phi \\ -\cos\phi \end{pmatrix}$$

satisfy

$$\psi_i(\theta+\pi, \pi-\phi) = -\psi_i(\theta,\phi).$$

[21] For $N = 2$, decompose the tensor product into symmetric and antisymmetric elements. For $N > 2$, consider a matrix representing a transposition. Then again decompose it into the symmetric and antisymmetric parts with respect to those two factors.

This is all we needed. This comes from considerations in the spin-$\frac{1}{2}$ representation of $SO(3)$. Namely, the matrices $A(\theta, \phi)$ with columns ψ_1 and ψ_2, represent a rotation of angle π about the axis specified by (θ, ϕ). The essential fact is that we have a double-valued representation, i.e. $A(\theta + \pi, \pi - \phi) = -A(\theta, \phi)$.

The following works for $N = 2$. The idea is to decompose each bundle into a sum of Bose and Fermi lines. We then take pairs of Fermi lines and trivialize them. If there are an even number of Fermi lines, then we have trivialized the whole bundle as the sum of trivial bundles is trivial. An odd number implies that it is nontrivial, a fact we have already established. We start with the Bose bundles. Since we have only two particles, the bundle is the sum of a symmetric part and an antisymmetric part. The symmetric subbundle is trivial (a sum of Bose lines), while the antisymmetric part is a sum of Fermi lines. Thus, the Bose bundle is trivial if and only if the number of Fermi lines is even i.e., iff $\binom{2s+1}{2}$, the dimension of the antisymmetric part, is even. As for the Fermi bundle, its triviality depends upon the parity of $2s + 1 + \binom{2s+1}{2}$, the dimension of the symmetric part, since upon tensoring with a Fermi line, the Fermi lines become Bose lines and the Bose lines become Fermi lines. The results follow.

5.7 Another viewpoint for identical particles

We have described how a theory about identical particles should be formed in Bohmian mechanics. In this section, we shall pursue a different, very speculative, idea. The idea suggests how Bohmian mechanics can choose bosons and fermions even if we start on the wrong space. Furthermore, it demonstrates the difficulties in satisfying the intuition that permutation symmetry leads to only two classes of wave functions. It is also useful in revealing a certain strength of Bohmian mechanics. But we are not suggesting that this is the appropriate story for identical particles; that story has already been presented. Unless otherwise stated, we will be discussing complex-valued wave functions even though spin does not substantially change the discussion. The wave functions will be functions on $\mathbb{R}^{dN}$ and the configurations will be elements of $\mathbb{R}^{dN}$.

We have a system of labelled particles. In this context, a system of identical particles

requires two things. The first is to define an action of the permutations on the state of the system. Once one has that, then the particles are identical iff the action of the permutation group commutes with the evolution of the system. In other words, if A_t is the evolution operator for the system, χ is any initial condition, and σ is a permutation, then we require that $\sigma(A_t(\chi)) = A_t(\sigma\chi)$.

The second condition is straightforward to understand, but the first condition can be tricky. The action of a permutation on the particle positions is, of course, to exchange the positions. One must decide how the other objects in the theory should change. For this, we have two classes of objects. Roughly, objects considered as laws follow the particles while state objects follow the positions. As that was a bit vague, let us look at classical mechanics. In classical mechanics, the state of the system is given by the position and velocity of the particles. When a permutation acts on the sate, the tangent vector associated with the given position stays associated with the same position. But both the position and tangent vector are now associated to a different particle. In contrast, the law is given by the force which is a function of the state of the system. We do not redefine the force in order to undo the permutation; rather, we take the state as it is after the action of the permutation and evaluate the force at that position. We would therefore generally expect the initial force vectors to have changed when we act upon the initial state by a permutation. We can then see that the second condition will be satisfied in classical mechanics iff the force is invariant under permutations, i.e. we need, say for two particles, that $F(\boldsymbol{q}_a, \dot{\boldsymbol{q}}_a, \boldsymbol{q}_b, \dot{\boldsymbol{q}}_b) = F(\boldsymbol{q}_b, \dot{\boldsymbol{q}}_b, \boldsymbol{q}_a, \dot{\boldsymbol{q}}_a)$.

How do permutations act in Bohmian mechanics? That is the question which is central to this section. There are three objects in the theory: $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \in \mathbb{R}^{dN}$, ψ, H. The status of Q is clear and permutations act on it by permuting the positions, i.e. $(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \mapsto \boldsymbol{Q}_{\sigma 1}, \ldots, \boldsymbol{Q}_{\sigma N}$. The Hamiltonian, being a law, is unchanged as an object acting on the wave functions. The question is how ψ should change. The standard view of Bohmian mechanics, as stated in Section 2.1, treats ψ as part of the state of the system. Thus, we have the freedom, and the obligation, to have the permutations act on it. Since ψ generates the velocity field, we want the permutation action to be such that the velocity vector at $\boldsymbol{q}$ is unchanged when we exchange the

particles. That is to say, we want $v^{\sigma\psi}_{\sigma^{-1}i}(\boldsymbol{q}_{\sigma 1}, \ldots \boldsymbol{q}_{\sigma N}) = v^{\psi}_{i}(\boldsymbol{q}_1, \ldots \boldsymbol{q}_N)$. To understand the formula, note that if $k = \sigma^{-1}i$, then $\boldsymbol{q}_i = \boldsymbol{q}_{\sigma k}$ implying that the k^{th} particle has position $\boldsymbol{q}_i$ after the action of the permutation; hence, the velocity vector of the k^{th} particle, after the permutation, should be the same as the velocity vector of the i^{th} particle before the permutation. As can be readily checked, if $\sigma\psi := \psi \circ \sigma^{-1}$, then the above equality holds. Defining the permutation action on wave functions in this manner, we see that if H is invariant under permutations, then the system satisfies the conditions for it to be a system of identical particles. We do not arrive at any constraint on the wave function. From this viewpoint, the experimental evidence of particles being either bosons or fermions must be incorporated as an additional postulate and is not a result of the particles being identical. Succinctly, the problem is that the wave function changes covariantly under symmetries.[22]

The remedy of this problem, motivated by other reasons, is to consider the wave function as part of the law. It is our contention that the wave function should be thought of as an analog of the Hamiltonian; that is to say, the wave function defines the law of motion and can be given before the evolution of the particles is considered.[23] For details on the wave function as a law, please see [26]. This viewpoint implies that the velocity field is not a consequence of the dynamics, but rather defines the dynamics in the same way as the force does in classical mechanics. The only initial conditions are the initial positions of the particles. Accepting our assertion leads to the conclusion that the velocity field should be appropriately invariant under permutations of the configurations. This eventually leads to concluding that we have either bosons or fermions.

A natural challenge to our invoking "the wave function is a law" argument is why we

[22] As far as we can tell, this is also a problem in standard quantum mechanics as evidenced by the symmetrization postulate. In the standard story, the wave function should change in the same way, resulting in the probability density changing in just the right way so as to be insensitive to a relabelling.

[23] This is true for isolated Bohmian systems or, more appropriately, for the universal wave function. For systems that are interacting with their environment, the system's wave function, defined to be the composite wave function (system plus environment) evaluated at the actual configuration of the environment, can evolve in a way different from the Schrödinger evolution. This is the case when a wave function collapses [25]. Effectively, a subsystem is governed by two complementary evolutions: the Schrödinger evolution and an evolution based on the environment's configuration changing.

would still adhere to the usual policy of changing the wave function under a rotation or gauge transformation, but not under a permutation symmetry. The answer lies with the conditional wave function. If we have a wave function governing a system and its environment, we can write it as a function of two variables; namely, the wave function is a function of the configuration of the system and the configuration of the environment: $\psi(\text{sys, env})$. We then define the conditional wave function of the system to be $\psi(\cdot, \text{ENV})$ where ENV stands for the actual configuration of the environment. In certain situations, the conditional wave function will satisfy its own Schrödinger's equation and give rise to a Bohmian system whose trajectories for the system will agree with the actual Bohmian trajectories. See [25] for more details about the conditional wave function.

If we have a fundamental symmetry of the universal wave function, then there is, in general, no reason to assume that symmetry for a conditional wave function. For example, consider rotations. If our universal wave function is invariant under rotations of physical space, then rotating our initial universal configuration produces an equivalent motion. But now let us consider a smaller system. If we rotate the system's configuration, but not the environment's configuration, then there is no reason to assume invariance. Invariance only holds when both system and environment are transformed, not when just one of them is transformed.

Symmetry under exchange of particle positions is quite different. This symmetry is inherited by conditional wave functions. This is obvious; the exchange symmetries work for any subcollection of identical particles. Thus, if we believe that the particles are fundamentally identical, then we may assume this property for all systems. It is relevant to our earthly domain of physics and it is why our argument in this section ultimately works.

It should be quite clear that this is a Bohmian argument. In sharp contrast to the other approaches presented in this chapter, we can see no way of doing this argument in standard quantum mechanics.

5.8 Classical Mechanics and identical particles

It is interesting to contrast the Bohmian story of identical particles with that of the classical story. For identical, classical particles, we would again use the natural configuration space. The initial conditions are the positions and velocities of the particles. The law is specified by an acceleration field on configuration space.

If we demand that the acceleration field comes from a potential, we might expect the covering space to be relevant. But lifting to $\mathbb{R}^{d,N}_{\neq}$ does not achieve any greater generality. To explain why $\mathbb{R}^{d,N}_{\neq}$ is not useful, consider the periodicity condition for the proposed potential V:

$$V(\sigma\hat{q}) = V(\hat{q}) + \gamma_\sigma.$$

This is the general condition that will lead to the same gradient at each level of the covering fiber. The map $\sigma \mapsto \gamma_\sigma$ forms an additive representation of the fundamental group. It is impossible to form an additive representation, over the reals, of a finite group. For example, consider a transposition. Writing down the periodicity conditions leads to $2\gamma_\sigma = 0$. There is only one solution to that equation.

To put it succinctly, all closed 1-forms on ${}^N\mathbb{R}^d$, $d \geq 3$, are exact. In classical mechanics, the covering space of identical particles seems irrelevant except as a convenient setting for analysis. Incidentally, the natural configuration space is useful in classical mechanics and statistical mechanics. Indeed, it answers the Gibbs paradox of thermodynamics; see [39] for details.

5.9 Distinguished particles

The question arises as to what the proper description of distinguished particles should be. The typical viewpoint is that the parameters are attached to the particles. In other words, there are electrons, protons, etc. with various charges and masses and these are different types of particles. In a typical approach, one would take the Cartesian product of the natural configuration spaces for the various particles, possibly removing coincident points, and tensor the appropriate bundles together to form the bundle whose sections are the wave functions.

In classical mechanics, the Cartesian product seems to be the only option. We see no other way of giving the points different fundamental properties. But in quantum mechanics, we do have another option, as hinted at by Bell in [9]. The idea is that the particle differences are all incorporated into the wave function itself. We shall describe the setup here. See [35] for more discussion.

We start with the scalar wave functions and the usual theory on $\mathbb{R}^{d,N}_{\neq}$. In general, the velocity field generated by a wave function on $\mathbb{R}^{d,N}_{\neq}$ will have no particular symmetry properties. The simple idea is to symmetrize the probability current as well as the probabilties and then define the Bohmian velocity from that. We then have a well-defined velocity field on ${}^N\mathbb{R}^d$. Furthermore, if the wave function only has support in one "level" of $\mathbb{R}^{d,N}_{\neq}$, then this velocity field gives the usual motion for distinguished particles. In particular, the particles will retain their distinctive motions. We also note that the corresponding equivariant density for this new motion is the symmetrized density; one could argue that this is the only position density that one has access to. Indeed, based on the position of particles at a single time, it would be hard, if not impossible, to distinguish an electron from a proton if they were indeed point particles. It is only through their paths that we see their differences.

We can write the above theory in a better way. In fact, on ${}^N\mathbb{R}^d$, we form the $N!$-dimensional bundle obtained by summing over the covering fibers. More precisely, at $q \in {}^N\mathbb{R}^d$, we form the bundle $\bigoplus_{\rho^{-1}(q)} \mathbb{C}$. We have in mind that a wave function $\hat{\psi}$ on $\mathbb{R}^{d,N}_{\neq}$ defines $\psi(q) := \frac{1}{N!} \oplus_{\hat{q} \in \rho^{-1}(q)} \psi(\hat{q})$. Each component satisfies a Schrödinger equation and the sum over the probability currents leads to the correct symmetrized Bohmian velocity of $\hat{\psi}$.

We now give a much more general story about distinguishable particles. Fix K to be the number of different types of particles; we shall use k to index the different particles. Let $(E_k, \nabla_k, m_k, \boldsymbol{S}_k)$ be the data for particle type k and we take e_k to be the dimension of the fiber. Then the one-particle bundle is $E := \bigoplus_{k=1}^K E_k$ with the connection induced from a direct sum, e.g. $\nabla(\psi_1 \oplus \psi_2) := \nabla_1\psi_1 \oplus \nabla_2\psi_2$. The Schrödinger Hamiltonian is defined as $(G \otimes M + \sum_k \boldsymbol{S}_k) \circ D^2\psi$ where M is a diagonal matrix with the masses along the diagonal. In each component of the sum, it will look like the usual Pauli equation.

Bohm's equation is dealt with in the same way, i.e. use $G \otimes M$ to make the conversion from a 1-form to a tangent vector. It is necessary to do this before taking the inner product.

For many particles, the configuration space will be ${}^N\mathbb{R}^d$. Let $\mathcal{K}$ denote the set $\{1, \ldots K\}$. Let $A_{q/K}$ be the set of partitions of q into K sets. We shall use ν to denote a partition which may also be viewed as a map $\nu : q \to \mathcal{K}$. We denote the set of positions of type k, which is the set $\nu^{-1}(k)$, by q_k^ν. The bundle for a theory just about bosons is $E^{\otimes \mathcal{Q}}$. The fiber at q may be written as

$$E_q^{\otimes \mathcal{Q}} := \otimes_{\boldsymbol{q} \in q} E_{\boldsymbol{q}} \cong \bigoplus_{\nu \in A_{q/K}} (\bigotimes_{k \in \mathcal{K}} \{E_k^{\otimes q_k^\nu}\}).$$

The connection is defined in the same way as the spin bundle's connection: it is induced by the set bijections which paths in ${}^N\mathbb{R}^d$ provide. Parallel transport in terms of the sum over partitions requires taking into consideration that the bijection also maps one partition to another.

For a theory involving fermions, we must construct the appropriate bundles first. Given a subset r of the point $q \in \mathcal{Q}$, we define the vector space

$$F_r := \bigwedge_r \mathbb{C}^{|r|} := (\mathbb{C}^{|r|})^{\otimes r}_{Alt}.$$

We can therefore define $\bigwedge^i_{\mathcal{Q}}$ as the bundle whose fiber is the direct sum of the F_r's where $|r| = i$. If we have any path σ in ${}^N\mathbb{R}^d$ which gives a bijection between r and itself, then we have that parallel transport around that path will be $(-1)^{\alpha_r}(1)^{\alpha_{q \setminus r}}$. That is to say, it is the sign of the permutation induced on the set r by the bijection σ. In other words, we have, in a loose sense, i fermions and $N - i$ bosons. As one can easily see, the subspace of the points in r being fermions. is not invariant whereas the number of fermions is invariant under the parallel transport. In three dimensions, this is equivalent to the $3i$-form bundle of Section 5.5. The direct sum over i leads to the bundle of all mixed bosons and fermions; we denote it by $\bigwedge_{\mathcal{Q}}$.

The situation where the first j of the K type of particles are fermionic and the other types are bosonic may be easily described using these restricted Fermi lines. The

appropriate bundle is J and it is defined at a point q to be

$$J_q := \bigoplus_{\nu \in A_{q/K}} (\{\bigotimes_{k=1}^{j} F_{q_k^\nu} \otimes (E_k)^{\otimes q_k^\nu}\} \otimes \{ \bigotimes_{k=j+1}^{K} E_k^{\otimes q_k^\nu}\}).$$

Its dimension is

$$\sum_{\nu \in A_{q/K}} \binom{N}{l_1 \cdots l_K} \{\prod_{k \in \mathcal{K}} e_k^{l_k}\} = (\sum_{k \in \mathcal{K}} e_k)^N.$$

The evolution equations are the same as they always have been. Explicitly, we have

$$\frac{d\boldsymbol{Q}_t}{dt} = \hbar \mathrm{Im} \frac{(\psi_t, (G_{\boldsymbol{Q}_t} \otimes M)^{\oplus\otimes} \circ D_{\boldsymbol{Q}_t} \psi_t)}{(\psi_t, \psi_t)}(Q_t) \quad \text{for each } \boldsymbol{Q}_t \in Q_t, \tag{5.8a}$$

$$i\hbar \frac{\partial \psi_t}{\partial t}(q) = -\frac{\hbar^2}{2} \sum_{\boldsymbol{q} \in q} \{(G_{\boldsymbol{q}} \otimes M + \sum_{k \in \mathcal{K}} \boldsymbol{S}_{k,\boldsymbol{q}})^{\oplus\otimes} \circ D_{\boldsymbol{q}}^2 \psi_t(q)\} + V_q \psi_t(q). \tag{5.8b}$$

If A is an operator acting on $E_{\boldsymbol{q}}$, then by $A^{\oplus\otimes}$ we mean an operator defined to be the the direct sum over $\nu \in A_{q/K}$ of the operators defined on the tensor product as A acting on the $\boldsymbol{q}^{th}$ component and the identity on the other components. We can also write the evolution equations as

$$\frac{d\boldsymbol{Q}_t}{dt} = \frac{\hbar}{(\psi_t, \psi_t)} \sum_{\nu \in A_{q/K}} \frac{1}{m_{\nu(\boldsymbol{q})}} \mathrm{Im}(\psi_t^\nu, \nabla_{\boldsymbol{Q}_t} \psi_t^\nu)(Q_t) \quad \text{for each } \boldsymbol{Q}_t \in Q_t, \tag{5.9a}$$

$$i\hbar \frac{\partial \psi_t^\nu}{\partial t}(q) = -\frac{\hbar^2}{2} \sum_{\boldsymbol{q} \in q} \{\frac{1}{m_{\nu(\boldsymbol{q})}} \Delta_{\boldsymbol{q}} + \boldsymbol{S}_{\nu(\boldsymbol{q})} \circ (B_{\boldsymbol{q}}) \psi_t^\nu(q)\} + (V_q \psi_t(q))^\nu \tag{5.9b}$$

for each partition $\nu \in A_{q/K}$.

Chapter 6

Taylor expansions of mappings between manifolds

6.1 Introduction

This chapter is somewhat different from the other chapters. The first half of the chapter is needed for Chapter 7. The second half is answering a natural question that arose while exploring the first half.

Let M and N be two smooth manifolds with connections. Let $f : M \to N$ be smooth. In this chapter, we shall define how to expand f in the best polynomial expansion using the linear structure at hand. A connection provides a notion of a local linear structure on a manifold. Using this linear structure, one can view f as a map between open subsets of vector spaces. Thus, we can expand f using Taylor polynomials to obtain the Taylor expansion of f. But a connection also provides a method for differentiating objects. The question then arises as to whether using the covariant derivatives in the usual Taylor formula will provide us with the Taylor expansion of f. The answer is no, but it does when the range is suitably flat.

Given a manifold M and a connection on M, ∇^M, we have a natural identification between the tangent space of a point p and small neighborhoods about p via the exponential map. When we wish to view the neighborhood $U_p \subset M$ as a subset of T_pM using this identification, then we shall refer to U_p as a *linearized neighborhood*; putting linear coordinates on the tangent space leads to normal coordinates for our neighborhood. The identification allows us to discuss the linearity of functions. Indeed, let us assume that we have a map $f : M \to N$ and that N carries a connection, ∇^N. Let U_p be a linearized neighborhood about p and let $U_{f(p)}$ be a linearized neighborhood about $f(p)$. Then we can discuss linear mappings, quadratic mappings, and, in general, multilinear mappings between the neighborhoods. It therefore makes sense to expand

f itself into a Taylor series expansion. The expansion is a sum of multilinear functions, or *tensors*, evaluated along the diagonal; symbolically we have $f(v) \approx \sum D^n_T([v]^n)$. We define the n^{th} *Taylor derivative*[1] to be the tensor $D^n_T f$. The n^{th} covariant derivative, $D^n_C f$, and $D^n_T f$ are both sections of $(T^*M)^n \otimes f_*(TN)$, where $f_*(TN)$ is the pullback of the tangent bundle of N to be explained later. It makes sense to ask whether the symmetric parts agree. This would imply that using the covariant derivatives in the formulas from multi-variable calculus would be legitimate. If we have agreement, then we say that f satisfies *covariant normality* or is *covariantly normal*.

The following will be proven:

1. If N is flat, then all smooth mappings into it are covariant normality;

2. If the smooth mapping f maps geodesics to geodesics, then f is covariantly normal; and

3. Let $N = V$ be a vector bundle carrying a bundle connection ∇^V over a base manifold M carrying the manifold connection ∇^M. Then those two connections induce a manifold connection, ∇^N, on N. Using ∇^N, all smooth sections of the vector bundle are covariantly normal.

Results that are about Taylor expansions of sections of a vector bundle are in the literature, but viewing them as Taylor expansion of mappings between manifolds is not. Even allowing for this weak connection, this author could find few papers with a similar flavor to the material in this chapter. An alternative idea to the Taylor expansions for tensors is the notion of affine extensions. Roughly, an affine extension of a tensor is defined by choosing normal coordinates and differentiating the coefficients in that coordinate system. One then assembles new tensors out of this construction. This is different from what we ultimately do in such contexts in the sense that our basis for the tensor spaces can differ from the basis that the affine extension concept is based upon. It was first pursued in the 1920's and 1930's by O. Veblen, T. Thomas, and H. Ruse. More recently, it was discussed in [30] in the context of symplectic manifolds

[1]This is in honor of the usual Taylor expansions and not in honor of the author.

with connections. If one is interested in this concept, then that recent paper is the paper to learn the material from. It contains references to the original works as well.

