Electrodynamic Absorber Theory
A Mathematical Study

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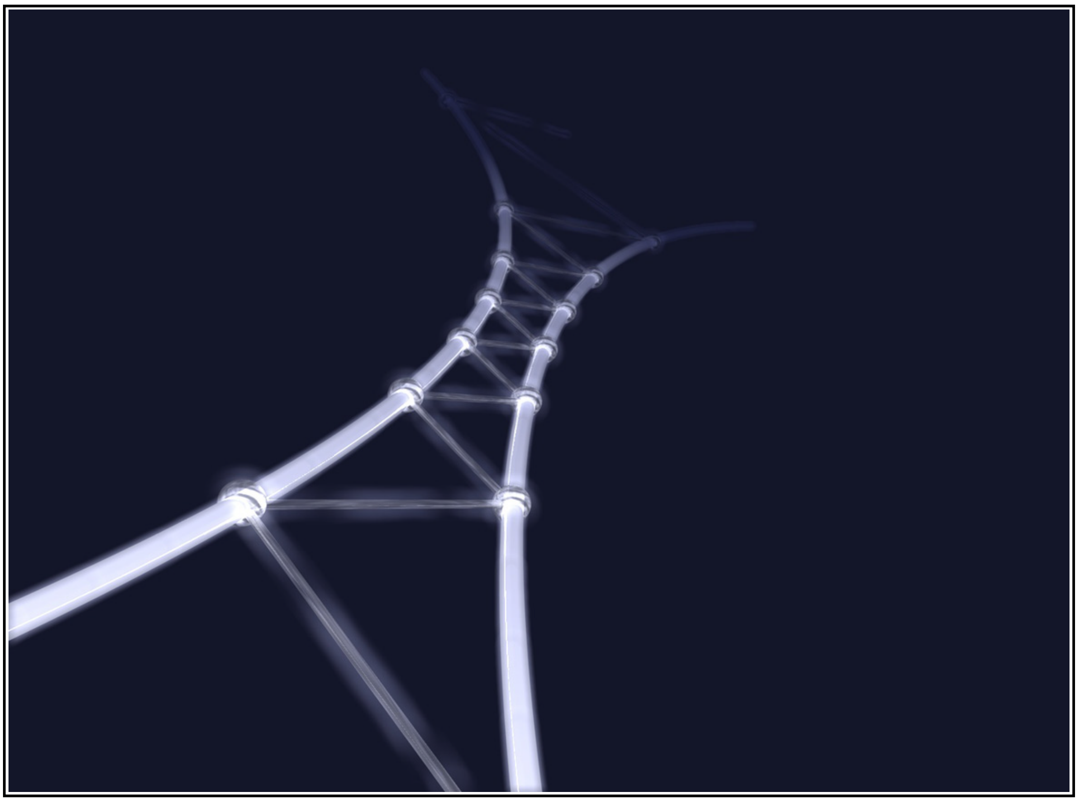

Doctoral Thesis

Electrodynamic Absorber Theory
A Mathematical Study

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Abstract

This work treats mathematical questions which arise in classical and quantum electrodynamics when describing the phenomena of *radiation reaction* and *pair creation*. It consists of two major parts:

In the first part classical dynamics are studied which allow for radiation reaction: For classical, non-rotating, rigid charges we give explicit representation formulas of strong solutions to the Maxwell equations, prove global existence and uniqueness of strong solutions to the Maxwell-Lorentz equations in a general setting in which we also allow for a class of infinite energy solutions, negative masses of the charges, and for both cases, Maxwell-Lorentz equations of motion with self-interaction (ML+SI) and without (ML-SI). We prove that possible solutions to the Wheeler-Feynman equations with bounded accelerations and momenta are also solutions to the ML-SI equations. This gives rise to a unique characterization of Wheeler-Feynman solutions by position, momenta and their Wheeler-Feynman fields at one instant of time. Moreover, we give a reformulation of Wheeler-Feynman electrodynamics for rigid and non-rotating charges in terms of an initial value problem for Newtonian Cauchy data. With it we prove existence of Wheeler-Feynman solutions on finite time intervals corresponding to the Newtonian Cauchy data. We discuss how this method could yield global existence of solution to the Wheeler-Feynman equations.

In the second part quantum dynamics are studied for systems of an infinite number of Dirac electrons which interact only with a prescribed external field that allows for pair creation: We construct the time-evolution for the second quantized Dirac equation subject to a smooth and compactly supported, time-dependent electrodynamic four-vector field. Earlier works on this (Ruijsenaars) observed the Shale-Stinespring condition and showed that the one-particle time-evolution can be lifted to Fock space if and only if the external field has zero magnetic components. The basic obstacle in the construction is that there is neither a distinguished Dirac sea, i.e. Fock space vacuum, nor a distinguished polarization. Therefore, the key idea (suggested already by Fierz and Scharf) is to implement this time-evolution between time-varying Fock spaces. We show that this implementation is unique up to a phase. All induced transition amplitudes are unique and finite.

In a last part we give a brief outlook on our perspective of a divergence free, electrodynamical theory for point-like charges which accounts for both phenomena, radiation reaction as well as pair creation. It is based on the idea that the Dirac sea represents the absorber medium proposed by Wheeler and Feynman. The presented mathematical results can be considered as first steps towards it.

Keywords: Absorber Electrodynamics, Radiation Reaction, Pair Creation, Wheeler-Feynman Solutions, Maxwell Solutions, Liénard-Wiechert Fields, Maxwell-Lorentz Solutions, Functional Differential Equations, Infinite Wedge Spaces, Second-Quantized Dirac Time-Evolution, Quantum Electrodynamics, Quantum Wheeler-Feynman Interaction

Zusammenfassung (Translation of the Abstract)

Diese Arbeit behandelt mathematische Fragen, die im Zusammenhang mit der Strahlungsrückwirkung und der Paarerzeugung in der klassischen Feldtheorie sowie in der Quantenfeldtheorie stehen. Sie besteht aus zwei Hauptteilen:

Im ersten Teil werden klassische Dynamiken studiert, die es ermöglichen den Effekt der Strahlungsrückwirkung zu beschreiben: Für klassische, nicht-rotierende, starre Körper geben wir explizite Darstellungsformeln für starke Lösungen der Maxwell Gleichungen an und beweisen die globale Existenz und Eindeutigkeit von starken Lösungen der Maxwell-Lorentz Gleichungen in einem allgemeinen Rahmen. Dieser erlaubt es eine Klasse von Lösungen unendlicher Energien, negativen Massen der Ladungen, sowie beide Fälle, Maxwell-Lorentz Gleichungen mit Selbstwechselwirkung (ML+SI) und ohne (ML-SI) zu behandeln. Wir beweisen weiter, dass mögliche Wheeler-Feynman Lösungen mit beschränkten Beschleunigungen und Impulsen auch Lösungen der ML-SI Gleichungen darstellen. Dies ermöglicht eine eindeutige Charakterisierung der Wheeler-Feynman Lösungen anhand von Ort, Impuls, und ihrer Wheeler-Feynman Felder zu einem bestimmten Zeitpunkt. Zudem geben wir eine Umformulierung der Wheeler-Feynman Elektrodynamik für starre und nicht-rotierende Ladungen in ein Anfangswertproblem für Newtonsche Cauchy Daten an und beweisen damit die Existenz von Wheeler-Feynman Lösungen auf endlichen Zeitintervallen entsprechend Newtonscher Cauchy Daten. Wir diskutieren wie mit Hilfe dieser Vorgehensweise die globale Existenz von Lösungen zu den Wheeler-Feynman Gleichungen gezeigt werden könnte.

Im zweiten Teil studieren wir Quantendynamiken für Systeme mit unendlich vielen Dirac Elektronen, die nur mit einem vorgeschriebenen äußeren Feld wechselwirken, welches Paarerzeugung ermöglicht: Wir konstruieren die Zeitentwicklung für die zweitquantisierte Dirac Gleichung in Abhängigkeit von einem glatten und kompakten, zeitabhängigen, elektrodynamischen Viervektorfeld. In früheren Arbeiten auf diesem Gebiet (Ruijsenaars) wurde die Shale-Stinespring Bedingung berücksichtigt, und es wurde gezeigt, dass die Einteilchenzeitentwicklung genau dann auf den Fock Raum gehoben werden kann, wenn die magnetischen Komponenten des äußeren Feldes null sind. Das wesentliche Hindernis bei dieser Konstruktion ist, dass es weder einen ausgezeichneten Dirac See, d.h. Fock Raum Vakuum, noch eine ausgezeichnete Polarisation gibt. Aus diesem Grund ist die Schlüsselidee (wie bereits von Fierz und Scharf vorgeschlagen), diese Zeitentwicklung zwischen zeitlich variierenden Fock Räumen umzusetzen. Wir zeigen, dass diese Umsetzung bis auf eine Phase eindeutig ist. Alle sich dadurch ergebenden Übergangsraten sind eindeutig und endlich.

In einem letzten Teil geben wir aus unserem Blickwinkel einen kurzen Ausblick auf eine divergenzfreie, elektrodynamische Theorie über Punktladungen, welche beide Phänomene, sowohl Strahlungsrückwirkung als auch Paarerzeugung beschreibt. Sie basiert auf der Idee, dass der Dirac See das von Wheeler und Feynman eingeführte Absorbermedium darstellt. Die präsentierten mathematischen Resultate könnten als erste Schritte in diese Richtung angesehen werden.

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The mathematical results of Part II of this work emerged from a collaboration with Prof. Dr. Detlef Dürr, Prof. Dr. Franz Merkl and Prof. Dr. Martin Schottenloher. In this regard I especially need to thank Prof. Dr. Franz Merkl for our regular computation sessions on Wednesday afternoons. Thanks also to Sören Petrat and Torben Krüger for discussions and for proofreading.

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Style of Writing

Although this doctoral thesis is written by only one author, the chosen form of writing employs the use of first person plural throughout the work for two reasons: First, research is never done by a single person alone. In this sense phrases like “we conclude” are used to recall all people who contributed to a “conclusion” in one way or another. Second, for an interested reader phrases like “we prove” are also meant in the sense that the author and the reader go through a “proof” together to check if it is correct.

Contents

Chapter 1

Preface

1.1 Interaction between Light and Matter

Although the interaction between *light* and *matter* determines most of our daily experience (the other bigger part being due to gravitation), a detailed mathematical study of this natural phenomena had to wait until James C. Maxwell published his *Maxwell equations* in the Royal Society in 1864 and Hendrik A. Lorentz formulated his *Lorentz force law* 28 years later. In their equations the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are used to represent the physical entity, which we call light, while the part of matter that may interact with light is modeled by the charge source density ρ and the charge current density $\boldsymbol{j}$. Maxwell's equations are supposed to describe the static as well as the time-dependent interaction of the electric and magnetic fields with prescribed charge sources and charge currents while, in turn, the Lorentz force law rules the classical motion of charges which are subject to prescribed electric and magnetic fields. One of the great innovations of Maxwell's equations was the description of *radiation* which was later verified by Heinrich R. Hertz in 1888. With its help Nikola Tesla invented a radio device capable of wireless communication in 1894, and two years later Alexander S. Popov made the use of radiation part of everyday life in form of the radio receiver at the All-Russia exhibition. The interplay of mathematics and physics along the works of Lorentz, Hermann Minkowski, Albert Einstein and many others in the late 19th and early 20th century led to the unification of the electric and magnetic forces into one, the electromagnetic force. This development reduced the initially more than 20 equations to two and gave a further insight into the symmetries and the structure of space-time. The electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ were no longer regarded as separate entities but as one, namely the electromagnetic field F. Also in the early 20th century, based on the work of Paul A. M. Dirac, Vladimir A. Fock, Werner K. Heisenberg and Wolfgang E. Pauli, an embedding into quantum mechanics was accomplished which initiated the field of quantum electrodynamics. Quantum electrodynamics was the prototype of modern field theory that gave rise to the *Standard Model* which is supposed to describe the elementary properties of matter and their interaction mediated by its gauge fields among which light, the photon field, is one of them. Abdus Salam, Sheldon L. Glashow and Steven Weinberg succeeded in the further unification of the *weak* and the electromagnetic interaction into the *electroweak* interaction. This again initiated the dream of a grand unification of all elementary interactions (the electroweak, the strong and the gravitational) into one theory.

Despite the undoubtedly enormous successes of classical and quantum field theory in the description of the interaction between light and matter, it turned out that there were two phenomena whose mathematical description approved to be problematic and does not yet stand on solid mathematical grounds: *radiation reaction*, the deceleration of radiating charges, and *pair creation*, the process of creation of a particle-antiparticle pair induced by an electromagnetic field:

Radiation Reaction. Early on Max Abraham and Lorentz himself tried to include the effect of radiation reaction into an *electrodynamic* theory by coupling the Maxwell equations to the Lorentz force law. This way the Maxwell equations govern the time-evolution and the emission and absorption of the electromagnetic field F by the charge source and current density $(\rho,\boldsymbol{j})$ generated by charged matter, while simultaneously, the Lorentz force law rules the motion of the charged matter under the influence of the same field F. In these models whenever charged matter is accelerated, an electromagnetic field (e.g. like the electrons in a radio antenna) will be emitted. However, these just emitted field reacts back on its source to decelerate it such that the emitting charge has to pay for the radiated energy by loss of its own kinetic energy; hence the name radiation reaction or *radiation damping*. Unfortunately, these models work only for extended charges and fail for point-like charges. At first sight the reason for this is only mathematical. The field strength at position $\boldsymbol{x}$ of the electromagnetic field induced by a point charge behaves roughly like one over the square of the distance between the position of the charge and the point $\boldsymbol{x}$, giving infinity (i.e. undefined) if $\boldsymbol{x}$ equals the position of the point charge. But right there the Lorentz force needs to evaluate the field strength of F to compute the back reaction on the charge. So for point charges the coupled Maxwell and Lorentz equations, however physically reasonable, make mathematically no sense. Nevertheless, from a physical point of view, as for example discussed in [Fre25] and [Dir38], it would be very desirable to have an electrodynamic theory for point charges for the following reasons: First, the electrons which should be described by such an electrodynamic theory are considered to be elementary particles having no further structure. Even if in the future we might learn that electrons too have an inner structure, at least on a classical scale, any electrodynamic theory should be robust enough to survive the limit from extended charges to point charges (the classical electron radius lives on a length scale of $10^{-15}m$!). Second, rigid extended charges, even if correctly Lorentz-boosted, are incompatible with special relativity [Nod64]. And third, the shape of the charge distribution introduces an unwanted arbitrariness into the theory of electrodynamics. Therefore, numerous attempts have been made to reformulate the electrodynamic theory for point charges in order to obtain a mathematically well-defined theory. Examples are Dirac's mass renormalization program [Dir38], the Born-Infeld theory [BI34] and Wheeler-Feynman electrodynamics [WF45, WF49]. Dirac suggested to keep the Maxwell equations but to replace the Lorentz force law by the so-called Lorentz-Dirac equation which is motivated by a mass renormalization recipe. This approach is plagued by the fact that the Lorentz-Dirac equation allows for unphysical (dynamically unstable, so-called run-away) solutions which arise as an artefact from Dirac's renormalization program [Dir38, Roh94, Spo04] in which the masses of the charges acquire a negative sign [BD01]. Concerning the second approach, Born and Infeld proposed to keep the Lorentz force law but to replace the Maxwell equations by a nonlinear variant. The role of the nonlinearity is to smoothen out the aforementioned singularities of the electromagnetic fields on the trajectories of the point charges. Though the regularity of the electrodynamic fields suffices to yield finite self-energies and even to formulate a Hamilton-Jacobi theory, it is insufficient to render the Lorentz force law well-defined [Kie04]. In contrast to these two attempts, which in some sense present ad hoc cures of the ill-defined system of Maxwell-Lorentz equations and which are plagued by new difficulties, Wheeler-Feynman electrodynamics (WFED) presents a conceptually new formulation of electrodynamics. It is based on an old and well-known idea of the action-at-a-distance (free of electromagnetic fields) which is by definition free of divergences and fully compatible with special relativity. The action-at-a-distance idea actually goes back to Carl F. Gauß [Gau45], Karl Schwarzschild [Sch03], Hugo M. Tetrode [Tet22] and Adriaan D. Fokker [Fok29]. In the case of prescribed charge trajectories the equations of motion of another charge in WFED turns into the Lorentz force law. Furthermore, one can show for large many particle systems that their trajectories are also solutions to the aforementioned Lorentz-Dirac equation if they fulfill a certain thermodynamical condition, the so-called *absorber assumption*. This is important as, according to the present state of knowledge, some so-called physical solutions to the Lorentz-

Dirac equation (the ones which do not exponentially fast approach speed of light) are believed to correctly account for classical radiation reaction – although somehow surprisingly it has not been thoroughly tested in experiments yet [Spo04]. It must be stressed that as in WFED the Lorentz-Dirac equation is only fulfilled effectively whenever the absorber assumption holds, the charge trajectories always obey the Wheeler-Feynman equations. This way unphysical solutions are unexpected in WFED as no renormalization procedure is involved and in particular as the masses of the charges stay untouched. The price for these advantages is a mathematically more difficult equation of motion which includes state-dependent advanced and delayed arguments and lacks a physically satisfactory existence and uniqueness theory.

Pair Creation. In the early 20th century it was soon realized that the Dirac equation which governs the quantum mechanical motion of electrons gives rise to a major problem. It allows for electrons with positive and negative kinetic energy, and, even worse, transitions from the positive to the negative part of the energy spectrum and vice versa. This aggravates the interpretation of its solutions as wave functions describing electrons problematic. So either the Dirac equation was not completely correct or one had to find an explanation why we do not see electrons with negative kinetic energy. Dirac himself proposed a solution in 1930 [Dir30] which was later republished in [Dir34]. He assumed that all negative energy states for which the Dirac equation allows are occupied by electrons which make up the *Dirac sea*. Then the Pauli exclusion principle would prevent a positive energy state to make a transition to a negative one since they are all filled. The Dirac sea, however, cannot be observed due to its homogeneity. Only a departure from this homogeneity, for example, in form of a transition of a Dirac sea electron to the positive kinetic energy under the influence of a nearby nuclei should be observable. Each of such transitions would leave a hole in the Dirac sea, which then effectively behaves like an electron but with opposite charge. The electron with positive kinetic energy together with the hole are then called an electron-positron pair. The reunion of the hole and the electron is referred to as pair annihilation. Three years later Carl D. Anderson verified Dirac's predictions experimentally [And33]. However, though powerful, the description of pair creation with the help of the Dirac sea comes at the cost of having to deal with an infinitely many particle system in which quantities such as the charge current diverge and, apart from the scattering theory, the time-evolution of the Dirac sea on a fixed Fock space was only constructed for external fields with zero magnetic components [SS65, Rui77].

As it stands, though there exist physical recipes (renormalization programs of all kinds) that yield predictions which are in great agreement with the according experiments, a mathematically sound and physically satisfactory description of radiation reaction and pair creation has not been established yet.

1.2 Scope of this Work

This brings us to the mathematical contributions of this work which are guided by the following two objectives:

1. Existence and Uniqueness results for classical dynamics which are capable of describing radiation reaction.
2. Existence and Uniqueness results for quantum dynamics which are capable of describing pair creation.

The reason why we study the first objective classically is the following: In simple toy models of quantum electrodynamics like the Nelson model it is not difficult to see that the ultraviolet divergences (the mentioned singularities of the electrodynamic field) are inherited from classical field theory [Dec04]. Therefore, we believe that it is necessary that one first understands radiation reaction classically before considering its quantum analogue – especially as quantum theory did not provide any mechanism that could possibly cure the problem, which has been wishful thinking for a long time (in fact, quantum theory brought along an additional problem, the so-called infrared divergence).

The first objective is content of Part I$_{\text{p.7}}$. After an introduction into the classical description of radiation reaction in Chapter 2$_{\text{p.7}}$, the following two chapters give an existence and uniqueness result for global solutions to the Maxwell-Lorentz equations with and without self-interaction, cf. Chapter 3$_{\text{p.15}}$, means to speak about Wheeler-Feynman solutions in terms of initial values of the Maxwell-Lorentz equations without self-interaction and an existence result for solutions to the Wheeler-Feynman equations on finite time intervals, cf. Chapter 4$_{\text{p.43}}$.

The second objective is treated in Part II$_{\text{p.105}}$ where we construct a time-evolution for systems with an infinite number of Dirac electrons subject to a classical external field and identify the degree of freedom in the construction.