6.2 Covariant derivatives of mappings between manifolds

We start by discussing the standard notions of the pullback bundle and the pullback connection. Given a bundle E over N, $E \xrightarrow{\pi} N$, and a manifold M with a smooth map $f : M \to N$, then the pullback bundle, denoted by $f_*(E)$, is defined by $f_*(E)_q := E_{f(q)}$. It can be shown that this has the appropriate smooth structure. Notice that a section of E pulls back to a section of $f_*(E)$, but the reverse is only guaranteed if f is a diffeomorphism. If E has a connection, ∇^E, then the pullback connection, ∇^{fE}, is defined by demanding that the connection acting on a pulled back section in the direction $X \in TM$ should agree with the connection on E acting on the section in the direction $Df(X)$; $Df : TM \to TN$ is the differential of the map f. Schematically, the definition demands

$$\{\nabla^{fE}_X [f_*(e)]\}_p := \{\nabla^E_{Df(X)}[e]\}_{f(p)} \tag{6.1}$$

where e is a section of E and $f_*(e)$ is the pull-back of e. By choosing a local frame field on E and pulling the frame back, one can establish what the connection must be in these coordinates. One then must show, straightforwardly, that the different coordinate expressions describe the same object. An informative example is when f maps M into a single point of N. Then the pullback bundle is a trivial bundle and the pullback connection is a trivial connection.

Now that we have the general definition of the pullback of a bundle, we can specialize to the tangent bundles. Connections on the tangent bundles TM and TN of the manifolds M and N will be denoted by ∇^M and ∇^N. Note that if $M \xrightarrow{f} N$ is a smooth map between M and N, then $Df \in T^*\mathcal{Q} \otimes f_*(TN)$. The Leibniz (product) rule allows us to define the tensor product of two connections. Thus, we can differentiate Df to obtain an element of $T^*\mathcal{Q} \otimes T^*\mathcal{Q} \otimes f_*(TN)$. Repeating, we can compute the derivatives of all orders of DF.[2] This material is from [28].

[2]It is useful to point out that this is different then repeatedly taking differentials. Indeed, the second

The next section discusses the product rule, chain rule, and the abuses of the tensor world. In section 6.4 we discuss the key observations that makes the entire chapter doable. We then finally discuss various versions of Taylor expansions of mappings between manifolds.

6.3 The chain rule

The chain rule for derivatives rests on two defining properties of derivatives: the Leibniz rule and the commutativity of contractions and derivatives. We first state the main results of this section and then explain the notations and proofs.

The results of this section are:

Theorem 3 (Usual Chain Rule). *Let $M \xrightarrow{F} N \xrightarrow{f} P$ be smooth maps between connection carrying manifolds. Let G denote their composition. Then*

$$D^n[G](v_1,\dots,v_n) = D^n[G](\overset{n}{\underset{i=1}{\otimes}} v_i) = \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[f](\underset{r\in R_Z}{\otimes}\{D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\})$$
$$= \sum_{k=1}^{n} D^k[f](\sum_{Z\in\mathcal{Z}(n,k)} \underset{r\in R_Z}{\otimes}\{D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\}).$$

Theorem 4 (General Chain Rule). *Let $\mathcal{W}$ be a vector bundle over M, $\mathcal{V}$ a bundle over N, and $\mathcal{X}$ a bundle over P and assume that all the bundles have a connection. Let $M \xrightarrow{F} N \xrightarrow{f} P$ be smoot maps between the base manifolds. Assume $B \in \Gamma\{\mathcal{W}\otimes F_*(\mathcal{V})\}$ and $A \in \Gamma\{\mathcal{V}^*\otimes f_*(\mathcal{X})\}$. Then $D[A(B)] = D[A](B, DF) + A(D[B])$.*

Those are the two main theorems. We will use the following two defining facts.

Statement 1 *(Tensor Input).* If A is a multilinear functional acting on $V\times W$, then A may be viewed as a linear functional acting on $V\otimes W$. Furthermore, the input (v,w) into the multilinear A is the same as $v\otimes w$ into the linear version of A.

Statement 2 *(The Product Rule).* If we have connections on the bundles E and F, both over M, then we define the tensor product connection on $E\otimes F$ by demanding

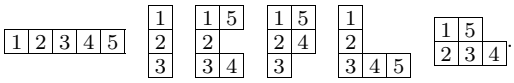
differential is a map between $T(TM)$ and $T(TN)$ whereas the second covariant derivative goes from $TM\otimes TM$ to TN. Using principal bundles, it can be show that there is a relation between the two versions of derivatives.

the Leibniz rule and commutation with contractions. Specifically, if $A \in \Gamma\{E\}$ and $B \in \Gamma\{F\}$ are any two sections, then

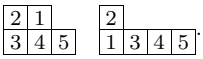
1. Leibniz Rule: $\nabla_X[A \otimes B] = \nabla_X[A] \otimes B + A \otimes \nabla_X[B]$
2. Contraction Commutation: If $C(A \otimes B)$ indicates any kind of contraction, then $\nabla_X[C(A \otimes B)] = C(\nabla_X[A \otimes B])$.

Diagrams. It is convenient to introduce the notion of a chain rule diagram or cr-diagram. It is a diagram similar to a Young tableaux, but there are important differences. In a cr-diagram, we left justify the diagrams so that the first entries lie in a column. The number of boxes in a row has no particular relation to the other rows i.e. the number can increase, decrease, or stay the same as we move from row to row. Furthermore, it is a numbered diagram. We demand that the entries in the first column strictly increase as we go down the column and the entries in a row to strictly increase as we move to the right. Thus, the number in the first box in the first column is 1. The numbers in a column other than the first column are not constrained to behave in a certain way. In particular, the numbers in the second column can either increase or decrease.

Here are several cr-diagrams:



Here are two objects which are not cr-diagrams:



We now define terminologies and notations which will be extremely helpful. A cr-diagram will be denoted by Z. The length of Z, denoted by $|Z|$, is the number of boxes in the diagram. The collection of all cr-diagrams will be denoted by $\mathcal{Z}$ and the set of cr-diagrams of length n will be denoted by $\mathcal{Z}(n)$. The height of the cr-diagram Z is the number of rows and it will be denoted by $|Z|_R$. The set of cr-diagrams of length n and height m, shall be denoted by $\mathcal{Z}(n, m)$. The entry in row r and column c will be denoted by z_{rc}. The length of row r will be denoted by $|r|$. The set of rows in Z will

be denoted by R_Z. The set of columns in row r will be C_r. The diagram $Z = \begin{array}{|c|c|c|c|}\hline 1&4&7\\\hline 2&3&5&6\\\hline\end{array}$ has the following properties: $|Z| = 7$, $|Z|_R = 2$, $z_{12} = 4$. Note that the cr-diagram of length n and height n is unique and is a single column. Also unique is the single row diagram which is of length n and height 1.

A product rule diagram (pr-diagram) is a much simpler diagram. Each row must be strictly increasing, but there are no constraints for the first column. We also need to allow for rows with no entries. We do that by putting one box for the row and putting a zero in that entry. We use the same notations as before for the cr-diagram except we shall denote such objects by PR and the set of product rule diagrams with m rows as $\mathcal{PR}_m$. Notice that the number of rows, which includes empty rows, does not change as one increases the number of derivatives.

Notations. We shall use [ ] to denote an object which is being differentiated. Hence, functions should not freely pass through. Parentheses shall be mainly used to indicate some kind of contraction. The simplest case is $A(B) = B(A)$ when $A \in E^*$ and $B \in E$, then this is just usual composition. This also defines the kernel of the contraction operation. Namely,

$$C(A \otimes B) := A(B).$$

As contractions can be more general and ambiguous, caution and excellent notation is required. A discussion of what we shall eventually need is in the appendices. For other uses of parenthetical expressions, we shall use {}.

We shall often employ $D^n[A]$ to denote the n^{th} covariant derivative acting on $A \in E$, where E is some vector bundle. Note that to define the higher covariant derivatives, we essentially need a connection on the base space. Namely, $D[A] \in E \otimes T^*\mathcal{Q}$ and therefore $D^n[A]$ is defined via the Leibniz rule if we have a connection on the bundle and its base space. Note that $D[A](X) := \nabla^E_X[A]$ and $D[A](B \otimes X) := B(\nabla^E_X[A]) = \nabla^E_X[A](B)$. Furthermore, $D_p[A]$ will denote the covariant derivative evaluated at $p \in M$. It is useful to remark that $D^2[A](X \otimes Y) \neq \nabla_Y[\nabla_X[A]]$ as the left-hand side is a tensor in both X and Y whereas the right-hand side is only a tensor in Y.

The set of sections of a bundle E shall be denoted by $\Gamma\{E\}$. Recall that a section of

a bundle is a smooth map from the base space into the bundle such that the projection map acting on the section is the identity map on the base space.

The tensor product will occur often and the ordering of the elements will be crucial. Therefore, unless otherwise noted, when we write the tensor product over an ordered set, we shall mean the tensor product in that order. In other words,

$$\bigotimes_{i=1}^{n} v_i := v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

Comment on tensor input. The fact to note is that the tensor takes over the role of multilinearity, enabling the function A to be viewed as linear. Notice that once we view the input as a tensor product, we can use the Leibniz rule on it.

Comment on product rule. The product rule allows us to define the covariant derivative on the dual bundle since, for $A \in \Gamma\{E^*\}$ and $B \in \Gamma\{E\}$, they must satisfy $D[A](B) = D[A(B)] - A(D[B])$. As $A(B)$ should be a function over M, we know its derivative and, by assumption, the derivative is defined on E. In particular a connection on the tangent bundle provides a connection on all of the tensor product bundles involving only the tangent and cotangent bundles of M.

Proof of the general chain rule. Since A is a section of a bundle over N, we can pull it back to the pull-back bundle over M. Thus,

$$F_*(A) \in F_*(\Gamma\{\mathcal{V}^* \otimes f_*(\mathcal{X})\}) \subset \Gamma\{F_*(\mathcal{V})^* \otimes (fF)_*(\mathcal{X})\}.$$

Then $A(B)$ may be understood as contracting over the pulled back $\mathcal{V}$ bundles. In particular, we have that $\nabla_X^M[A(B)] = \nabla_X^M[F_*(A)](B) + A(\nabla_X^M[B])$. This is just the product rule for tensor product bundles. Since $\nabla_X^M[F_*(A)] := \nabla^N_{D[F](X)}[A]$, we actually have our result. □

Proof of the usual chain rule. The starting case is the usual well-known chain rule.[3]

[3]The method of proof depends on the definition of tangent vector used. Coordinates and the multivariable chain rule can always be used. If we use the notion of tangent vectors as equivalence classes

Our inductive step goes as follows. We assume

$$D^n[G](\overset{n}{\underset{i=1}{\otimes}} v_i) = \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[f](\underset{r\in R_Z}{\otimes}\{D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\})$$

Let us take the v_i to be vector fields and differentiate the resultant scalar field. The terms involving the derivatives of the vector field will cancel while the other remaining terms will be what we are interested in. The calculations follows from the product rule and general chain rule. We start with what we want to find which is

$$\begin{aligned}\nabla_{v_{n+1}}[D^n[G](\overset{n}{\underset{i=1}{\otimes}} v_i)] &= \nabla_{v_{n+1}}[D^n[G]](\overset{n}{\underset{i=1}{\otimes}} v_i) + D^n[G](\nabla_{v_{n+1}}[\overset{n}{\underset{i=1}{\otimes}} v_i]) \\ &= D^{n+1}[G](\overset{n+1}{\underset{i=1}{\otimes}} v_i) + D^n[G](\nabla_{v_{n+1}}[\overset{n}{\underset{i=1}{\otimes}} v_i]);\end{aligned}$$

the second term is an extra complication which will also appear in the following:

$$\begin{aligned}\nabla_{v_{n+1}}[D^n[G](\overset{n}{\underset{i=1}{\otimes}} v_i)] &= \sum_{Z\in\mathcal{Z}(n)} \nabla_{v_{n+1}}[D^{|Z|_R}[f](\underset{r\in R_Z}{\otimes}\{D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\})] \\ &= \sum_{Z\in\mathcal{Z}(n)} \nabla_{v_{n+1}}[D^{|Z|_R}[f]](\underset{r\in R_Z}{\otimes}\{D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\}) \\ &\quad + \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[f](\sum_{\tilde{r}\in R_Z}\{\underset{r\in R_Z, r\neq\tilde{r}}{\otimes} D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\} \\ &\qquad\qquad \otimes \nabla_{v_{n+1}}[D^{|\tilde{r}|}[F]](\underset{c\in C_{\tilde{r}}}{\otimes} v_{z_{\tilde{r}c}})) \\ &\quad + \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[f](\sum_{\tilde{r}\in R_Z}\{\underset{r\in R_Z, r\neq\tilde{r}}{\otimes} D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\} \\ &\qquad\qquad \otimes D^{|\tilde{r}|}[F](\underset{c\in C_{\tilde{r}}}{\otimes} \nabla_{v_{n+1}}[v_{z_{\tilde{r}c}}])) \\ &= \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R+1}[f](\{\underset{r\in R_Z}{\otimes} D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\} \otimes D[F](v_{n+1})) \\ &\quad + \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[f](\sum_{\tilde{r}\in R_Z}\{\underset{r\in R_Z, r\neq\tilde{r}}{\otimes} D^{|r|}[F](\underset{c\in C_r}{\otimes} v_{z_{rc}})\} \\ &\qquad\qquad \otimes \nabla_{v_{n+1}}[D^{|\tilde{r}|}[F]](\{\underset{c\in C_{\tilde{r}}}{\otimes} v_{z_{\tilde{r}c}}\} \otimes v_{n+1})) \\ &\quad + D^n[G](\nabla_{v_{n+1}}[\overset{n}{\underset{i=1}{\otimes}} v_i]).\end{aligned}$$

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of curves, then $D[F]$ is defined by taking $v = [\gamma] \to [F(\gamma)] := D[F](v)$; it is necessary to check that this is independent of the representative γ. But then the basic chain rule is simply composition: $[\gamma] \to [G(\gamma))] = D[G](v)$ is the same as $[\gamma] \to [F\gamma] = D[F](v) \to [f(F(\gamma))] = D[f](D[F](v))$.

The first term, in the set of three terms, requires the use of the general chain rule. That set of terms corresponds to cr-diagrams of length $n+1$ obtained from those of length n by attaching a new row at the bottom; the new row contains exactly one box whose entry is $n+1$. The second set of terms requires the use of the product rule. It corresponds to modifying the old cr-diagrams by appending a new box to the end of a row. For every cr-diagram of length n and height l, we obtain l new cr-diagrams with the same height but with length $n+1$ and $n+1$ is again the entry in the new box. The third set of terms is an extraneous set for which the product rule and linearity is of crucial importance for the manipulation of those terms. □

6.4 The primary tools

In this section, we prove that a certain group of 1-forms, associated to a point p, have the property that the fully symmetrized version of the covariant derivatives of those forms are zero at the point p. These forms are the coordinate forms in normal coordinates. We then use them to prove that the fully symmetrized covariant derivatives of any $(0,l)$ tensor is given by Euclidean differentiation in normal coordinates.

Theorem 5. *Let γ_v be a geodesic. Assume that both $\nabla_v[A(B)] = 0$ and $\nabla_v[B] = 0$ along γ_v. Then $D^n[A](B, v^n) = 0$ along γ_v.*

Corollary 1. *Let γ_v be a geodesic. Assume that $\nabla_v[A] = 0$ along γ_v. Then $D^n[A](v^n) = 0$ along γ_v.*

Corollary 2. *Let (M, ∇^M) be a smooth manifold with connection. Let $(x_1, \ldots, x_m)$ be the coordinate functions for normal coordinates centered at $p \in M$. Then $D^n_p[dx^i](v^{n+1}) = 0$.*

Corollary 3. *Let (M, ∇^M) be a smooth manifold with connection. Let $(x_1, \ldots, x_m)$ be the coordinate functions for normal coordinates centered at $p \in M$. Let ω be a $(0,l)$ tensor. If we view the coordinates as providing a linear structure, we can compute the n^{th} Euclidean derivative of ω, $\tilde{D}^n_p[\omega]$. Then $D^n_p[\omega](v^{l+n+1}) = \tilde{D}^n_p[\omega](v^{l+n+1})$.*

Proof of Theorem 5. It is a simple induction using the product rule. Assume that $D^{n-1}[A](B, v^{n-1}) = 0$ along γ_v. Notice $n = 1$ is our assumption. Then

$$\begin{aligned}(\nabla_v[D^{n-1}[A]])(B, v^{n-1}) =& \nabla_v[(D^{n-1}[A](B, v^{n-1}))] \\ & - D^{n-1}[A](\nabla_v[B], v^{n-1}) - D^{n-1}[A](B, \nabla_v[v^{n-1}]).\end{aligned}$$

By assumption, the objects being differentiated along the geodesic is zero along the geodesic; thus, the first two terms vanish. first and second terms are zero. Since, along a geodesic, $\nabla_v[v] = 0$ and v^{n-1} is the $(n-1)$-fold tensor power of v with itself, we have that the third term is zero by the Leibniz rule. It is the vanishing of the third term which requires γ_v to be a geodesic. □

Proof of Corollary 1. Since $A(\mathrm{Id}) = A$, we may apply Theorem 5 with $B = \mathrm{Id}$. All we need to establish is that $D[\mathrm{Id}] = 0$. The quick statement is that this must be true for consistency's sake. Indeed,

$$D[\mathrm{Id}(\mathrm{Id})] = D[\mathrm{Id}](\mathrm{Id}) + \mathrm{Id}(D[\mathrm{Id}].$$

Since it is the identity, this says that

$$D[\mathrm{Id}] = 2D[\mathrm{Id}]$$

which immediately implies that it is a zero tensor.[4] □

Proof of Corollary 2. In normal coordinates, the image of $v \in T_pM$ under the exponential map $\exp_p : T_pM \to M$ is labelled v. We also have $tv = \exp_p tv = \gamma_{tv}(1) = \gamma_v(t)$, (t between 0 and 1). In particular, $\dot{\gamma}_v(t)$ is simply v where γ_v is a geodesic starting

[4] A coordinate proof would go as follows. Pick a frame field, say $\{e_i\}$, and its dual frame field $\{e^i\}$. Then we have that $e^i(e_j)$ is a constant function in the neighborhood and therefore its derivative is zero. Thus, the product rule implies that $\nabla_v[e^i](e_j) = -e^i(\nabla_v[e_j])$. Since $I = e^i \otimes e_i$, we have

$$\begin{aligned}D[\mathrm{Id}](e_j, v) =& \\ &= \sum_i \{(D[e^i](e_j, v)) \otimes e_i + (e^i)(e_j) \otimes (D[e_i](v))\} \\ &= \sum_i \{-e^i(D[e_j](v)) \otimes e_i\} + (D[e_j](v)) \\ &= -\mathrm{Id}(D[e_j](v)) + D[e_j](v) = 0.\end{aligned}$$

Thus we have that the identity tensor always has a zero covariant derivative.

at p with $\dot{\gamma}_v(0) = v$. As $\dot{\gamma}_v(t)$ is constant along γ_v, it can be extended to a constant vector field in the normal neighborhood. Thus, $dx^i(v)$ is a constant function. We have $\nabla_v[dx^i(v)] = 0$ and $\nabla_v[v] = 0$. Since $\nabla_v[dx^i](v) = \nabla_v[(dx^i(v))] + dx^i(\nabla_v[v]) = 0$, we are in the situation where Theorem 5 applies. Thus, we have that $D^n[dx^i](v, v^n) = 0$ along γ_v and, in particular, at p. □

Warning. Although the parallel transport of v along its geodesic is simply a constant in these coordinates, in general the other coordinate vector fields will not have this property on γ_v. If they did, then this would be a locally trivial connection. Indeed, if the connection was the Levi-Civita connection for a metric, then, as the metric is preserved by parallel transport, the metric would be a flat metric.

Proof of Corollary 3. Using coordinates, we have $\omega := \omega_{i_1 \cdots i_n} dx^{i_1} \cdots dx^{i_n}$ where summation is implied over repeated indices. To compute the derivatives, we use the product rule repeatedly. Indeed, we can take the above coordinate expression and rewrite it as $\omega_{i_1 \cdots i_n} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_n}$. From Corollary 2, we have that when we evaluate $D^n[\omega](v^{n+l+1})$, the terms involving the derivatives of the coordinate forms vanish. Thus, we are only left with the derivative of the coefficient, which is exactly what the Euclidean derivative is. □

6.5 Taylor expansions of mappings between finite dimensional vector spaces

In this section, we shall formulate the question properly. We start by recalling the usual facts about the Taylor expansion in vector spaces which may be found, for example, in [1]. The n^{th} Taylor function of f, $T^n f$, is the best n-polynomial approximation to f up to terms of order $o(\epsilon^n)$.

This means, precisely, the following. We start with the finite dimensional vector spaces V, W, and a smooth map $f : V \to W$. Then the Taylor function $T^n f$ at x_0

should be an n^{th}-degree polynomial, i.e.

$$T^n f \in \{\bigoplus_{i=0}^{n} (V^*)^{\otimes i}\} \otimes W.^5$$

We require that $T^n f$ satisfies

$$\lim_{x \to x_0} \frac{\|f(x) - T^n f(\oplus_{i=0}^n \{x - x_0\}^i)\|_W}{\|x - x_0\|_V^n} = 0 \tag{6.2}$$

This statement is independent of which norm is chosen since, in finite dimensions, all norms are equivalent.We shall take x_0 to be 0 and write v for the vector $x - x_0$.

We first establish that the symmetric part is unique. Indeed, let A and B be two polynomials satisfying (6.2). We will be done if we show that for any vector $v \in V$,

$$(A - B)(\oplus_{i=1}^n \{v\}^i) = (A - B)(v \oplus \{v \otimes v\} \oplus \cdots \oplus \{\otimes_{i=1}^n v\}) = 0.$$

But this is easy. We first note that

$$\lim_{v \to 0} \frac{\|(A - B)(v)\|_W}{\|v\|_V^n} = 0; \tag{6.3}$$

this follows by inserting 0 in the form of $f - f$ into the numerator, using the triangle inequality, and then using (6.2). Let A_i and B_i be the i-linear parts of A and B, respectively. By replacing v with hv, $h \in \mathbf{R}$, we have

$$0 = \lim_{h \to 0} \frac{\|\sum_{i=1}^n (A_i - B_i)(\{hv\}^i)\|_W}{\|hv\|_V^n} = \lim_{h \to 0^+} \frac{\|\sum_{i=1}^n \frac{1}{h^{n-i}} (A_i - B_i)(v^i)\|_W}{\|v\|_V^n}. \tag{6.4}$$

Thus, $\|(A_i - B_i)(v^i)\|_W$ must be zero for every i which implies uniqueness.

We now prove that

$$T^n f(v) = \sum_{i=0}^{n} \frac{1}{i!} D_0^i[f](v).$$

All we need is the fundamental theorem of calculus and integration by parts. The fundamental theorem of calculus is

$$f(u + v) - f(u) = \int_0^1 D_{u+tv}[f](v) dt.$$

[5] A vector space raised to the 0^{th}-power is defined to be the field itself. Also, polynomials are always taken to be symmetric objects. That is to say, one mainly wants to evaluate them on the diagonal.

We now use integration by parts which is the reverse of the product rule. Specifically, we claim that

$$\int_0^1 \frac{(1-t)^{i-1}}{(i-1)!} D^i_{u+tv}[f](v^i)dt = \frac{1}{i!} D^i_u[f](v^i) + \int_0^1 \frac{(1-t)^i}{i!} D^{i+1}_{u+tv}[f](v^{i+1})dt.$$

This follows immediately from

$$\begin{aligned} 0 - (-\frac{1}{i!} D^i_u[f](v^i)) &= \int_0^1 \frac{d}{dt}[\frac{-(1-t)^i}{i!} D^i_{u+tv}[f](v^i)]dt \\ &= \int_0^1 \frac{(1-t)^{i-1}}{(i-1)!} D^i_{u+tv}[f](v^i)dt \\ &\quad + \int_0^1 \frac{-(1-t)^i}{i!} D^{i+1}_{u+tv}[f](v^{i+1})dt. \end{aligned}$$

Since f's derivatives are continuous both as operator-valued functions over M, we have that

$$\|D^l_q[f](v^l)\|_W \leq C_l \|v^l\|_V$$

for q in a small enough neighborhood. This immediately leads to

$$\|f(x) - \sum_{i=0}^{n} \frac{1}{i!} D^i_0[f](v)\|_W = \|\int_0^1 \frac{(1-t)^n}{n!} D^{n+1}_{u+tv}[f](v^{n+1})dt\|_W \leq K_{n+1} \|v\|^{n+1}_V$$

for some constant K_{n+1} and v in the appropriate neighborhood of the origin. Thus, we have our limit.

Having discussed the vector space case, we now formulate the question for manifolds. If M and N have connections, then there is a canonical identification via the exponential map between a neighborhood of $0 \in T_pM$ and a neighborhood of $p \in M$. The same is true for N. Thus any mapping f between M and N can be viewed as a mapping between some neighborhood of $0 \in T_pM$ and $T_{f(p)}N$. We can then Taylor expand that map. Since these are vector spaces, their tangent spaces are canonically identifiable with themselves. Thus, we end up with a polynomial mapping from the tangent space of M to that of the tangent space of N. The Taylor polynomial of f is the polynomial between the tangent spaces obtained through the Taylor expansion of f as a mapping between the two tangent spaces. The i^{th} monomial of the polynomial is indepedent of n; we shall call this the i^{th} Taylor derivative. We could also construct a polynomial out of the covariant derivatives of these mappings. Are these polynomials the same?

That is the question. The answer is that, loosely speaking, they agree when the range is appropriately flat.

A map f is *covariantly normal at* p iff its symmetrized covariant derivatives at p agree with the Taylor derivatives. If f is covariantly normal at all points of M, then f is *covariantly normal*.

6.6 Taylor expansions of mappings into flat spaces

In this section we discuss mappings into a flat space. As we shall prove, these maps are always covariantly normal.

Let (N, ∇^N) be a flat cc-manifold. This means that both the torsion and the curvature of the connection vanishes. In this section, the following standard lemma is of crucial importance.

Lemma 1. *Let (N, ∇^N) be a flat cc-manifold and $(U, (x^1, \dots x^n))$ be normal coordinates for U centered at p. Then $\nabla^N[\frac{\partial}{\partial x^i}] = 0$ everywhere in U for every i, i.e. the coordinate vector fields are parallel.*

Proof. Let $(U, (y_1, \dots, y_n))$ be a coordinate neighborhood such that the coordinate vector fields are parallel and such that $\frac{\partial}{\partial y^i}|_p = \frac{\partial}{\partial x^i}|_p$. Existence of such a neighborhood follows from flatness as in the proof of Theorem 7.3 in the text [38]. The claim is that $y^i = x^i$. As the x^i's are normal coordinates, we know, by definition, that in these coordinates $\gamma(t) := t\sum_{i=1}^n a_i x^i$ is a geodesic such that $\gamma(0) = 0 (= p)$ and $\dot\gamma(0) = a_i \frac{\partial}{\partial x^i}|_p$. But if we define $\phi(t) := t\sum_{i=1}^n a_i y^i$, then we claim that this is a geodesic. Once we establish that claim, then we are done by uniqueness results since γ and ϕ do have the same initial conditions; hence, the coordinates are the same (the point represented by $\sum a_i x^i$ is the same, through the geodesics, as $\sum a_i y^i$).

We now need to establish the $\phi(t)$ is a geodesic. We thus need to show that $\nabla^N_{\dot\phi}[\dot\phi] = 0$. We know that $\dot\phi(t) = \sum_{i=1}^n a_i \frac{\partial}{\partial y^i}|_{\phi(t)}$. Thus the vector field along the curve agrees with the vector field $\sum_{i=1}^n a_i \frac{\partial}{\partial y^i}$. And therefore, by definition, $\nabla^N_{\dot\phi}[\dot\phi] = \nabla^N_{\sum_{i=1}^n a_i \frac{\partial}{\partial y^i}}[\sum_{i=1}^n a_i \frac{\partial}{\partial y^i}]$. But this is zero as all of the $\frac{\partial}{\partial y^i}$ are parallel and the a_i are constants. □

The above proof was rather indirect. Indeed, in the proofs of the existence of the y coordinates, all of them that the author has seen requires the construction of a new coordinate system out of a given coordinate system. Even more galling, in [38], one starts with some coordinate system; that author suggests that normal coordinates is a possible choice. Then the proof constructs that coordinate system using theorems about integrability of systems. But, as we just showed, this constructs normal coordinates. Thus, one uses an elaborate machinery to construct coordinates that one already has.

With the lemma, we may now prove the following theorem.

Theorem 6. *Let f be a map from (M, ∇^M) into the flat cc-manifold (N, ∇^N). Then f is covariantly normal.*

Proof. Choose normal coordinate neighborhoods about $p \in M$ and $f(p) \in N$. Using these coordinates, we write $Df = df^j \partial_j$ where df^j is a $(0,1)$ tensor and ∂_j is the pull-back of the coordinate fields on N. By repeatedly applying the product rule, we have, for all l,

$$D^l[f] = D^{l-1}[df^j]\partial_j + \sum_{\substack{Z \in \mathcal{PR}_2(l-1) \\ r_1 \neq l-1}} D^{r_1}[df^j] \otimes D^{r_2}[\partial_j]$$

where the derivative on the coordinate vector fields on N is the pull-back derivative. In any event, the terms in the sum vanish by applying Lemma 1. Applying Corollary 3, we find that we have the Euclidean derivatives of Df when we evaluate the derivative acting on v^n at p. By viewing the coordinates as providing the linear structure, we have that this is also what appears in the Taylor formula. Thus, f is covariantly normal. □

6.7 The exponential map and its derivatives

Let M be a cc-manifold. Recall that the exponential map at $p \in M$ maps the tangent space at p into M, i.e. $\exp_p : T_pM \to M$. It is actually restricted to a neighborhood about the origin such that its restriction is a diffeomorphism. Its inverse, $\exp_p^{-1}$, is a map from an open subset of M into a flat space.

Theorem 7. *Let M be a cc-manifold. Then for every $p \in M$, we have, for $n > 1$, that $D_p^n[\exp_p^{-1}](v^n) = 0$ and $D_0^n[\exp_p](v^n) = 0$. For $n = 1$, the maps are essentially the*

identity.

Proof. By definition, $\exp_p^{-1} : M \to T_pM$ is defined such that, when expressed in normal coordinates, it is the identity map. By Theorem 6, we have that $D_p[\exp_p^{-1}(p)](v) = v$ and, for $n > 1$, $D_p^n[\exp_p^{-1}(p)](v^n) = 0$. It is immediate that $D_p[\exp_p(0)](v) = v$.

To obtain the $n > 1$ result for $\exp_p$, we shall use the chain rule. We start with the defining relation $\exp_p(\exp_p^{-1}(q)) = q$. Thus, $D^n[(\exp_p \exp_p^{-1})] = D^n[\mathrm{Id}] = 0$. We expand using the chain rule. We end up with

$$0 = D_0^n[(\exp_p(\exp_p^{-1}))](v^n) = \sum_{k=1}^{n} D^k[\exp_p](\sum_{Z\in\mathcal{Z}(n,k)} \bigotimes_{r\in R_Z} D^{|r|}[\exp_p^{-1}](v^{|r|})).$$

If $|r| > 1$, then $D^{|r|}[\exp_p^{-1}](v^{|r|}) = 0$ as we just established. Just as importantly, if $|r| = 1$, then $D^{|r|}[\exp_p^{-1}](v^{|r|}) = v$. The only cr-diagram to survive is the one such that each row has exactly one column. We therefore find that the only term we have left is $D^n[(\exp_p \exp_p^{-1})](v^n)$ which therefore must be zero. □

To put it succinctly, the symmetric part of the higher derivatives of the exponential map vanish at the point whose tangent space is being exponentiated.

Although we do not need the following, it may be useful to remark that the antisymmetric part of the second derivative of the exponential map is the torsion. The third derivative decomposes into the symmetric part, the derivative of the torsion, and the curvature tensor. Both facts can be seen by writing the map in coordinates and looking at the appropriate symmetry types. We should also mention that if the space is flat, all the derivatives of the exponential map vanish everywhere. As before, it is not clear how to prove this statement directly.

6.8 Geodesic submanifolds

We need one more ingredient before we can give much broader sufficient conditions for covariant normality. In this section, we discuss submanifolds that have a connection on them which agree with the ambient manifold's connection. It is rare for a submanifold to have this property. For example, the sphere in $\mathbb{R}^3$ does not have this property. For a Riemannian manifold, submanifolds inherit a metric and a connection although the

inherited connection may be quite different as is the case for the sphere as the flat metric in $\mathbb{R}^3$ leads to a curved metric on the sphere.

For the rest of this section, let M be a cc-manifold.

Definition 1. *A* geodesic submanifold Q *of a cc-manifold* M *is a submanifold which carries a connection that agrees with that of* M*. In other words, if* $v \in \Gamma\{TQ\}$ *and* $X \in T_pQ$*, then* $\nabla^Q_X[v] = \nabla^M_X[v] \in T_pQ$*.*

This notion is explained in [48] under the name of totally geodesic submanifold and in the context of semi-Riemannian manifolds. At the heart of the definition is the idea that if $\nabla^M_x[v] \in T_pQ$ for every $v \in TQ$, then the connection on Q making it into a geodesic submanifold is just the restriction. In contrast, if the connection on M does not preserve the tangent space of Q in the above sense, then one could try to fix this up by using a projection operator onto the tangent space of Q. But, of course, there is a choice of such projections; a metric does provide such a choice.

Theorem 8. Q *is a geodesic submanifold iff given any curve* γ *in* Q*, parallel transport along* γ *using* M*'s connection is a map from* $T_{\gamma(0)}Q$ *into* $T_{\gamma(t)}Q$*.*

Proof. This is almost immediate. If Q is geodesic, then the defining equations for parallel transport in M along γ are exactly the same as parallel transport in Q along γ. Thus, the solutions are the same. As for the other direction, since the covariant derivative may be defined by the difference quotient using parallel transport, we see that Q is geodesic. □

Theorem 9. *If* Q *is a geodesic submanifold, then*

1. *for every* $v \in T_pQ$*, the geodesic in* γ_v *in* M *with initial tangent vector* v *is contained in* Q*, at least for a short while;*

2. Q *can be recovered locally by the exponentiation of neighborhoods in its tangent spaces.*

Proof. The tangent vector field of a curve in Q is contained in its tangent space. By looking at what the acceleration of a curve is, it is clear that both M and Q give the

same acceleration. Thus, we define the geodesics using the connection on Q and then note that these are geodesics under M's connection. The exponentiation statement follows immediately from the geodesic result. □

Definition 2. *Let $p \in M$. Then a* flat subspace of T_pM *is a subspace V_p such that $D^n[\exp_p]$ vanishes on $V_p^{\otimes n}$.*

Definition 3. *A* flat submanifold *is a geodesic submanifold whose tangent spaces are flat.*

We mention a few examples. All 1-dimensional subspaces are flat. A geodesic is a geodesic submanifold which is flat. M is flat iff every tangent space to M is flat. This is the remark we made at the end of the previous section. For vector bundles as manifolds, as we shall explain later, the vertical fiber direct summed with a 1-dimensional horizontal fiber is a flat subspace. This corresponds to the flat submanifold of a geodesic on the base space coupled with the fiber above its points.

Lemma 2. *If Q is a geodesic submanifold such that the curvature and torsion of the induced connection vanishes everywhere, then Q is a flat submanifold.*

Proof. This follows by regarding Q as a manifold. Hence any tangent space of Q is flat. □

Conceptually, one way to establish that a subspace V_p is flat is to exponentiate a neighborhood about 0 and ask whether it is a flat submanifold. If it is, then V_p is flat. Note that the exponential image need not even be a geodesic submanifold. How well this captures the picture of a flat subspace is not known to the author.

We conclude with a reasonable lemma which we shall use in the next section.

Lemma 3. *If γ is a curve in a geodesic submanifold Q, then all of its derivatives are contained in the tangent bundle of Q.*

Proof. By definition, the n^{th} derivative of γ is $\nabla_{\dot\gamma} \cdots \nabla_{\dot\gamma}\dot\gamma$. Since this is the connection evaluated on sections of TQ, we have, by assumption of Q being a geodesic submanifold, that it is contained in the tangent space of Q. □

6.9 Covariant normality

We may now formulate the positive results of covariant normality.

Lemma 4. *Let M, N be cc-manifolds and f be a smooth map from M into N. Then f is covariantly normal at p if, for each $v \in T_pM$, the subspace $V(f,v) := span\{D_p^n[f](v^n)\}$ is a flat subspace of $T_{f(p)}N$.*

Proof. Let $\tilde{f} : T_pM \to T_{f(p)}N$ be the following map induced on the tangent spaces using the exponential maps and f:

$$\tilde{f}(v) := \exp_{f(p)}^{-1}(f(\exp_p v)).$$

Thus, $f(q) = \exp_{f(p)}(\tilde{f}(\exp_p^{-1} q))$. Essentially by definition, $D^n[\tilde{f}]$ are the Taylor derivatives. Furthermore, we also know that $D_0^n[\exp_{f(p)}(\tilde{f})](v^n) = D_p^n[f](v^n)$ since

$$\begin{aligned} D^n[\exp_{f(p)}(\tilde{f})](v^n) =& D^n[f(\exp_p v)](v^n) \\ =& \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[f](\bigotimes_{r\in R_Z} D^{|r|}[\exp_p](v^{|r|})) \\ =& D^n[f](\{D[\exp_p](v)\}^n) = D^n[f](v^n) \end{aligned}$$

where the vanishing of the terms is due to applying Theorem 7. We therefore only need to compute the derivatives of $\exp_{f(p)}(\tilde{f})$. We need to show that

$$D^n[\exp_{f(p)}(\tilde{f})](v^n) = D^n[\tilde{f}](v^n).$$

We again use the chain rule. In particular, we have that

$$\begin{aligned} D^n[\exp_{f(p)}(\tilde{f})](v^n) =& \sum_{Z\in\mathcal{Z}(n)} D^{|Z|_R}[\exp_{f(p)}](\bigotimes_{r\in R_Z} D^{|r|}[\tilde{f}](v^{|r|})) \\ =& D[\exp_{f(p)}](D^n[\tilde{f}](v^n)) \\ &+ \sum_{i=2}^{n} D^i[\exp_{f(p)}](\sum_{Z\in\mathcal{Z}(n,i)} \bigotimes_{r\in R_Z} D^{|r|}[\tilde{f}](v^{|r|})) \end{aligned}$$

Since $D[\exp_{f(p)}] = \mathrm{Id}$, the first term is what we want. Thus, the sum must vanish in order to obtain the result. In general, this will not happen as the input into the derivatives of the exponential map are not symmetric. However, by the hypothesis, the derivatives $D^i[\tilde{f}](v^i)$ are all elements of the flat subspace $V(f,v)$. Thus, as the

n^{th} derivative of the exponential map vanishes on $V(f,v)^{\otimes n}$, we have that the sum vanishes. □

Lemma 5. *Fix $p \in M$. Assume that for every geodesic γ in M starting at p, there is a* **flat** *submanifold Q_γ of N such that f maps the image of γ into Q_γ, at least for some interval about $t = 0$. Then f is covariantly normal at p.*

Proof. Let $v \in T_pM$ be given and γ_v be the unique geodesic corresponding to v. Let V_v be the vector space at p tangent to the flat submanifold Q_{γ_v} which contains the image of γ_v. Note V_v is a flat subspace implying that all we need to establish is that, for each n, $D^n[f](v^n)$ is an element of V. Note that $f(\gamma_v)$ is a curve in a flat submanifold. Thus its derivatives, of all orders, are elements of V_v i.e. $\frac{d^n}{dt^n}[f(\gamma_v)] \in V_v$. This follows from Lemma 3. We also have

$$\begin{aligned}\frac{d^n f(\gamma_v)}{dt^n} &= \sum_{Z \in \mathcal{Z}(n)} D^{|Z|_R}[f](\bigotimes_{r \in R_Z} D^{|r|}[\gamma_v](\frac{\partial}{\partial t}^{|r|})) \\ &= D^n[f](\dot{\gamma_v}^n)\end{aligned}$$

as the higher derivatives of a geodesic vanish. Since V_v is a flat subspace and v was arbitrary, the lemma applies and we have covariant normality of f. □

Theorem 10. *Assume that for every geodesic γ in M, there is a flat submanifold Q_γ of N such that f maps the image of γ into Q_γ. Then f is covariantly normal.*

Proof. Apply the lemma at every point p in the manifold. □

Corollary 4. *If N is flat, then all maps $f : M \to N$ are covariantly normal.*

Proof. This was actually deduced before and is actually used in the theorem. Nevertheless, as a means to unification, it is convenient to note that this follows immediately from the theorem. □

Corollary 5. *If $N = E$ is a vector bundle over M and f is a section, then f is covariantly normal.*

Proof. The proof is delayed until the next section. The main insight is that the flat submanifold that the geodesic γ is mapped into is the submanifold consisting of the fiber over γ. □

Corollary 6. *If f maps geodesics into geodesics, then f is covariantly normal.*

Proof. A geodesic is a flat submanifold and therefore this is an immediate application of the theorem. □

Corollary 7. *Let $f : M \to N$. Assume that for every $p \in M$ and $v \in T_pM$, f satisfies $D_p^2[f](v^2) = \alpha_2(v)D_p[f](v)$ for some $\alpha_2 : T_pM \to \mathbb{R}$. Then f is covariantly normal.*

Proof. This will follow immediately from Lemma 4, where the subspace will be 1-dimensional and therefore flat. We shall show that $D^n[f](v^n) = \alpha_n(v^{n-1})D[f](v)$.

We shall use induction. Fix $p \in M$ and $v_p \in T_pM$. Extend v_p to a vector field whose covariant derivative vanishes at p; this can be done by parallel transporting along the geodesics originating at p. We start with $D^{n+1}[f](v^{n+1}) := \nabla_v[D^n[f]](v^n)$. By using the product rule, we find

$$\begin{aligned}
\nabla_v[D^n[f]](v^n) &= \nabla_v[D^n[f]v^n] - D^n[f]\nabla_v[v^n] \\
&= \nabla_v[\alpha_n(v^{n-1})D[f](v)] - 0 \\
&= D[\alpha_n](v^{n-1}, v)D[f](v) + 0 + \alpha_n(v^{n-1})D^2[f](v, v) + 0 \\
&= (D[\alpha_n](v^{n-1}, v) + \alpha_n(v^{n-1})\alpha_2(v))D[f](v) \\
&:= \alpha_{n+1}(v^n)D[f](v).
\end{aligned}$$

This is all we needed to show. Incidentally, the α's are multilinear functionals. □

We now show that Corollary 6 and Corollary 7 actually describe the same kind of map.

Lemma 6. *[28] A map $f : M \to N$ satisfies $D^2[f]([v]^2) = \alpha_2(v)D[f](v)$ if and only if f maps geodesics of M into geodesics on N. Furthermore, α_2 is identically zero everywhere if and only if the mapping preserves parametrizations.*

Proof. The proof is just the chain rule. Let γ_v be a geodesic on M such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$. Then define ϕ to be the curve on N which is the image of γ_v. By the chain rule, we have

$$\ddot{\phi} = D^2[f](\dot{\gamma_v}, \dot{\gamma_v}) + D[f](\ddot{\gamma_v}).$$

Since γ_v is a geodesic, the second term vanishes. The first term vanishes iff the symmetric part of the second covariant derivative vanishes. For the full statement, one has to understand that a curve is a reparametrization of a geodesic iff its acceleration vector is a multiple of its tangent vector. More precisely, if $\ddot{\phi}(t) = \alpha(t)\dot{\phi}(t)$, then there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $\phi(h(s))$ is a geodesic and conversely. The proof follows by differentiating using the chain rule to arrive at

$$\ddot{\phi}(h) = \dot{h}^2\ddot{\phi}(h) + \ddot{h}\dot{\phi}(h).$$

The only way to have that vanish, without g being a constant, is for $\ddot{\phi}$ to be a multiple of $\dot{\phi}$ and, furthermore, g needs to satisfy $\ddot{h} = -\dot{h}^2\alpha(t)$, $\dot{h}(0) = 1$, $h(0) = 0$. Assuming that α is differentiable, we do have local existence of the solution. That is all we needed to claim that the symmetric part of the second derivative had to be a multiple of the first derivative. □

6.10 The failure of covariant normality

Having established certain situations in which we have covariant normality, it is rather important to investigate situations in which we do not have covariant normality which we shall call covariant abnormality. We shall content ourselves with showing that there are manifolds and maps between those manifolds which are not covariantly normal. When one looks at what needs to vanish in order to have covariant normality, one gets the feeling that a covariant abnormality should be the generic case. Unfortunately, it becomes quite complicated to take a brute force approach as there are many terms that cancel and/or are immediately zero. Nevertheless, I would conjecture that given a manifold map $f : M \to N$, if there exists a tangent vector $v \in T_pM$ such that the span of $\{D^n[f](v)\}$ is not a flat subspace, then f is covariantly abnormal at p. This is a conjecture with no evidence, but it seems a natural guess.