Besides the relevance of the presented mathematical results in contemporary classical and quantum field theory we regard them also as first steps towards a different electrodynamic theory which is content of Part III$_{\text{p.159}}$ of this work. There we conclude with a brief outlook and discussion of an informal and yet not quite complete proposal for a divergence free electrodynamic theory for point-like charges, the so-called *electrodynamic absorber theory*, in which the Dirac sea represents the absorber medium proposed by Wheeler and Feynman so that radiation reaction as well as pair creation emanate from one and the same physical origin.

The mathematics in the first two parts are self-contained and can be read independently while the physical content will converge towards the end of this work. The third and last part contains solely physics only on an informal but conceptual level. A summary of the used notation can be found at the end.

Part I

Radiation Reaction

Chapter 2

Complete Absorption and Radiation Reaction

The purpose of this chapter is to put the later presented mathematical results of this part in perspective to their application in classical electrodynamics. Classical electrodynamics is usually seen [Jac98, Roh94] as a theory about the interaction between light and charged matter. Light is represented in the theory as the electrodynamic field which is an antisymmetric second-rank tensor F on 3 + 1 dimensional Minkowski space $\mathbb{M} := (\mathbb{R} \times \mathbb{R}^3, g)$ for which we use the metric tensor $g = \operatorname{diag}(1, -1, -1, -1)$, while charged matter is represented by the four-vector current density j on $\mathbb{M}$. For prescribed j the time-evolution of the electrodynamic field F is ruled by the Maxwell equations

Maxwell equations

$$\partial_\nu F^{\mu\nu}(x) = -4\pi j^\mu(x) \qquad \text{and} \qquad \partial^\gamma F^{\alpha\beta}(x) + \partial^\alpha F^{\beta\gamma}(x) + \partial^\beta F^{\gamma\alpha}(x) = 0.$$

Throughout this exposition we use Einstein's summation convention for Greek indices, i.e. $x^\mu y_\mu = \sum_{\mu,\nu=0}^{3} x^\mu g^{\mu\nu} y^\nu$, and physical units such that the speed of light is $c = 1$. As addressed in the preface, Maxwell's equations give rise to a description of electro- and magneto-statics as well as radiation.

Moreover, given an electromagnetic field F the motion of N point-like charges, which are represented by $\mathbb{R}$-parameterized world lines $\tau \mapsto z_i^\mu(\tau)$ for labels $1 \le i \le N$ in Minkowski space $\mathbb{M}$, obey the Lorentz force law

Lorentz force law

$$m_i \ddot{z}_i^\mu(\tau) = e_i F^{\mu\nu}(z_i(\tau))\, \dot{z}_{i,\nu}(\tau)$$

where $m_i \neq 0$ denotes the mass and e_i is a coupling constant (their charge). The overset dot denotes a differentiation with respect to the parametrization τ of the world line. In other words, the Lorentz force describes the motion of test charges in an external field.

While for great many physical applications it suffices to consider only either the Lorentz force law for a given electrodynamical field or vice versa the Maxwell equations for a given four-vector charge current density only, radiation reaction can only be described by a fully interacting system of charges and fields. The first approach to include radiation reaction in the Maxwell-Lorentz equations is usually done in the following way. One defines the four-vector current density j_i^μ as induced by the world line of the ith given by

Four-vector current density induce by ith charge trajectory

$$j_i^\mu(x) = e_i \int_{z_i} dz_i^\mu\, \delta^4(x - z) = e_i \int_{\mathbb{R}} d\tau\, \dot{z}_i^\mu(\tau) \delta^4(x - z_i(\tau)).$$

The integral over dz_i^μ denotes the integral over the world line of the ith charge and δ^4 denotes the four-dimensional Dirac delta distribution. Due to the linearity of the Maxwell equations we may split up the electromagnetic F into a sum of fields F_i, each induced by the ith four-vector current density, so that the coupled equations, usually called the Maxwell-Lorentz equations with self-interaction (ML+SI equations), read

ML+SI equations

$$\partial_\nu F_i^{\mu\nu}(x) = -4\pi j_i^\mu(x) \qquad \text{and} \qquad \partial^\gamma F_i^{\alpha\beta}(x) + \partial^\alpha F_i^{\beta\gamma}(x) + \partial^\beta F_i^{\gamma\alpha}(x) = 0 \tag{2.1}$$

together with

$$m_i \ddot{z}_i^\mu(\tau) = e_i \sum_{j=1}^{N} F_j^{\mu\nu}(z_i(\tau))\, \dot{z}_{i,\nu}(\tau) \tag{2.2}$$

for indices $1 \le i \le N$. At first sight it seems appealing that the $j = i$ summand in the Lorentz force law lets the ith field, which is induced by the ith charge, react back on it. When looking informally at the constants of motion of this system of N charges and fields, it appears that whenever a charge accelerates, the energy put into the radiation fields has to be paid for by loss of kinetic energy of the charge. However, at second sight one realizes a serious problem. For its discussion we need the explicit form of solutions to the Maxwell equations (2.1) for prescribed four-vector current densities j_i. It is convenient to write a field F as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2.3}$$

for some four-vector field A on Minkowski space. The existing freedom in the choice of A can be restricted further to fulfill the gauge condition $\partial_\mu A^\mu = 0$. Then the left-hand side of the Maxwell equations can be written as

$$\Box A_i^\mu = -4\pi j_i^\mu \tag{2.4}$$

where $\Box = \partial_\mu \partial^\mu$ denotes the d'Alembert operator. Let us denote the advanced, respectively, retarded Green's function by $\triangle^\pm(x)$ whereas "+" stands for advanced, respectively, "−" stands for retarded. These Green's functions are uniquely determined by the condition $\triangle^\pm(x) = 0$ for $\pm x^0 > 0$. With the help of these Green's functions one finds two special solutions to the Maxwell equations

Liénard-Wiechert potentials

$$A_{i,\pm}^\mu(x) = \int_{\mathbb{M}} d^4y\ \triangle^\pm(x-y) j_i^\mu(y) = e_i \int_{\mathbb{M}} d^4y\ \triangle^\pm(x-y) \int_{z_i} dz_i^\mu\ \delta^4(y - z_i), \tag{2.5}$$

the so-called Liénard-Wiechert potentials. Written more explicitly they read

$$A_{i,\pm}^\mu(x) = e_i \frac{\dot{z}_i^\mu(\tau_{i,\pm}(x))}{(x - z_i(\tau_{i,\pm}(x)))_\nu\ \dot{z}_i^\nu(\tau_{i,\pm}(x))} \tag{2.6}$$

for world line parameters $\tau_{i,\pm}$ which are implicitly defined through

$$z_i^0(\tau_{i,+}(x)) = x^0 + \|\boldsymbol{x} - \boldsymbol{z}_i(\tau_{i,+}(x))\| \qquad \text{and} \qquad z_i^0(\tau_{i,-}(x)) = x^0 - \|\boldsymbol{x} - \boldsymbol{z}_i(\tau_{i,-}(x))\|$$

where we use the notation $x = (x^0, \boldsymbol{x})$ for a space-time point in Minkowski space, x^0 being the time and $\boldsymbol{x}$ the space component with Euclidian norm $\|\cdot\|$. By linearity of the wave equation (2.4) any solution A_i is of the form

$$A_i^\mu = A_{i,0}^\mu + \frac{1}{2}\left(A_{i,+}^\mu + A_{i,-}^\mu\right) \tag{2.7}$$

where $A_{i,0}$ fulfills the free wave equation $\Box A_{i,0}^\mu = 0$. Hence, by the form of the equations (2.6) the electromagnetic A_i which is induced by the ith charge trajectory is singular on this trajectory. Translating the potentials A_i back into fields F_i by (2.3) while keeping the appropriate indices therefore yields that also F_i is singular on the ith charge trajectory. But exactly there the Lorentz force law (2.2) needs to evaluate them in order to compute the back reaction on the charge. In other words, the coupled set of equations, i.e. the Maxwell equations (2.1) together with the Lorentz force law (2.2) are ill-defined and there is no way to make sense of of the Maxwell-Lorentz equations for point-like charges.

This fact compels us to consider a different formulation of electrodynamics which is based on the absorber idea of Wheeler and Feynman and wich we shall call *electrodynamic absorber theory*:

The electrodynamic absorber theory is a theory about the motion of N charges and their N electrodynamic fields which obey the Maxwell equations (2.1p.8) ML-SI equations

$$\partial_\nu F_i^{\mu\nu}(x) = -4\pi j_i^\mu(x) \qquad \text{and} \qquad \partial^\gamma F_i^{\alpha\beta}(x) + \partial^\alpha F_i^{\beta\gamma}(x) + \partial^\beta F_i^{\gamma\alpha}(x) = 0 \tag{2.1}$$

together with a different Lorentz force law

$$m_i \ddot{z}_i^\mu(\tau) = e_i \sum_{j\neq i} F_j^{\mu\nu}(z_i(\tau))\, \dot{z}_{i,\nu}(\tau). \tag{2.8}$$

We shall refer to these equations as Maxwell-Lorentz equations without self-interaction which we abbreviate with: ML-SI equations. Before we go on let us capture four important points:

1. As long as charge trajectories do not cross, there is no reason to expect ill-defined equations of motion as the self-interaction summand $j = i$ is explicitly excluded in the Lorentz force law.

2. The equations are relativistically invariant and the charges are mathematically treated as point particles.

3. With prescribed fields the world lines of the charges obeying the ML-SI equations agree with the ones of usual electrodynamics.

4. With prescribed charge current densities the fields obeying the ML-SI equations agree with the one of usual electrodynamics.

Now, since self-interaction was explicitly excluded the question is how this theory is capable of describing radiation reaction? The answer is: By special initial conditions which we shall describe now. Supposing the total number of charges N is very large and they are distributed in a sufficiently homogeneous way, then it is conceivable that for a test charge at space-time point x in some distance to all of the N particles the net force acting on it is only due to the free fields $F_{i,0}$, cf. (2.7p.8). We shall refer to such a homogeneous sea of N charges as the *absorber* or *absorber medium* and whenever we say *outside* of the absorber we mean in the space-time region that is in a sufficient spatial distance to all the charges of the absorber medium. In mathematical terms according to the Lorentz force (2.8), the condition of a test charge at x outside of the absorber being subject to free fields only reads

Radiation reaction

Absorber assumption

$$\sum_{i=1}^{N} F_i(x) \approx \sum_{i=1}^{N} F_{i,0}(x) \tag{2.9}$$

which in analogy to Wheeler-Feynman electrodynamics we shall call the *complete absorption assumption* or simply *absorber assumption*. Here, "≈" should remind us that equality holds only in a thermodynamic limit $N \to \infty$ and in sufficient distance to the absorber. However, in the following we will for simplicity just replace the "≈" by an "=". By (2.7p.8) this equation turns into

$$\sum_{i=1}^{N} (F_{i,+} + F_{i,-}) = 0 \qquad \text{i.e.} \qquad \sum_{i=1}^{N} (A_{i,+} + A_{i,-}) = 0$$

which is an equation that has been studied by Wheeler and Feynman in [WF45, Equation (37)]. They argued that since $\sum_{i=1}^{N} F_{i,+}$, respectively $\sum_{i=1}^{N} F_{i,-}$, represents a converging, respectively outgoing, wave at large distances to the absorber which cannot interfere destructively for all times. Therefore, outside of the absorber it must hold that $\sum_{i=1}^{N} A_{i,+} = 0 = \sum_{i=1}^{N} A_{i,-}$ and, hence also that

$$\sum_{i=1}^{N} (A_{i,-} - A_{i,+}) = 0. \tag{2.10}$$

This expression is a solution to the free wave equation as $\square(A_{i,-} - A_{i,+}) = 0$ by (2.4p.8). Therefore, we may finally conclude that (2.10) holds everywhere. Translating this result back in terms of fields (2.9p.9) implies

$$\sum_{i=1}^{N} (F_{i,-} - F_{i,+}) = 0 \tag{2.11}$$

Effective Lorentz force under complete absorption

everywhere in Minkowski space. We can now use this condition together with the Lorentz force (2.8p.9) to derive the *effective Lorentz force* on the ith charge in the absorber

$$m_i \ddot{z}_i^{\mu}(\tau) = e_i \sum_{j\neq i} F_j^{\mu\nu}(z_i(\tau))\dot{z}_{i,\nu}(\tau) = e_i \left[\sum_{j\neq i} F_{j,0} + \sum_{j\neq i} F_{j,-} + \frac{1}{2}\left(F_{i,-} - F_{i,+}\right) \right]^{\mu\nu} (z_i(\tau))\ \dot{z}_{i,\nu}(\tau). \tag{2.12}$$

Thus, according to the three terms on the right-hand side, the ith particle *effectively* feels a Lorentz force due to the free fields of all other charges, the retarded fields of all other charges and a third term which turns out to be the same that Dirac made responsible for radiation reaction in [Dir38]. He computed its value to be

$$\frac{1}{2}\left(F_{i,-}^{\mu\nu} - F_{i,+}^{\mu\nu}\right)(z_i(\tau)) = \frac{2}{3} e_i^2 \left(\dddot{z}_i^{\mu}(\tau)\dot{z}_i^{\nu}(\tau) - \dddot{z}_i^{\nu}(\tau)\dot{z}_i^{\mu}(\tau) \right) \tag{2.13}$$

which by involving the third derivative displays its dissipative behavior from which the name *radiation damping* comes from. Another remarkable feature of the ML-SI equations is that in contrast to any other theories describing radiation damping, the ML-SI equations still allow for bound states (the Schild solutions [Sch63]). Let us capture the discussed features:

1. An accelerating charge effectively feels a damping due to the sum of the advanced fields of the other absorber charges acting on it.
2. No renormalization program like in [Dir38] was used to derive the radiation damping term so that model parameters like the mass stay unchanged.
3. The description of radiation damping with the ML-SI equations does not rule out bound states.

Before we continue we need to make three remarks with regard to the effective Lorentz force (2.12):

No "unphysical" run-away solutions

First, if the complete absorption assumption (2.9p.9) holds, then any solution to the Lorentz force equation (2.8p.9) is also a solution to the effective Lorentz force equation (2.12). However, since by equation (2.13) the effective Lorentz force involves three derivatives with respect to the world line parametrization τ it naturally allows for more solutions than the Lorentz force (2.8p.9) which involves only two such derivatives. Therefore, not every solution to the effective Lorentz force equation can be considered as being physical, simply because they do not necessarily obey the fundamental equation (2.8p.9) of the theory. It is expected that such unphysical solutions are, for example, the so-called run-away solutions which were already studied by Dirac in [Dir38] as they are due to the fact that the negative bare masses which are needed for the renormalization procedure lead to dynamical instability as argued in [BD01].

Second, we observe that both terms $\sum_{j\neq i} F_{j,0}$ as well as $\frac{1}{2}(F_{i,-} - F_{i,+})$ are free fields which raises the question if they could (at least partially) cancel each other for all times so that radiation damping cannot be observed. Though one might be able to find initial conditions to the ML-SI equations such that this is the case (which is not trivial as a change of $F_{i,0}$ implies a change of all world lines which in turn implies a change of $\frac{1}{2}(F_{i,-} - F_{i,+})$), for general initial conditions this seems to be a rather improbable conspiracy. In case of doubt, the term $\sum_{j\neq i} F_{j,0}$ can be measured in the vicinity of a world line of a charge by a precise measurement of the curvature of the world line of the ith charge as one can create situations in which a charge with constant velocity feels no force (e.g. in a Faraday cage or Millikan experiment) such that we may set

$$\sum_{j\neq i}\left(F_{j,0} + F_{j,-}\right) \approx 0.$$

If the charge is then accelerated, for example by gravitation, one would have direct access to measure the damping term $\frac{1}{2}(F_{i,-} - F_{i,+})$. There are of course more sophisticated ways to measure this term [Spo04, Chapter 9.3]. However, it has to be noted that until today no quantitative measurement of the radiation damping term has been conducted. Only qualitative measurements via the energy loss predicted by Lamor's formula have been experimentally verified.

Third, the effective Lorentz force is still time-symmetric because from (2.11p.10) it also follows that

$$m_i\ddot{z}_i^\mu(\tau) = e_i\sum_{j\neq i} F_j^{\mu\nu}(z_i(\tau))\dot{z}_{i,\nu}(\tau) = e_i\left[\sum_{j\neq i} F_{j,0}^{\mu\nu} + \sum_{j\neq i} F_{j,+}^{\mu\nu} + \frac{1}{2}\left(F_{i,+}^{\mu\nu} - F_{i,-}^{\mu\nu}\right)\right](z_i(\tau))\dot{z}_{i,\nu}(\tau).$$

This is not surprising because both the ML-SI equations as well as the absorber assumption are completely time-symmetric. The irreversible phenomena of radiation must therefore be attributed to special initial conditions as it is always the case in statistical mechanics.

To continue the discussion we finally note that the Lorentz force (2.8p.9) can informally be derived by variation principle for a given external field. The construction of the action principle is as follows. Let $A_{i,0}(x)|_{x^0=0}$ be given at time $x^0 = 0$. Then compute with this initial value the corresponding unique solution to the free wave equation $\Box A_{i,0}^\mu(x) = 0$. The world line of the N charges then obey the principle of minimal action for the action integral

Action principle for ML-SI

$$S[z_1,\ldots,z_N] = -\sum_{i=1}^N m_i\int_{z_i}\sqrt{dz_i^\mu dz_{i,\mu}} - \frac{1}{2}\sum_{i=1}^N e_i\int_{z_i} dz_{i,\mu}\sum_{j\neq i}^N e_j\int_{z_j} dz_j^\mu\,\delta((z_i - z_j)_\mu(z_i - z_j)^\mu) + \\ -\sum_{i=1}^N e_i\sum_{j\neq i}^N\int_{z_i} dz_{i,\mu}A_{j,0}^\mu(z_i)$$

Minimization with respect to the world lines $z_1,\ldots,z_n$ yields

$$m_i\ddot{z}_i^\mu(\tau) = e_i\sum_{j\neq i}\left(F_{j,0}^{\mu\nu} + \frac{1}{2}\left[F_{j,+}^{\mu\nu} + F_{j,-}^{\mu\nu}\right]\right)(z_i(\tau))\dot{z}_{i,\nu}(\tau) \tag{2.14}$$

where we used (2.3p.8) and the Liéard-Wiechert potentials (2.6p.8). By construction, the world lines $z_1,\ldots,z_N$ obeying this equation are a solution to the ML-SI equations, i.e. the Lorentz force (2.8p.9) and the Maxwell equations (2.1p.8), for any initial conditions such that the initial fields are given by

$$F_i(x)\big|_{x^0=0} = \left[F_{i,0} + \frac{1}{2}\left(F_{i,+} + F_{i,+}\right)\right](x)\Big|_{x^0=0}$$

where we have used (2.3p.8) again.

This formulation suggests to look at an important special case of the solution to the ML-SI equations: Consider initial conditions such that

$$F_{i,0}(x)|_{x^0=0} = 0 \text{ for some time } x^0 = 0. \tag{2.15}$$

Because $F_{i,0}$ is a free field it follows that it must be zero for all times. Hence, the Lorentz force (2.14p.11) simplifies to

Wheeler-Feynman equations of motion

$$m_i \ddot{z}_i^{\mu}(\tau) = e_i \frac{1}{2}\left[F_{i,+} + F_{i,-}\right]^{\mu\nu}(z_i(\tau))\, \dot{z}_{i,\nu}(\tau) \tag{2.16}$$

which is the well-known equation of motion of Wheeler-Feynman electrodynamics. Note that since the fields $F_{i,\pm}$ are functionals of the world line z_i as defined in (2.6p.8), the fields are no dynamical degrees of freedom anymore. In the special case of initial conditions (2.15) we wind up with a theory only about world lines of charges. Hence, on the one hand, as we have already begun, the ML-SI equations can be regarded as the fundamental equations of a theory about charges and fields which includes Wheeler-Feynman electrodynamics, a theory only about charges, as a special case (provided that initial conditions (2.15) exist). On the other hand, one could regard Wheeler-Feynman electrodynamics as the fundamental theory because it is only about charges and therefore certainly more minimal and maybe more appealing. The reason why we started with the ML-SI equations instead of the Wheeler-Feynman equations is that the Wheeler-Feynman equations are more subtle than they look and it is not clear at all what a corresponding existence and uniqueness theory of solutions would look like (furthermore, according to the above reasoning everything we say for the ML-SI equations holds also for the Wheeler-Feynman equations). Since the fields on the right-hand side (2.16) are functionals of the charge trajectories as defined in (2.6p.8), the Wheeler-Feynman equations form a coupled set of differential equations with state-dependent delayed and advanced arguments. This type of equation cannot be treated with conventional mathematical tools from the theory of nonlinear partial differential equations and has been sparsely studied in the literature. While some special solutions to the Wheeler-Feynman equations of motions were found [Sch63], general existence of solutions to these equations has only been settled in the case of restricted motion of two equal charges on a straight line in $\mathbb{R}^3$ [Bau97]. An even greater, outstanding problem is the question how the solutions can be uniquely characterized, and above all if it is possible to pose a well-defined initial value problem for Wheeler-Feynman electrodynamics by specifying Newtonian Cauchy data, i.e. position and momenta of the charges at time zero.