To establish what we want, we shall use normal coordinates. Let $f : M \to N$ be a map between cc-manifolds. Let $p \in M$ and take a neighborhood about p small enough so that it can be identified with the tangent space at p. Take a basis $\{\partial_i\}$ of the tangent space leading to a coordinatization of the neighborhood in M. We shall use the summation convention of summing over repeated indices and shall omit summation symbols. Then the coordinate representation of f is $f =: f^i\partial_i$. Then the first four covariant derivatives of f expressed in this coordinate system and evaluated on a given vector v, are

$$
\begin{aligned}
D[f](v) =& D[f^i](v)\partial_i \\
D^2[f](v^2) =& D^2[f^i](v^2)\partial_i + D[f^i](v)D[\partial_i](D[f](v)) \\
D^3[f](v^3) =& D^3[f^i](v^3)\partial_i + 2D^2[f^i](v^2)D[\partial_i](D[f](v)) \\
&+ D[f^i](v)D[\partial_i](D^2[f](v^2)) + D[f^i](v)D^2[\partial_i](\{D[f](v)\}^2) \\
D^4[f](v^4) =& D^4[f^i](v^4)\partial_i + 3D^3[f^i](v^3)D[\partial_i](D[f](v)) \\
&+ 3D^2[f^i](v^2)D[\partial_i](D^2[f](v^2)) \\
&+ 3D^2[f^i](v^2)D^2[\partial_i](D[f](v) \otimes D[f](v)) \\
&+ D[f^i](v)D^2[\partial_i](2D^2[f](v^2) \otimes D[f](v) + D[f](v) \otimes D^2[f](v^2)) \\
&+ D[f^i](v)D[\partial_i](D^3[f](v^3)) + D[f^i](v)D^3[\partial_i](\{D[f](v)\}^3)
\end{aligned}
$$

In order to have covariant normality, we need all of the terms to vanish except the first term, i.e. the only term that should survive is what one would have from Euclidean differentiation in this coordinate system.

In order to clean this up a bit, we shall assume that, at p, $D[f](v) =: \partial_1$ and

$D^2[f](v^2) =: \partial_2$.[6] Then the above equations become

$$D_p[f](v) = \partial_1 \tag{6.5a}$$

$$D_p^2[f](v^2) = \partial_2 \tag{6.5b}$$

$$\begin{aligned} D_p^3[f](v^3) =& D_p^3[f^i](v^3)\partial_i + 2D_p[\partial_2](\partial_1) \\ &+ D_p[\partial_1](\partial_2) + D_p^2[\partial_1](\partial_2^2) \end{aligned} \tag{6.5c}$$

$$\begin{aligned} D_p^4[f](v^4) =& D_p^4[f^i](v^4)\partial_i + 3D_p^3[f^i](v^3)D[\partial_i](\partial_1) \\ &+ 3D_p[\partial_2](\partial_2) + 3D_p^2[\partial_2](\partial_1^2) \\ &+ D_p^2[\partial_1](2\partial_2 \otimes \partial_1 + \partial_1 \otimes \partial_2) \\ &+ D_p[\partial_1](D_p^3[f](v^3)) + D_p^3[\partial_1](\{\partial_1\}^3) \end{aligned} \tag{6.5d}$$

Please note that since $D_p[\partial_1](\partial_1) = 0$ at p, we have that (6.5b) implies

$$D_p^2[f^i](v^2) = \begin{cases} 0 & \text{if } i \neq 2 \\ \\ 1 & \text{if } i = 2 \end{cases}.$$

Furthermore, we have that $D_p^n[w](w^n) = 0$ for any vector field w which is a sum of the coordinate vector fields with constant coefficients. This is what normal coordinates accomplish. Taking this into account in the above expressions, we are left with

$$\begin{aligned} D_p^3[f](v^3) =& D_p^3[f^i](v^3)\partial_i \\ &+ D_p[\partial_2](\partial_1) + D_p^2[\partial_1](\partial_2^2) \end{aligned} \tag{6.6a}$$

$$\begin{aligned} D_p^4[f](v^4) =& D_p^4[f^i](v^4)\partial_i + 3D_p^3[f^i](v^3)D[\partial_i](\partial_1) \\ &+ 3D_p^2[\partial_2](\partial_1^2) + D_p^2[\partial_1](\partial_2 \otimes \partial_1) \\ &+ D_p[\partial_1](D_p^3[f](v^3)) \end{aligned} \tag{6.6b}$$

From this point on, we leave generalities. We shall choose f to have the properties that we wish it to have. Our goal is to establish that there exists a map which is not covariantly normal. At some point, we shall also make a further assumption on the manifold, but that has not come yet. The argument is consideration of certain cases;

[6] The only loss of generality is assuming that $D^2[f](v^2)$ is not a multiple of $D[f](v)$. If we cannot find a point p and a vector $v \in T_pM$ which satisfies this assumption, then the map is, in fact, covariantly normal.

this is a bit distasteful and may largely be the result of the author's ignorance as to what constraints are actually placed upon the derivatives of these objects. Nevertheless, whatever the facts are, we shall be successful in establishing that which we have claimed.

We first show that if torsion does not vanish, then we can choose f such that it is covariantly abnormal. In normal coordinates, we have that $\tau(x,y) = 2D[x](y)$ where x and y are sums of the coordinate vector fields (constant coefficients), and the equality is only up to a sign.[7] Strangely, the most efficient analysis of the cases involves focussing on the fourth derivative and assuming that the second covariant derivative vanishes at p when evaluated at v. When one makes this assumption, then the third covariant derivative is the coordinate derivative. The fourth derivative reduces to

$$D^4_p[f](v^4) = D^4_p[f^i](v^4)\partial_i + 3D[\partial_3](\partial_1) + D_p[\partial_1](\partial_3)$$

where $\partial_3 := D^3[f](v^3)$. After applying the vanishing of symmetric expressions, we have

$$D^4_p[f](v^4) = D^4_p[f^i](v^4)\partial_i + 2D[\partial_3](\partial_1)$$

Choosing $D_p[f](v) = \partial_1$ and $D^3_p[f](v^3)\partial_3$ to be such that the torsion acting on them does not vanish, we see that the fourth covariant derivative is not the fourth coordinate derivative. Hence f is not covariantly normal. Note that the span of the covariant derivatives is not a flat subspace of the tangent space.

If torsion does vanish, then the third covariant derivative will be the coordinate derivative if and only if $D^2_p[\partial_1](\partial_2^2) = 0$. I do not know if this vanishes or not. If it does not, then we are done. If it does vanish, then we must go to the fourth derivative. For the fourth derivatve to be the coordinate derivative, we find, after taking into account the terms we have assumed to vanish, that we need $D^2_p[\partial_1](\partial_2 \otimes \partial_1) = 0$. Due to the symmetries already discussed, this is the curvature operator applied to those vectors. Although such an expression could vanish, if we assume that N is a curved, two dimensional manifold, then this cannot vanish.

---

[7] The ingredients to establish this are: $\tau(x,y) := D[x](y) - D[y](x) + [x,y]$; the Lie bracket of coordinate vector fields vanish; and normal coordinates imply that the fully symmetric expression for covariant derivatives of the coordinate vector fields vanish which in turn implies that $D[x](y) = -D[y](x)$.

As an example, take the sphere and take a curve on the sphere. Then its acceleration in normal coordinates is the covariant acceleration. But the higher covariant derivatives of the curve will not, in general, match the coordinate derivatives. In particular, either the third or fourth derivative will differ if the acceleration is not zero. Thus, we have a map which is not covariantly normal.

As I said in the beginning, I expect that much stronger statements can be made, but I do not know how to prove such assertions.

6.11 Vector bundles as manifolds

We now explain why sections of a bundle are covariantly normal. The first item to explain is how a connection on a vector bundle and the base space give a connection on the vector bundle as a manifold. Once we have that, then we can view a section as a map from the base manifold into the bundle manifold and use linear approximations induced by the normal coordinates.

We start by recalling the standard fact that on a vector bundle, there are three equivalent notions of a connection. The first is that of covariant differentiation and the second is that of parallel transport. These we have used before. The new one that we need is that a connection is a choice of horizontal spaces, as we shall now explain. A reference for this is Addendum 3 of Chapter 8 in [55]. Let the vector bundle be $B := (E, \pi, M, V)$ where E is the total space and it is a manifold; M is the base space; $\pi : E \to M$ is the projection map; and V is the fiber, i.e. for every p, $\pi^{-1}(p)$ is isomorphic to V. For a vector bundle, we also have the (smooth) multiplication map: for every $\alpha \in \mathbb{R}$, we have $\tilde{\alpha} : E \to E$ defined by $(p, v) \mapsto (p, \alpha v)$. For each $\alpha \neq 0$, this mapping is an automorphism, i.e. a diffeomorphism of E which preserves the bundle structures.

Fix a point $e \in E$ and define $p := \pi(e)$. Then the map $D_e\pi : T_eE \to T_pM$ is a map between two linear spaces and thus it has a kernel. We define the vertical space at e, V_e, to be the kernel. It is natural to try to decompose each tangent space into the direct sum $T_eE = V_e \oplus H_e$ where H_e should be the horizontal space at e. We note that

any complementary space of V_e is isomorphic to T_pM under $D_e\pi$. The problem is that we have no canonical way of finding a complementary subspace to the vertical space. A connection is the assignment of a horizontal space for each point. More specifically, a distributional connection H in E is a C^∞ distribution such that $T_eE = H_e \oplus V_e$ and that $D_e\tilde{\alpha}(H_e) = H_{\tilde{\alpha}(e)}$ for every point $e \in E$ and non-zero $\alpha \in \mathbb{R}$. As for notation, if $X \in T_eE$, then we define $\tilde{h}_X \in H_e$ and $\tilde{v}_X \in V_e$ to be the horizontal and vertical projections of X, respectively; thus, $X = \tilde{h}_X + \tilde{v}_X$. By using the isomorphism $D_e\pi$, we can view $\tilde{h}_X$ as a tangent vector. We shall generally do this and write h_X for that.

For vector bundles, as opposed to more general fiber bundles, we are able to abuse the v term as well. Indeed, $V_p := \pi^{-1}(p)$ is a vector space which is also a submanifold of E. At the point $e := (p, v) \in E$, the tangent space of the fiber is V_e. But we also know that a vector space is canonically identifiable with its tangent space. Let $F_e : V_e \to V_p$ be this canonical isomorphism. We shall use the notation v_X for $F_e(\tilde{v}_X)$.

Having setup the basic notations and definitions, we are in a position to sketch the equivalence of the connections.

Theorem 11. *The following notions are equivalent:*

1. *A connection as covariant derivative* $(\nabla : T_pM \times \Gamma\{B\} \to \Gamma\{B\})$ *;*
2. *A connection as parallel transport* $(P_{\gamma_t} : V_{\gamma_0} \to V_{\gamma_t})$*;*
3. *A connection as a distribution of horizontal spaces* $(e \mapsto H_e)$*.*

Sketch of proof. We shall only highlight what the source of the relation is; the details may be found in [55].

$(1 \Rightarrow 2)$ Given a curve γ and a vector v to be parallel transported, the following ODE, for the function $v_t \in V_{\gamma_t}$, defines the parallel transport operator: $\dot{v}_t = 0$; roughly written this means $\nabla_{\dot{\gamma}_t}[v(\gamma_t)] = 0$.

$(2 \Rightarrow 1)$ Given $v \in \Gamma\{B\}$ and a tangent vector $y \in T_pM$, let γ be a curve through p whose tangent vector at p is y. Then the covariant derivative is defined to be $\lim_{t\to 0} \frac{1}{t}(P_{\gamma_t}^{-1} v(\gamma_t) - v_0)$.

($1 \Rightarrow 3$) Let $(p, v) := e \in E$ and $\sigma \in \Gamma\{B\}$ such that $\sigma(p) = v$ and $\nabla_y[\sigma](p) = 0$ for all $y \in T_pM$. This can be done (at least locally) by parallel transporting v along the geodesics emanating from p. Then H_e is the image of T_pM under $D\sigma : T_pM \to T_eE$.

($3 \Rightarrow 1$) Given a tangent vector $y \in T_pM$ and a section $\sigma \in \Gamma\{B\}$, then $D[\sigma](y) = \tilde{v}_e + \tilde{h}_e$; we need to have chosen the horizontal space in order to define $\tilde{v}_e$.[8] Thus, the covariant derivative is v_e, the canonical isomorphic image of $\tilde{v}_e$.

□

With this information, it is relatively straightforward to setup a manifold connection on E. We shall do this by describing how to parallel transport tangent vectors. Let $\tilde{\gamma}$ be a curve in E going from the point e_0 to e_1. Define $\gamma := \pi(\tilde{\gamma})$. This is a curve in M (although it may only be piecewise smooth, i.e. its tangent vector may vanish). Let $X \in T_{e_0}E$ be the tangent vector we wish to parallel transport. Let $X = \tilde{h}_X + \tilde{v}_X$, the direct sum decomposition of X. Of course, we now use the isomorphisms to obtain h_X and v_X. We parallel transport those objects along γ. We do this piecewise. We need both connections in this step. Once we have the transported objects at e, we use the isomorphisms to return to a tangent vector of E. Schematically, we have

$$X \overset{H_{e_0} \oplus V_{e_0}}{\longmapsto} \tilde{h}_X + \tilde{v}_X \overset{D_{e_0}\pi, F_{e_0}}{\longmapsto} (h_X, v_X) \overset{\gamma}{\mapsto} (g := P_\gamma^M h_X, w := P_\gamma^B v_x) \mapsto \tilde{g} + \tilde{w} \in T_{e_1}E.$$

One can easily verify linearity, smoothness, invertibility, etc., of this operation which we define to be the parallel transport operator P_γ^E.

What are the geodesics of E? The short answer is that these are the parallel transport of the bundle's vectors along the geodesics of M.

Lemma 7. *Let $e = (p, z) \in E$, $X \in T_eE$ and γ the geodesic in M with tangent vector h_X. Then the curve defined by $\alpha(t) = P_{p,\gamma(t)}(tv_X + z)$ is a geodesic starting at e whose initial tangent vector is X.*

[8] Only if $\tilde{h}_e = 0$ would this be canonical; this, however, never occurs for a section as p is what varies.

Proof. We shall use the following coordinates. Let U be a neighborhood in M centered at p and use normal coordinates. Parallel transport V_p along the geodesics emanating from p. This will locally trivialize the bundle. With these coordinates, we have $\alpha(t) = (th_X, tv_X + z)$. Its time derivative is $\dot{\alpha}(t) = (h_X, v_X)$. We must now ask whether this vector is the parallel transport of (h_X, v_X) along α. By definition of the coordinate system, v_X is transported to v_X along the geodesic $\gamma(t) = th_X$. And the tangent vector of a geodesic is the parallel transport of itself along the geodesic, so h_X is correct as well. And this is exactly how we defined parallel transport for the manifold E. □

Thus, we know the normal neighborhood about a point $e \in E$ and the exponential map. In fact, we have $\exp(p, z)(\tilde{h}, 0)$ is the parallel transport of z along the geodesic starting at p with tangent vector h. This is the same as $\exp(p, 0)(\tilde{h}, \tilde{z})$.

We may finally establish covariant normality of sections of B using the viewpoint that they are maps from M into E. Let $\sigma \in \Gamma\{B\}$. We need to show that it maps geodesics of M into flat submanifolds of E. Then we may apply 10.

Lemma 8. *Let Q be a geodesic submanifold of M. Then $E_Q := \cup_{q \in Q} E_q$ is a geodesic submanifold of E.*

Proof. First note that E_Q is a submanifold as well as a subbundle of B. Let $\tilde{\alpha}$ be a curve in E_Q and let α be its projection to M which is actually a curve in Q. Parallel transport of a tangent vector of Q along α stays within the tangent bundle of Q by assumption of Q being geodesic. Parallel transporting a vector in E_p to q is definitely an operation which takes the vector into an element of E_q. By looking at the definition of parallel transport of tangent vectors to E, we see that the parallel transport does preserve TE_Q; thus E_Q is geodesic. □

Lemma 9. *Let $Q := \gamma$ be a geodesic. Then E_Q is a flat submanifold.*

Proof. We shall show that parallel transport in E_Q is independent of the path. First note that parallel transport is independent of the parametrization of the path. Given a path $\tilde{\alpha}$ in E_Q, Then the projection of α is γ parametrized in a different way; in particular, α can change directions along γ. The parallel transport operators multiply

together and even can cancel each other if the path goes forwards and backwards. Thus, the parallel transport of a tangent vector along $\tilde{\alpha}$ is just the parallel transport of the appropriate objects along γ from the beginning point of α to the terminating point. □

In contrast, if we assume Q is some higher dimensional flat submanifold of M, we do not have this result unless the curvature of the bundle connection vanishes on B_Q.

Noting that the image of the geodesic γ, under the manifold map $\sigma \in \Gamma\{B\}$, is contained in the flat submanifold E_γ, i.e. the set of fibers over γ, we find that sections of a bundle are covariantly normal by applying 10.

We make a further remark that the n^{th} Taylor approximation of the section is the following:

$$T_p^n[\sigma](q,\psi) = P^B_{\gamma_{p\to q}}(\sum_{i=0}^{n} D_p^i[\sigma](\{\exp_p^{-1} q\}^i)$$

where $P^B_{\gamma_{p\to q}}$ is the parallel transport operator along the unique geodesic connecting p and q in this small neighborhood and the covariant derivatives of σ are the bundle's covariant derivatives. This claim follows from considering the normal coordinates for the manifold E; these coordinates correspond to the basis field described in Appendix B.

Chapter 7

Two algorithms for exact quantum dynamics from trajectories

In this chapter, we shall discuss a new idea for solving Schrödinger's equation. The results are few, but the potential for more is extremely high. We start by rewriting Schrödinger's equation into the usual pair of the modified Hamilton-Jacobi equation and the continuity equation. Then a natural iterative process suggests itself. Namely, ignore the quantum potential, solve the H-J equation, use the continuity equation to get a new density, use that density to obtain an approximate quantum potential, and repeat. The fixed point of this scheme is the solution. We are particularly interested in an equivalent formulation involving trajectories. An important fact is that if we had the Bohmian trajectories and the initial wave function, then we could produce the solution to Schrödinger's equation. But to find the Bohmian trajectories, we need to already have the solution to Schrödinger's equation. The two algorithms we will be discussing provide a separation. We start with an initial set of trajectories, using them to construct an approximate wave function. We then use that new wave function to find new trajectories. We continue. Pictorially, we have

$$Q_0 \to \psi_1 \to Q_1 \to \cdots \to Q_n \to \psi_{n+1} \to Q_{n+1} \to \cdots$$

Convergence is a central question of any such proposal. Unfortunately, we have no general answer to that question. We do, however, have some examples of beautiful convergence. Unfortunately, these are restricted to only quadratic potentials in 1 dimension. Additionally, we have used the process on the fixed point to derive a PDE that the family of Bohmian trajectories satisfies. That is to say, we can actually find the Bohmian trajectories before solving Schrödinger's equation. We explicitly do this in the special case of when the Bohmian trajectories are linear in the initial configuration;

we do this only in 1 dimension.

7.1 Background material

In chapter 3, we derived the Hamilton-Jacobi equation and the continuity equation from Schrödinger's equation. In this section, we write down what the solutions are to these equations in terms of the characteristics. It is convenient to start with some definitions and notations. Appendix F contains the proofs of these assertions as well as a discussion of the derivatives of the determinant of the differential of a map.

We start with some definitions. By *family of trajectories*, we shall mean essentially a flow although we do not make assumptions about invertibility. If Q is a family of trajectories, then we shall write $Q(x_0, t)$ to denote the configuration at time t of a system whose initial configuration was x_0. We also define the action functional. Given a family of trajectories Q, a potential U, and an initial function T, we define

$$\mathcal{S}[Q, U, T](x_0, t) := T(x_0) + \int_0^t [\frac{\dot{Q}^2}{2}(x_0, s) - U(Q(x_0, s), s)]ds.$$

Our last definition is for the amplitude functional. Given the family Q and an initial function P, we define

$$\mathcal{R}[Q, P](x_0, t) = P(x_0)|DQ(x_0, t)|^{-\frac{1}{2}}.$$

As we shall discuss, these are essentially the solutions to the Hamilton-Jacobi and continuity equations, respectively.

In this chapter, all wave functions will be complex-valued wave functions over a given manifold $\mathcal{Q}$; the examples will all be for the simple manifold $\mathcal{Q} = \mathbb{R}$. We generally demand these functions to be smooth elements of $L^2(\mathcal{Q})$. We start by writing the wave function, ψ, in polar form; $\psi = Re^{\frac{iS}{h}}$, where the amplitude $R \geq 0$, and the phase, S, is real. The quantum density, ρ, is $|\psi|^2 = R^2$. We first think about the setup for N particles moving in $\mathbb{R}^k$; we shall use the usual setup for this rather than the one we discussed in Chapter 5. Solving Schrödinger's equation is equivalent to solving the

coupled equations

$$\frac{\partial S}{\partial t} = -\sum_{k=1}^{N} \frac{1}{2m_k}(\nabla_k S \cdot \nabla_k S) - V + \sum_{k=1}^{N} \frac{\hbar^2}{2m_k}\frac{\Delta_k R}{R}$$
$$\frac{\partial \rho}{\partial t} = -\sum_{k=1}^{N} \nabla_k \cdot (\rho \frac{1}{m_k}\nabla_k S).$$

For a general manifold, one immediately has the same thing using the Riemannian metric. One can also formulate N particles moving in $\mathcal{Q}$. The masses can be absorbed into the metric as described in Chapter 3. Thus, we can drop the masses and the summations. We could also write the above equations explicitly using the metric:

$$\frac{\partial S}{\partial t} = -\frac{1}{2}G(DS, DS) - V + \frac{\hbar^2}{2}\frac{G \circ D^2 R}{R} = -\frac{1}{2}\nabla S \cdot \nabla S - V + \frac{\hbar^2}{2}\frac{\Delta R}{R} \qquad \text{(H-J)}$$
$$\frac{\partial \rho}{\partial t} = -G \circ (D(\rho DS)) = -\nabla \cdot (\rho \nabla S). \qquad \text{(C)}$$

We shall absorb the mass into the metric except possibly when we do the examples.