We summarize the last facts:

1. In the electrodynamic absorber theory, radiation reaction is a statistical mechanical, multi-particle phenomena.

2. When the complete absorption assumption holds the radiation reaction on ith charge due to the $(N-1)$ other charges is completely determined by the motion of the ith charge.

3. ML-SI can be formulated via a variational problem and includes Wheeler-Feynman electrodynamics as a special case.

Overview of the mathematical results

The physical reasoning so far gave rise to two promising ways to account for radiation reaction, via the ML-SI equations or via the Wheeler-Feynman equations. Here starts our mathematical work: In Chapter 3p.15 we discuss the issue of existence and uniqueness of solutions to the Maxwell-Lorentz equations. This will also provide a characterization of possible Wheeler-Feynman solutions and a way to reformulate the question of the existence of Wheeler-Feynman solutions for Newtonian Cauchy data. Both topics are content of Chapter 4p.43 where we also show existence of solutions to the Wheeler-Feynman solutions on finite time intervals for Newtonian Cauchy data. However, in order to circumvent the issue of crossing trajectories we treat only the Maxwell-Lorentz and Wheeler-Feynman equations for classical, rigid, non-rotating charge densities instead of point-like charges, even though this clashes with our reasoning that a physical theory should be about point-like charges argued in the preface. The limit to point-like charges remains open but note that in contrast to the Maxwell-Lorentz equations with self-interaction, the ML-SI as well as the Wheeler-Feynman equations bare no mathematical obstacles for such a limit; e.g. two charges of equal sign surely do not lead to a crossing of trajectories – even with charges of opposite sign a crossing of the trajectories is prohibited for almost all initial conditions by angular momentum conservation. In other words, the limit to point-like sources in the case of ML-SI and Wheeler-Feynman equations is then only mathematical work but does not require any further physical input like Dirac's renormalization program. In particular, we expect that besides a change of the used norms the presented proofs will conceptually not change much for point-like sources.

Chapter 3

Maxwell-Lorentz equations of Motion

3.1 Chapter Overview and Results

This chapter treats the initial value problem of the so-called *Maxwell-Lorentz equations* for rigid charges. These equations are essentially the equations (2.1p.8) and (2.2p.8) discussed in the preceding chapter with the difference that the charges are smeared out by some smooth and compactly supported functions on $\mathbb{R}^3$. These equations then describe a system of $N \in \mathbb{N}$ classical, non-rotating, rigid charges interacting with their electromagnetic fields. Using non-relativistic notation in a special reference frame (cf. Section 5.1p.95) they are formed by coupling the *Lorentz force law*:

Lorentz force law

$$\begin{aligned}\partial_t \boldsymbol{q}_{i,t} &= \boldsymbol{v}(\boldsymbol{p}_{i,t}) := \frac{\sigma_i \boldsymbol{p}_{i,t}}{\sqrt{m_i^2 + \boldsymbol{p}_{i,t}^2}} \\ \partial_t \boldsymbol{p}_{i,t} &= \sum_{j=1}^{N} e_{ij} \int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_{i,t}) \left[\boldsymbol{E}_{j,t}(\boldsymbol{x}) + \boldsymbol{v}_{i,t} \wedge \boldsymbol{B}_{j,t}(\boldsymbol{x})\right],\end{aligned} \tag{3.1}$$

to the *Maxwell equations*:

Maxwell equations including the constraints

$$\begin{aligned}\partial_t \boldsymbol{E}_{i,t} &= \nabla \wedge \boldsymbol{B}_{i,t} - 4\pi \boldsymbol{v}(\boldsymbol{p}_{i,t})\varrho_i(\cdot - \boldsymbol{q}_{i,t}) & \nabla \cdot \boldsymbol{E}_{i,t} &= 4\pi\varrho_i(\cdot - \boldsymbol{q}_{t,i}) \\ \partial_t \boldsymbol{B}_{i,t} &= -\nabla \wedge \boldsymbol{E}_{i,t} & \nabla \cdot \boldsymbol{B}_{i,t} &= 0.\end{aligned} \tag{3.2}$$

The equations on the right-hand side are commonly called the *Maxwell constraints*. We denote the partial derivative with respect to time t by ∂_t, the divergence by $\nabla\cdot$ and the curl by $\nabla\wedge$. Vectors in $\mathbb{R}^3$ are written as bold letters, e.g. $\boldsymbol{x} \in \mathbb{R}^3$. At time t the ith charge for $1 \le i \le N$ is situated at position $\boldsymbol{q}_{i,t}$ in space $\mathbb{R}^3$ and has momentum $\boldsymbol{p}_{i,t} \in \mathbb{R}^3$. It carries the classical mass $m_i \in \mathbb{R} \setminus \{0\}$. The geometry of the rigid charge is given in terms of a charge distribution (i.e. form factor) ϱ_i which is assumed to be an infinitely often differentiable function $\mathbb{R}^3 \to \mathbb{R}$ with compact support. The factors $\sigma_i := \text{sign}(m_i)$ are introduced to allow also for negative masses, while the matrix coefficients $e_{ij} \in \mathbb{R}$ for $1 \le i, j \le N$ allow to adjust the action of the jth field on the ith particle. Each charge is coupled to an own electric and magnetic field $\boldsymbol{E}_{i,t}$ and $\boldsymbol{B}_{i,t}$, which are $\mathbb{R}^3$ valued functions on $\mathbb{R}^3$. Whereas in the classical literature only one electric and magnetic field is considered, we have given every charge its own field which will later permit us to explicitly exclude self-interaction.

While the existence and uniqueness theory build in this chapter will not depend on a particular choice of the coupling matrix $(e_{ij})_{1\le i,j\le N}$ we still want to point out relevant choices for classical electrodynamics: The first is

$$e_{ij} = 1 \qquad \text{for} \qquad 1 \le i, j \le N. \tag{ML+SI}$$

Note that for this choice the ith field generated by the ith charge is allowed to act back on the charge. We refer to this case as the Maxwell-Lorentz equations with self-interaction: *ML+SI equations*. By linearity of the Maxwell equations this case is equivalent to having only one electric $\boldsymbol{E}_t = \sum_{i=1}^N \boldsymbol{E}_{i,t}$ and one magnetic field $\boldsymbol{B}_t = \sum_{i=1}^N \boldsymbol{B}_{i,t}$. Several well-known difficulties such as dynamical instability and unavoidable divergences in the point particle limit are connected to this approach. Recovering the mass renormalized Lorentz-Dirac equation [Dir38] by an appropriate point particle limit as well as recovering the Maxwell-Vlasov equations in an appropriate hydrodynamic scaling limit are considered outstanding problems. The second and with respect to our argument in the preceding chapter more important choice is

$$e_{ij} = 1 - \delta_{ij} = \begin{cases} 1 & \text{for } i \neq j \\ 0 & \text{otherwise} \end{cases} \qquad \text{for} \qquad 1 \leq i, j \leq N. \tag{ML-SI}$$

Here the self-interaction is explicitly excluded. This choice we shall refer to as Maxwell-Lorentz equations without self-interaction: *ML-SI equations*.

Sketch of the mathematical results in this chapter

Let us sketch the mathematical results in this chapter. For the discussion it is convenient to write equations (3.1p.15) and (3.2p.15) in a more compact form:

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t) \tag{3.3}$$

where we use the notation $\varphi_t := (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\leq i\leq N}$, a linear operator A given by the expression

$$A\varphi_t := (0, 0, \nabla \wedge \boldsymbol{B}_{i,t}, -\nabla \wedge \boldsymbol{E}_{i,t})_{1\leq i\leq N}$$

and a nonlinear operator J given by the expression

$$J(\varphi_t) := \left(\boldsymbol{v}(\boldsymbol{p}_{i,t}), \sum_{j=1}^N e_{ij} \int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_{i,t}) \left[\boldsymbol{E}_{j,t}(\boldsymbol{x}) + \boldsymbol{v}_{i,t} \wedge \boldsymbol{B}_{j,t}(\boldsymbol{x}) \right], -4\pi \boldsymbol{v}(\boldsymbol{p}_{i,t}) \varrho_i(\cdot - \boldsymbol{q}_{i,t}, 0 \right)_{1\leq i\leq N}.$$

At first we regard (3.3) as an abstract nonlinear partial-differential equation for solutions $t \mapsto \varphi_t$ taking values in an abstract Banach space and work out conditions on A and J which suffice for a global existence and uniqueness theorem of solutions for this Banach space. We use the standard methods of nonlinear functional analysis. First, we demand the linear operator A to generate a group $(W_t)_{t\in\mathbb{R}}$ on its domain so that we are able to rewrite (3.3) in its integral form as

$$\varphi_t = W_t\varphi + \int_0^t ds\, W_{t-s} J(\varphi_s) \tag{3.4}$$

for any initial value $\varphi_t|_{t=0} = \varphi$. Furthermore, we demand that the nonlinear operator J is Lipschitz continuous in its argument. For small enough times t this leads to the fact that the integral equations (3.4) read as a fixed point map on an appropriate Banach space is a contraction map which provides local existence of solutions by Banach's fixed point theorem. Under the assumption of an a priori bound on solutions to (3.4) local existence can be extended to global existence for all times. Uniqueness is simply implied by the uniqueness assertion of Banach's fixed point theorem. The precise existence and uniqueness assertion together with its proof is given in Section 3.3p.21.

Having this abstract result the next question is in which Banach space solutions $t \mapsto \varphi_t$ of the Maxwell-Lorentz equations take their values. Usually one would allow solutions to the ML+SI equations to have electric and magnetic fields in $L^2(\mathbb{R}^3, \mathbb{R}^3)$, the space of square integrable $\mathbb{R}^3$ valued functions on $\mathbb{R}^3$, which would imply that the energy of the system is finite at all times. However, as we have discussed in the introductory Chapter 2p.7 Wheeler-Feynman solutions are also solutions to the ML-SI equations and in order to exploit this feature we need to choose the

Banach space of solutions big enough to allow for all physically expected Wheeler-Feynman solutions – in other words we need to allow for electric and magnetic fields which are of the Liénard-Wiechert form of the charge trajectories. By the form of the Liénard-Wiechert fields (2.6p.8), which are in our case also smeared out by the charge densities ϱ_i, one can compute that it is sufficient for Liénard-Wiechert fields to be in L^2 if the corresponding charge trajectories have accelerations that decay for times $t \to \pm\infty$ as well as bounded velocities with a bound smaller than one. At the present state of knowledge not much is known about Wheeler-Feynman solutions except the analytic Schild solutions [Sch63] and Bauer's existence theorem on the straight line [Bau97]. Bauer's result states that the Wheeler-Feynman solutions on a straight line are exactly of that form but, unfortunately, the Schild solution which describes two charges of opposite sign moving in circular orbits around each other implies that the modulus of the acceleration of the charges does not decay for times $t \to \pm\infty$. Hence, the corresponding Liénard-Wiechert fields are not necessarily in L^2. As we want to allow at least for solutions which behave as badly as the Schild solution we consider the space L^2_w for the fields which comprises all functions $\boldsymbol{F}$ such that $\sqrt{w}\boldsymbol{F}$ is in L^2 for some smooth weight function w. This way we deliver the missing decay of the accelerations of the charges. The properties of these weighted spaces and their corresponding Sobolev spaces are the content of Section 3.4p.25. Note that weighted function spaces appear naturally in the Fourier characterization of Sobolev spaces with weight functions w that increase at spatial infinity in order to make the functions more regular. However, we use them for weight functions w that decay at spatial infinity in order to allow for more irregular functions.

With this consideration in mind we specify the Banach space in which the possible solutions $t \mapsto \varphi_t$ take their values to be

$$\mathcal{H}_w := \bigoplus_{i=1}^{N} \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus L^2_w(\mathbb{R}^3, \mathbb{R}^3) \oplus L^2_w(\mathbb{R}^3, \mathbb{R}^3)$$

for any smooth and non-zero weight function decaying slower than exponentially at spatial infinity. Last but not least it is left to show that the abstract existence of uniqueness theorem for (3.3p.16) holds also on $\mathcal{H}_w$. First, we need to check if A generates a group $(W_t)_{t\in\mathbb{R}}$ on its natural domain in $\mathcal{H}_w$. Because of the weight the measure in L^2_w is not translational invariant anymore and we cannot expect A to be self-adjoint. Therefore, we make use of the Hille-Yosida theorem to show the existence of the group $(W_t)_{t\in\mathbb{R}}$ which relies on the resolvent properties of A. Second, the Lipschitz continuity of the nonlinear operator J must be checked which is a long but straight-forward computation. And finally, we need an a priori estimate for solutions to (3.4p.16) which is the most sensitive part of the proof as we do not have any constants of motion (for the ML-SI equations there are no known constants of motion on equal time hypersurfaces whereas for the ML+SI equations the energy is not well-defined for general fields in L^2_w). Fortunately, it turns out that the norm of $J(\varphi_t)$ can be bounded by a constant that only depends on the position of the charges at time t times the norm of φ_t. As the speed of light is smaller than one, we infer from this together with Gronwall's Lemma that the norm of any solution to (3.4p.16) is bounded uniformly on each compact time interval which is the needed a priori estimate. This yields global existence and uniqueness of solutions to (3.3p.16) for initial conditions in the domain of the operator A.

Furthermore, by the form of the Maxwell-Lorentz equations one sees that the Maxwell constraints are respected for all times by the time-evolution. This fact allows us to show that the regularity of solutions depends on the regularity of the initial conditions and we show that the Maxwell-Lorentz equations admit strong global solutions. This is the main content of Section 3.5p.30.

The main result of this chapter is: Main results

- Existence, uniqueness and regularity of solutions to the Maxwell-Lorentz equations for N rigid charges, possibly negative masses and infinite energy initial conditions and both cases with and without self-interaction, given in Theorem 3.5p.20 and Theorem 3.6p.20.

Literature review

In order to compare our results with the contemporary work in this field, we would like to call attention to the following literature on the topic of Maxwell-Lorentz equations: In the case of $\sigma_i = 1$, $e_{ij} = 1$ and spherical symmetric charge densities ϱ_i for $1 \le i, j \le N$, these equations are usually referred to as the Abraham Model; see [Spo04] which provides an excellent and comprehensive overview of the topic of charge distributions interacting with their own electromagnetic fields. The question of existence and uniqueness of solutions to these equations for one particle interacting with its own fields, i.e. $N = 1$, $\sigma_1 = 1$, $e_{11} = 1$, $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ and square integrable initial fields $\boldsymbol{E}_{1,t}|_{t=0}, \boldsymbol{B}_{1,t}|_{t=0}$, has been settled by two different techniques: While in [KS00] one exploits the energy conservation to gain an a priori bound needed for global existence, a Gronwall argument was used in [BD01] which made it possible to allow also for negative masses $m_1 < 0$. Recent works also provided results on the long-time behavior of solutions in [KS00] as well as in [IKS02] for soliton-like solutions, on the dynamical instability for negative masses in [BD01] and on conservation laws in [Kie99]. Furthermore, a generalization to a spinning extended charge was treated in [AK01] which also contains an exhaustive list of references on the subject.

3.2 Solutions to the Maxwell-Lorentz equations of Motion

In this section we formalize and state the global existence and uniqueness theorem for solutions to the Maxwell-Lorentz equations (4.34p.60) with the following choice of parameter: $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$, $e_{ij} \in \mathbb{R}$ and $m_i \in \mathbb{R} \setminus \{0\}$ for $1 \le i, j \le N$. As motivated in the introduction we aim at an initial-value problem for given positions and momenta $\boldsymbol{p}_i^0, \boldsymbol{q}_i^0 \in \mathbb{R}^3$ as well as electric and magnetic fields $\boldsymbol{E}_i^0, \boldsymbol{B}_i^0 : \mathbb{R}^3 \to \mathbb{R}^3$ at time $t_0 \in \mathbb{R}$ general enough to allow for Liénard-Wiechert fields produced by any strictly time-like charge trajectory, see Definition 4.5p.60, with bounded acceleration, i.e. there exists an $a_{max} < \infty$ such that $\sup_{t\in\mathbb{R}} \|\partial_t \boldsymbol{v}(\boldsymbol{p}_t)\| \le a_{max}$, as Cauchy data. Theorem 4.18p.66 states that such electric and magnetic Liénard-Wiechert fields are in $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$, and Corollary 4.22p.69 gives us their decay behavior $O([1 + \|\boldsymbol{x} - \boldsymbol{q}_0\|]^{-1})$ for $\|\boldsymbol{x}\| \to \infty$. Hence, in general these fields are not in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ which forces us to regard the initial value problem for the following, bigger class of fields:

Class of fields and weight functions

Definition 3.1. *Let*

$$\mathcal{W} := \left\{ w \in C^\infty(\mathbb{R}^3, \mathbb{R}^+ \setminus \{0\}) \,\middle|\, \exists\, C_w \in \mathbb{R}^+, P_w \in \mathbb{N} : w(\boldsymbol{x} + \boldsymbol{y}) \le (1 + C_w\|\boldsymbol{x}\|)^{P_w} w(\boldsymbol{y}) \right\} \tag{3.5}$$

be the class of weight functions. For any $w \in \mathcal{W}$ and $\Omega \subseteq \mathbb{R}^3$ we define the space of weighted square integrable functions $\Omega \to \mathbb{R}^3$ by

$$L^2_w(\Omega, \mathbb{R}) := \left\{ \boldsymbol{F} : \Omega \to \mathbb{R}^3 \,\middle|\, \int d^3x\, w(\boldsymbol{x}) \|\boldsymbol{F}(\boldsymbol{x})\|^2 < \infty \right\}.$$

For global regularity arguments we need more conditions on the weight functions which for $k \in \mathbb{N}$ gives rise to the definitions:

$$\mathcal{W}^k := \left\{ w \in \mathcal{W} \,\middle|\, \exists\, C_\alpha \in \mathbb{R}^+ : |D^\alpha \sqrt{w}| \le C_\alpha \sqrt{w}, |\alpha| \le k \right\} \tag{3.6}$$

and

$$\mathcal{W}^\infty := \{ w \in \mathcal{W} \mid w \in \mathcal{W}^k \text{ for any } k \in \mathbb{N} \}.$$

The choice of $\mathcal{W}$ is quite natural (compare [HÖ5]) as it provides a tool to treat the new measure $w(\boldsymbol{x})d^3x$ almost as if it were translational invariant which it is not unless w equals a constant. Applying its definition twice we obtain for all $w \in \mathcal{W}$ the estimate

$$(1 + C_w\|\boldsymbol{y}\|)^{-P_w} w(\boldsymbol{x}) \leq w(\boldsymbol{x}+\boldsymbol{y}) \leq (1 + C_w\|\boldsymbol{y}\|)^{P_w} w(\boldsymbol{x}) \tag{3.7}$$

which also states that $w \in \mathcal{W} \Rightarrow w^{-1} \in \mathcal{W}$. In particular, the weight $w(\boldsymbol{x}) = (1 + \|\boldsymbol{x}\|^2)^{-1}$ is in $\mathcal{W}$ because

$$w^{-1}(\boldsymbol{x}+\boldsymbol{y}) := 1 + \|\boldsymbol{x}+\boldsymbol{y}\|^2 \leq 1 + (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2 \leq (1 + \|\boldsymbol{x}\|^2)(1 + \|\boldsymbol{y}\|)^2, \tag{3.8}$$

and therefore the desired Liénard-Wiechert fields, which are, as mentioned, of order $O([1 + \|\boldsymbol{x} - \boldsymbol{q}_0\|]^{-1})$ for $\|\boldsymbol{x}\| \to \infty$, are in $L^2_w(\mathbb{R}^3, \mathbb{R}^3)$ for $w(\boldsymbol{x}) := (1 + \|\boldsymbol{x}\|^2)^{-1}$.