Equation (H-J) is almost the Hamilton-Jacobi equation for classical motion under the potential V; it differs by the quantum potential, $-\frac{\hbar^2}{2}\frac{\Delta R}{R}$. Classically, the velocity field is ∇S, which is just the Bohmian velocity field. Equation (C) is the continuity equation; this describes how ρ evolves under the flow corresponding to configurations moving with the Bohmian velocities. The trajectory of the system is a single integral curve of the Bohmian velocity field.

Let $Q(x_0, t)$ be the family of trajectories solving $\dot{Q}(x_0, t) = \nabla S(Q(x_0, t))$ with inverse Q^{-1} and $|DQ^{-1}| \neq 0$. Let ρ_0 and S_0 be the amplitude and phase of the initial wave function. Then in the appendix, we show that $(\mathcal{R}[Q, \rho_0](Q^{-1})(x, t)))^2$ is the solution to (C) while $\mathcal{S}[Q, V - \frac{\hbar^2}{2}\frac{\Delta R}{R}, S_0](Q^{-1}(x, t))$ is the solution to (H-J).

7.2 The algorithms

We start by describing the algorithms in terms of the approximating PDEs. We shall begin with algorithm A1. At the beginning of the n^{th} step, we have $\rho_{n-1} := R_{n-1}^2$ and

S_{n-1}. We demand that S_n satisfies

$$\frac{\partial S_n}{\partial t} = -\nabla S_n \cdot \nabla S_{n-1} + \frac{1}{2}\nabla S_{n-1} \cdot \nabla S_{n-1} - V + \frac{\hbar^2 \Delta R_{n-1}}{2R_{n-1}} \qquad \text{(H-J:A1, } n\text{)}$$

and that ρ_n satisfies

$$\frac{\partial \rho_n}{\partial t} = -\nabla \cdot (\rho_n \nabla S_n). \qquad \text{(C:A1, } n\text{)}$$

For algorithm A2, we have a similar setup, but we demand that S_n satisfies

$$\frac{\partial S_n}{\partial t} = -\frac{1}{2}\nabla S_n \cdot \nabla S_n - V + \frac{\hbar^2 \Delta R_{n-1}}{2R_{n-1}} \qquad \text{(H-J:A2, } n\text{)}$$

and that ρ_n satisfies

$$\frac{\partial \rho_n}{\partial t} = -\nabla \cdot (\rho_n \nabla S_n). \qquad \text{(C:A2, } n\text{)}$$

We denote by U_{n-1} the potential $V - \frac{\hbar^2}{2}\frac{\Delta R_{n-1}}{R_{n-1}}$.

There are several comments that may be made. First, the fixed point of both algorithms provides the solution of Schrödinger's equation. Second, both equations in A1 are linear, understood as solved in the appropriate order, whereas (H-J:A2) is quadratic in the spatial derivative. Third, in the limit $\hbar \to 0$, A2 becomes classical mechanics whereas A1 does not. Indeed, if $\hbar = 0$, A2 does not iterate and the only step becomes a quasiclassical approximation. But for A1, this is not the case. In fact, if $\hbar = 0$, then A1 still iterates. It becomes an algorithm for computing the solution to the classical H-J equation.

On the n^{th} step of either algorithm, we define $\dot{Q}_n := \nabla S_n$. With this definition, Q_n corresponds to the characteristics of (H-J:A1, $n+1$), (C:A1, n), (H-J:A2, n) and (C:A2, n); *characteristics* are trajectories that convert a PDE to an ODE. Indeed, in A1, it can be shown that

$$S_n(x,t) = \mathcal{S}[Q_{n-1}, U_{n-1}, S(\cdot,0)](Q_{n-1}^{-1}(x,t),t)$$

and

$$R_n = \mathcal{R}[Q_n, R(\cdot,0)](Q_n^{-1}(x,t),t).$$

In A2, we have

$$S_n(x,t) = \mathcal{S}[Q_n, U_{n-1}, S(\cdot,0)](Q_n^{-1}(x,t),t)$$

and

$$R_n = \mathcal{R}[Q_n, R(\cdot, 0)](Q_n^{-1}(x,t),t).$$

From this perspective, A1 is close to Bohmian mechanics in spirit. At every stage, we start with S_n and compute the integral curves of its gradient thereby forming the family $Q_n(x_0,t)$. Then we compute the Jacobian of Q_n to obtain R_{n+1} and the action integral using the new potential $U_{n+1}$to obtain S_{n+1}. In contrast, A2 is closer to the spirit of classical mechanics. To find Q_n, we actually solve the classical equations with the force being $-\nabla U_{n-1}$. We then use the Jacobian of Q_n to define R_n and U_n. This sets us up to calculate the next set of trajectories. In A2, we do not need to compute S_n in order to complete the step.

In algorithm A2, we encounter the issue that S_n may not be well-defined for all time; this generally corresponds to Q_n not being invertible. Following [6, pages 438–445], we define an approximate wave function by

$$\psi_n(x,t) := \sum_{\{y_j | Q_n(y_j,t)=x\}} \mathcal{R}[Q_n, R(\cdot,0)](y_j,t) e^{i(\mathcal{S}[Q_n, U_{n-1}, S(\cdot,0)](y_j,t) - \frac{\pi}{2}\mu_j)}.$$

We then obtain R_n and S_n from the polar form of that wave function. The Maslov index μ_j gives us an appropriate phase for the summation. Appendix F gives a slighly more informative discussion of this.

7.2.1 The algorithms in short

We now describe the algorithms in a more mechanical fashion. Given $T(x)$, $P(x)$, $V(x,t)$, and $Q(x,t)$, the two algorithms may be viewed as generating different ODEs to solve involving the function $q(x,t)$ once the initial x is fixed.

We shall denote the following procedure as $A1(T,P,V,Q)$; it produces a family of trajectories $q(x,t)$.

1. Compute $H := Q^{-1}$.
2. Define $R := P(H)|DQ^{-1}|^{\frac{1}{2}}$.
3. Compute $\tilde{S} := T + \int_0^t [\frac{\dot{Q}^2}{2} - V(Q,s) - \frac{\hbar^2}{2}\frac{\Delta R}{R}(Q,s)]ds$.

4. Define $S := \tilde{S}(H)$.

5. Define $q(x,t) := \gamma_x(t)$ where γ_x the solution of $\dot{\gamma}_x(s) = (\nabla S)(\gamma_x(s), s)$ with initial condition $\gamma_x(0) = x$.

For an algorithm, fix T, P, V and define $B(Q) := A1(T, P, V, Q)$. The n^{th} step of A1 is then the same as $B^n(Q_0)$. To start, one can choose a best guess, $Q_0(x,t) = x$, or the solution to $\dot{Q}_0(x,t) = \nabla S_0(Q)$.

We do a similar iteration for A2. We first describe the version when Q is invertible. We shall denote the following procedure as $A2(T, P, V, Q)$; it produces a family of trajectories $q(x,t)$.

1. Compute $H := Q^{-1}$.

2. Define $R := P(H)|DQ^{-1}|^{\frac{1}{2}}$.

3. Define $q(x,t) := \gamma_x(t)$ where γ_x the solution of $\ddot{\gamma}_x(s) = -(\nabla(V - \frac{\hbar^2}{2m}\frac{\Delta R}{R}))(\gamma_x(s), s)$ with initial condition $\gamma_x(0) = x$, $\dot{\gamma}_x(0) = \nabla T(x)$.

For an algorithm, fix T, P, V and define $C(Q) := A2(T, P, V, Q)$. The n^{th} step of A1 is then the same as $C^n(Q_0)$. To start, one can choose a best guess, $Q_0(x,t) = x$, the solution to $\dot{Q}_0(x,t) = \nabla S_0(Q)$, or the classical trajectories using the initial S to for the initial conditions.

Finally, we have the case when the trajectories are not invertible, but are generated in a classical way. In particular, we shall let U be the potential which generated the trajectories. In the above algorithm, it was not necessary to have this information. As the above is mostly a special case of what is below, we shall denote the following procedure as $A2(T, P, V, Q, U)$; it produces a family of trajectories $q(x,t)$ and the potential u that generated those trajectories.

1. Define the set of inverse points $H_{y,t} := Q^{-1}(y,t)$.

2. Define $\mu(x,t)$ to be the Maslov index of the curve $Q(x,t)$ under the flow of Q. In particular, it is the number of signed zeroes that the trajectory has crossed of the function $|DQ|(x,t)$.

3. Compute $S(x,t) := T(x) + \int_0^t [\frac{\dot{Q_x}^2}{2}(s) - U(Q_x(s),s)]ds - \frac{\pi}{2}\mu(x,t)$ where U is the potential which generated the trajectories.

4. Define $\tilde{R}(x,t) := P(x)|DQ|^{-\frac{1}{2}}(x,t)$.

5. Define

$$R(y,t) := \sqrt{\sum_{x\in H_{y,t}} \tilde{R}^2(x,t) + 2 \sum_{\{x_1,x_2\}\subset H_{y,t}} \tilde{R}(x_1,t)\tilde{R}(x_2,t)\cos(S(x_1,t)-S(x_2,t))}.$$

6. Define $u := V - \frac{\hbar^2}{2}\frac{\Delta R}{R}$

7. Define $q(x,t) := \gamma_x(t)$ where γ_x the solution of $\ddot{\gamma}_x(s) = -(\nabla u)(\gamma_x(s),s)$ with initial condition $\gamma_x(0) = x$, $\dot{\gamma}_x(0) = \nabla T(x)$.

For an algorithm, fix T, P, V and define $C(Q,U) := A2(T,P,V,Q,U)$. The n^{th} step of A1 is then the same as $C^n(Q_0,U_0)$. Perhaps the most natural possibility for Q_0 in this setting is the set of classical trajectories although one could also use a free motion or a best guess.

7.3 Simple examples

We shall now work out an example. The potential is $V(x) = \frac{1}{2}x^2$ with initial wave function $\psi(x,0) := e^{-\frac{1}{2}(x-c)^2+iax}$. Thus, $S(x,0) = ax$ and $R(x,0) = e^{-\frac{1}{2}(x-c)^2}$. We take $\hbar = 1$ throughout this section and it is more convenient to reindex the iteration in such a way that Q_N generates R_N and S_N. The actual family of Bohmian trajectories for this system is $Q(x,t) = x_0 + c(\cos(t)-1) + a\sin(t)$

Assume that $Q_N(x,t) = x + f_N(t)$ for $f_N \in C^\infty(\mathbb{R})$ such that $f_N(0) = 0$ and $\dot{f_N}(0) = a$. Immediately, we have $Q_N^{-1}(x,t) = x - f_N(t)$. Whether we use A1 or A2, we compute R_N in the same way. We find that, since $DQ_N^{-1} = 1$,

$$R_N(x,t) := R(Q^{-1}(x,t),0)|DQ^{-1}(x,t)|^{\frac{1}{2}} = e^{-\frac{1}{2}(x-f_N(t)-c)^2}.$$

The corresponding quantum potential is

$$-\frac{\Delta R_N}{2R_N} = -\frac{1}{2}((x-f_N(t)-c)^2+1) = -\frac{1}{2}(x^2 - 2x(f_N(t)+c) + (f_N(t)+c)^2 + 1).$$

Thus, we have that the new potential is

$$U_N = \frac{1}{2}(2x(f_N(t)+c) - (f_N(t)+c)^2 - 1).$$

For A1, we compute the action integral along Q_N using the potential U_N. We have

$$\begin{aligned}\mathcal{S}[Q_N, U_N, S(x,0)](x_0,t) &= ax_0 + \frac{1}{2}\int_0^t [\dot{f_N}^2(s) - 2(x_0 + f_N(s))(f_N(s)+c) \\ &\quad + (f_N(s)+c)^2 - 1]ds \\ &= ax_0 + \frac{1}{2}\int_0^t [-2x_0(f_N(s)+c) + \dot{f_N}^2(s) \\ &\quad - f_N(s)^2 + c^2 - 1]ds.\end{aligned}$$

To obtain $S_N(x,t)$, we insert $Q_N^{-1}(x,t)$ for x_0 in the above formula. To calculate Q_{N+1}, we take the gradient of S_N. We compute that $\nabla S_N(x,t) = a - \int_0^t (f_N(s)+c)ds$

Let $f_{(i),N}$ be the i^{th} antiderivative of f_N such that $f^{(j)}_{(i),N}(0) = 0$ for all $0 \leq j < i$. We find that

$$\dot{Q}_{N+1}(x,t) = \nabla S_N(Q_{N+1}(x,t),t) = a - f_{(1),N}(t) - ct.$$

The solution to the above equation, with initial condition $Q_{N+1}(x,0) = x$, is

$$Q_{N+1}(x,t) = x + at - f_{(2),N}(t) - \frac{1}{2}ct^2$$

As for A2, we do not compute S_N at all. Instead, we compute the trajectories directly from having the quantum potential. Indeed, we need to solve

$$\ddot{Q}_{N+1}(x,t) = -\nabla U_N(Q_{N+1}(x,t),t) = -(f(t)+c)$$

with the initial conditions $Q_{N+1}(x,0) = x$ and $\dot{Q}_{N+1}(x,0) = a$. The solution is

$$Q_{N+1}(x,t) = x + at - f_{(2),N}(t) - \frac{1}{2}ct^2$$

In this example, we have that the two algorithms produce the same trajectories. Furthermore, the produced solutions are again of the same form. Starting with $Q_0(x,t) = x_0 + f(t)$ and iterating this scheme, we find

$$Q_N(x,t) = x + c\sum_{m=1}^{N}\frac{(-1)^m t^{2m}}{(2m)!} + a\sum_{m=0}^{N}\frac{(-1)^m t^{2m+1}}{(2m+1)!} + (-1)^N f_{(2N)}(t).$$

The convergence of the algorithm in terms of trajectories is that of uniform convergence on $\mathbb{R}\times[0,T]$ to the actual Bohmian trajectories.

7.4 Propagators and the classical algorithm

We now discuss an application of A2. The idea is to compute the propagator, or Green's function, for Schrödinger's equation using this algorithm. If we have a quadratic potential, then A2 will compute the propagator exactly. The fixed point trajectories are the classical trajectories. We exemplify this with the one-dimensional system whose potential is $V(x,t) = \frac{1}{2}a(t)x^2 + b(t)x + c(t)$.

The Green's function is the integral kernel, $G(x,t;y_0,0)$, for which

$$\psi(x,t) := \int G(x,t;y_0,0)\psi(y_0,0)dy_0$$

is a solution to the Schrödinger's equation with initial condition $\psi(x,0)$. Finding $G(x,t;y_0,0)$ is equivalent to solving Schrödinger's equation with the initial wave function being $\delta(x-y_0)$. Such a wave function may be thought of as representing a swarm of particles located at y_0 with a uniform spread of initial velocities, then we may attempt to use A2 to solve the problem. Let the uniform distribution be $R(v,0) = \sqrt{\gamma}$ where γ is a constant to be discussed later.

Let the initial family of trajectories, which are now functions of the initial velocity, be of the form

$$Q_0(v,t) = f(t)v + g(t)y_0 + h(t) \tag{7.1a}$$

$$f(0) = \dot{g}(0) = h(0) = \dot{h}(0) = 0 \tag{7.1b}$$

$$\dot{f}(0) = g(0) = 1 \tag{7.1c}$$

wher f, g, and h are independent of y_0. Then

$$Q_0^{-1}(x,t) = \frac{1}{f(t)}(x - y_0 g(t) - h(t)).$$

Ignoring the issue for now of when $f(t) = 0$, we find that

$$R_0(x,t) := R(Q_0^{-1}(x,t),0)|\frac{\partial Q_0}{\partial v}|^{-\frac{1}{2}}(Q_0^{-1}(x,t)) = \sqrt{\frac{\gamma}{f(t)}}.$$

We therefore have that the quantum potential vanishes as R_0 does not depend on x. Thus, U_0 is the classical potential and Q_1 will be the classical trajectories. For a quadratic potential, the classical trajectories are of the same form as Q_0. Thus, the

classical trajectories are the fixed point of the algorithm. Therefore, let f, g, and h be the appropriate functions for the solutions to the classical system according to (7.1a). The equations they satisfy, in addition to the initial conditions (7.1b), (7.1c) are

$$m\ddot{g}(t) = -a(t)g(t) \tag{7.2a}$$

$$m\ddot{f}(t) = -a(t)f(t) \tag{7.2b}$$

$$m\ddot{h}(t) = -h(t)a(t) - b(t). \tag{7.2c}$$

Note that h vanishes if b vanishes.

Given f, we can immediately write down R for the Green's function. It is possible for f to have zeros. A zero of f, which is a very singular caustic, can be dealt with as we suggested before. We are essentially using the square root to see the correction in the phase appearing as the Maslov index tells us. Since the Maslov index will increase by 1 through a generic caustic, if we start in the middle of a caustic, then we take the index to be $\frac{1}{2}$. What we need to do is to compute the action S; once we have that, then the solution will be

$$G(x,t,y_0,0) = R(t)e^{-i\pi 4 - \mu(t)\frac{\pi}{2}} e^{i\frac{S_{y_{t_0}}}{\hbar}}.$$

We can do this using either the action integral or the fact that ∇S is the velocity field, i.e. $\nabla S(x,t) = m\dot{Q}(Q^{-1}(x,t),t)$. Define $k_t := y_0 g(t) + h(t)$ and let f_t denote $f(t)$. The second method yields

$$S(x,t) = m(\tilde{S}_{y_0}(t) + \frac{\dot{f}_t}{f_t}\frac{x^2}{2} - \frac{\dot{f}_t}{f_t}(k_t x + (\dot{k}_t x))$$

where $\tilde{S}_{y_0}$ is an unknown function for which one needs the first method to compute. To do so, we write down the action integral, which is

$$\int_0^t [\frac{m}{2}(\dot{f}_s v + \dot{k}_s)^2 - (\frac{1}{2}a_s(f_s v + k_s)^2 + b_s(f_s v + k_s) + c_s)]ds.$$

Replacing v with $\frac{x}{f_t} - \frac{k_t}{f_t}$, simplifying, and only retaining the terms not involving x, we find

$$\int_0^t [\frac{m}{2}(-\dot{f}_s\frac{k_t}{f_t} + \dot{k}_s)^2 - \frac{1}{2}a(s)(-f_s\frac{k_t}{f_t} + k_s)^2 + b(s)(-f_s\frac{k_t}{f_t} + k_s) + c(s)]ds \tag{7.3}$$

In summary, we have reduced the computation of the propagator to computing the above integral and the set of ODEs (7.2). And, to simplify even more, the terms not

involving y_0 are common to all of the Green's functions and do not have to be computed. Indeed, $c(s)$ is largely irrelevant and could even be thought of as a free parameter to be chosen to cancel such terms.

There is also an argument, based on quantum mechanical heuristics, which establishes what γ should be. We want a measure on the velocity space which corresponds to a delta function in the position space. We therefore take the Fourier transform of the delta function, finding that $\frac{1}{\sqrt{2\pi}}e^{iky} =: \hat{\psi}$ with density $|\hat{\psi}|^2 = \frac{1}{2\pi}$. But this is momentum space and the relation to the velocity space is $k = \frac{mv}{\hbar}$. Taking the differential of that relation, we find that $\frac{dk}{2\pi} = \frac{mdv}{2\pi\hbar}$ implying that $\gamma = \frac{m}{2\pi\hbar}$.

We finish by explicitly computing the propagators for the free potential and the harmonic oscillator. The free potential, $V(x) = 0$, has $f(t) = t$, $g(t) = 1$, and $h(t) = 0$. We find that the propagator is

$$G(x, t, y_0, 0) = \sqrt{\frac{\gamma}{t}} e^{-i\frac{\pi}{4}} e^{i\frac{m}{\hbar}\frac{(x-y_0)^2}{2t}}$$

for $t > 0$. For $t < 0$ multiply the above by $e^{i\frac{\pi}{2}}$ as the Maslov index indicates. It is a fact that

$$\delta(x - y_0) = \lim_{a \to +\infty} \frac{a}{\sqrt{\pi}} e^{i(a^2(x-y_0)^2 - \frac{\pi}{4})}.$$

Thus, this confirms that the initial condition is satisfied, that the Maslov index is contributing the correct factor, and that γ is $\frac{m}{2\hbar\pi}$.

As for the harmonic oscillator, the following comes out. Let the potential be $V(x) =$

$\frac{m\omega^2}{2}x^2$. Then $f(t) = \frac{1}{\omega}\sin(\omega t)$, $g(t) = \cos(\omega t)$, and $h(t) = 0$. The integral (7.3) becomes

$$\begin{aligned}
\tilde{S}_{y_0}(t) = &\int_0^t [\frac{m}{2}(\cos(\omega s)\frac{\omega y_0 \cos(\omega t)}{\sin(\omega t)} + y_0\omega\sin(\omega s))^2 \\
&- \frac{m\omega^2}{2}(-\frac{\sin(\omega s)}{\omega}\frac{\omega y_0\cos(\omega t)}{\sin(\omega t)} + y_0\cos(\omega s))^2]ds \\
=&\frac{m\omega^2 y_0}{2}\int_0^t [(\cos^2(\omega s) - \sin^2(\omega s))\cot(\omega t)^2 \\
&+ 4(\cos(\omega s)\sin(\omega s))\cot(\omega t) \\
&- (\cos^2(\omega s) - \sin^2(\omega s))]ds
\end{aligned}$$

$$\begin{aligned}
=&\frac{m\omega^2 y_0}{2}(\frac{1}{\omega}(\sin(\omega t)\cos(\omega t)(\cot(\omega t)^2 - 1) + 2(\frac{1}{\omega}\sin^2(\omega t))\cot(\omega t) \\
=&\frac{m\omega y_0}{2}\cot(\omega t)
\end{aligned}$$

Thus, when we put it all together, we find

$$G(x,t,y,0) = \sqrt{|\frac{\gamma\omega}{\sin(\omega t)}|}e^{-i(\frac{\pi}{4}+\mu(t)\frac{\pi}{2})}e^{i\frac{m\omega}{2\hbar}(x^2+y^2)\cot(\omega t) - 2xy\csc(\omega t)}.$$

The Maslov index, $\mu(t)$, starts at 0 and is incremented by one at the end of every half period; this incrementation corresponds to when all of the trajectories come together and form a very singular caustic. The Maslov index produces the correct phase as the trajectories move through the singularity. Explicitly, we have $\mu(t) = k$ for $t \in [k\frac{\pi}{\omega}, (k+1)\frac{\pi}{\omega})$. Computing the limits as t approaches a singular point, we find that

$$G(x, k\frac{\pi}{\omega}, y, 0) = e^{i\frac{\pi}{2}k}\begin{cases}\delta(x-y) & \text{if } k \text{ is even} \\ \delta(x+y) & \text{if } k \text{ is odd}\end{cases}.$$

This is a complete description of the propagator for the harmonic oscillator.

7.5 Some calculations involving the chain rule

We wish to provide a succinct formula for computing the quantities in these algorithms. The basic quantity to compute is the quantum force and this involves three derivatives of an object created by composition. These are all simple calculations using the chain rule. We shall use the notations of Section 7.2.1

We start with R. This is the object for which much computation must be done. First, we define $\tilde{R}(x,t) := P(x)|DQ|^{-\frac{1}{2}}$ and then $R := \tilde{R}(H)$. We will have to compute up to three derivatives of R. Using the chain rule, we find that

$$\begin{aligned}
DR(v_1) =& D\tilde{R}(DH(v_1)) \\
D^2R(v_1 \otimes v_2) =& D^2\tilde{R}(DH(v_1) \otimes DH(v_2)) + D\tilde{R}(D^2H(v_1 \otimes v_2)) \\
D^3R(v_1 \otimes v_2 \otimes v_3) =& D^3\tilde{R}(DH(v_1) \otimes DH(v_2) \otimes DH(v_3)) \\
&+ D^2\tilde{R}\{(D^2H(v_1 \otimes v_2) \otimes DH(v_3)) \\
&\quad + (D^2H(v_1 \otimes v_3) \otimes DH(v_2)) \\
&\quad + D_3H(v_1) \otimes (D^2H(v_2 \otimes v_3))\} \\
&+ D\tilde{R}(D^3H(v_1 \otimes v_2 \otimes v_3)).
\end{aligned}$$

In our equations, we need to compute the quantum force which is the gradient of the Laplacian of R divided by R. We have that

$$D(\frac{\Delta R}{R})(v) = \frac{D^3R(G \otimes v)}{R} - \frac{D^2R(G)DR(v)}{R^2}$$

where $G \in TM^{\otimes 2}$ is the dual of g; in coordinates, $G = \sum_{(i,j)\in \mathfrak{n}^2} G^{ij} v_i \otimes v_j$ and G^{ij} are the components of the inverse matrix of g_{ij}.

The derivatives of $\tilde{R}$ can in turn be computed using the product rule, the chain rule and the differentiation formulas for the determinant developed in Appendix F.

Both for implementing the algorithms and for an application of A2 described in the next section, we need to compute $D^i_{Q(x,t)}[Q^{-1}]$ in terms of the derivatives of DQ. Yet again we use the chain rule. We need it up to the third order. Let $y := Q(x)$; alternatively, define $x := Q^{-1}(y)$. We also define $B := (D_x[Q])^{-1}$. For $i = 1, 2, 3$, let

$v_i \in T_yM$ be arbitrary. We then find that

$$\begin{aligned}
Q^{-1}(Q(x)) =& \mathrm{Id} && (7.4)\\
D_y[Q^{-1}](v_1) =& B(v_1) && (7.5)\\
D_y^2[Q^{-1}](v_1 \otimes v_2) =& -B(D_x^2[Q](B(v_1) \otimes B(v_2))) && (7.6)\\
D_y^3[Q^{-1}](v_1 \otimes v_2 \otimes v_3) =& -B(D_x^3[Q]\{B(v_1) \otimes B(v_2) \otimes B(v_3)\}) \\
&+ B(D_x^2[Q](B(D_x^2[Q](B(v_1) \otimes B(v_2))) \otimes B(v_3) \\
&+ B(D_x^2[Q](B(v_1) \otimes B(v_3))) \otimes B(v_2) \\
&+ B(v_1) \otimes B(D_x^2[Q](B(v_2) \otimes B(v_3)))) && (7.7)
\end{aligned}$$

These are routine calculations. For example, to compute the second derivative, the chain rule leads to $D_Q^2[Q^{-1}](D_x[Q] \otimes D_x[Q]) + D_Q[Q^{-1}](D_x^2 Q) = 0$. One then solves for the second derivative of the inverse and then composes with the inverse of the first derivative using the equation for the first derivative to substitute in B.

In total, we can see what we need. Before starting the algorithms, we compute the first derivative of T, i.e. the 1-form dual to the initial Bohmian velocity field. We also compute three derivatives of P. The rest of the information is mainly computed from Q. We need to compute four derivatives of Q although the fourth derivative appears only in the determinant derivative. To finish the computations, Q^{-1} and DQ^{-1} need to be computed. We then simply plug the objects into the correct places.

7.6 The Bohmian velocity partial differential equation

From the previous section, we discussed some ways in which the essential computations can be organized. We apply these ideas here, at the fixed point, to derive a PDE for the family of Bohmian trajectories. One advantage of this is that it is not necessary to compute the inverse of a given family of trajectories; the disadvantage is that we are again facing a PDE.

The idea is that one writes down the family of ODE's generated by the algorithm A2. One then assumes that the fixed point has been achieved. The ODE for A2 is

$$g(\ddot{q}(x,t),) = -dV(q(x,t)) + QF(Q^{-1}(q(x,t),t),t)$$

where the quantum force, QF, is thought of as being a function of the initial x_0. When we evaluate along the fixed point, we have that Q^{-1} and Q cancel, just leaving the quantum force evaluated at x.

We have the following PDE for Q, the family of Bohmian trajectories for a given $\psi(x,0)$ and potential V. We write it in terms of 1-forms acting on a vector v. C is the determinant of DQ, B is $(D[Q])^{-1}$. We sometimes employ subscripts to denote which objects are being contracted. A rough schematic of the quantum force expression is quite helpful in understanding the PDE; the schematic is

$$\frac{1}{\tilde{R}}\{D^3[Rq](\cdot) + D^2[\tilde{R}](\cdot) + D[\tilde{R}](\cdot)\} - \frac{1}{\tilde{R}^2}\{D^2[\tilde{R}](\cdot) + D[\tilde{R}](\cdot)\}\{D[\tilde{R}](\cdot)\}.$$

One task is to compute, somewhat explicitly, the derivatives of $\tilde{R}$. The other task is to compute the various different tensors that are composed into the $\cdot$ slots. It can get quite messy.

$$
\begin{aligned}
g(\ddot{Q}(x,t),v) =& - dV_Q(v) + \frac{\hbar^2}{2}(\frac{1}{P}\{ \\
& (D^3P - \frac{15}{8}PC^{-3}D^3C \\
& - \frac{1}{2}C^{-1}D^2P_{12}\otimes DC_3 - \frac{1}{2}C^{-1}D^2P_{13}\otimes DC_2 - \frac{1}{2}C^{-1}DC_1\otimes D^2P_{23} \\
& + \frac{3}{4}C^{-2}D^2C_{12}\otimes DP_3 + \frac{3}{4}C^{-2}D^2C_{13}\otimes DP_2 + \frac{3}{4}C^{-2}DP_1\otimes D^2C_{23}) \\
& \quad \circ (\{(B\otimes B)\circ G\}\otimes B(v)) \\
& + (D^2P + \frac{3}{4}PC^{-2}D^2C \\
& - \frac{1}{2}C^{-1}DP\otimes DC - \frac{1}{2}C^{-1}DC\otimes DP) \\
& \quad \circ (-\{B\circ(D^2[Q])\circ(B\otimes B)\circ G\}\otimes B(v) \\
& \quad\quad - \{B\circ(D^2[Q])\circ(B(G_1)\otimes B(v))\}\otimes B(G_2) \\
& \quad\quad - B(G_1)\otimes\{B\circ(D^2[Q])\circ(B(G_2)\otimes B(v))\}) \\
& + (C^{-\frac{1}{2}}DP - \frac{1}{2}PC^{-1}DC) \\
& \quad \circ (-B\circ(D^3[Q])\circ\{(B\otimes B)\circ G\}\otimes B(v) \\
& \quad\quad + B\circ D^2[Q]\circ B\otimes B\circ(\{D^2[Q]\circ(B\otimes B)\circ G\}\otimes v \\
& \quad\quad\quad\quad + \{D^2[Q]\circ(B(G_1)\otimes B(v))\}\otimes G_2 \\
& \quad\quad\quad\quad + G_1\otimes\{D^2[Q](B(G_2)\otimes B(v)\}))\} \\
& - \frac{1}{P^2}\{(D^2P - \frac{1}{2}C^{-1}DP\otimes DC - \frac{1}{2}C^{-1}DC\otimes DP + \frac{3}{4}PC^{-2}D^2C) \\
& \quad \circ (B\otimes B)\circ G \\
& + (DP - \frac{1}{2}PC^{-1}DC)\circ(-B)\circ(D^2[Q])\circ(B\otimes B)\circ G\} \\
& \{DP - \frac{1}{2}PC^{-1}DC)(B(v))\}
\end{aligned}
$$

with the initial conditions

$$
\begin{aligned}
Q(x,0) =& x \\
\dot{Q}(x,0) =& \nabla T(x).
\end{aligned}
$$

For the algorithms, the only change in the quantum force is the evaluation which takes place at $Q^{-1}(q(x,t))$ instead of at x; appearances of Q outside the quantum force term are replaced with q.

There is not much to say about this PDE. Notice that if C is constant, then many of the terms vanish. The equation, however is still rather unapproachable.

A key simplification occurs when $D^2Q = 0$ everywhere which geometrically is when the mapping Q preserves geodesics as proven in Chapter 6. The PDE becomes the relatively simple

$$\begin{aligned} g(\ddot{Q}(x,t),v) = &- dV_Q(v) + \frac{\hbar^2}{2}(\frac{1}{P}D^3P \circ (\{(B \otimes B) \circ G\} \otimes B(v)) \\ &- \frac{1}{P^2}\{D^2P \circ (B \otimes B) \circ G\}\{DP(B(v))\}. \end{aligned}$$

Notice that C has disappeared entirely from the equation.

7.6.1 The one-dimensional case

If M is just $\mathbb{R}$, then G becomes $\frac{1}{m}$, the derivatives become numbers, and we use primes to denote spatial differentiation. The full PDE simplifies slightly, but it is still brutal.

If $Q'' = 0$ identically, then we can solve the equation. We have

$$m\ddot{Q}(x,t) = -V_Q' + \frac{\hbar^2}{2}(\frac{P'''}{mP(Q')^3} - \frac{P''P'}{mP^2(Q')^3}.)$$

Furthermore, $Q'' = 0$ implies that $Q(x,t) = f(t)x + k(t)$.

Let $H_n(x)$ denote the n^{th} Hermite polynomial. We claim that if $V(x) = \frac{m}{2}\omega^2x^2 + m\gamma x$ and

$$\psi(x,0) = H_n(\sqrt{md/\hbar}(x-c))e^{-\frac{md}{2\hbar}(x-c)^2}e^{i\frac{m}{\hbar}(\frac{a}{2}(x-c)^2+b(x-c))},$$

$(d > 0)$, then the trajectory solutions are of the above affine form and this PDE can be solved quite explicitly; it actually becomes two decoupled ODEs. We shall use that $y(x) := H_n(x)e^{-\frac{x^2}{2}}$ satisfies $y'' = (x^2 - 2n - 1)y$; this comes from the Schrödinger equation for the harmonic oscillator. For us, the useful imphlication of the Hermite differential equation is

$$yy''' - y''y' = 2xy^2 + (x^2 - 2n - 1)y'y - (x^2 - 2n - 1)yy' = 2xy^2$$

When we use it, x will be $\sqrt{md/\hbar}(x-c)$ and we shall also be using the chain rule which produces a multiplicative factor of $\sqrt{md/\hbar}^3$.

It is convenient to take $Q(x,t) = f(t)(x-c) + k(t)$. Then we have that

$$\frac{PP''' - P''P'}{P^2} = 2(\sqrt{md/\hbar}(x-c)\sqrt{md/\hbar}^3$$

The PDE becomes

$$\begin{aligned} m(\ddot{f}(t)(x-c) + \ddot{k}(t)) =& \frac{\hbar^2}{2mf(t)^3}(2(md/\hbar)^2(x-c)) \\ & - m\omega^2(f(t)(x-c)+k(t)) - m\gamma \\ =& \frac{md^2}{f(t)^3}(x-c) - m\omega^2 f(t)(x-c) - m\omega^2 k(t) - m\gamma. \end{aligned}$$

This decouples into the two independent ODEs

$$\ddot{f} = \frac{d^2}{f^3} - \omega^2 f \tag{7.8}$$

$$\ddot{k} = -\omega^2 k - \gamma \tag{7.9}$$

with the initial conditions

$$f_0 = 1, \dot{f}_0 = a, k_0 = c, \dot{k}_0 = b.$$

The ODE (7.9) is an easy equation to solve; its solution is

$$k(t) = (c + \frac{\gamma}{\omega^2})\cos(\omega t) + \frac{b}{\omega}\sin(\omega t) - \frac{\gamma}{\omega^2}.$$

The other ODE is not as easy to solve. The inspiration is that (7.8) is actually the ODE that the radius function satisfies in a 2-dimensional, classical, harmonic oscillator system. More precisely, let u and v satisfy

$$\ddot{u} = -\omega^2 u, \quad \ddot{v} = -\omega^2 v$$

$$u(0) = 1, \dot{u}(0) = a, \quad v(0) = 0, \dot{v}(0) = d.$$

Define $r := \sqrt{u^2 + v^2}$. Then, by differentiating and using the conservation of $J^2 = (\dot{u}v - \dot{v}u)^2 = d^2$, we find that r satisfies

$$\ddot{r} = \frac{d}{r^3} - \omega^2 r$$

with the initial conditions

$$r(0) = 1, \dot{r}(0) = a.$$

Thus, r and f satisfy the same equations implying that they are the same function. But we know how to solve the harmonic oscillator. Doing so and finding the radial function, we have

$$f(t) = r(t) = \sqrt{\cos^2(\omega^2 t) + 2\frac{a}{\omega}\cos(\omega t)\sin(\omega t) + \frac{a^2+d^2}{\omega^2}\sin^2(\omega t)}.$$

For both f and k, if we take the limit as ω goes to zero, then the correct solution for $\omega = 0$ is the limiting solution.

We have just computed the Bohmian trajectories. An immediate consequence is that we have the amplitude. Namely,

$$\begin{aligned} R(x,t) =& P(\frac{x-k(t)}{f(t)} + c)|f(t)|^{-\frac{1}{2}} \\ =& \frac{1}{\sqrt{f(t)}} H_n(\sqrt{md/\hbar}(\frac{x-k(t)}{f(t)}))e^{-\frac{md}{2\hbar}(\frac{x-k(t)}{f(t)})^2}. \end{aligned}$$

The phase is

$$S(x,t) = \frac{m}{\hbar}(\frac{1}{2}\dot{f}(t)(\frac{x-k(t)}{f(t)})^2 + \dot{k}(t)(\frac{x-k(t)}{f(t)}) + h(t))$$

where $h(t)$ is a function for which one needs to do the action integral in order to determine it. We shall not compute $h(t)$.

Appendix A

The set-indexed tensor product

Let T be a finite set. For each element $a \in T$, we associate a vector space W_a to that element. We define the T-indexed Cartesian product of these spaces to be

$$\underset{a\in T}{\times} W_a := \{f | f : T \to \bigcup_{a\in W} W_a \text{ such that } f(a) \in W_a\}.^1$$

Endowed with the usual vector space structure, this is the T-indexed direct sum of the spaces.[2] We define the T-indexed tensor product $\bigotimes_{a\in T} W_a$ to be the set of multilinear functionals defined on $\times_{a\in T} W_a^*$, the T-indexed Cartesian product of the dual spaces.

If, as will be our case, the vector spaces W_a are the same vector space, say W, then simplifications occur. Notationally, W^T denotes the Cartesian product; $W^{\otimes T}$ denotes the tensor product. As for the definition of the Cartesian product, the union can be replaced with the vector space itself. The Cartesian product becomes

$$W^T = \{f | f : T \to W\}.$$

From this point on, we will assume that we have only one vector space. The more general case does not substantially change any statements below, but it is irrelevant to the purpose of this paper.

Let us consider a different approach to $W^{\otimes T}$. We will use the standard equivalence class approach to tensors. We start by introducing some notation. For $f \in W^T$, we write $\otimes_{a\in T} w_a$, where $w_a := f(a)$. Given a decomposition of T into the subsets T_α, $\alpha \in A$, for some A, then we certainly take as true

$$\underset{\alpha\in A}{\otimes}\,(\underset{a\in T_\alpha}{\otimes} w_a) = \underset{a\in T}{\otimes} w_a = f.$$

[1] We could also say that the Cartesian product is the set of sections of the bundle over the discrete space T whose fiber at a is W_a.

[2] As an example, the tangent space of ${}^N\mathbb{R}^3$ is the q-indexed direct sum of the tangent spaces in physical space.

A special case of the above, and one with a special abuse of notation, is when A is a two element subset and one of the sets in the decomposition has only one element, say a'. Then we would write $(\otimes_{\substack{a\in T\\ a\neq a'}} w_a)\otimes w_{a'}$ which is equal to $\otimes_{a\in T} w_a$ as well as $w_{a'}\otimes(\otimes_{\substack{a\in T\\ a\neq a'}} w_a)$. We can then view the T-indexed tensor product as the equivalence classes of all formal linear combinations of the elements in the set W^T under the relations

$$c(\underset{a\in T}{\otimes} w_a) = (\underset{\substack{a\in T\\ a\neq a'}}{\otimes} w_a)\otimes cw_{a'}$$

$$(\underset{\substack{a\in T\\ a\neq a'}}{\otimes} w_a)\otimes x_{a'} + (\underset{\substack{a\in T\\ a\neq a'}}{\otimes} w_a)\otimes y_{a'} = (\underset{\substack{a\in T\\ a\neq a'}}{\otimes} w_a)\otimes (x+y)_{a'}.$$

These relations should hold for every $a'\in T$, $c\in\mathbb{C}$ and w_a $w_{a'}$, $x_{a'}$, $y_{a'}\in W$. We should note the standard fact that the multilinear functionals acting on W^T extend to linear functionals acting on $W^{\otimes T}$; this is a third way of viewing $W^{\otimes T}$.

If W has an inner product, then there is a natural inner product on $W^{\otimes T}$. Namely, for elements in W^T, we define the inner product to be

$$(\underset{a\in T}{\otimes} w_a, \underset{a\in T}{\otimes} v_a) := \prod_{a\in T}(w_a, v_a).$$

We then extend this definition to $W^{\otimes T}$ by using the appropriate linearities.

If we have a linear operator C acting on W, then C^a shall denote a linear operator acting on $W^{\otimes T}$ such that, when acting on product elements, it applies C to the a^{th} factor while applying the identity to the other factors, i.e.

$$C^{a'}(\underset{a\in T}{\otimes} w_a) := (\underset{\substack{a\in T\\ a\neq a'}}{\otimes} w_a)\otimes Cw_{a'}$$

We finish with a word about contractions. The set-indexed tensor product is excellent for use with contractions. Indeed, given $w\in\bigotimes_{a\in\alpha} E\otimes\bigotimes_{b\in\beta} E^*$, then the contraction of w over the index pair $(\tilde{a},\tilde{b})$ is denoted by $\mathcal{C}(\tilde{a},\tilde{b})w$ and is an element of $\bigotimes_{a\in\{\alpha\setminus\{\tilde{a}\}\}} E\otimes\bigotimes_{b\in\{\beta\setminus\{\tilde{b}\}\}} E^*$. This allows one to notationally keep track of which indices were contracted. Using numbers as indices eliminates this information. Furthermore, one can also see that a type of unit analysis may be invoked. In other words, the expression $\mathcal{C}(\tilde{a},\tilde{b})w + \mathcal{C}(a',b')w$ would seem to make no sense whereas $\mathcal{C}(\tilde{a},\tilde{b})\mathcal{C}(a',b')w + \mathcal{C}(\tilde{a},b')\mathcal{C}(a',\tilde{b})w$ would make sense. With numbers, this would be

hard to indicate. As such, the set-indexed tensor product may be as helpful in formulating meaningful expressions as dimensional analysis has historically been.

Appendix B

A very useful frame field

In this part, we shall explain how to find sections satisfying constraints on the symmetric part of their covariant derivative. Let M be a manifold with connection and E be a vector bundle over M whose fiber is either isomorphic to $\mathbb{R}^k$ or $\mathbb{C}^k$. Fix $p \in M$. This appendix relies upon the work of Chapter 6 and is used in Chapter 3

Choose a coordinate neighborhood about p such that for every point q in the neighborhood, there is a unique geodesic between p and q. Choose a basis $\{e_i\}_{i=1}^k$ of the fiber E_p for the rank k bundle E over M. Parallel transport these vectors along the geodesics emanating from p, defining a basis field also denoted by $\{e_i\}_{i=1}^k$.

Let q be any point in the neighborhood. Take γ_v to be the unique geodesic between p and q. Then, by definition, for each i, $\nabla_v[e_i] = 0$ along γ_v. Thus, Corollary 1 applies and we may conclude that the **symmetric** part of $D_p^n[e_i] = 0$ for all i.[1] We emphasize that this is only true at p.

Now define $A := \sum_{i=1}^k f^i e_i$ where the f^i are scalar functions defined on the neighborhood. Then the symmetric part of the covariant derivatives of A are the symmetric covariant derivatives of the f^i's. Using Corollary 3, we know that the symmetric part of the higher order covariant derivatives of the f^i's are the Euclidean derivatives when expressed in normal coordinates. Thus, it is reasonably easy to specify the symmetric part of various higher order covariant derivatives.

The other parts of the covariant derivatives are part of the geometric structure of the connection. For example, the second covariant derivative's antisymmetric part is

[1] It is true that the diagonal part of a tensor determines its symmetric values. One proof is based on the following. Define $f(a_1, \ldots, a_k) := A(\{\sum_i a_i e_i\}^n)$. Notice that we know f for all values of the a_i's. Expand the right-hand side. Taking the appropriate n^{th}-order partial derivatives with respect to the a_i's and evaluating the derivatives at $(0, \ldots, 0)$, we can generate any symmetric expression in the e_i's of length n that we wish.

the curvature tensor, at least if the connection on M has no torsion. More specifically, we claim

$$\nabla_x[\nabla_y[A]] - \nabla_y[\nabla_x[A]] - \nabla_{[x,y]}[A] = D^2[A](y \otimes x - x \otimes y) - D[A](\tau(x,y))$$

where τ is the torsion; torsion, acting on a function f, is

$$\tau(x,y)f = (\nabla_x[y] - \nabla_y[x] - [x,y])f = D^2[f](x \otimes y - y \otimes x).$$

These assertions may be verified using the chain rule. A typical fact to use, for example, is

$$\nabla_x[\nabla_y[A]] = D^2[A](y \otimes x) + D[A](\nabla_x[y]).$$

The formulas involving the differentiation of the vector fields are the usual formulas for the curvature and torsion of a connection. Writing it in this fashion demonstrates that curvature is independent of the connection on the manifold. But writing it using the higher covariant derivatives allows us to immediately see that the objects are tensors in the tangent vector slots involving x and y. As for the curvature being a tensor in A, that is why the torsion term is necessary. Indeed, multiply A by a function f, and one sees that cross terms cancel when computing the antisymmetric part leaving the expression in A that we want plus a torsion term. Subtracting the torsion therefore eliminates the contribution of f.