The space of initial values is then given by:

Definition 3.2. *We define the Newtonian phase space $\mathcal{P} := \mathbb{R}^{6N}$, the field space* Phase Space for the Maxwell-Lorentz equations of Motion

$$\mathcal{F}_w := L^2_w(\mathbb{R}^3, \mathbb{R}^3) \oplus L^2_w(\mathbb{R}^3, \mathbb{R}^3)$$

and the phase space for the Maxwell-Lorentz equation of motion

$$\mathcal{H}_w := \mathcal{P} \oplus \mathcal{F}_w.$$

Any element $\varphi \in \mathcal{H}_w$ consists of the components $\varphi = (\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N}$, i.e. positions $\boldsymbol{q}_i$, momenta $\boldsymbol{p}_i$ and electric and magnetic fields $\boldsymbol{E}_i, \boldsymbol{B}_i$ for each of the $1 \leq i \leq N$ charges.

Wherever not explicitly noted otherwise, any spatial derivative will for the rest of this section be understood in the distribution sense, and the Latin indices shall run over the charge labels $1\ldots N$. We shall also need the weighted Sobolev spaces $H^{curl}_w(\mathbb{R}^3, \mathbb{R}^3) := \{\boldsymbol{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3) \mid \nabla \wedge \boldsymbol{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3)\}$ and $H^k_w(\mathbb{R}^3, \mathbb{R}^3) := \{\boldsymbol{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3) \mid D^\alpha \boldsymbol{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3)\ \forall\ |\alpha| \leq k\}$ for any $k \in \mathbb{N}$. Furthermore, we define the following operators:

Definition 3.3. *Let $\mathtt{A}$ and A be given by the expressions* Operator A

$$A\varphi = \Big(0, 0, \mathtt{A}(\boldsymbol{E}_i, \boldsymbol{B}_i)\Big)_{1\leq i\leq N} := \Big(0, 0, -\nabla \wedge \boldsymbol{E}_i, \nabla \wedge \boldsymbol{B}_i)\Big)_{1\leq i\leq N}.$$

for a $\varphi = (\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N}$. The natural domain is given by

$$D_w(A) := \bigoplus_{i=1}^{N} \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus H^{curl}_w(\mathbb{R}^3, \mathbb{R}^3) \oplus H^{curl}_w(\mathbb{R}^3, \mathbb{R}^3) \subset \mathcal{H}_w.$$

Furthermore, for any $n \in \mathbb{N} \cup \{\infty\}$ we define

$$D_w(A^n) := \{\varphi \in D_w(A) \,\big|\, A^k\varphi \in D_w(A)\, for\ k = 0, \ldots, n-1\}.$$

Definition 3.4. *Let $m_i \neq 0$, $\sigma_i := \mathrm{sign}(m_i)$ and $e_{ij} \in \mathbb{R}$, $1 \leq i, j \leq N$. Together with $\boldsymbol{v}(\boldsymbol{p}_i) := \frac{\sigma_i \boldsymbol{p}_i}{\sqrt{\boldsymbol{p}_i^2 + m^2}}$ we define $J : \mathcal{H}_w \to D_w(A^\infty)$ by the expression* Operator J

$$J(\varphi) = \left(\boldsymbol{v}(\boldsymbol{p}_i), \sum_{j=1}^{N} e_{ij} \int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\Big(\boldsymbol{E}_j(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(x)\Big), -4\pi \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i), 0\right)_{1\leq i\leq N}$$

for a $\varphi = (\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N} \in \mathcal{H}_w$.

Note that J is well-defined because $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. With these definitions the Lorentz force law (3.1p.15), the Maxwell equations (3.2p.15), neglecting the Maxwell constraints, can be collected in the form

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t).$$

The two main theorems of this section are:

Global existence and uniqueness for the Maxwell-Lorentz equations

Theorem 3.5. *Let the space $\mathcal{H}_w$ and the operators $A : D_w(A) \to \mathcal{H}_w$, $J : \mathcal{H}_w \to D_w(A^\infty)$ be the ones introduced in Definitions 3.2, 3.3 and 3.4. Let the weight function $w \in \mathcal{W}^1$ and let $n \in \mathbb{N}$ and $\varphi^0 = (\boldsymbol{q}_i^0, \boldsymbol{p}_i^0, \boldsymbol{E}_i^0, \boldsymbol{B}_i^0)_{1\le i\le N} \in D_w(A^n)$ be given. Then the following holds:*

(i) (global existence) *There exists an n times continuously differentiable mapping*

$$\varphi_{(\cdot)} : \mathbb{R} \to \mathcal{H}_w, \qquad t \mapsto \varphi_t = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N}$$

such that $\frac{d^j}{dt^j}\varphi_t \in D_w(A^{n-j})$ for all $t \in \mathbb{R}$ and $0 \le j \le n$, which solves

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t) \tag{3.9}$$

for initial value $\varphi_t|_{t=0} = \varphi^0$.

(ii) (uniqueness) *The solution φ is unique in the sense that if for any interval $\Lambda \subset \mathbb{R}$ there is any once continuously differentiable function $\widetilde{\varphi} : \Lambda \to D_w(A)$ which solves the Equation (3.9) on Λ and there is some $t^* \in \Lambda$ such that $\widetilde{\varphi}_{t^*} = \varphi_{t^*}$ then $\varphi_t = \widetilde{\varphi}_t$ holds for all $t \in \Lambda$. In particular, for any $T \ge 0$ such that $[-T, T] \subseteq \Lambda$ there exist $C_1, C_2 \in$ `Bounds` such that*

$$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \le C_1\left(T, \|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \le i \le N\right) \|\varphi^0\|_{\mathcal{H}_w}. \tag{3.10}$$

and

$$\sup_{t\in[-T,T]} \|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} \le C_2(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w})\|\varphi_{t_0} - \widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w}. \tag{3.11}$$

(iii) (constraints) *If the solution $t \mapsto \varphi_t = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N}$ obeys the Maxwell constraints*

$$\nabla \cdot \boldsymbol{E}_{i,t} = 4\pi\varrho(\cdot - \boldsymbol{q}_{i,t}), \qquad \nabla \cdot \boldsymbol{B}_{i,t} = 0 \tag{3.12}$$

for one $t = t^* \in \mathbb{R}$, then they are obeyed for all times $t \in \mathbb{R}$.

Regularity of the Maxwell-Lorentz solutions

Theorem 3.6. *Assume the same conditions as in Theorem 3.5 hold. In addition, let $w \in \mathcal{W}^2$. Let $t \mapsto \varphi_t = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N}$ be the solution to the Maxwell equations 3.9 for initial value $\varphi_t|_{t=0} = \varphi^0 \in D_w(A^n)$. Now let $n = 2m$ for some $m \in \mathbb{N}$, then for all $1 \le i \le N$:*

(i) It holds for any $t \in \mathbb{R}$ that $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t} \in \mathcal{H}_w^{\triangle^m}$.

(ii) The electromagnetic fields viewed as maps $\boldsymbol{E}_i : (t, \boldsymbol{x}) \mapsto \boldsymbol{E}_{i,t}(\boldsymbol{x})$ and $\boldsymbol{B}_i : (t, \boldsymbol{x}) \mapsto \boldsymbol{B}_{i,t}(\boldsymbol{x})$ are in $L^2_{loc}(\mathbb{R}^4, \mathbb{R}^3)$ and have a representative in $C^{n-2}(\mathbb{R}^4, \mathbb{R}^3)$ in their equivalence class, respectively.

(iii) For $w \in \mathcal{W}^k$ for $k \ge 2$ and every $t \in \mathbb{R}$ we have also $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t} \in H^n_w$ and $C < \infty$ such that:

$$\sup_{\boldsymbol{x}\in\mathbb{R}^3} \sum_{|\alpha|\le k} \|D^\alpha \boldsymbol{E}_{i,t}(\boldsymbol{x})\| \le C\|\boldsymbol{E}_{i,t}\|_{H^k_w} \quad \text{and} \quad \sup_{\boldsymbol{x}\in\mathbb{R}^3} \sum_{|\alpha|\le k} \|D^\alpha \boldsymbol{B}_{i,t}(\boldsymbol{x})\| \le C\|\boldsymbol{B}_{i,t}\|_{H^k_w}. \tag{3.13}$$

For their proofs we will need tools for the study of the $L^2_w(\mathbb{R}^3, \mathbb{R}^3)$ and corresponding weighted Sobolev spaces, which we discuss in subsection 3.4p.25. But first we establish suitable conditions in which we expect a well-defined initial value problem for (3.14p.21) on Banach spaces in subsection 3.3p.21. The proofs of the above theorems then follow by straightforward computations which we show in 3.5p.30.

3.3 A Global Existence and Uniqueness Result on Banach Spaces

For this subsection we assume $\mathcal{B}$ to be only a Banach space, denote its norm by $\|\cdot\|_{\mathcal{B}}$, and assume the existence of operators A and J with the following properties:

Definition 3.7. *Let $A : D(A) \subseteq \mathcal{B} \to \mathcal{B}$ be a linear operator with the properties:*

Abstract operator A

(i) A is closed and densely defined.

(ii) There exists a $\gamma \geq 0$ such that $(-\infty, -\gamma) \cup (\gamma, \infty) \subseteq \rho(A)$, the resolvent set of A.

(iii) The resolvent $R_\lambda(A) = \frac{1}{\lambda - A}$ of A with respect to $\lambda \in \rho(A)$ is bounded by $\frac{1}{|\lambda|-\gamma}$, i.e. for all $\phi \in \mathcal{B}, |\lambda| > \gamma$ we have $\|R_\lambda(A)\phi\|_{\mathcal{B}} \leq \frac{1}{|\lambda|-\gamma}\|\phi\|_{\mathcal{B}}$.

For $n \in \mathbb{N} \cup \{\infty\}$ we define $D(A^n) := \{\varphi \in D(A) \mid A^k\varphi \in D(A) \text{ for } k = 0, \ldots, n-1\}$.

Definition 3.8. *For an $n_J \in \mathbb{N}$ let $J : D(A) \to D(A^{n_J})$ be a mapping with the properties:*

Abstract operator J

(i) For all $0 \leq n \leq n_J$ and $\varphi, \widetilde{\varphi} \in D(A)$ there exist $C_3{}^{(n)}, C_4{}^{(n)} \in \texttt{Bounds}$ such that

$$\|A^n J(\varphi)\|_{\mathcal{B}} \leq C_3{}^{(n)}(\|\varphi\|_{\mathcal{B}}) \qquad \|A^n(J(\varphi) - J(\widetilde{\varphi}))\|_{\mathcal{B}} \leq C_4{}^{(n)}(\|\varphi\|_{\mathcal{B}}, \|\widetilde{\varphi}\|_{\mathcal{B}})\ \|\varphi - \widetilde{\varphi}\|_{\mathcal{B}}.$$

(ii) For all $0 \leq n \leq n_J$ and $T > 0$, $t \in (-T, T)$ and any $\varphi_{(\cdot)} \in C^n((-T,T), D(A^n))$ such that $\frac{d^k}{dt^k}\varphi_t \in D(A^{n-k})$ for $k \leq n$, the operator J fulfills for $j + l \leq n - 1$:

(a) $\frac{d^j}{dt^j} A^l J(\varphi_t) \in D(A^{n-1-j-l})$ and

(b) $t \mapsto \frac{d^j}{dt^j} A^l J(\varphi_t)$ is continuous on $(-T, T)$.

With these operators we shall prove:

Theorem 3.9. *Let A and J be the operators introduced in Definitions (3.7) and (3.8) then:*

Abstract global existence and uniqueness

(i) (local existence) For each $\varphi^0 \in D(A^n)$ with $n \leq n_J$, there exists a $T > 0$ and a mapping $\varphi_{(\cdot)} \in C^n((-T,T), D(A^n))$ which solves the equation

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t) \tag{3.14}$$

for initial value $\varphi_t|_{t=0} = \varphi^0$. Furthermore, $\frac{d^k}{dt^k}\varphi_t \in D(A^{n-k})$ for $k \leq n$ and $t \in (-T, T)$.

(ii) (uniqueness) If $\widetilde{\varphi}_{(\cdot)} \in C^1((-\widetilde{T}, \widetilde{T}), D(A))$ for some $\widetilde{T} > 0$ is also a solution to (3.14) and $\widetilde{\varphi}_t|_{t=0} = \varphi_t|_{t=0}$, then $\varphi_t = \widetilde{\varphi}_t$ for all $t \in (-T,T) \cap (-\widetilde{T}, \widetilde{T})$.

(iii) (global existence) Assume in addition that for any solution $\varphi_{(\cdot)}$ of Equation (3.14) with $\varphi_t|_{t=0} \in D(A^n)$ and a $T < \infty$ there exists a constant $C_5 = C_5(T) < \infty$ such that

$$\sup_{t \in [-T,T]} \|\varphi_t\|_{\mathcal{B}} \leq C_5(T) \tag{3.15}$$

then (i) and (ii) holds for any $T \in \mathbb{R}$.

The proof uses the idea of the one in [BD01]. However, here we consider general operators A and J on an abstract Banach space. This generality will later allow us to prove existence and uniqueness of solutions to the Maxwell-Lorentz equations for any N particle system even in the case of infinite energy initial conditions which generalizes the result in [BD01], where only one particle with finite energy initial conditions has been considered. The strategy is to use the Hille-Yosida theorem to show that A generates a contraction group $(W_t)_{t\in\mathbb{R}}$. Definition 3.7 collects all properties needed for A to be a generator. Then, with the help of $(W_t)_{t\in\mathbb{R}}$ we shall show the existence and uniqueness of local solutions to the integral equation

$$\varphi_t = W_t\varphi^0 + \int_0^t W_{t-s}J(\varphi_s)\, ds$$

via Banach's fixed point theorem. All these solutions, as we shall show, are readily also solutions to the Equation (3.14p.21). Global existence is then achieved with the help of the assumed a priori bound.

Abstract contraction group

Lemma 3.10. *A introduced in Definition 3.7p.21 generates a γ-contractive group $(W_t)_{t\in\mathbb{R}}$ on $\mathcal{B}$, i.e. a family of linear operators $(W_t)_{t\in\mathbb{R}}$ on $\mathcal{B}$ with the properties that for all $\varphi\in D(A),\phi\in\mathcal{B}$ and $s,t\in\mathbb{R}$:*

(i) $\lim_{t\to0} W_t\,\phi = \phi$

(ii) $W_{t+s}\phi = W_tW_s\phi$

(iii) $W_t\varphi \in D(A)$

(iv) $AW_t\varphi = W_tA\varphi$

(v) $W_{(\cdot)}\varphi \in C^1(\mathbb{R}, D(A))$

(vi) $\frac{d}{dt}W_t\varphi = AW_t\varphi$

(vii) $\|W_t\phi\|_{\mathcal{B}} \le e^{\gamma|t|}\|\phi\|_{\mathcal{B}}$.

Proof. A simple application of the Hille-Yosida Theorem [HP74]. □

Next we prove local existence and uniqueness of solutions:

Proof of Theorem 3.9p.21

Proof of Theorem 3.9p.21. (i) Since we want to apply Banach's fixed point theorem, we define a Banach space on which we later define our self-mapping. For $T>0$ let

$$X_{T,n} := \Big\{\varphi_{(\cdot)} : [-T,T]\to D(A^n)\ \Big|\ t\mapsto A^j\varphi_t \in C^0((-T,T),D(A^n)) \text{ for } j\le n \\ \text{and } \|\varphi\|_{X_{T,n}} := \sup_{T\in[-T,T]}\sum_{j=0}^n \|A^j\varphi_t\|_{\mathcal{B}} < \infty\Big\}.$$

$(X_{T,n},\|\cdot\|_{X_{T,n}})$ is a Banach space because it is normed and linear by definition, and complete because A on $D(A)$ is closed (see Computation in Appendix 5.1p.97). Note that the mapping $t\mapsto W_t\varphi^0$ is an element of $X_{T,n}$ because for all $t\in\mathbb{R}$ and $j\le n$) we have $t\to A^jW_t\varphi^0 = W_tA^j\varphi^0$ which is continuous and $\|W_tA^j\varphi^0\|_{\mathcal{B}} \le e^{\gamma|t|}\|A^j\varphi^0\|_{\mathcal{B}}$ by Lemma 3.10 and because $\varphi^0\in D(A^n)$. Let

$$M_{T,n,\varphi^0} := \Big\{\varphi_{(\cdot)}\in X_{T,n}\ \Big|\ \varphi_t|_{t=0}=\varphi^0, \|\varphi_{(\cdot)} - W_{(\cdot)}\varphi^0\|_{X_{T,n}} \le 1\Big\},$$

which clearly is a closed subset of $X_{T,n}$. Next we show that

$$S_{\varphi^0} : M_{T,n,\varphi^0}\to M_{T,n,\varphi^0} \qquad \varphi_{(\cdot)}\mapsto S_{\varphi^0}[\varphi_{(\cdot)}] := W_t\varphi^0 + \int_0^t W_{t-s}J(\varphi_s)\,ds \tag{3.16}$$

is a well-defined, contracting self-mapping provided T is chosen sufficiently small. The following estimates are based on the fact that for all $\varphi_{(\cdot)}\in M_{T,n,\varphi^0}$ we have the estimate $\|\varphi_t\|_{\mathcal{B}}\le$

$1 + \|W_t\varphi^0\|_{\mathcal{B}} \leq 1 + e^{\gamma|t|}\|\varphi^0\|_{\mathcal{B}} \leq 1 + e^{\gamma T}\|\varphi^0\|_{\mathcal{B}}$ for each $t \in [-T, T]$. Let also $\widetilde{\varphi}_{(\cdot)} \in M_{T,n,\varphi^0}$, then the properties of J, see Definition 3.8p.21, yield the helpful estimates for all $t \in [-T, T]$:

$$\|A^j J(\varphi_t)\| \leq C_6(T) \qquad \text{and} \qquad \|A^j(J(\varphi_t) - J(\widetilde{\varphi}_t))\| \leq C_7(T)\|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{B}}. \tag{3.17}$$

for

$$\begin{aligned} C_6(T) &:= C_3{}^{(j)}(\|\varphi_t\|_{\mathcal{B}}) \leq C_3{}^{(j)}(1 + e^{\gamma|t|}\|\varphi^0\|_{\mathcal{B}}) \text{ and} \\ C_7(T) &:= C_4{}^{(j)}(1 + e^{\gamma|t|}\|\varphi^0\|_{\mathcal{B}}, 1 + e^{\gamma|t|}\|\varphi^0\|_{\mathcal{B}}). \end{aligned} \tag{3.18}$$

Hence, $C_6(T), C_7(T)$ depend continuously and non-decreasingly on T.

We show now that S_{φ^0} is a self-mapping. Since $t \mapsto W_t\varphi^0$ is in M_{T,n,φ^0}, it suffices to show that the mapping $t \mapsto A^j \int_0^t W_{t-s}J(\varphi_s)\, ds$ is $D(A^{n-j})$ valued, continuous and that its $\|\cdot\|_{X_{T,n}}$ norm is finite for $j \leq n$. Consider $\varphi_{(\cdot)} \in M_{T,n,\varphi^0}$, so for some $h > 0$ we get

$$\begin{aligned} &\|A^j W_{t-(s+h)}J(\varphi_{s+h}) - A^j W_{t-s}J(\varphi_s)\|_{\mathcal{B}} \\ &\leq e^{\gamma|t-(s+h)|}\|A^j(J(\varphi_{s+h}) - J(\varphi_s))\|_{\mathcal{B}} + e^{\gamma|t-s|}\|(1 - W_h)A^j J(\varphi_s)\|_{\mathcal{B}} \\ &\leq e^{\gamma|t-(s+h)|}C_7\|\varphi_{s+h} - \varphi_s\|_{\mathcal{B}} + e^{\gamma|t-s|}\|(1 - W_h)A^j J(\varphi_s)\|_{\mathcal{B}} \xrightarrow[h\to 0]{} 0 \end{aligned}$$

by continuity of $t \to \varphi_t$, estimate (3.17) and properties of $(W_t)_{t\in\mathbb{R}}$. We may thus define

$$\sigma^{(j)}(t) := \int_0^t A^j W_{t-s}J(\varphi_s)\, ds$$

as $\mathcal{B}$ valued Riemann integrals. Let $\sigma_N^{(j)}(t) := \frac{t}{N}\sum_{k=1}^N A^j W_{t-\frac{t}{N}k}J(\varphi_{\frac{t}{N}k})$ be the corresponding Riemann sums. Clearly $\sigma_N^j(t) \in D(A^{n-j})$ since $J : D(A) \to D(A_J^n)$ and $\lim_{N\to\infty} A^j\sigma_N(t) = \sigma^{(j)}(t)$ for all $t \in \mathbb{R}$ and $j \leq n$. But A is closed which implies $\sigma^0(t) \in D(A^n)$ and $\sigma^j(t) = A^j\sigma^0(t)$. Next we show continuity. With estimate (3.17) we get for $t \in (-T, T)$:

$$\begin{aligned} &\|A^j\sigma(t + h) - A^j\sigma(t)\|_{\mathcal{B}} = \|\sigma^j(t + h) - \sigma^j(t)\|_{\mathcal{B}} \\ &\leq \int_t^{t+h} \left\|W_{t+h-s}A^j J(\varphi_s)\right\|_{\mathcal{B}} ds + \int_0^t \left\|W_{t-s}(W_h - 1)A^j J(\varphi_s)\right\|_{\mathcal{B}} ds \\ &\leq e^{\gamma|h|}\int_t^{t+h} \left\|A^j J(\varphi_s)\right\|_{\mathcal{B}} + e^{\gamma T}\int_0^t \left\|(W_h - 1)A^j J(\varphi_s)\right\|_{\mathcal{B}} ds \end{aligned}$$

For $h \to 0$ the right-hand side goes to zero as the integrand of the second summand $\|(W_h - 1)A^j J(\varphi_s)\|_{\mathcal{B}}$ does, which by (3.17) is also bounded by $(1 + e^{\gamma T})C_6(T)$ so that dominated convergence can be used. The self-mapping property is ensured by (3.17):

$$\begin{aligned} &\|S_{\varphi^0}[\varphi_{(\cdot)}] - W_{(\cdot)}\varphi^0\|_{X_{T,n}} = \sup_{t\in[-T,T]}\sum_{j=0}^n \left\|A^j\int_0^t W_{t-s}J(\varphi_s)\, ds\right\|_{\mathcal{B}} \\ &\leq e^{\gamma T}\sup_{t\in[-T,T]}\sum_{j=0}^n\int_0^t \|A^j J(\varphi_s)\|_{\mathcal{B}}\, ds \leq Te^{\gamma T}C_6(T)(n + 1). \end{aligned}$$

On the other hand for some $\widetilde{\varphi}_{(\cdot)} \in M_{T,n,\varphi^0}$ we find

$$\begin{aligned} &\|S_{\varphi^0}[\varphi_{(\cdot)}] - S_{\varphi^0}[\widetilde{\varphi}_{(\cdot)}]\|_{X_{T,n}} = \sup_{t\in[-T,T]}\sum_{j=0}^n \|A^j\int_0^t W_{t-s}[J(\varphi_s) - J(\widetilde{\varphi}_s)]\, ds\|_{\mathcal{B}} \\ &\leq e^{\gamma T}\sup_{t\in[-T,T]}\sum_{j=0}^n\int_0^t \|A^j[J(\varphi_s) - J(\widetilde{\varphi}_s)]\|_{\mathcal{B}}\, ds \leq Te^{\gamma T}C_7(T)\|\varphi_{(\cdot)} - \widetilde{\varphi}_{(\cdot)}\|_{X_{T,n}}. \end{aligned}$$

Since $T \mapsto C_6(T)$ and $T \mapsto C_7(T)$ are continuous and non-decreasing, there exists a $T > 0$ such that

$$Te^{\gamma T}[C_6(T)(n+1) + C_7(T)] < 1. \tag{3.19}$$

Thus, for this choice of T, S_{φ^0} is a contracting self-mapping on the closed set M_{T,n,φ^0} so that due to Banach's fixed point theorem S_{φ^0} has a unique fixed point $\varphi_{(\cdot)} \in M_{T,\varphi^0}$.

Next we study the differentiability of this fixed point, in particular of $t \to A^j\varphi_t$ on $(-T, T)$ for $j \le (n-1)$. As $\varphi_{(\cdot)} = S_{\varphi^0}[\varphi_{(\cdot)}]$, Definition (3.16p.22), and $\varphi^0 \in D(A^n)$ we have

$$\frac{A^j\varphi_{t+h} - A^j\varphi_t}{h} = \frac{W_{t+h} - W_t}{h} A^j\varphi^0 + \frac{\sigma^j(t+h) - \sigma^j(t)}{h} =: \boxed{1} + \boxed{2}.$$

By the properties of $(W_t)_{t\in\mathbb{R}}$ we know $\lim_{h\to 0} \boxed{1} = A^{j+1}\varphi^0$. Furthermore,

$$\boxed{2} = \frac{1}{h}\int_t^{t+h} W_{t+h-s}A^jJ(\varphi_s)\,ds + \int_0^t W_{t-s}\frac{W_h - 1}{h}A^jJ(\varphi_s)\,ds$$

For $h \to 0$ the first term on the right-hand side converges to $A^jJ(\varphi_t)$ because of

$$\frac{1}{h}\left\|\int_t^{t+h} W_{t+h-s}A^jJ(\varphi_s)\,ds - A^jJ(\varphi_t)\right\|_{\mathcal{B}} = \sup_{s\in(t,t+h)} \left\|W_{t+h-s}A^jJ(\varphi_s) - A^jJ(\varphi_t)\right\|_{\mathcal{B}}$$

and the continuity of $W_{t+h-s}A^jJ(\varphi_s)$ in h and s. For $h \to 0$ the second term converges to $\int_0^t W_{t-s}A^{j+1}J(\varphi_s)\,ds$ by dominated convergence as the integrand converges to $W_{t-s}A^{j+1}J(\varphi_s)$, and the following gives a convenient bound of it:

$$\left\|W_{t-s}\frac{W_h - 1}{h}A^jJ(\varphi_s)\right\|_{\mathcal{B}} = \left\|\frac{1}{h}\int_0^h W_{t-s}W_{h'}A^{j+1}J(\varphi_s)\,dh'\right\|_{\mathcal{B}} \le e^{\gamma(T+1)}\|A^{j+1}J(\varphi_s)\|_{\mathcal{B}}.$$

Collecting all terms, we have shown that

$$\frac{d}{dt}A^j\varphi_t = A^jW_t\varphi^0 + A^jJ(\varphi_t) + A^{j+1}\int_0^t W_{t-s}J(\varphi_s)\,ds = A^{j+1}\varphi_t + A^jJ(\varphi_t).$$

Note that the right-hand side is continuous because $j \le (n-1)$, $\varphi_{(\cdot)} \in M_{T,n,\varphi^0}$ and (3.17p.23). Hence $A^j\varphi_{(\cdot)} \in C^1((-T,T), D(A^{n-j}))$ and $\frac{d}{dt}A^j\varphi_t \in D(A^{n-j-1})$ for all $t \in (-T,T)$. Next we prove for $t \in (-T,T)$ and $k \le n$ that $\varphi_{(\cdot)} \in C^n((-T,T), D(A^n))$, $\frac{d^k}{dt^k}\varphi_t \in D(A^{n-k})$ by induction. We claim that

$$\frac{d^k}{dt^k}\varphi_t = A^k\varphi_t + \sum_{l=0}^{k-1}\frac{d^{k-1-l}}{dt^{k-1-l}}A^lJ(\varphi_t)$$

holds, is continuous in t on $(-T,T)$ and in $D(A^{n-k})$. We have shown before that this holds for $k = 0$. Assume it is true for some $(k-1) \le n-1$. We compute

$$\frac{d}{dt}\frac{d^{k-1}}{dt^{k-1}}\varphi_t = A^k\varphi_t + A^{k-1}J(\varphi_t) + \sum_{l=0}^{k-2}\frac{d^{k-1-l}}{dt^{k-1-l}}A^lJ(\varphi_t) = A^k\varphi_t + \sum_{l=0}^{k-1}\frac{d^{k-1-l}}{dt^{k-1-l}}A^lJ(\varphi_t).$$

The first term on the right-hand side is continuous in t on $(-T,T)$ and in $D(A^{n-k})$ as shown before. Now Definition (3.8p.21)(iip.21), where we have defined the operator J, was chosen to guarantee that these properties hold also for the second term.

(ii) Clearly, $\varphi_{(\cdot)}$ and $\widetilde{\varphi}_{(\cdot)}$ are both in $X_{T_1,1}$ for any $0 < T_1 \le \min(T, \widetilde{T})$ because they are at least once continuously differentiable. Since $\varphi_t|_{t=0} = \widetilde{\varphi}_t|_{t=0}$ holds, we can choose $T_1 > 0$ sufficiently small such that $\varphi_{(\cdot)}$ and $\widetilde{\varphi}_{(\cdot)}$ are also in $M_{T_1,1,\varphi^0}$ and in addition that S_{φ^0} is a contracting self-mapping on $M_{T_1,1,\varphi^0}$. As in (i) we infer that there exists a unique fixed point $\varphi^1_{(\cdot)} \in M_{T_1,1,\varphi^0}$ of S_{φ^0} which solves (3.14p.21). Since $\varphi_{(\cdot)}$ and $\widetilde{\varphi}_{(\cdot)}$ also solve (3.14p.21), it must hold that $\varphi_t = \varphi^1_t = \widetilde{\varphi}_t$ on $[-T_1, T_1]$. Let $\overline{T}$ be the supremum of all those T_1 and let us assume that $\overline{T} < \min(T, \widetilde{T})$. We can repeat the above argument with e.g. initial values $\varphi_t|_{t=T_1} = \widetilde{\varphi}_t|_{t=T_1}$ at time $t = T_1$. Again we find a $T_2 > 0$ and a fixed point $\varphi^2_{(\cdot)} \in M_{T_2,1,\varphi_{\overline{T}}}$ of $S_{\varphi_{\overline{T}}}$ so that $\varphi_t = \varphi^2_{t-\overline{T}} = \widetilde{\varphi}_t$ on $[\overline{T} - T_2, \overline{T} - T_2]$. The same can be done for initial values $\varphi_t|_{t=-T_1} = \widetilde{\varphi}_t|_{t=-T_1}$ at time $t = -T_1$. This yields $\varphi_t = \widetilde{\varphi}_t$ for $t \in [\overline{T} - T_2, \overline{T} + T_2]$ and contradicts the maximality of $\overline{T}$. Hence, $\varphi_{(\cdot)}$ equals $\widetilde{\varphi}_{(\cdot)}$ on $[-T, T] \cap [-\widetilde{T}, \widetilde{T}]$.

(iii) Fix any $\widetilde{T} > 0$. The a priori bound (3.15p.21) tells us that if any solution $\varphi : (-\widetilde{T}, \widetilde{T}) \to D(A^n)$ with $\varphi_t|_{t=0} = \varphi^0 \in D(A^n)$ exists, then $\sup_{t\in[-\widetilde{T},\widetilde{T}]} \|\varphi_t\|_{\mathcal{B}} \leq C_5(\widetilde{T}) < \infty$. By looking at equations (3.19p.24) and (3.18p.23) we infer that there exists a $T_{min} > 0$ such that for each $t \in [-\widetilde{T}, \widetilde{T}]$ the time span T for which S_{φ_t} on M_{T,n,φ_t} fulfills $T_{min} \leq T$. Let $\varphi_{(\cdot)}$ be the fixed point of S_{φ^0} on M_{T_1,n,φ^0} for $T_1 > 0$, and let $\overline{T}$ be the supremum of such T_1. Assume $\overline{T} < \widetilde{T}$. By taking an initial value $\varphi_{\pm(\overline{T}-\epsilon)}$ for $0 < \epsilon < T_{min}$ near to the boundary, (i) and (ii) extends the solution beyond $(-\overline{T}, \overline{T})$ and contradicts the maximality of $\overline{T}$. □

REMARK 3.11. *Definition 3.8p.21(iip.21) is only needed if one aims at two or more times differentiable solutions.*

3.4 The Spaces of Weighted Square Integrable Functions

In this subsections we collect all needed properties of the introduced weighted spaces. The following assertions, except Theorem 3.21p.29, are independent of the space dimension. That is why we often use the abbreviation $L^2_w = L^2_w(\mathbb{R}^3, \mathbb{R}^3)$ and $C^\infty_c = C^\infty_c(\mathbb{R}^3, \mathbb{R}^3)$. With $w \in \mathcal{W}$ the L^2_w analogues of almost all results of the L^2 theory which do not involve the Fourier transform can be proven with only minor modifications. For open $\Omega \subseteq \mathbb{R}^3$, $L^2_w(\Omega, \mathbb{R}^3)$ is clearly a linear space and has an inner product:

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle_{L^2_w} := \int_\Omega d^3x\, w(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{f}, \boldsymbol{g} \in L^2_w(\Omega, \mathbb{R}^3). \tag{3.20}$$

Theorem 3.12. *For any $w \in \mathcal{W}$, open $\Omega \subseteq \mathbb{R}^3$, $L^2_w(\Omega, \mathbb{R}^3)$ with (3.20) is a Hilbert space and $C^\infty_c(\Omega, \mathbb{R}^3)$ lies dense.* Properties of L^2_w

Proof. (see Proof in Appendix 5.2p.97) This is a standard result as $\sqrt{w}d^3x$ is an absolute continuous measure with respect to the Lebesgue measure d^3x on $\mathbb{R}^3$. □

Note as in any Hilbert space the Schwarz inequality holds, i.e. for all $\boldsymbol{f}, \boldsymbol{g} \in L^2_w$ we have $|\langle \boldsymbol{f}, \boldsymbol{g}\rangle_{L^2_w}| = \left|\left\langle \sqrt{w}\boldsymbol{f}, \sqrt{w}\boldsymbol{g}\right\rangle_{L^2}\right| \leq \|\sqrt{w}\boldsymbol{f}\|_{L^2}\|\sqrt{w}\boldsymbol{g}\|_{L^2} = \|\boldsymbol{f}\|_{L^2_w}\|\boldsymbol{g}\|_{L^2_w}$. We shall also need:

Definition 3.13. *For all $w \in \mathcal{W}$, $\Omega \subseteq \mathbb{R}^3$ and $k \geq 0$ we define* Weighted Sobolev spaces

$$H^k_w(\Omega, \mathbb{R}^3) := \left\{\boldsymbol{f} \in L^2_w(\Omega, \mathbb{R}^3) \,\middle|\, D^\alpha \boldsymbol{f} \in L^2_w(\Omega, \mathbb{R}^3), |\alpha| \leq k\right\},$$
$$H^{\triangle^k}_w(\Omega, \mathbb{R}^3) := \left\{\boldsymbol{f} \in L^2_w(\Omega, \mathbb{R}^3) \,\middle|\, \triangle^j \boldsymbol{f} \in L^2_w(\Omega, \mathbb{R}^3) \text{ for } 0 \leq j \leq k\right\},$$
$$H^{curl}_w(\Omega, \mathbb{R}^3) := \left\{\boldsymbol{f} \in L^2_w(\Omega, \mathbb{R}^3) \,\middle|\, \nabla \wedge \boldsymbol{f} \in L^2_w(\Omega, \mathbb{R}^3)\right\}$$

which are equipped with the inner products

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle_{H^k_w} := \sum_{|\alpha|\le k} \langle D^\alpha \boldsymbol{f}, D^\alpha \boldsymbol{g}\rangle_{L^2_w(\Omega)}, \qquad \langle \boldsymbol{f},\boldsymbol{g}\rangle_{H^\triangle_w(\Omega)} := \sum_{j=0}^{k} \left\langle \triangle^j \boldsymbol{f}, \triangle^j \boldsymbol{g}\right\rangle_{L^2_w(\Omega)}$$

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle_{H^{curl}_w(\Omega)} := \langle \boldsymbol{f},\boldsymbol{g}\rangle_{L^2_w(\Omega)} + \langle \nabla\wedge \boldsymbol{f}, \nabla\wedge\boldsymbol{g}\rangle_{L^2_w(\Omega)},$$

respectively. In the following we use a superscript #, e.g. $H^\#_w$*, as a placeholder for k,* $\triangle^k$*, and curl for any* $k\in\mathbb{N}$*, respectively. We shall also need the local versions, i.e.* $H^\#_{loc} := \{\boldsymbol{f}\in L^2_{loc} \mid \boldsymbol{f}\in H^\#(K)$*, for every open* $K \subset\subset \mathbb{R}^3\}$*, where* $\subset\subset$ *is short for compactly contained. Usually we abbreviate* $H^k_w = H^k_w(\mathbb{R}^3,\mathbb{R}^3)$*, write* $L^2_w = H^0_w$ *and drop the subscript w if* $w = 1$*.*

$H^\#_w$ are Hilbert spaces

Theorem 3.14. *For any* $w\in\mathcal{W}$*,* $H^\#_w$ *are all Hilbert spaces.*

Proof. The proof is the same for all three cases. We only show it for the case of H^k_w, $k\in\mathbb{N}$: $\mathbb{R}$-Linearity, the inner product and the norm are inherited from L^2_w and the definition of the weak derivative. Let $\left(\boldsymbol{f}^0_n\right)_{n\in\mathbb{N}}$ be a Cauchy sequence in H^k_w, then $\left(D^\alpha \boldsymbol{f}^0_n\right)_{n\in\mathbb{N}}$ are Cauchy sequences in L^2_w for every $\alpha\le k$. By completeness of L^2_w there exist $\boldsymbol{f}^\alpha \in L^2_w$ such that $\|\boldsymbol{f}^\alpha - \boldsymbol{f}^\alpha_n\|_{L^2_w} \to 0$ for $n\to\infty$. Now for all $\varphi\in C^\infty_c(\mathbb{R}^3,\mathbb{R})$ we have

$$\begin{aligned} D^\alpha \boldsymbol{f}^0[\varphi] :&= (-1)^{|\alpha|}\int d^3x\, \boldsymbol{f}^0(\boldsymbol{x}) D^\alpha\varphi(\boldsymbol{x}) = (-1)^{|\alpha|}\lim_{n\to\infty}\int d^3x\, \boldsymbol{f}^0_n(\boldsymbol{x}) D^\alpha\varphi(\boldsymbol{x}) \\ &= \lim_{n\to\infty}\int d^3x\, D^\alpha \boldsymbol{f}^0_n(\boldsymbol{x})\varphi(\boldsymbol{x}) = \int D^\alpha \boldsymbol{f}^\alpha_n(\boldsymbol{x})\varphi(\boldsymbol{x}) =: \boldsymbol{f}^\alpha[\varphi] \end{aligned}$$

as the L^2_w convergence implies the convergence in distribution sense. Hence, $\boldsymbol{f}^\alpha_n = D^\alpha \boldsymbol{f}^0_n$ almost everywhere on $\mathbb{R}^3$. □

Relation between $H^\#_w$ and $H^\#_{loc}$

Lemma 3.15. *Let* $w\in\mathcal{W}$*, then: (i) For an open* $O\subset\subset\mathbb{R}^3$*, i.e. a compactly contained subset of* $\mathbb{R}^3$*, one has* $H^\#_w(O) = H^\#(O)$ *in the sense of normed spaces. (ii) A function* $\boldsymbol{f}$ *is in* $H^\#_w(K)$ *for every open* $K\subset\subset\mathbb{R}^3$ *if and only if* $\boldsymbol{f}\in H^\#_{loc}$*.*

Proof. (i) Given $w\in\mathcal{W}$ Equation (3.7p.19) ensures that there are two finite and non-zero constants $0 < C_8 := \inf_{\boldsymbol{x}\in O} w(\boldsymbol{x})$, $C_9 := \sup_{\boldsymbol{x}\in O} w(\boldsymbol{x}) < \infty$. Thus, we get $C_8\|\boldsymbol{f}\|_{H^\#(O)} \le \|\boldsymbol{f}\|_{H^\#_w(O)} \le C_9\|\boldsymbol{f}\|_{H^\#(O)}$. This yields also (ii). □

Properties of weights in $\mathcal{W}^k$

Lemma 3.16. *Let* $w\in\mathcal{W}^k$*, then for every multi-index* $|\alpha|\le k$ *there also exists constants* $0\le C^\alpha<\infty$ *such that* $|D^\alpha w|\le C^\alpha w$ *on* $\mathbb{R}^3$*.*

Proof. We compute

$$\begin{aligned} D^\alpha w &= D^\alpha \sqrt{w}\sqrt{w} = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}(\sqrt{w}\sqrt{w}) = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\sum_{l_3=0}^{\alpha_3}\binom{\alpha_3}{l_3}(\partial_3^{\alpha_3-l_3}\sqrt{w})(\partial_3^{l_3}\sqrt{w}) \\ &= \sum_{l_1,l_2,l_3=0}^{\alpha_1,\alpha_2,\alpha_3}\binom{\alpha_1}{l_1}\binom{\alpha_2}{l_2}\binom{\alpha_3}{l_3}(D^{(\alpha_1-l_1,\alpha_2-l_2,\alpha_3-l_3)}\sqrt{w})(D^{(l_1,l_2,l_3)}\sqrt{w}). \end{aligned}$$