Another way to view the subtraction of the torsion is to think of torsion as a type of curvature and that multiplication of a scalar function times a section is actually the tensor product of the section of E with a section of the trivial one-dimensional bundle over M. Then, as is easily seen, the curvature of the tensor product connection is a sum of the two curvatures. Thus, we subtract the function's curvature to obtain the curvature of the bundle E.

In any event, when we use our special basis, we are only left with the antisymmetric part as the symmetric part of the basis fields vanish at p. Thus the second covariant derivatives of these fields is the curvature tensor applied to field that we are differentiating, assuming that there is no torsion.

As an application of this, we can prove that if E is a complex bundle with a connection and a parallel inner product $(,)$, then the curvature tensor is anti-self-adjoint. Let R be the curvature tensor. Then, to establish our claim, we need to show that

$$(\psi, R\phi) = -(R\psi, \phi)$$

for any sections ψ and ϕ of E. Fix a point $p \in M$. Given $\psi_p, \phi_p \in E_p$, define ψ and ϕ by parallel transporting the value at p to all the other points in an appropriately small neighborhood about p. As just explained, $D^2\psi = R\psi$ and similarly for ϕ where we use a torsionless connection on M to compute the second covariant derivative. Since the inner product is parallel, we have that $(\psi, \phi)_q = (\psi, \phi)_p$ for all q in the neighborhood. More to the point, we have $D[(\psi, \phi)] = 0$ and, more importantly,

$$D^2[(\psi, \phi)] = 0.$$

But notice that

$$D_p^2[(\psi, \phi)] = (D_p^2[\psi], \phi) + (\psi, D_p^2[\phi]) = (R\psi, \phi)_p + (\psi, R\phi)_p;$$

note that only at p do we have the vanishing of the first derivative terms in addition to the second derivative terms being the curvature tensor. As one can see, we have our result.

Appendix C

The issue of self-adjointness

There are technical issues that we have avoided. In particular, the formal Hamiltonian may not be essentially self-adjoint when defined on the domain of smooth, compactly supported wave functions on $\mathcal{Q}$. In [16], it is shown that for two identical particles moving in two dimensions, there is a one-parameter family of self-adjoint extensions for each periodicity condition except the one corresponding to fermions, for which the extension is unique. That paper argues for a certain regularity condition which does select out the usual extension.

In [2, 52], the issue of self-adjointness for the Aharonov-Bohm effect is explored. Modelling on the plane with the origin removed, one finds that, in addition to the topological choice, there is a family of extensions specified by four real parameters. They use the bundle approach and specify the connection first before considering the self-adjoint extensions.

In general, if $\mathcal{Q}$'s Hamiltonian is not an essentially self-adjoint operator on the usual small domain of wave functions, then one needs to appeal to the particulars of the situation to make the appropriate selection. With the bundle approach, it seems necessary to first make the topological choices and then ask for extensions. For the cut approach, it would seem to be reasonable to do both at the same time. In the case of the covering space it seems possible that we could first pick the extension on the covering space and then choose among the compatible periodicity conditions.

Finally, from the perspective of Bohmian evolutions, it is sufficient to choose the wave functions which can serve as initial wave functions. That is to say, for our theory we need to specify what are the allowed initial conditions. The wave functions which are allowed are those which provide a Bohmian evolution for typical configurations. This is

a dense set invariant under the Schrödinger evolution; hence it is a domain of essential self-adjointness. Thus, if $\mathcal{A}$ denotes the set of Bohmian wave functions, then our theory is completely specified with the following information: $\{\mathcal{Q}, g, E, \nabla, (,), \boldsymbol{S}, V, \gamma, \mathcal{A}\}$.

Appendix D

On the question of the triviality of the Bose and Fermi bundles

The notation of this section is that $\mathcal{B}(N, s)$ is the Bose bundle for spin-s particles and $\mathcal{F}(N, s)$ is the Fermi bundle. This is the notation for $3 = 3$. For $3 > 3$, we replace s with the dimension of the complex vector space. So s is replaced with $2s + 1$.

Here are the results of this section:

1. For $N = 2$, $3 = 3$, the triviality of the bundles are as follows:

	$s = 2l$	$s = 2l + \frac{1}{2}$	$s = 2l + 1$	$s = 2l + \frac{3}{2}$
$\mathcal{B}(2, s)$	trivial	nontrivial	nontrivial	trivial
$\mathcal{F}(2, s)$	nontrivial	nontrivial	trivial	trivial

   for $l \in \mathbb{N}$.

2. For $N \geq 2$, we have $\mathcal{B}(N, 2l + 1)$ is nontrivial and $\mathcal{F}(N, 2l)$ is nontrivial. We do not know the triviality or nontriviality of the other bundles.

3. The configuration space of ${}^N\mathbb{R}^3$ is orientable if and only if 3 is even.

Throughout, $3 \geq 3$. The nontriviality results apply for any such 3. The triviality results only apply for $N = 2, 3 = 3$ and for the Bose line in any dimension.

Lemma 10. *A bundle is trivial iff it has a flat connection with trivial holonomy.*

Proof. If it is trivial, say $M \times V$, then any basis of V translates into a global basis. Hence, the coordinates provide the way to do trivial transport. If a bundle has trivial holonomy, then construct a basis by parallel transporting it. As parallel transport along any closed path is, by assumption, the identity operation, the constructed basis is well-defined and smooth. Note that having trivial holonomy is the only way that a connection can have a global parallel basis. □

Theorem 12. *If L is a real or complex line bundle over $\mathcal{Q} := {}^N\mathbb{R}^3$, then it is nontrivial if there exists a flat connection ∇^L on L such that its holonomy is the alternating character of $\pi_1(\mathcal{Q})$.*

Proof. The heart of the proof is the following idea. The difference between two connections is a 1-form and for both of them to be flat, the 1-form must be closed. Furthermore, to have a nontrivial holonomy, the 1-form must be non-exact. Since all closed 1-forms on $\mathcal{Q}$ are exact, we are done.

The details are the following. We need to recall a few basic bundle facts.

1. The difference of two connections on the same bundle is a matrix-valued 1-form. For complex (real, respectively) line bundles, this means complex-valued (real-valued) 1-forms. All we need to check is that it satisfies

$$(\nabla^1 - \nabla^2)(f\psi) = \nabla^1 f\psi - \nabla^2 f\psi = f(\nabla^1\psi - \nabla^2\psi),$$

   i.e. it needs to be a tensor. This follows from the Leibniz rule imposed on all connections; namely $\nabla f\psi = df\psi + f\nabla\psi$.

2. Let P^2_γ and P^1_γ be the holonomy operators for ∇^1, ∇^2 along the path γ. By definition, $\nabla^i_{\dot\gamma} P^i_\gamma = 0$. We define C_γ to satisfy $P^2_\gamma = C_\gamma P^1_\gamma$. In particular, $C_\gamma = P^2_\gamma (P^1_\gamma)^{-1}$. If we further define $\omega := \nabla^2 - \nabla^1$ then we have

$$\nabla^1_{\dot\gamma} C_\gamma = -\omega(\dot\gamma) C_\gamma. \tag{D.1}$$

3. If a connection is flat, then its parallel transport operators depend only on the homotopy class of the curve.

4. For a line bundle, ω is a 1-form and C_γ is just a number. Then the solution to equation (D.1) is $C_\gamma = e^{-\int_\gamma \omega}$. If we define $f(\gamma) = -\int_\gamma \omega$ and if γ and σ are in the same homotopy class, then $f(\gamma) - f(\sigma)$ must be an integral multiple of $2\pi i$ in order that $C_\gamma = C_\sigma$. But the difference function is a continuous function with a discrete set of values. Hence, it must be constant on any connected component. In particular, the integrals depend only on the homotopy class.

Assume that L is trivial and let ∇^T be the implied trivial flat connection on L. Then ω defined by $\omega := (\nabla^L - \nabla^T)$ is a 1-form on $\mathcal{Q}$. We have $P^L_\gamma = e^{\int_\gamma -\omega} P^T_\gamma$. By assumption, if σ is a path in $\mathcal{Q}$ exchanging two particles, then $\int_\sigma -\omega \equiv i\pi (\mathrm{mod}\ 2\pi)$, i.e. it is non-zero. But

$$0 = \int_{\mathrm{Id}} -\omega = \int_{\sigma\sigma} -\omega = 2\int_\sigma -\omega.$$

Thus, we have that the integral must be zero which contradicts nontrivial holonomy. □

Corollary 8. *The Fermi line on ${}^N\mathbb{R}^3$ is nontrivial.*

Proof. The Fermi line's holonomy is the alternating character. Thus it is a nontrivial line bundle. □

Corollary 9. *The configuration space of ${}^N\mathbb{R}^3$ is orientable if and only if 3 is even.*

Proof. A space is orientable if and only if the determinant line of the tangent space is trivial. That is to say, we need to compute $\Lambda^{3N}(T\mathcal{Q})$. Let $\{e_i\}_{i=1}^3$ be the standard basis in $\mathbb{R}^3$. Then a basis element of $\Lambda^{3N}(T\mathcal{Q})$ is $\bigwedge_{i=1}^{3}(\bigwedge_{\boldsymbol{q}\in q} e_{i,\boldsymbol{q}})$; this is defined up to a sign. In order to define a global basis, we must be able to choose the sign consistently. Rather than do that, we shall compute the holonomy of this bundle:

$$\bigwedge_{i=1}^{3}(\bigwedge_{\boldsymbol{q}\in q} e_{i,\boldsymbol{q}}) \overset{\Gamma_\sigma}{\mapsto} \bigwedge_{i=1}^{3}(\bigwedge_{\boldsymbol{q}\in q} e_{i,\sigma\boldsymbol{q}}) = (-1)^{3|\sigma|} \bigwedge_{i=1}^{3}(\bigwedge_{\boldsymbol{q}\in q} e_{i,\boldsymbol{q}}).$$

That is to say, the induced flat connection has trivial holonomy if and only if 3 is even. By the above results, we therefore have that $\Lambda^{3N}(T\mathcal{Q})$ is trivial if and only if 3 is even. □

Lemma 11. *The following relations hold:* $\mathfrak{b}\otimes\mathfrak{b} = \mathfrak{b}$, $\mathfrak{f}\otimes\mathfrak{f} = \mathfrak{b}$, $\mathfrak{f}\otimes\mathfrak{b} = \mathfrak{f}$

Proof. It immediately follows from the fact that the holonomy operators of a tensor product are the tensor products of the original holonomies. In other words, the characters multiply under tensoring of the bundles. □

The following statement and argument is for $3 = 3$.

Lemma 12. *Let F be the Whitney sum (direct sum) of the 2 particle Fermi line with itself. Then $F = \mathfrak{f} \oplus \mathfrak{f} \simeq {}^2\mathbb{R}^3 \times \mathbb{C}^2$.*

Proof. We start by coordinatizing the covering space of $\mathcal{Q} := {}^2\mathbb{R}^3$:

$$\widehat{\mathcal{Q}} := \mathbb{R}^{2,3}_{\neq} = \mathbb{R}^3 \times \mathbb{R}^+ \times S^2.$$

That is to say, we coordinatize the particle system by the center of mass, the distance between the two particles, and the direction from particle 1 to particle 2. To create the base space, we identify antipodal points on S^2 which corresponds to not knowing which particle is particle 1. Our plan is to create two non-zero sections of the trivial bundle $\widehat{\mathcal{Q}} \times \mathbb{C}^2$ that satisfy the alternating periodicity condition. We first do this on the sphere and then extend it to the whole space by constancy in the other coordinates.

Although we shall write it down explicitly, the map from the sphere into sections of $\mathbb{C}^2$ will be the procedure of taking the columns of the unitary matrix which represents, in the spin-$\frac{1}{2}$ representation, a rotation about the line in the sphere, by an angle π. The point is that the representation is a double cover of the rotation group and the two differ by a minus sign. Since rotation about π is the same rotation element for a point on the sphere and its antipode, we expect that this is what we want. Indeed, let the usual representation with the Pauli spin matrices, as generators of the representation, be given. Then a rotation about an angle α about the line specified by the point in the sphere, written as a (x, y, z) vector with the sphere parametrized by the angles $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$, is $\boldsymbol{p} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ and it maps to the unitary matrix, expressed in the basis of σ_z eigenstates,

$$\begin{aligned} e^{i\frac{\alpha}{2}\boldsymbol{\sigma}\cdot\boldsymbol{p}} &= \cos\frac{\alpha}{2} + i\boldsymbol{\sigma}\cdot\boldsymbol{p}\sin\frac{\alpha}{2} \\ &= \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\begin{pmatrix} \cos\phi & \sin\phi\cos\theta - i\sin\phi\sin\theta \\ \sin\phi\cos\theta - i\sin\phi\sin\theta & \cos\phi \end{pmatrix}. \end{aligned}$$

Thus, if we use the angle $\alpha = \pi$ and take the columns as the sections, we have, on $S^2 \times \mathbb{C}^2$,

$$\psi_1(\theta,\phi) = \begin{pmatrix} \cos\phi \\ e^{i\theta}\sin\phi \end{pmatrix} \quad \psi_2(\theta,\phi) = \begin{pmatrix} e^{-i\theta}\sin\phi \\ -\cos\phi \end{pmatrix}$$

Being the columns of a unitary matrix, or by direct computation, the norm of ψ_1 and ψ_2 are both 1, they are orthogonal vectors, they are smooth sections, and they satisfy

$$\psi_i(\theta + \pi, \pi - \phi) = -\psi_i(\theta, \phi).$$

More to the point, if we define a new bundle by $(p, v) \sim (-p, -v)$ where p and $-p$ are antipodal points, then $[\psi_1]$ and $[\psi_2]$ form orthonormal sections of the new bundle.[1]

To go back to the original problem, we define $\Psi_i(x, r, [p]) := [\psi_i]([p])$. These are sections of F. Since they form a basis at every point, F is trivial. □

Lemma 13. *Let C be the Whitney sum of the Fermi line for N particles with the trivial bundle ${}^N\mathbb{R}^3 \times \mathbb{C}^l$. Then C is not a trivial bundle, i.e. $C = \mathfrak{f} \oplus \mathbb{C}^l \ncong {}^N\mathbb{R}^3 \times \mathbb{C}^{l+1}$.*

Proof. This follows from the fact that the determinant line of a tensor product is the tensor product of the determinant lines, i.e $\Lambda^{mn}(E \otimes F) \cong \Lambda^n(E) \otimes \Lambda^m(F)$ where E has dimension n and F has dimension m.[2] Since the determinant line of a trivial bundle is $\mathfrak{b}$, we have that the determinant line of $C \cong \mathfrak{f} \otimes \mathfrak{b}$. That is to say, $\Lambda^{l+1}(C) \cong \mathfrak{f}$. Hence it is nontrivial as the determinant line is nontrivial and any trivial bundle has a trivial determinant line. □

Theorem 13. *For $N = 2$, the triviality of the bundles are as follows:*

	$s = 2l$	$s = 2l + \frac{1}{2}$	$s = 2l + 1$	$s = 2l + \frac{3}{2}$
$\mathcal{B}(2, s)$	*trivial*	*nontrivial*	*nontrivial*	*trivial*
$\mathcal{F}(2, s)$	*nontrivial*	*nontrivial*	*trivial*	*trivial*

for $l \in \mathbb{N}$.

Proof. The idea of this proof is to decompose each bundle into a sum of Bose and Fermi lines. We then pair the Fermi lines to make trivial bundles. We are then either left with a completely trivial bundle or the sum of a trivial bundle with a single Fermi line.

[1] There is another tempting map which is to take the eigenspinors of the spin angular momentum operator in the direction represented by p. This fails since under antipodal exchange, the vectors are exchanged. We wanted to only change the sign.

[2] This follows from the general identity

$$\Lambda^k(E \otimes F) \cong \sum_{i=0}^{k} \Lambda^i(E) \otimes \Lambda^{k-i}(F)$$

where $\Lambda^0(\cdot)$ is defined to be the trivial line bundle.

Let us start with the Bose bundles. Since we have only two particles, the bundle is the sum of a symmetric part and an antisymmetric part. The symmetric subbundle is trivial (a sum of Bose lines), while the antisymmetric part is a sum of Fermi lines. Thus, the Bose bundle is trivial if and only if the number of Fermi lines is even, i.e. if the dimension of the antisymmetric part is even. As for the Fermi bundle, its triviality depends upon the parity of the dimension of the symmetric part of the Bose bundle since upon tensoring with a Fermi line, the Fermi lines become Bose lines and the Bose lines become Fermi lines.

More specifically, the Bose bundle $\mathcal{B}(N,s)$ may be written as, using the basis w^i of $\mathbb{C}^{2s+1}$,

$$\mathcal{B}(N,s) = \bigoplus_{i=1}^{2s+1} sp\{w_q^i \otimes w_{q'}^i\} \oplus \bigoplus_{1\le i<j\le 2s+1} (sp\{w_q^i \otimes w_{q'}^j + w_q^j \otimes w_{q'}^i\} \oplus sp\{w_q^i \otimes w_{q'}^j - w_q^j \otimes w_{q'}^i\})$$

where $sp\{v\}$ means the linear span of the set of vectors; in this case these are all 1-dimensional subspaces. Thus it is the sum of $2s+1+\binom{2s+1}{2}$ Bose lines and $\binom{2s+1}{2}$ Fermi lines. Since $\binom{2s+1}{2}$ is odd exactly when $s = 2l+\frac{1}{2}$ and $s = 2l+1$, we have the Bose bundle results. For the Fermi bundle $\mathcal{F}(N,s)(=\mathfrak{f}\otimes\mathcal{B}(n,s))$ we have that it is a sum of $\binom{2s+1}{2}$ Bose lines and $2s+1+\binom{2s+1}{2}$ Fermi lines as tensoring distributes over sums. Since $2s+1+\binom{2s+1}{2}$ is odd exactly when $s = 2l$ and $s = 2l+\frac{1}{2}$, we have our results. □

Lemma 14. *The determinant line $\Lambda^n(E)$ of a flat bundle E has an induced connection whose holonomy representation is multiplication by the determinant of the corresponding holonomy operator on E.*

Proof. The claim follows immediately from the standard fact that, for any matrix $A : V^n \to V^n$, we have that $\det A$ satisfies

$$\bigwedge_{i=1}^{n} Ae_i = \det A(\bigwedge_{i=1}^{n} e_i)$$

where each $e_i \in V$. Thus, if $\{e_i\}_{i=1}^n$ forms a basis, then $\det A$ is determined by the above equation. Thus, the determinant line's induced holonomy is given by the determinant map of the holonomy representation. □

Theorem 14. *For $N \geq 2$ and $3 \geq 3$, we have that $(\mathbb{C}^{4k+2})^{\otimes \mathcal{Q}}$ and $\mathfrak{f} \otimes (\mathbb{C}^{4k+1})^{\otimes \mathcal{Q}}$ are nontrivial bundles.*

Proof. We will establish that their determinant lines are nontrivial. In fact, we will show that the determinant of the parallel transport operator which exchanges two particles is -1. By the above lemma and the first theorem, that is sufficient to establish our result. Fix the two positions to be exchanged. Let l be the dimension of the factor in the tensor product. Ignoring the Fermi line factor, symmetrize and antisymmetrize on the two factors that are being exchanged. Count how many of each kind there are. The total number of symmetric elements is $(l + \binom{l}{2}) * (l)^{N-2}$ while the total number of antisymmetric elements is $\binom{l}{2} * (l)^{N-2}$. Thus, the bundle without the Fermi line involved is nontrivial if l is odd and $\binom{l}{2}$ is odd. This occurs when $l = 4k+2$; for $N = 2$, we also have $l = 4k + 3$. Tensoring with the Fermi line means the resultant bundle is nontrivial if l is odd and $(l + \binom{l}{2})$ is odd. This occurs when $l = 4k + 2$; for $N = 2$, we also have $l = 4k + 1$. We are done. □

Appendix E

On bijective correspondences between Bohmian velocity fields and wave functions

The PDE for the Bohmian trajectories in Chapter 7 simplified greatly in the situations in which the second derivative of Q vanished. Indeed, we find in that situation that we can have multiple initial wave functions which have the same Bohmian velocity fields. They must have the same phase and their amplitudes must satisfy $\frac{\Delta R_1}{R_1} = \frac{\Delta R_2}{R_2} + C$, where C is a constant. It is possible to have multiple L^2 solutions of this equation. These include all known cases in which the system has the property that the same velocity field corresponds to different initial wave functions.

If the second derivative of Q does not vanish, then the PDE becomes pretty intractable. The right hand side has to be the same in order to produce the same quantum forces. But now we see that the requirement on R is more intensive than just that equation above, which still must be satisfied. We therefore could speculate that it can no longer happen. Although we do not see how to derive it from this PDE, we do know of some other arguments. Two arguments have been done by others; see [49, 7]. We shall give yet another argument. Although lacking in rigor, it may give an appropriate picture as to when and why the claim holds.

First, for two positive functions to satisfy the same stationary state equation, we essentially need one of them to have codimension one with respect to the other. This follows from energy arguments, i.e. the ground state is positive and the only such wave function. If there were two, then they could be combined in such a way as to lower the energy. Actually, one first argues that they must have the same energy; that is to say, there can only be one energy value with positive solutions. This is easy; indeed, the functions must be orthogonal if they are eigenstates for the same energy value.

But two positive functions cannot be orthogonal. That is essentially it. Then, one can demonstrate that at the same eigenvalue, one can create states of lesser energy. Therefore, if the potential, the one defining the stationary state equation, is such that it has a lowest energy state, then we are done.