Using $|D^\alpha\sqrt{w}| \le \sqrt{w}$ we find $|D^\alpha| \le C^\alpha w$ for

$$C^\alpha := \sum_{l_1,l_2,l_3=0}^{\alpha_1,\alpha_2,\alpha_3}\binom{\alpha_1}{l_1}\binom{\alpha_2}{l_2}\binom{\alpha_3}{l_3} C_\alpha|_{\alpha=(\alpha_1-l_1,\alpha_2-l_2,\alpha_3-l_3)} C_\alpha|_{\alpha=(l_1,l_2,l_3)}.$$

□

Lemma 3.17. *Let $k \in \mathbb{N}$ and $w \in \mathcal{W}^k$, then in the sense of sets $\sqrt{w}H_w^\# = H^\#$.*

Set equivalence $\sqrt{w}H_w^\# = H^\#$

Proof. The proof is the same for all three spaces. We only show it for the case of H_w^k: Recall (3.7p.19) from which we deduce $w \leq w(0)$ and therefore $f \in H^k \Rightarrow \frac{f}{\sqrt{w}} \in H_w^k \Rightarrow f \in \sqrt{H}_w^k$. On the other hand, let $f \in H_w^k$ then

$$\begin{aligned}
\|\sqrt{w}\boldsymbol{f}\|_{H^n}^2 &:= \sum_{|\alpha|\leq n} \|D^\alpha(\sqrt{w}\boldsymbol{f})\|_{L^2}^2 = \sum_{|\alpha|\leq k} \|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}(\sqrt{w}\boldsymbol{f})\|_{L^2}^2 \\
&= \sum_{|\alpha|\leq k} \|\partial_1^{\alpha_1}\partial_2^{\alpha_2} \sum_{l_3=0}^{\alpha_3} \binom{\alpha_3}{l_3} (\partial_3^{\alpha_3-l_3}\sqrt{w})(\partial_3^{l_3}\boldsymbol{f})\|_{L^2}^2 \\
&\leq \sum_{|\alpha|\leq n} 2^{|\alpha|} \sum_{l_1,l_2,l_3=0}^{\alpha_1,\alpha_2,\alpha_3} \binom{\alpha_1}{l_1}^2 \binom{\alpha_2}{l_2}^2 \binom{\alpha_3}{l_3}^2 \left\| \left|\partial_1^{\alpha_1-l_1}\partial_2^{\alpha_2-l_2}\partial_3^{\alpha_3-l_3}\sqrt{w}\right| \partial_1^{l_1}\partial_2^{l_2}\partial_3^{l_3}\boldsymbol{f}\right\|_{L^2}^2.
\end{aligned}$$

But as $w \in \mathcal{W}^k$, there is some finite constant C_{10} such that $\left|\partial_1^{\alpha_1-l_1}\partial_2^{\alpha_2-l_2}\partial_3^{\alpha_3-l_3}\sqrt{w}\right| \leq C_{10}w$ and, hence,

$$\begin{aligned}
\ldots &\leq \sum_{|\alpha|\leq k} 2^{|\alpha|} \sum_{l_1,l_2,l_3=0}^{\alpha_1,\alpha_2,\alpha_3} \binom{\alpha_1}{l_1}^2 \binom{\alpha_2}{l_2}^2 \binom{\alpha_3}{l_3}^2 C_{10} \left\|\partial_1^{l_1}\partial_2^{l_2}\partial_3^{l_3}\boldsymbol{g}\right\|_{L_w^2}^2 \\
&\leq C_{11}\|\boldsymbol{f}\|_{H_w^n}^2 < \infty
\end{aligned}$$

for

$$C_{11} := \sum_{|\alpha|\leq k} 2^{|\alpha|} \sum_{l_1,l_2,l_3=0}^{\alpha_1,\alpha_2,\alpha_3} \binom{\alpha_1}{l_1}^2 \binom{\alpha_2}{l_2}^2 \binom{\alpha_3}{l_3}^2 C_{10}.$$

This implies $\sqrt{w}H_w^k \ni \sqrt{w}\boldsymbol{f} \in H^k$. □

Theorem 3.18. *Let $k \in \mathbb{N}$ and $w \in \mathcal{W}^k$, then C_c^∞ is a dense subset of $H_w^\#$.*

C_c^∞ dense in $H_w^\#$

Proof. For all three cases the proof is essentially the same. Only the case of $H_w^\triangle$ is a bit more involved as one needs to estimate the derivatives $\partial_i\partial_j$, $1 \leq i, j \leq 3$, in terms of the Laplacian. Therefore, we only prove the latter case. Let $\boldsymbol{f} \in H_w^\triangle$, then we need to show that for every $\epsilon > 0$ there is a $\boldsymbol{g} \in C_c^\infty$ such that $\|\boldsymbol{f} - \boldsymbol{g}\|_{H_w^\triangle} < \epsilon$. Take a $\varphi \in C_c^\infty(\mathbb{R}^3, [0,1])$ such that $\varphi(\boldsymbol{x}) = 1$ for $\|\boldsymbol{x}\| \leq 1$. Define $\varphi_n(\boldsymbol{x}) := \varphi\left(\frac{\boldsymbol{x}}{n}\right)$. We get

$$\|\boldsymbol{f} - \boldsymbol{f}\varphi_n\|_{L_w^2}^2 \leq \int_{\mathbb{R}^3\setminus B_n(0)} d^3x\, w(\boldsymbol{x})\|f(\boldsymbol{x})\|^2 \xrightarrow[n\to\infty]{} 0 \tag{3.21}$$

and

$$\|\triangle\boldsymbol{f} - \triangle(\boldsymbol{f}\,\varphi_n)\|_{L_w^2}^2 \leq \|\triangle\boldsymbol{f} - \varphi_n\,\triangle\boldsymbol{f}\|_{L_w^2} + \frac{1}{n^2}\|\triangle\varphi_n\,\boldsymbol{f}\|_{L_w^2} + \frac{1}{n}\left\|\sum_{i=1}^3 \partial_i\varphi_n\,\partial_i\boldsymbol{f}\right\|_{L_w^2}. \tag{3.22}$$

With $C_{12} := \sup_{n\in\mathbb{N},\boldsymbol{x}\in\mathbb{R}^3} \sum_{|\alpha|\leq 3} \left|D^\alpha\varphi\left(\frac{\boldsymbol{x}}{n}\right)\right|$, which is finite, we have $\|\triangle\varphi_n\,\boldsymbol{f}\|_{L_w^2} \leq C_{12}\|\boldsymbol{f}\|_{L_w^2}$ and the first two terms on the right-hand side go to zero for $n \to \infty$. On the other hand, on compact sets $K \subset \mathbb{R}^3$ we have $H_w^\triangle(K,\mathbb{R}^3) = H^\triangle(K,\mathbb{R}^3)$ by Lemma 3.17 and therefore $\boldsymbol{f} \in H_{loc}^\triangle$. Thus, we can apply partial integration and, using the abbreviation $\omega = w\sum_{i=1}^3(\partial_i\varphi_n)^2$, yield

$$\left\|\sum_{i=1}^3 \partial_i\varphi_n\,\partial_i\boldsymbol{f}\right\|_{L_w^2}^2 \leq \sum_{i=1}^3 \int d^3x\,\omega(\boldsymbol{x})(\partial_i\boldsymbol{f}(\boldsymbol{x}))^2 \leq \left|\sum_{i=1}^3 \int d^3x\,\left[\partial_i\omega\,\partial_i\boldsymbol{f}\boldsymbol{f} + \omega\,\partial_i^2\boldsymbol{f}\boldsymbol{f}\right](\boldsymbol{x})\right|.$$

In the first terms on the right-hand side we apply the chain rule $f\,\partial_i f = \frac{1}{2}\partial_i f^2$ and integrate by parts again so that

$$\dots = \left|\sum_{i=1}^{3}\int d^3x\,\left[\frac{1}{2}\partial_i^2\omega\,f^2 + \omega\,\partial_i^2 f\,f\right](x)\right| \le \frac{1}{2}\sum_{i=1}^{3}\left\|\sqrt{|\partial_i^2\omega|}\,f\right\|_{L^2}^2 + \left|\left\langle\sqrt{|\omega|}\,\triangle f,\,\sqrt{|\omega|}\,f\right\rangle_{L^2}\right|$$

By definition of $\mathcal{W}^k$ and Lemma 3.16p.26 we have $|D^\alpha w| \le C^\alpha w$ on $\mathbb{R}^3$. Let $C_{13} := \sum_{|\alpha|\le 2} C^\alpha$, then $|\partial_i^2\omega| \le 27C_{12}{}^2C_{13}w$ and $|\omega| \le 3C_{12}{}^2w$ uniformly in n on $\mathbb{R}^3$. We get

$$\dots \le \frac{81}{2}C_{12}{}^2C_{13}\|f\|_{L_w^2}^2 + 3C_{12}\left|\langle\triangle f, f\rangle_{L_w^2}\right|$$

and finally using Schwarz's inequality

$$\left\|\sum_{i=1}^{3}\partial_i\varphi_n\,\partial_i f\right\|_{L_w^2}^2 \le C_{12}{}^2\left(\frac{81}{2}C_{13} + 3\right)\|f\|_{H_w^\triangle}^2$$

uniformly in n. Going back to Equation (3.22p.27) we then find that also the last term on the right-hand side goes to zero as $n \to \infty$. Combining equations (3.21p.27) and (3.22p.27) we conclude that there is an $h \in H_w^\triangle$ with compact support and $\|f - h\|_{H_w^\triangle} \le \frac{\epsilon}{2}$. Now let $\psi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ and define $\psi_n(x) := n^3\psi(nx)$. It is a standard analysis argument that $\|h - h * \psi_n\|_{H^2} \to 0$ for $n \to \infty$ so that for n large enough $\|h - h * \psi_n\|_{H_w^\triangle} < \frac{\epsilon}{2}$. Since h and ψ_n have compact support $g := h * \psi_n \in C_c^\infty$. With that $\|f - g\|_{H_w^\triangle} \le \|f - h\|_{H_w^\triangle} + \|h - h * \psi_n\|_{H_w^\triangle} < \epsilon$ which concludes the proof. □

REMARK 3.19. *By the standard approximation argument this theorem allows us to make use of partial integration in the spaces $H_w^\#$ with respect to the appropriate differential operators.*

Hilbert space equivalence $H_w^{\triangle^k} = H_w^{2k}$

Theorem 3.20. *Let $w \in \mathcal{W}^2$, then for any $k \in \mathbb{N}$ in the sense of Hilbert spaces $H_w^{\triangle^k} = H_w^{2k}$.*

Proof. First we prove $H_w^\triangle = H_w^2$. Now $f \in H_w^2$ implies $\|f\|_{H_w^\triangle} \le \|f\|_{H_w^2}$ and therefore $f \in H_w^\triangle$. Next let $g \in C_c^\infty$. By definition

$$\|g\|_{H_w^\triangle}^2 = \|f\|_{L_w^2}^2 + \sum_{i=1}^{3}\|\partial_i g\|_{L_w^2}^2 + \sum_{i,j=1}^{3}\|\partial_i\partial_j g\|_{L_w^2}^2. \tag{3.23}$$

Using partial integration in the second term on the right-hand side we obtain

$$\sum_{i=1}^{3}\|\partial_i g\|_{L_w^2}^2 = \sum_{i=1}^{3}\int d^3x\,\left[w\,(\partial_i g)^2\right](x) = -\sum_{i=1}^{3}\int d^3x\,\left[\partial_i w\,\partial_i g\,g + w\,\partial_i^2 g\,g\right].$$

The chain rule $\partial_i g\,g = \frac{1}{2}\partial_i(g)^2$ and another partial integration in the first term on the right-hand side yields

$$\dots = \sum_{i=1}^{3}\int d^3x\,\left[\frac{1}{2}\partial_i^2 w\,(g)^2 - w\,\partial_i^2 g\,g\right] \le C_{13}\left(\frac{3}{2}\|g\|_{L_w^2}^2 + \left|\langle\triangle g, g\rangle_{L_w^2}\right|\right)$$

for the finite constant C_{13} defined in the proof of Theorem 3.18p.27. By Schwarz's inequality

$$\sum_{i=1}^{3}\|\partial_i g\|_{L_w^2}^2 \le \frac{5C_{13}}{2}\|g\|_{H_w^\triangle}. \tag{3.24}$$

Next we estimate the last term in Equation (3.23). By partial integration we get

$$\sum_{i,j=1}^{3} \|\partial_i \partial_j \boldsymbol{g}\|^2_{L^2_w} = \sum_{i,j=1}^{3} \int d^3x \left[w\, (\partial_i \partial_j \boldsymbol{g})^2 \right](\boldsymbol{x})$$
$$= -\sum_{i,j=1}^{3} \int d^3x \left[\partial_j w\, \partial_i \partial_j \boldsymbol{g}\, \partial_i \boldsymbol{g} + w\, \partial_i \partial_j^2 \boldsymbol{g}\, \partial_i \boldsymbol{g} \right](\boldsymbol{x})$$
$$= \sum_{i,j=1}^{3} \int d^3x \left[\partial_i \partial_j w\, \partial_j \boldsymbol{g}\, \partial_i \boldsymbol{g} + \partial_i w\, \partial_j^2 \boldsymbol{g}\, \partial_i \boldsymbol{g} + \partial_j w\, \partial_j \boldsymbol{g}\, \partial_i^2 \boldsymbol{g} + w\, \partial_j^2 \boldsymbol{g}\, \partial_i^2 \boldsymbol{g} \right](\boldsymbol{x})$$
$$\leq C_{13} \left[\sum_{i,j=1}^{3} \left| \left\langle \partial_j \boldsymbol{g}, \partial_i \boldsymbol{g} \right\rangle_{L^2_w} \right| + \sum_{i=1}^{3} \left| \langle \triangle \boldsymbol{g}, \partial_i \boldsymbol{g} \rangle_{L^2_w} \right| + \sum_{j=1}^{3} \left| \left\langle \partial_j \boldsymbol{g}, \triangle \boldsymbol{g} \right\rangle_{L^2_w} \right| + \|\triangle \boldsymbol{g}\|^2_{L^2_w} \right].$$

Now Schwarz's inequality and the estimate (3.24p.28) gives the estimate

$$\sum_{i,j=1}^{3} \|\partial_i \partial_j \boldsymbol{g}\|^2_{L^2_w} \leq 14 C_{13} \|\boldsymbol{g}\|^2_{H^\triangle_w}.$$

and therewith

$$\|\boldsymbol{g}\|_{H^2_w} \leq \sqrt{14 C_{13}} \|\boldsymbol{g}\|_{H^\triangle_w}. \tag{3.25}$$

Let now $\boldsymbol{f} \in H^\triangle_w$. According to Theorem 3.18p.27, there is a sequence $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$ in C^∞_c such that $\|\boldsymbol{f} - \boldsymbol{f}_n\|_{H^\triangle_w} \to 0$ as $n \to \infty$. Estimate (3.25) implies that $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in H^2_w. Thus, subject to Theorem 3.14p.26, there is a $\boldsymbol{h} \in H^2_w$ such that for $|\alpha| \leq 2$, $\|D^\alpha \boldsymbol{f}_n - D^\alpha \boldsymbol{h}\|_{L^2_w} \to 0$ as $n \to \infty$. For $\varphi \in C^\infty(\mathbb{R}^3, \mathbb{R})$ we find

$$\boldsymbol{f}[\varphi] := \int d^3x\, \varphi(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}) = \lim_{n\to\infty} \int d^3x\, \varphi(\boldsymbol{x}) \boldsymbol{f}_n(\boldsymbol{x}) = \int d^3x\, \varphi(\boldsymbol{x}) \boldsymbol{h}(\boldsymbol{x}) =: \boldsymbol{h}[\varphi]$$

and we conclude that $\boldsymbol{f} = \boldsymbol{h}$ almost everywhere and $H^\triangle_w = H^2_w$ in the sense of sets. Furthermore, the estimate (3.25) for $\boldsymbol{g} = \boldsymbol{f}_n$ states in the limit $n \to \infty$ the equivalence of the norms in $H^\triangle_w$ and H^2_w, i.e. $\|\boldsymbol{f}\|_{H^\triangle_w} \leq \|\boldsymbol{f}\|_{H^2_w} \leq \sqrt{14 C_{13}} \|\boldsymbol{f}\|_{H^2_w}$. Hence, $H^\triangle_w = H^2_w$ in the sense of normed spaces.

The equivalence of $H^{\triangle^k}_w = H^{2k}_w$ for any $k > 1$ is still left to prove. We prove this by induction. Let us assume the claim is true for some $k \in \mathbb{N}$. Let $\boldsymbol{f} \in H^{\triangle^{k+1}}_w$ which implies that for $j \leq k+1$, $\triangle^j \boldsymbol{f} \in L^2_w$, i.e. for $j \leq k$ we have $\triangle^j \boldsymbol{f} \in H^\triangle_w = H^2_w$. Hence, for $j \leq k$ and $|\beta| \leq 2$ it is true that $D^\beta \triangle^j \boldsymbol{f} \in L^2_w$, i.e. $D^\beta \boldsymbol{f} \in H^{\triangle^k}_w$. According to the induction hypothesis, $D^\beta \boldsymbol{f} \in H^{2k}_w$ for $|\beta| \leq 2$ which implies $\boldsymbol{f} \in H^{2k+2}_w$ and concludes the proof. □

Theorem 3.21. *Let $O \subset\subset \mathbb{R}^3$ be open, $w \in \mathcal{W}$ and $k \geq 2$, then:*

Sobolev's Lemma and Morrey's Inequality for weighted spaces

(i) $\boldsymbol{f} \in H^k_w(O, \mathbb{R}^3)$ implies that there is a $\boldsymbol{g} \in C^l(O, \mathbb{R}^3)$, $0 \leq l \leq k-2$, such that almost everywhere $\boldsymbol{f} = \boldsymbol{g}$ on O.

(ii) $\boldsymbol{f} \in H^k_w(O, \mathbb{R}^3)$ for all $O \subset\subset \mathbb{R}^3$ implies that there is a $\boldsymbol{g} \in C^l(\mathbb{R}^3, \mathbb{R}^3)$, $0 \leq l \leq k-2$, such that almost everywhere $\boldsymbol{f} = \boldsymbol{g}$ on $\mathbb{R}^3$.

(iii) Let $O = \mathbb{R}^3$ and $w \in \mathcal{W}^k$. Then for each k there is a $C < \infty$ such that

$$\sup_{\boldsymbol{x}\in\mathbb{R}^3} \sum_{|\alpha|\leq k} \|D^\alpha \boldsymbol{f}(\boldsymbol{x})\| \leq C \|\boldsymbol{f}\|_{H^k_w}. \tag{3.26}$$

Proof. (i) For any compactly contained open set $O \subset\subset \mathbb{R}^3$, $\boldsymbol{f} \in H^k_w(O, \mathbb{R}^3)$ implies $\boldsymbol{f} \in H^k(O, \mathbb{R}^3)$ subject to Lemma 3.15p.26. Sobolev's Lemma [SR75, IX.24] states that there is then a $\boldsymbol{g} \in C^l(O, \mathbb{R}^3)$ for $0 \leq l < n - \frac{3}{2}$ with $\boldsymbol{f} = \boldsymbol{g}$ almost everywhere on O. (ii) Applying (i) we get for every open $O \subset\subset \mathbb{R}^3$ a $g_O \in C^l(O, \mathbb{R}^3)$ such that almost everywhere $\boldsymbol{f} = \boldsymbol{g}_O$. Let $O_1, O_2 \subset\subset \mathbb{R}^3$ be two such sets with the corresponding functions $\boldsymbol{g}_{O_1}$ and $\boldsymbol{g}_{O_2}$, respectively. Assume $O_1 \cap O_2 \neq \emptyset$. By (i) we know that except on a set, let us say $M \subset O_1 \cap O_2$, of measure zero, it holds that $\boldsymbol{g}_{O_1} = \boldsymbol{f} = \boldsymbol{g}_{O_2}$ on $O_1 \cap O_2$. Due to the continuity for all $\boldsymbol{x} \in O_1 \cap O_2$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{y}\|_{\mathbb{R}^3} < \delta$ implies $\|\boldsymbol{g}_{O_i}(\boldsymbol{x}) - \boldsymbol{g}_{O_i}(\boldsymbol{y})\|_{\mathbb{R}^3} < \frac{\epsilon}{2}$, for $i = 1, 2$. Let $\boldsymbol{x} \in M$ and choose $\epsilon > 0$, then there is an $\boldsymbol{y} \in (O_1 \cap O_2) \setminus M$ such that $\boldsymbol{g}_{O_1}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{g}_{O_2}(\boldsymbol{y})$ and, hence,

$$\begin{aligned}\|\boldsymbol{g}_{O_1}(\boldsymbol{x}) - \boldsymbol{g}_{O_2}(\boldsymbol{x})\|_{\mathbb{R}^3} &\leq \|\boldsymbol{g}_{O_1}(\boldsymbol{x}) - \boldsymbol{g}_{O_1}(\boldsymbol{y})\|_{\mathbb{R}^3} + \|\boldsymbol{g}_{O_1}(\boldsymbol{y}) - \boldsymbol{g}_{O_2}(\boldsymbol{y})\|_{\mathbb{R}^3} + \|\boldsymbol{g}_{O_2}(\boldsymbol{y}) - \boldsymbol{g}_{O_2}(\boldsymbol{x})\|_{\mathbb{R}^3} \\ &= \|\boldsymbol{g}_{O_1}(\boldsymbol{x}) - \boldsymbol{g}_{O_1}(\boldsymbol{y})\|_{\mathbb{R}^3} + \|\boldsymbol{g}_{O_2}(\boldsymbol{y}) - \boldsymbol{g}_{O_2}(\boldsymbol{x})\|_{\mathbb{R}^3} < \epsilon\end{aligned}$$

and therefore $\boldsymbol{g}_{O_1} = \boldsymbol{g}_{O_2}$ on $O_1 \cap O_2$. This permits us to define a function $\boldsymbol{g} \in C^l(\mathbb{R}^3, \mathbb{R}^3)$ by setting $\boldsymbol{g} = \boldsymbol{g}_O$ for every open $O \subset\subset \mathbb{R}^3$. (iii) For $w \in \mathcal{W}^k$ we know subject to Lemma 3.17p.27 that $\boldsymbol{f} \in H^k(\mathbb{R}^3, \mathbb{R}^3)$. Applying Sobolev's lemma as in (i) we yield the same result for $O = \mathbb{R}^3$ which provides the conditions for Morrey's inequality (3.26p.29) to hold, see [Lie01, Chapter 8, Theorem 8.8(iii), p.213]. □

REMARK 3.22. *Note that this is the only result that depends on the dimension of* $\mathbb{R}^3$.

3.5 Proof of Main Theorem and Regularity

Proof of Theorem 3.5p.20

Proof of Theorem 3.5p.20. Assertion (i) and (ii). We intend to use the general local existence and uniqueness Theorem 3.9p.21 for $\mathcal{B} = \mathcal{H}_w$. In order to do so we need to show that the operators A and J from Definitions 3.3p.19 and 3.4p.19 have the properties given in Definitions 3.7p.21 and 3.4p.19, respectively. This will be done in Lemma 3.23 and Lemma 3.26p.32. Furthermore, we need to establish an a priori bound for the local solutions such that Theorem 3.9p.21(iii) can be applied.

A fulfills the abstract requirements

Lemma 3.23. *The operator A introduced in Definition 3.3p.19 on $D_w(A)$ with weight $w \in \mathcal{W}^1$ fulfills all properties of Definition 3.7p.21 with $\mathcal{B} = \mathcal{H}_w$ and $\gamma = C_\nabla$ for a constant C_∇, fulfilling $\|\nabla w\| \leq C_\nabla w$ on $\mathbb{R}^3$, e.g. $C_\nabla := \sqrt{\sum_{|\alpha|=1}(C^\alpha)^2}$.*

Proof. By Definition 3.3p.19 the operator A was given for all $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} \in D_w(A) := \bigoplus_{i=1}^N \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus H^{curl} \oplus H^{curl}_w$ by the expression $A(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} = (0, 0, \mathtt{A}(\boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n}$. Let us first regard the operator $\mathtt{A}$ on $\mathcal{D} := H^{curl}_w \oplus H^{curl}_w$. We abbreviate the Hilbert space direct sum $\mathcal{L} := L^2_w \oplus L^2_w$ and write vectors $f \in \mathcal{L}$ in components as $f = (f_1, f_2)$.

First, we prove that $\mathtt{A}$ is closed and densely defined: According to Theorem 3.14p.26, H^{curl}_w is a Hilbert space so that $\mathcal{D}$ is a Banach space with respect to the norm $\|\varphi\|_{\mathcal{D}} := \|\varphi\|_{\mathcal{L}} + \|\mathtt{A}\varphi\|_{\mathcal{L}}$. This means any sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $(u_n)_{n\in\mathbb{N}}$ and $(\mathtt{A}u_n)_{n\in\mathbb{N}}$ converges in $\mathcal{L}$ to u and v, respectively, $(u_n)_{n\in\mathbb{N}}$ converges also with respect to $\|\cdot\|_{\mathcal{D}}$. This implies $u \in \mathcal{D}$ and $v = \mathtt{A}u$, i.e. $\mathtt{A}$ is closed. According to Theorem 3.18p.27, we know also that C^∞_c lies dense in H^{curl}_w, so it follows that $C^\infty_c \times C^\infty_c \subset \mathcal{D}$ lies dense in $\mathcal{L}$. Thus, the operator $\mathtt{A}$ is densely defined.

Next we prove that there exists a $\gamma \geq 0$ such that $(-\infty, -\gamma) \cup (\gamma, \infty) \subset \rho(\mathtt{A})$ which means that for all $|\lambda| > \gamma$

$$(\lambda - \mathtt{A}) : \mathcal{D} \to \mathcal{L} \tag{3.27}$$

is a bijection: Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ denote the Schwartz space of infinitely often differentiable $\mathbb{R}^3$ valued functions on $\mathbb{R}^3$ with faster than polynomial decay, and let $\mathcal{S}^*$ denote the dual of $\mathcal{S}$. On

$\mathcal{S}^* \times \mathcal{S}^*$ we regard in matrix notation

$$(\lambda - \mathtt{A})\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} \lambda & -\nabla\wedge \\ \nabla\wedge & \lambda \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = 0$$

for $T_1, T_2 \in \mathcal{S}^*$ and $\lambda \in \mathbb{R}$. With the use of the Fourier transformation $\widehat{\cdot}$ and its inverse $\widetilde{\cdot}$ on $\mathcal{S}^*$ we get

$$\begin{pmatrix} \lambda & -\nabla\wedge \\ \nabla\wedge & \lambda \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}[\boldsymbol{u}] = \begin{pmatrix} \lambda T_1[\boldsymbol{u}] - \nabla \wedge T_2[\boldsymbol{u}] \\ \lambda T_2[\boldsymbol{u}] + \nabla \wedge T_1[\boldsymbol{u}] \end{pmatrix} = \begin{pmatrix} \widetilde{T_1}[\lambda\widehat{\boldsymbol{u}}] - \widetilde{T_2}[\boldsymbol{k} \mapsto i\boldsymbol{k} \wedge \widehat{\boldsymbol{u}}(\boldsymbol{k})] \\ \widetilde{T_2}[\lambda\widehat{\boldsymbol{u}}] + \widetilde{T_1}[\boldsymbol{k} \mapsto i\boldsymbol{k} \wedge \widehat{\boldsymbol{u}}(\boldsymbol{k})] \end{pmatrix} = 0$$

for all $u \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$. By plugging the second equation into the first for $\lambda \neq 0$, one finds

$$0 = \widetilde{T_1}\left[\boldsymbol{k} \mapsto (\lambda^2 + |\boldsymbol{k}|^2)\widehat{\boldsymbol{u}}(\boldsymbol{k}) - \boldsymbol{k}(\boldsymbol{k} \cdot \widehat{\boldsymbol{u}}(\boldsymbol{k}))\right] =: R_1[\widehat{\boldsymbol{u}}]$$

for all $\boldsymbol{u} \in \mathcal{S}$. However, for all $\boldsymbol{v} \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ we find a $\boldsymbol{u} \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)$ according to

$$\widehat{\boldsymbol{u}}(\boldsymbol{k}) = \frac{\lambda^2\widehat{\boldsymbol{v}}(\boldsymbol{k}) + \boldsymbol{k}(\boldsymbol{k} \cdot \widehat{\boldsymbol{v}}(\boldsymbol{k}))}{\lambda^2(\lambda^2 + |\boldsymbol{k}|^2)}$$

such that $T_1[\boldsymbol{v}] = \widetilde{T_1}[\widehat{\boldsymbol{v}}] = R_1[\widehat{\boldsymbol{u}}] = 0$, which means that $T_1 = 0$ and hence also $T_2 = 0$ on $\mathcal{S}^*$. We have thus shown that for $\mathrm{Ker}(\lambda - \mathtt{A}) = \{0\}$ since $H_w^{curl} \times H_w^{curl} \subset \mathcal{S}^* \times \mathcal{S}^*$, and therefore that the map (3.27p.30) is injective for $\lambda \neq 0$.

We shall now see that there exists a $\gamma > 0$ such that for all $|\lambda| > \gamma$ this map is also surjective. Therefore, we intend to show that for such λ, $\mathrm{Ran}(\lambda - \mathtt{A}) = \mathcal{L}$. Let us assume $v \in \mathrm{Ran}(\lambda - \mathtt{A})^\perp$. Since C_c^∞ is dense in H_w^{curl}, we can use partial integration so that for all $u \in \mathcal{D}$ we obtain

$$0 = \langle(\lambda - \mathtt{A})u, v\rangle_{\mathcal{L}} = \int d^3x\; w(\boldsymbol{x})u(\boldsymbol{x}) \cdot \frac{((\lambda + \mathtt{A})w(\boldsymbol{x})v(\boldsymbol{x}))}{w(\boldsymbol{x})} =: \langle u, (\lambda - \mathtt{A})^* v\rangle_{\mathcal{L}}$$

On the other hand, we have shown that $\mathrm{Ker}(\lambda - \mathtt{A}) = \{0\}$ for all $\lambda \neq 0$, hence wv must be zero which implies that $v = 0$ since $w \in \mathcal{W}^1$. Thus, we have shown that $\mathrm{Ran}(\lambda - \mathtt{A})$ is dense, so that $\mathcal{L} = \overline{\mathrm{Ran}(\lambda - \mathtt{A})}$.

As $(\lambda - \mathtt{A}) : \mathcal{D} \to \mathrm{Ran}(\lambda - \mathtt{A})$ is bijective, we can define $R_\lambda(\mathtt{A})$ as the inverse of this map. In the following it suggests that we at first show the boundedness of $R_\lambda(\mathtt{A})$ and use this property to show the closedness of $\mathrm{Ran}(\lambda - \mathtt{A})$. Let $f \in \mathrm{Ran}(\lambda - \mathtt{A})$, then there is a unique $u \in \mathcal{D}$ which solves $(\lambda - \mathtt{A})u = f$. The inner product with u gives $\langle u, (\lambda - \mathtt{A})u\rangle_{\mathcal{L}} = \langle u, f\rangle_{\mathcal{L}}$ and with the Schwarz inequality and the symmetry of the inner product it implies

$$|\lambda|\; \|u\|_{\mathcal{L}}^2 - \frac{1}{2}\left|\langle u, \mathtt{A}u\rangle_{\mathcal{L}} + \langle u, \mathtt{A}u\rangle_{\mathcal{L}}\right| \leq \|f\|_{\mathcal{L}}\|u\|_{\mathcal{L}}. \tag{3.28}$$

As said before, $C_c^\infty \times C_c^\infty$ lies dense in $\mathcal{D} \subset \mathcal{L}$ so that we may apply partial integration which yields

$$\begin{aligned}
\left|\langle u, \mathtt{A}u\rangle_{\mathcal{L}} + \langle u, \mathtt{A}u\rangle_{\mathcal{L}}\right| &= \left|\int d^3x \begin{pmatrix} 0 & -\nabla w(\boldsymbol{x})\wedge \\ \nabla w(\boldsymbol{x})\wedge & 0 \end{pmatrix} u(\boldsymbol{x}) \cdot u(\boldsymbol{x})\right| \\
&\leq \int d^3x\; |(\nabla w(\boldsymbol{x}) \wedge \boldsymbol{u}_2(\boldsymbol{x})) \cdot \boldsymbol{u}_1(\boldsymbol{x})) - (\nabla w(\boldsymbol{x}) \wedge \boldsymbol{u}_1(\boldsymbol{x})) \cdot \boldsymbol{u}_2(\boldsymbol{x}))| \\
&\leq 2C_\nabla \int d^3x\; w(\boldsymbol{x})\, |\boldsymbol{u}_1(\boldsymbol{x})) \cdot \boldsymbol{u}_2(\boldsymbol{x}))| \leq 2C_\nabla \|u\|_{\mathcal{L}}^2.
\end{aligned}$$

using the notation $u = (u_1, u_2)$ and Schwarz's inequality in the last step. Let us define $\gamma := C_\nabla$. Then for $|\lambda| > \gamma$ the estimate (3.28) gives

$$\|R_\lambda(\mathtt{A})f\|_{\mathcal{L}} = \|u\|_{\mathcal{L}} \leq \frac{1}{|\lambda| - \gamma}\|f\|_{\mathcal{L}}. \tag{3.29}$$

As $\mathrm{Ran}(\lambda - \mathtt{A})$ is dense, there is a unique extension of $R_\lambda(\mathtt{A})$ that we denote by the same symbol $R_\lambda(\mathtt{A}) : \mathcal{L} \to \mathcal{D}$ which obeys the same bound (3.29) on whole $\mathcal{L}$.

In order to finally show that $\mathrm{Ran}(\lambda - \mathtt{A})$ is closed, regard a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Ran}(\lambda - \mathtt{A})$ which converges in $\mathcal{L}$ for $|\lambda| > \gamma$. Define $u_n := R_\lambda(\mathtt{A})f_n$ for all $n \in \mathbb{N}$. By (3.29) we immediately infer convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ to some u in $\mathcal{L}$. Thus, $(u_n, (\lambda - \mathtt{A})u_n) = (u_n, f_n)$ converge to (u, f) in $\mathcal{L} \oplus \mathcal{L}$, and because $\mathtt{A}$ is closed, $u \in \mathcal{D}$ and $(\lambda - \mathtt{A})u = f$. Hence, $f \in \mathrm{Ran}(\lambda - \mathtt{A})$ and $\mathrm{Ran}(\lambda - \mathtt{A})$ is closed. Since we have shown that $\mathrm{Ran}(\lambda - \mathtt{A})$ is closed, we have also $\mathrm{Ran}(\lambda - \mathtt{A}) = \mathcal{L}$. Hence, for all $|\lambda| > \gamma$ the map (3.27p.30) is a bijection.

Finally, we show that A inherits these properties from $\mathtt{A}$: Since $\mathtt{A}$ is closed on $\mathcal{D} = H_w^{curl} \oplus H_w^{curl}$, A is closed on $D_w(A) := \bigoplus_{i=1}^N \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathcal{D}$, and $\bigoplus_{i=1}^N \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus C_c^\infty \oplus C_c^\infty \subset D_w(A)$ lies dense in $\mathcal{H}_w$. This implies property (i) of Definition 3.7p.21. Furthermore, as for $|\lambda| > \gamma \geq 0$ and $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} \in D_w(A)$

$$(\lambda - A)(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} = (\lambda \boldsymbol{q}_i, \lambda \boldsymbol{p}_i, (\lambda - \mathtt{A})(\boldsymbol{E}_i, \boldsymbol{B}_i))_{1\leq i\leq n}.$$

As $\lambda \neq 0$, $(\lambda - A) : D_w(A) \to \mathcal{H}_w$ is a bijection and for $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} \in \mathcal{H}_w$ its inverse $R_\lambda(A)$ is given by

$$R_\lambda(A)(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} = \left(\frac{1}{\lambda}\boldsymbol{q}_i, \frac{1}{\lambda}\boldsymbol{p}_i, R_\lambda(\mathtt{A})(\boldsymbol{E}_i, \boldsymbol{B}_i)\right)_{1\leq i\leq n}.$$

Therefore, $(-\infty, -\gamma) \cup (\gamma, \infty)$ is a subset of the resolvent set $\rho(A)$ of A. This implies property (ii) of Definition 3.7p.21. Finally, by (3.29p.31) for any $\varphi = (\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq n} \in \mathcal{H}_w$ we have the estimate

$$\|R_\lambda(A)\varphi\|_{\mathcal{H}_w} = \sqrt{\sum_{i=1}^N \left(\frac{1}{\lambda^2}\|\boldsymbol{q}_i\|^2 + \frac{1}{\lambda^2}\|\boldsymbol{p}_i\|^2 + \|R_\lambda(\mathtt{A})(\boldsymbol{E}_i, \boldsymbol{B}_i)\|_{\mathcal{L}}^2\right)} \leq \frac{1}{|\lambda| - \gamma}\|\varphi\|_{\mathcal{H}_w}$$

which implies property (iii) of Definition 3.7p.21 and concludes the proof. □

This lemma together with Lemma 3.10p.22 states that A on $D_w(A)$ generates a γ-contractive group $(W_t)_{t\in\mathbb{R}}$ which gives rise to the next definition:

Free Maxwell Time-Evolution

Definition 3.24. *We denote by $(W_t)_{t\in\mathbb{R}}$ the γ-contractive group on $\mathcal{H}_w$ generated by A.*

REMARK 3.25. *The γ-contractive group $(W_t)_{t\in\mathbb{R}}$ comes with a standard bound $\|W_t\|_{\mathfrak{L}(L_w^2)} \leq e^{\gamma|t|}$, see Lemma 3.10p.22, which we shall use often. For the case that w is a constant, one finds $\gamma = 0$ and the whole proof collapses into an argument about self-adjointness on L^2. In this case, $(W_t)_{t\in\mathbb{R}}$ is simply the unitary group generated by the self-adjoint operator A. For non-constant w, $(W_t)_{t\in\mathbb{R}}$ does not preserve the norm. Imagine, for example, a weight w that decreases with the distance to the origin. Then, any wave packet moving towards the origin while retaining its shape (like e.g. solutions to the free Maxwell equations) has necessarily an L_w^2 norm that increases in time.*

J fulfills the abstract requirements

Lemma 3.26. *The operator J introduced in Definition 3.4p.19 with a weight $w \in \mathcal{W}$ fulfills all properties of Definition 3.8p.21 with $\mathcal{B} = \mathcal{H}_w$. Furthermore, there exists a constant $C_J \in$ Bounds such that*

$$\|J(\varphi)\|_{\mathcal{H}_w} \leq C_J\left(\|\varrho_i\|_{L_w^2}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \leq i \leq N\right) \sum_{i=1}^N (1 + C_w\|\boldsymbol{q}_i\|)^{\frac{P_w}{2}} \|\varphi\|_{\mathcal{H}_w} \tag{3.30}$$

for any $\varphi = (\boldsymbol{q}_i, \boldsymbol{P}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N} \in \mathcal{H}_w$ where C_w and $P_w P_w$ are only dependent on the weight w, according to (3.5p.18).