Second, typically a wave function's nodal set has codimension 2 since it requires the vanishing of both the real and imaginary parts. Therefore, for a typical evolution, one would expect the Schrödinger evolution to instantly take a codimension 1 wave function into a codimension 2 wave function. There can, of course, be exceptions, but these must be explained. We therefore expect that the full-time Bohmian velocity field is generically in bijective correspondence with the initial wave function.

Appendix F

Supplement to Chapter 7

We start with some useful notations. We denote the set of the first n numbers of the natural numbers, i.e. $\{1, \ldots n\}$, by $\mathfrak{n}$. We use $\mathfrak{n}^k_{\neq}$ for the set of k-tuples, each entry being distinct, with the entries being elements of $\mathfrak{n}$ while ${}^k\mathfrak{n}$ is the set of k-element subsets of $\mathfrak{n}$. We shall use the notation (I), where I is a subset of a canonically ordered set (such as the natural numbers), to indicate we wish to form an ordered set using the canonical ordering, e.g. $(\{1, 5, 3\}) := (1, 3, 5)$. Conversely, if we have an ordered set I, then we shall use $\{I\}$ to indicate the set itself. Thus, $\{\}$ is the inverse of $()$. In this section, they will be maps between $\mathfrak{n}^k_{\neq}$ and ${}^k\mathfrak{n}$. All tensor and wedge products are utilizing the idea of the set-based tensor product of appendix A. In particular, the ordering in which things are written down is not relevant; only the labelling is relevant.

We also define an anti-symmetric symbol which is crucial for component formulas. The symbol ϵ^n_I is defined to be the sign of the permutation putting the numbers in the correct order, i.e. taking $((\mathfrak{n}\backslash\{I\}), I) \to (\mathfrak{n})$ or $((\mathfrak{n}\backslash I), (I)) \to (\mathfrak{n})$ depending on whether I is an ordered set or unordered. Just to emphasize, we will be using this symbol both for ordered and unordered sets.

Let $A : V \to W$ be a map between two vector spaces of dimension n. Each vector space is endowed with a volume vector, say ν for V and ω for W. By a volume vector, say for V, we mean a non-zero element of $\bigwedge^n V$, which is the 1-dimensional vector space formed from the totally antisymmetric subspace of the n-fold self-tensor product of V. By $A^{\otimes n}$, we mean the linear operator acting on $V^{\otimes n}$ taking values in $W^{\otimes n}$ defined on product elements by

$$A^{\otimes n}(\underset{i\in\mathfrak{n}}{\otimes} e_i) = \underset{i\in\mathfrak{n}}{\otimes} A(e_i).$$

A key property of $A^{\otimes n}$ is that it preserves spaces invariant under permutations. That

is to say, if P_σ represents the action of a permutation σ on an n-fold self-tensor product induced by the permutation acting on the indices, then $A^{\otimes n}P_\sigma = P_\sigma A^{\otimes n}$. Thus, $A^{\otimes n}(\nu) \in \bigwedge^n V$ since $P_\sigma A^{\otimes n}(\nu) = -1^\sigma A^{\otimes n}(\nu)$ for every permutation σ. This characterizes a volume vector.

We define the determinant of A as the unique number satisfying

$$A^{\otimes n}(\nu) =: (\det A)\omega. \tag{F.1}$$

If we choose bases $\{v_i\}$ and $\{w_i\}$ such that the volume vectors are the wedge product of the bases in the given order, i.e. $\nu = \bigwedge_{i\in\mathfrak{n}} v_i$, then the determinant is the number satisfying

$$\bigwedge_{i\in\mathfrak{n}} A(v_i) =: (\det A) \bigwedge_{i\in\mathfrak{n}} w_i. \tag{F.2}$$

In its turn, we find the usual determinant formula

$$\det A = \sum_{\sigma:\mathfrak{n}\leftrightarrow\mathfrak{n}} -1^\sigma \prod_{i\in n} A_i^{\sigma(i)}.$$

We also note that if $V = W$, then we can choose $\nu = \omega$. This means that the determinant in that case is defined independently of the choice of ν since different choices correspond to different constants and these constants can be cancelled.

The trace of an operator can also be defined using volume vectors and tensors. But its definition requires that A is map from V to itself. Then $A^{(i)}$ is defined to be the operator acting on $V^{\otimes n}$ defined on product elements via

$$A^{(i)}(\underset{j\in\mathfrak{n}}{\otimes} e_j) = \{\underset{j\in\mathfrak{n}\setminus\{i\}}{\otimes} e_j\} \otimes A(e_i).$$

That is to say, it is the identity on each factor except the i^{th} factor for which A acts on that factor. The operator $\sum_{i\in\mathfrak{n}} A^{(i)}$ clearly commutes with permutations. Thus, one can define the trace of the operator A as the unique number $\mathrm{Tr}A$ satisfying

$$\sum_{i\in\mathfrak{n}} A^{(i)}(\nu) =: (\mathrm{Tr}A)\nu.$$

In coordinates, we see that

$$\sum_{i\in\mathfrak{n}} A^{(i)}(\nu) = \sum_{i\in\mathfrak{n}} \{\bigwedge_{j\in j\in\mathfrak{n}\setminus\{i\}} v_j\} \wedge A(v_i).$$

Since $A(v_i) = \sum_k A_i^k v_k$, we notice that in the i^{th} term of the sum, only $A_i^i v_i$ gives a non-zero contribution to the wedge product. Thus, we have that the trace is given, in components, by

$$\mathrm{Tr}A = \sum_i A_i^i$$

which is just the usual coordinate formula. Also notice that this definition is independent of the volume form chosen and hence any wedge product of a basis will yield the same result. In other words, the trace is canonically defined.[1]

We needed to choose a volume form on each space in order to define the determinant. One way to do this is to specify a metric. Then, up to a sign, we have a canonical volume vector. This is formed by taking an orthonormal basis and wedging all of the elements in the basis together. The sign ambiguity comes in by not knowing how to choose which order to wedge product the vectors together. But, if we want a positive number, then there is no ambiguity. In fact, given A, we can redefine one of the volume vectors so that the determinant is positive. Thus, $|\det A|$ or just $|A|$ for short, is well-defined. Note that if we had a map between V and itself, then given any volume vector on V, we would use the same volume vector on both sides. This gives an unambiguous determinant which may be negative.

Extending this to manifolds should now be clear. Our setup is that we have a mapping $f : M^k \to N^k$ between two Riemannian manifolds of the same dimension. Since $D_p f : T_pM \to T_{f(p)}N$, we can ask what is the determinant of Df as a mapping of the tangent spaces. The volume vectors that we choose are the Riemannian volume vectors. For M, let ν be the volume form and ω be the volume form for N. Then $|\det(D_p f)| =: |D_p f|$ satisfies

$$|D_p f|\omega_{f(p)} = (D_p f)^{\otimes k}(\nu_p)$$

[1]The reason that we need A to map V to itself, rather than some other vector space, is that the identity is crucial in this argument. On the other hand, if $A : V \to W$, then if we choose the volume vectors ν and ω as well as another map $B : V \to W$, then we can define

$$\mathrm{Tr}_B A(\omega) = (\sum_{i \in \mathbf{n}} \{ \bigotimes_{j \in \{\mathbf{n} \setminus i\}} B\} \otimes (A)_i)\nu$$

Taking $A = B$, we find that the above formula gives us n times the determinant of A. One can also envision a further generalization involving multiple operators appearing a specified number of times. This begins to resemble something remarkably similar to minors.

up to a sign. That completely determines the information. In any coordinate system, the volume form is $\sqrt{\det G^{ij}} \bigwedge_{i \in \mathfrak{n}} v_i$. Thus, the determinant in coordinates is

$$|D_p f| = |\frac{\sqrt{\det G_M^{ij}}}{\sqrt{\det G_N^{ij}}} \sum_{\sigma : \mathfrak{n} \leftrightarrow \mathfrak{n}} -1^{\sigma} \prod_{j \in \mathfrak{n}} \frac{\partial f^{\sigma j}}{\partial x^j}|$$

We do not have to worry about orientability as we are concerned only with local properties of objects which should be non-zero.[2]

F.1 Derivatives of the determinant

We need to compute derivatives of the volume vector. Our setup is that we have a map Q with differential DQ acting on the tangent space; we shall suppress t in the following discussion. The only important fact is that the Riemannian volume vector is parallel. The reason is that we can choose an orthonormal basis at a point $q \in \mathcal{Q}$ and then parallel transport the basis along geodesics. This gives us a basis of orthonormal vector fields at every point as the inner product is preserved by parallel transport. We have that the first covariant derivative of these vector fields is zero at q. Thus, at every point in $\mathcal{Q}$, the Riemannian volume form's first covariant derivative vanishes. This means that it is parallel and that the higher derivatives also vanish since these will be derivatives of a zero tensor.

With that trivial fact established, we can now differentiate the determinant. Let $A := DQ$. We start with

$$|A|\omega = A^{\otimes n} \circ \nu.$$

Recall that composition essentially means that we tensor these objects together and then contract the indices. Since covariant derivatives commute with contractions, we have that

$$D[|A|]\omega + |A|D[\omega] = D[A^{\otimes n}] \circ \nu + A^{\otimes n} \circ D[\nu].$$

[2]If we were concerned about the Maslov index and the determinant vanishing, we would choose an initial volume vector in a neighborhood about the initial point and parallel transport the volume vector along the curve. As we pass through a zero of the determinant, we would pickup a phase and appropriately order the volume vector to have a positive determinant.

Since the volume forms are parallel, this reduces to

$$D[|A|]\omega = D[A^{\otimes n}] \circ \nu.$$

Continuing in this fashion, we have

$$D^m[|A|]\omega = D^m[A^{\otimes n}] \circ \nu.$$

One can then use the product rule to express $D^m[A^{\otimes n}]$ in terms of the factors. If one does this and then expresses it in coordinates, we find

$$(D^{|i|}[\det(A)])_{k_1 \cdots k_i} = \sum_{PR \in \mathcal{PR}(i,n)} \sum_{\sigma : \mathfrak{n} \leftrightarrow \mathfrak{n}} -1^{\sigma} \prod_{j \in \mathfrak{n}} A^{\sigma_j}_{j k_{r_j(1)} \cdots k_{r_j(|r_j|)}}$$

where $\mathcal{PR}(i,n)$ is the set of product rule diagrams of length i with height n, R_{PR} is the set of nonempty rows, and r_j is the j^{th} row. The extra subscripts indicate covariant differentiation slots. This is the same formula as in the Euclidean case; it can also be written in terms of minors, although we shall not do that here.

F.2 The continuity equation

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

where $\rho : \mathcal{Q} \to \mathbb{R}$ and $\boldsymbol{v}$ is a time-dependent velocity field on $\mathcal{Q}$. We assume that $Q(x_0, t)$ describes the family of integral curves of the velocity field. Let ρ_0 be given. Then we claim that the solution to the continuity equation, as said in Chapter 7, is

$$\rho_t(x) = \rho_0(Q^{-1}(x,t))|DQ^{-1}(x,t)|.$$

To understand this, we note that under a mapping f, a density μ on the image is mapped to a density on the domain via $\tilde{\mu}(x) := \mu(f(x))|Df|(x)$. This is the definition which gives $\int_A \tilde{\mu} = \int_{f(A)} \mu$. The Jacobian factor is what is needed for the change of variables. This is the picture of why it is true.

Direct verification is as follows. Define $h(x,t) := Q^{-1}(x,t)$. Then $\rho(x,t) := \rho_0(h(x,t))|Dh|(x,t)$. Fix (x,t) and let ω be the Riemannian volume vector at $h(x,t)$ while ν is the volume vector at x. The full derivative of h is $D[h] : T_x\mathcal{Q} \oplus \mathbb{R} \to T_{h(x,t)}\mathcal{Q}$.

As this is a direct sum of two bundles, we define $D_x[h] : T_x\mathcal{Q} \to T_{h(x,t)}$ by restricting $D[h]$ to the subspace T_x. We define $\dot{h} = \frac{\partial h}{\partial t}$ in a similar fashion except that, by evaluating at 1, we can say that $\dot{h} \in T_{(h(x,t)}$. Additionally, please note that $\dot{Q}(h(x,t)) \in T_x\mathcal{Q}$. We shall also take the liberty of using various equivalences such as $\nabla f \cdot \boldsymbol{v} = D[f](\boldsymbol{v})$ without necessarily mentioning them.

By using the chain rule, we have

$$\frac{\partial \rho}{\partial t} = |Dh| D_h[\rho_0](\dot{h}) + \rho_0(h) \frac{\partial [|Dh|]}{\partial t}$$

and

$$\nabla \cdot [\rho \nabla [S]]) = |Dh| D_h[\rho_0] \circ D_x[h] \circ \boldsymbol{v} + \rho_0(h) D_x[|Dh|] \circ \boldsymbol{v} + \rho \nabla \cdot \boldsymbol{v}.$$

Since $h(Q(x_0,t),t) = x_0$ and $\dot{Q}(x_0,t) = \nabla S(Q)$ by definition , we can eliminate two terms by noting

$$\begin{aligned} \dot{h}(Q,t) + D_Q[h] \circ \dot{Q} &= 0 \\ \dot{h}(Q,t) &= -D_Q[h] \circ \boldsymbol{v}(Q) \\ |Dh| D_h[\rho_0] \circ \dot{h} &= -|Dh| D_h[\rho_0] \circ (D_x[h] \circ \boldsymbol{v}) \end{aligned}$$

Since contractions commute, we have the cancellation of the first terms.

Replacing Q with x and differentiating, we have that

$$D_x[\dot{h}] = -D_x^2[h](\boldsymbol{v}) - D_x[h](D_x \boldsymbol{v}).$$

Using the definition of determinant and following the discussion of the derivative of the determinant, we find that

$$\frac{\partial [|D_x[h]|]}{\partial t} \omega = \sum_i \{D_x[h]^{\otimes \mathfrak{n} \backslash i} \otimes D_x[\dot{h}]\} \nu.$$

Combining these two equations, we find,

$$\frac{\partial [|D_x[h]|]}{\partial t} \omega = \sum_i \{D_x[h]^{\otimes \mathfrak{n} \backslash i} \otimes \{-D_x^2[h](\boldsymbol{v}) - D_x[h](D_x \boldsymbol{v})\}\} \nu.$$

Again by a simple computation, we also have,

$$D_x[|D[h]|](\boldsymbol{v}) \omega = \sum_i \{D_x[h]^{\otimes \mathfrak{n} \backslash i} \otimes D_x^2[h](\boldsymbol{v})\} \nu$$

which replaces part of the above equation.

To eliminate the term with the derivative of $\boldsymbol{v}$, one should recall the definition of the trace of a matrix. Additionally, we shall view $D\boldsymbol{v}$ as a matrix as it takes in a tangent vector and produces a tangent vector. We therefore have

$$\begin{aligned}\sum_i \{D_x[h]^{\otimes \mathfrak{n}\setminus i} \otimes \{D_x[h](D_x\boldsymbol{v})\}\}\nu &= D_x[h]^{\otimes n}\sum_i (D\boldsymbol{v})^{(i)}(\nu)\\ &= D_x[h]^{\otimes n}(\mathrm{Tr}(D\boldsymbol{v})\nu)\\ &= \nabla\cdot\boldsymbol{v}\{D_x[h]^{\otimes n}\nu\}\\ &= \nabla\cdot\boldsymbol{v}\{|D[h]|\}\omega\}.\end{aligned}$$

We therefore conclude that

$$\frac{\partial[|D_x[h]|]}{\partial t} + D_x[|D[h]|](\boldsymbol{v}) + \nabla\cdot\boldsymbol{v}\{|D[h]|\} = 0$$

Thus, all of the terms in the continuity equation do vanish.

F.3 The Hamilton-Jacobi equation

We claim that the Hamilton-Jacobi equation's solution is the action integral integrated over the trajectories. More precisely, let V be the potential, Q the set of trajectories, and S_0 the initial action function. We are assuming that Q is invertible and that $\dot{Q}(x_0,0) = \nabla S_0(x_0)$, and $\ddot{Q}(x_0,t) = -\nabla V(Q(x_0,t))$.

Then we claim that

$$S(x,t) = S_0(Q^{-1}(x,t)) + \int_0^t [\frac{1}{2}g(\dot{Q},\dot{Q}) - V(Q)](Q^{-1}(x,t),s)ds \tag{F.3}$$

is the solution to the H-J equation. To establish this, first note that

$$S(Q(x_0,t),t) =: \tilde{S}(x_0,t) = S_0(x_0) + \int_0^t [\frac{1}{2}g(\dot{Q},\dot{Q}) - V(Q)](x_0,s)ds$$

satisfies, for fixed x_0,

$$\frac{d\tilde{S}}{dt}(x_0,t) = \mathcal{L}(x_0,t) := \{\frac{1}{2}g(\dot{Q},\dot{Q}) - V(Q)\}(x_0,t) \tag{F.4}$$

as a trivial application of the fundamental theorem of calculus establishes. On the other hand,

$$\frac{dS(Q(x_0,t),t)}{dt} = \frac{\partial S}{\partial t}(Q,t) + D_Q[S]\circ\dot{Q}(x_0,t). \tag{F.5}$$

The main claim, which requires effort to prove, is that

$$\nabla S(Q(x_0,t)) = \dot{Q}(x_0,t) \tag{F.6}$$

implying

$$\nabla S \cdot \nabla S = D_Q[S] \circ \dot{Q} d = g(\dot{Q},\dot{Q}).l \tag{F.7}$$

Combining (F.7), (F.4), and (F.5), we find

$$\{\frac{\partial S}{\partial t} + \nabla S \cdot \nabla S - (\frac{1}{2}\nabla S \cdot \nabla S) - V)\}(Q(x_0,t),t) = 0$$

which, after replacing Q with x, is the Hamilton-Jacobi equation demonstrating that our claimed solution is a local solution to the Hamilton-Jacobi equation whenever Q is locally invertible.[3]

All that remains to show is that if S is defined by (F.3) and that Q is the family of classical trajectories under the potential V whose initial velocity field is given ∇S, then (F.6) holds. That is to say, the constraint is preserved. To prove this, we use the chain rule. We start with noting that

$$D_{x_0}[\tilde{S}](x_0,t) = D_{x_0}S_0(x_0) + \int_0^t \{g(D_{x_0}[\dot{Q}],\dot{Q}) - D_Q[V] \circ D_{x_0}[Q]\}(x_0,s)ds$$

and applying the product rule

$$\frac{dg(\dot{Q}, D_{x_0}[Q])}{dt} = g(\ddot{Q}, D_{x_0}[Q]) + g(\dot{Q}, D_{x_0}[\dot{Q}])$$

setting us up for an integration by parts leading to

$$\begin{aligned} D_{x_0}[\tilde{S}](x_0,t) =& D_{x_0}S_0(x_0) + g(\dot{Q}, D_{x_0}[Q])|_0^t \\ &- \int_0^t g(\ddot{Q}, D_{x_0}[Q]) - D_Q[V] \circ D_{x_0}[Q](x_0,s)ds \\ &= (D_{x_0}S_0(x_0) - g(\dot{Q}(x_0,0), D_{x_0}[Q](x_0,0)) + g(\dot{Q}, D_{x_0}[Q])(x_0,t) \\ &- \int_0^t g(\ddot{Q}, D_{x_0}[Q]) - DV \circ D_{x_0}[Q](x_0,s)ds. \end{aligned}$$

[3] In Chapter 7, algorithm A1 requires the solution to an equation of the form $\frac{\partial S}{\partial t} + \nabla S \cdot \boldsymbol{v} = \frac{1}{2}g(\boldsymbol{v},\boldsymbol{v}) - V$. From the above discussion, one can immediately see that integrating the right hand side over the integral curves of $\boldsymbol{v}$ will yield the solution.

As $D_{x_0}[Q](x_0, 0) = \mathrm{Id}$, the initial conditions lead to the cancellation of the first two terms in the above equation. The differential of Q can be pulled out to the right in the integrand leading to

$$(g(\ddot{Q},) - DV) \circ D_{x_0}[Q](x_0, s) = 0 \circ D_{x_0}[Q](x_0, s)$$

since $Q_{x_0}(t)$ is a classical trajectory with potential V. The integral's vanishing leaves us with one term. To finish the claim, note that, evaluating at (x, t),

$$D_x[S] = D_{Q^{-1}}[\tilde{S}] \circ D_x[Q^{-1}] = g(\dot{Q}, D_{Q^{-1}}[Q] \circ D_x[Q^{-1}]) = g(\dot{Q}, \mathrm{Id}).$$

Taking $x = Q$, this establishes (F.6). Thus, S is a solution to the Hamilton-Jacobi equation under the invertibility assumption of Q.

F.4 Semi-classical asymptotics and the Maslov index

The idea of semi-classical asymptotics in quantum mechanics is quite well known and we shall be brief. A few references are [6, 41, 19] The basic idea is to take $\hbar \to 0$. Setting it equal to 0 leads to the classical Hamilton-Jacobi equation. The continuity equation can then be solved using the classical trajectories. When $\hbar$ is considered a small parameter in the problem, this is expected to be a good approximation to the solution. An immediate problem of this method is that the classical trajectories are not generically invertible. Thus, the Hamilton-Jacobi equation does not have a global solution. Undeterred by this minor inconvenience, one can do some Fourier analysis and, by applying the method of stationary phase, we arrive at

$$\psi_n(x, t) := \sum_{\{y_j | Q_n(y_j, t) = x\}} \mathcal{R}[Q_n, R(\cdot, 0)](y_j, t) e^{i(\mathcal{S}[Q_n, U_{n-1}, S(\cdot,0)](y_j,t) - \frac{\pi}{2}\mu_j)}.$$

In words, we are assuming that in a neighborhood of each point, the function Q is locally invertible. We look at the inverse image and generate the S along the trajectories using the action formula. We append the Jacobian correction factor and do a sum over the terms. It is important that the terms have the right phase which is accounted for by the Maslov index μ over the trajectory. Although it is a very important symplectic geometric object associated with Lagrangian submanifolds, we shall view it in a very

simple-minded way. The factor is just the number of zeroes of the determinant of Q that the trajectory crosses. The zeroes correspond to caustics and these are the places where invertibility of Q begins to fail. The reason this factor arises from our perspective is clear. Remove the absolute value signs around the determinant and the square root will then pick up factors of i (or $-i$) as a trajectory goes through a zero. Noticing the multiplicity arises by considering a slight perturbation of the surrounding flow and showing that the overall phase factor is unaffected by the perturbation.

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