Proof. As already remarked below Definition 3.4p.19, J is a mapping from $\mathcal{H}_w$ to $D_w(A^\infty)$. In order to verify the properties given in Definition 3.8p.21(i) we show for all $\varphi, \widetilde{\varphi} \in D_w(A)$ that there exist $C_3{}^{(n)}, C_4{}^{(n)} \in \texttt{Bounds}$ such that

$$\begin{aligned} \|A^n J(\varphi)\|_{\mathcal{H}_w} &\leq C_3{}^{(n)}(\|\varphi\|_{\mathcal{H}_w})\, \|\varphi\|_{\mathcal{H}_w}, \\ \|A^n(J(\varphi) - J(\widetilde{\varphi}))\|_{\mathcal{H}_w} &\leq C_4{}^{(n)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}. \end{aligned} \tag{3.31}$$

Choose $\varphi = (\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N}$ and $\widetilde{\varphi} = (\widetilde{\boldsymbol{q}}_i, \widetilde{\boldsymbol{p}}_i, \widetilde{\boldsymbol{E}}_i, \widetilde{\boldsymbol{B}}_i)_{1\leq i\leq N}$ in $\mathcal{H}_w$. According to Definition 3.4p.19, for any $n \in \mathbb{N}$ we have

$$J(\varphi) := \left(\boldsymbol{v}(\boldsymbol{p}_i), \sum_{j=1}^N e_{ij} \int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i) \left(\boldsymbol{E}_j(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(x) \right), -4\pi \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i), 0 \right)_{1\leq i\leq N} \tag{3.32}$$

and

$$\begin{aligned} A^{2n+1} J(\varphi) &:= \left(0, 0, 0, (-1)^n 4\pi (\nabla\wedge)^{2n+1} \left(\boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i)\right)\right)_{1\leq i\leq N}, \\ A^{2n+2} J(\varphi) &:= \left(0, 0, (-1)^n 4\pi (\nabla\wedge)^{2n+2} \left(\boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i)\right), 0\right)_{1\leq i\leq N}. \end{aligned} \tag{3.33}$$

Since $J(0) = 0$, inequality 3.31 for $\widetilde{\varphi} = 0$ gives $C_3{}^{(n)}(\|\varphi\|_{\mathcal{H}_w}) := C_4{}^{(n)}(\|\varphi\|_{\mathcal{H}_w}, 0)$. Therefore, it suffices to only prove 3.31. The only case involved therein is $n = 0$ as one needs to control the Lorentz force on each rigid charge, which for $n > 0$ is mapped to zero by any power of A. So for $n = 0$ we obtain:

$$\begin{aligned} \|J(\varphi) - J(\widetilde{\varphi})\|_{\mathcal{H}_w} \leq \sum_{i=1}^N \left\| \boldsymbol{v}(\boldsymbol{p}_i) - \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i) \right\|_{\mathbb{R}^3} + \\ + \sum_{i=1}^N \left\| \sum_{j=1}^N e_{ij} \int d^3x \left(\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\boldsymbol{E}_j(\boldsymbol{x}) - \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\widetilde{\boldsymbol{E}}_j(\boldsymbol{x}) + \right.\right. \\ \left.\left. + \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(\boldsymbol{x}) - \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\boldsymbol{v}(\widetilde{\boldsymbol{p}}_i) \wedge \widetilde{\boldsymbol{B}}_j(\boldsymbol{x}) \right) \right\|_{\mathbb{R}^3} + \\ + 4\pi \sum_{i=1}^N \left\| \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i)\varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i) - \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i) \right\|_{L^2_w} =: \boxed{3} + \boxed{4} + \boxed{5}. \end{aligned} \tag{3.34}$$

The following notation is now convenient: For any function $(f_i)_{1\leq i\leq m} = f : \mathbb{R}^n \to \mathbb{R}^m$ and $(x_j)_{1\leq j\leq n} = x \in \mathbb{R}^n$ we denote by Df the Jacobi matrix of f with entries $Df(x)|_{i,j} = \partial_{x_j} f_i(x)$ for $1 \leq i \leq m, 1 \leq j \leq n$ wherever the derivative makes sense. Furthermore, for any vector space V with norm $\|\cdot\|_V$ and operator T on V we write $\|T\|_V := \sup_{\|v\|_V\leq 1} \|T(v)\|_V$.

Recall also the coefficients $m \neq 0$, $|\sigma_i| = 1$ and $e_{ij} \in \mathbb{R}$ for $1 \leq i, j \leq N$ from Definition 3.4p.19 and define $e := \sup_{1\leq i,j\leq N} |e_{ij}|$. Without loss of generality, we may assume that $\varrho_i > 0$ and the possible signs being absorbed in the e_{ij} for $1 \leq i \leq N$.

By the mean value theorem for each index i there exists a $\lambda_i \in (0, 1)$ such that for $\boldsymbol{k}_i := \boldsymbol{p}_i + \lambda_i(\widetilde{\boldsymbol{p}}_i - \boldsymbol{p}_i)$ we obtain

$$\boxed{3} = \sum_{i=1}^N \left\| D_{\boldsymbol{k}_i}\boldsymbol{v}(\boldsymbol{k}_i) \cdot (\boldsymbol{p}_i - \widetilde{\boldsymbol{p}}_i) \right\|_{\mathbb{R}^3} \leq \sum_{i=1}^N \|D_{\boldsymbol{k}_i}\boldsymbol{v}(\boldsymbol{k}_i)\|_{\mathbb{R}^3} \|\boldsymbol{p}_i - \widetilde{\boldsymbol{p}}_i\|_{\mathbb{R}^3}.$$

Now with $\boldsymbol{k}_i = (\boldsymbol{k}_i)_{1\leq j\leq 3}$ we have $D_{\boldsymbol{k}_i}\boldsymbol{v}(\boldsymbol{k}_i)\Big|_{j,l} = \frac{\sigma_i}{\sqrt{m_i^2 + \boldsymbol{k}_i^2}} \left(\delta_{jl} - \frac{(\boldsymbol{k}_i)_j(\boldsymbol{k}_i)_l}{m_i^2 + \boldsymbol{k}_i^2} \right)$. Thus, it follows the estimate $\|D_{\boldsymbol{k}_i}\boldsymbol{v}(\boldsymbol{k}_i)\|_{\mathbb{R}^3} \leq K_{vel}$ for $K_{vel} := \sum_{i=1}^N \frac{2}{|m_i|}$ so that

$$\boxed{3} \leq K_{vel}\, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}. \tag{3.35}$$

Next we must get a bound on the Lorentz force. Let $B_R(z) := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x} - z\| < R\}$ be a ball of radius $R > 0$ around $z \in \mathbb{R}^3$. Choose $R > 0$ such that for $1 \le i \le N$ it holds $\operatorname{supp} \varrho_i \subseteq B_R(0)$. Define $I_i := B_R(\boldsymbol{q}_i) \cup B_R(\widetilde{\boldsymbol{q}}_i)$, then

$$\boxed{4} \le e \sum_{i,j=1}^{N} \left\| \int_{I_i} d^3x \left(\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\boldsymbol{E}_j(\boldsymbol{x}) - \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\widetilde{\boldsymbol{E}}_j(\boldsymbol{x}) \right) \right\|_{\mathbb{R}^3} +$$

$$+ e \sum_{i,j=1}^{N} \left\| \int_{I_i} d^3x \left(\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(\boldsymbol{x}) - \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\boldsymbol{v}(\widetilde{\boldsymbol{p}}_i) \wedge \widetilde{\boldsymbol{B}}_j(\boldsymbol{x}) \right) \right\|_{\mathbb{R}^3} =: \boxed{6} + \boxed{7}.$$

Let $z_i(\kappa) = \boldsymbol{q}_i + \kappa(\widetilde{\boldsymbol{q}}_i - \boldsymbol{q}_i)$ for each $1 \le i \le N$ and $\kappa \in \mathbb{R}$, then

$$\varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i) = \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i) + \int_0^1 d\kappa\, (\widetilde{\boldsymbol{q}}_i - \boldsymbol{q}_i) \cdot \nabla\varrho_i(\boldsymbol{x} - z_i(\kappa)).$$

Now $\left| \int_0^1 d\kappa\, (\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i) \cdot \nabla\varrho_i(x - z_i(\kappa)) \right| \le K_\varrho \|\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3}$ for $K_\varrho := \sqrt{3} \sum_{i=1,|\alpha|\le n+1}^{N} \|D^\alpha \varrho_i\|_{L^\infty}$ so that

$$\boxed{6} \le e \sum_{i,j=1}^{N} \int_{I_i} d^3x \left[\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\|\boldsymbol{E}_j(\boldsymbol{x}) - \widetilde{\boldsymbol{E}}_j(\boldsymbol{x})\|_{\mathbb{R}^3} + K_\varrho \|\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3} \|\widetilde{\boldsymbol{E}}_j(\boldsymbol{x})\|_{\mathbb{R}^3} \right].$$

We will use the following type of estimates in several places and therefore regard them for once separately. For any set M, $\mathbb{1}_M$ be its characteristic function and $\boldsymbol{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3)$, then

$$\int_{I_i} d^3x\, \|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} \le \left[(1 + C_w\|\boldsymbol{q}_i\|)^{P_w} + (1 + C_w\|\widetilde{\boldsymbol{q}}_i\|)^{P_w} \right]^{\frac{1}{2}} \left\| \frac{\mathbb{1}_{B_R(0)}}{\sqrt{w}} \right\|_{L^2} \|\boldsymbol{F}\|_{L^2_w}, \tag{3.36}$$

$$\int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} \le (1 + C_w\|\boldsymbol{q}_i\|)^{\frac{P_w}{2}} \left\| \frac{\varrho_i}{\sqrt{w}} \right\|_{L^2} \|\boldsymbol{F}\|_{L^2_w(\mathbb{R}^3)}. \tag{3.37}$$

where we used (3.7p.19) which states that $w^{-1} \in L^1_{loc}$ since $w^{-1}(\boldsymbol{x}) \le (1 + C_w\|\boldsymbol{x}\|)^{P_w} w^{-1}(0)$. The former inequality can be seen by

$$\int_{I_i} d^3x\, \|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} = \int_{I_i} d^3x\, \frac{\sqrt{w(\boldsymbol{x})}}{\sqrt{w(\boldsymbol{x})}} \|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} \le \left(\int_{I_i} d^3x\, w^{-1}(\boldsymbol{x}) \right)^{\frac{1}{2}} \|\boldsymbol{F}\|_{L^2_w}$$

$$\le \left(\int_{B_R(0)} d^3x\, (w^{-1}(\boldsymbol{x} - \boldsymbol{q}_i) + w^{-1}(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)) \right)^{\frac{1}{2}} \|\boldsymbol{F}\|_{L^2_w}$$

where we have used the Schwarz inequality. Using the weight estimate (3.7p.19) yields (3.36). Similarly the latter inequality can be seen by

$$\int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} = \int d^3x\, \frac{\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)}{\sqrt{w(\boldsymbol{x} - \boldsymbol{q}_i)}} \sqrt{w(\boldsymbol{x} - \boldsymbol{q}_i)}\|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3}$$

$$\le \left\| \frac{\varrho_i}{\sqrt{w}} \right\|_{L^2} \left(\int d^3x\, w(\boldsymbol{x} - \boldsymbol{q}_i)\|\boldsymbol{F}\|_{\mathbb{R}^3} \right)^{\frac{1}{2}}$$

and again using the weight estimate (3.7p.19). We abbreviate

$$f(x, y) := \left[(1 + C_w x)^{P_w} + (1 + C_w y)^{P_w} \right]^{\frac{1}{2}} \left\| \frac{\mathbb{1}_{B_R(0)}}{\sqrt{w}} \right\|_{L^2}, \qquad g(x) := (1 + C_w x)^{\frac{P_w}{2}} \sum_{i=1}^{N} \left\| \frac{\varrho_i}{\sqrt{w}} \right\|_{L^2}$$

so that (3.36) and (3.37) give

$$\int_{I_i} d^3x\, \|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} \le f\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w} \right) \|\boldsymbol{F}\|_{L^2_w}, \tag{3.38}$$

$$\int d^3x\, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\|\boldsymbol{F}(\boldsymbol{x})\|_{\mathbb{R}^3} \le g\left(\|\varphi\|_{\mathcal{H}_w} \right) \|\boldsymbol{F}\|_{L^2_w(\mathbb{R}^3)}. \tag{3.39}$$

We apply these estimates to the term

$$\begin{aligned}\boxed{6} &\leq e \sum_{i,j=1}^{N} g\left(\|\varphi\|_{\mathcal{H}_w}\right) \|\boldsymbol{E}_j - \widetilde{\boldsymbol{E}}_j\|_{L^2_w} + e \sum_{i,j=1}^{N} K_\varrho \|\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3} g\left(\|\varphi\|_{\mathcal{H}_w}\right) \|\widetilde{\boldsymbol{E}}_j\|_{L^2_w} \\ &\leq eNg\left(\|\varphi\|_{\mathcal{H}_w}\right) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} + eK_\varrho \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} f\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right) \|\widetilde{\varphi}\|_{\mathcal{H}_w}.\end{aligned}$$

Therefore, we yield the estimate

$$\boxed{6} \leq C_{14}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$$

for

$$C_{14}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) := e\left(Ng\left(\|\varphi\|_{\mathcal{H}_w}\right) + K_\varrho f\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right) \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right).$$

The same way we shall estimate:

$$\boxed{7} = e \sum_{i,j=1}^{N} \left\| \int_{I_i} d^3x \left(\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(\boldsymbol{x}) - \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\boldsymbol{v}(\widetilde{\boldsymbol{p}}_i) \wedge \widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\right)\right\|_{\mathbb{R}^3}.$$

First we apply the mean value theorem to the velocities as we did before such that

$$\begin{aligned}\ldots &\leq e \sum_{i,j=1}^{N} \int_{I_i} d^3x \left\| \boldsymbol{v}(\boldsymbol{p}_i) \wedge \left(\varrho_i(\boldsymbol{x} - \boldsymbol{q}_i)\boldsymbol{B}_j(\boldsymbol{x}) - \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\right)\right\|_{\mathbb{R}^3} \\ &\quad + e \sum_{i,j=1}^{N} \int_{I_i} d^3x \, K_{vel} \|\boldsymbol{p}_i - \widetilde{\boldsymbol{p}}_i\|_{\mathbb{R}^3} \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i) \|\widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\|_{\mathbb{R}^3}.\end{aligned}$$

Then we again rewrite the densities by the fundamental theorem of calculus and use this and $\|\boldsymbol{v}(\boldsymbol{p}_i)\|_{\mathbb{R}^3} \leq 1$ in order to obtain

$$\begin{aligned}\ldots &\leq e \sum_{i,j=1}^{N} \int_{I_i} d^3x \, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i) \|\boldsymbol{B}_j(\boldsymbol{x}) - \widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\|_{\mathbb{R}^3} + e \sum_{i,j=1}^{N} K_\varrho \|\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3} \int_{I_i} d^3x \, \|\widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\|_{\mathbb{R}^3} + \\ &\quad + e \sum_{i,j=1}^{N} K_{vel} \|\boldsymbol{p}_i - \widetilde{\boldsymbol{p}}_i\|_{\mathbb{R}^3} \int_{I_i} d^3x \, \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i) \|\widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\|_{\mathbb{R}^3}.\end{aligned}$$

Finally we apply the two estimates ($3.38_{\text{p.34}}$) and ($3.38_{\text{p.34}}$) to arrive at

$$\begin{aligned}\ldots &\leq e \sum_{i,j=1}^{N} g\left(\|\varphi\|_{\mathcal{H}_w}\right) \|\boldsymbol{B}_j - \widetilde{\boldsymbol{B}}_j\|_{L^2_w} + e \sum_{i,j=1}^{N} K_\varrho \|\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3} f\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right) \|\widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\|_{L^2_w} + \\ &\quad + e \sum_{i,j=1}^{N} K_{vel} \|\boldsymbol{p}_i - \widetilde{\boldsymbol{p}}_i\|_{\mathbb{R}^3} g\left(\|\varphi\|_{\mathcal{H}_w}\right) \|\widetilde{\boldsymbol{B}}_j(\boldsymbol{x})\|_{L^2_w}.\end{aligned}$$

and thus we obtain the estimate

$$\boxed{7} \leq C_{15}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$$

for

$$C_{15}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) := e\left(Ng\left(\|\varphi\|_{\mathcal{H}_w}\right) + K_\varrho f\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right) \|\varphi\|_{\mathcal{H}_w} + K_{vel} g\left(\|\varphi\|_{\mathcal{H}_w}\right) \|\varphi\|_{\mathcal{H}_w}\right).$$

It remains to estimate term $\boxed{5}$. However, we shall do this already for the general case of any fixed $n \in \mathbb{N}_0$. For the sake of readability we will not explicitly write out the n dependence in each term. Recall from Equation (3.33p.33) that

$$\boxed{5}_n := 4\pi \sum_{i=1}^{N} \|(\nabla\wedge)^n \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i) - (\nabla\wedge)^n \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i)\varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i)\|_{L^2_w}.$$

We begin with

$$\begin{aligned}\boxed{5}_n \leq 4\pi \sum_{i=1}^{N} \left\|(\nabla\wedge)^n \boldsymbol{v}(\boldsymbol{p}_i)\left(\varrho_i(\cdot - \boldsymbol{q}_i) - \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i)\right)\right\|_{L^2_w} + \\ + 4\pi \sum_{i=1}^{N} \left\|(\nabla\wedge)^n (\boldsymbol{v}(\boldsymbol{p}_i) - \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i))\varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i)\right\|_{L^2_w} =: \boxed{8}_n + \boxed{9}_n\end{aligned}$$

but before we continue we shall rewrite these terms in a more convenient form by using the following formulas: Let $\triangle^{-1} = 0$, then for all $\boldsymbol{v} \in \mathbb{R}^3$, $h \in C^\infty(\mathbb{R}^3, \mathbb{R})$ and all $m \in \mathbb{N}_0$ the following identities hold:

$$\begin{aligned}(\nabla\wedge)^{2m}(\boldsymbol{v}h) &= (-1)^{m-1}\left[\nabla\left(\boldsymbol{v}\cdot\nabla\triangle^{m-1}h\right) - \boldsymbol{v}\triangle^m h\right] \\ (\nabla\wedge)^{2m+1}(\boldsymbol{v}h) &= (-1)^m [\nabla\triangle^m h] \wedge \boldsymbol{v}\end{aligned}$$

This can be seen by induction. The formulas obviously hold for $m = 0$. Assuming them to be correct for some $m \in \mathbb{N}_0$, we find

$$\begin{aligned}(\nabla\wedge)^{2m+2}(\boldsymbol{v}g) &= \nabla\wedge(\nabla\wedge)^{2m+1}(\boldsymbol{v}g) = \nabla\wedge((-1)^m[\nabla\triangle^m h]\wedge\boldsymbol{v}) \\ &= (-1)^m \nabla\wedge(\nabla\wedge\boldsymbol{v}\triangle^m h) = (-1)^m\left[\nabla(\boldsymbol{v}\cdot\nabla\triangle^m h) - \boldsymbol{v}\triangle^{m+1}h\right], \\ (\nabla\wedge)^{2m+3}(\boldsymbol{v}g) &= \nabla\wedge(\nabla\wedge)^{2m+2}(\boldsymbol{v}g) = \nabla\wedge(-1)^m\left[\nabla(\boldsymbol{v}\cdot\nabla\triangle^m h) - \boldsymbol{v}\triangle^{m+1}h\right] \\ &= (-1)^{m+1}[\nabla\triangle^{m+1}h]\wedge\boldsymbol{v}.\end{aligned}$$

Let us begin with term $\boxed{8}_n$ for odd n. As before we write $\boldsymbol{z}_i(\kappa) = \boldsymbol{q}_i + \kappa(\widetilde{\boldsymbol{q}}_i - \boldsymbol{q}_i)$ for each $1 \leq i \leq N$ and $\kappa \in \mathbb{R}$ so that for

$$K_I\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right) := \sum_{i=1}^{N} \|\mathbb{1}_{I_i}\|_{L^2_w} = N\|\mathbb{1}_{B_R(0)}\|_{L^2_w}\left[(1 + C_w\|\varphi\|_{\mathcal{H}_w})^{P_w} + (1 + C_w\|\widetilde{\varphi}\|_{\mathcal{H}_w})^{P_w}\right]^{\frac{1}{2}} \tag{3.40}$$

we get

$$\begin{aligned}\boxed{8}_{n=2m+1} &= 4\pi \sum_{i=1}^{N} \left\|\nabla\triangle^m\left[\varrho_i(\cdot - \boldsymbol{q}_i) - \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i)\right]\wedge\boldsymbol{v}(\sigma_i, m_i, \boldsymbol{p}_i)\right\|_{L^2_w} \\ &\leq 4\pi \sum_{i=1}^{N} \left\|\int_0^1 d\kappa\, D[\nabla\triangle^m \varrho_i(\boldsymbol{x} - \boldsymbol{z}_i(\kappa))]\cdot(\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i)\right\|_{L^2_w} \leq 4\pi K_\varrho \sum_{i=1}^{N} \|\mathbb{1}_{I_i}\|_{L^2_w}\|\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3} \\ &\leq 4\pi K_\varrho K_I\left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}\right)\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} =: C_{16}(2m+1, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\,\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}.\end{aligned} \tag{3.41}$$

Similarly the term $\boxed{8}_n$ for even n gives:

$$\boxed{8}_{n=2m} = 4\pi \sum_{i=1}^{N} \Big\| \nabla \left(\boldsymbol{v}(\boldsymbol{p}_i) \cdot \nabla \triangle^{m-1} \left[\varrho_i(\cdot - \boldsymbol{q}_i) - \varrho_i(\cdot - \widetilde{\boldsymbol{p}}_i \right] \right) + \\ - \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i) \triangle^m \left[\varrho_i(\cdot - \boldsymbol{q}_i) - \varrho_i(\cdot - \widetilde{\boldsymbol{p}}_i \right] \Big\|_{L^2_w}$$

$$\leq 4\pi \sum_{i=1}^{N} \left(\Big\| \mathbb{1}_{I_i} \int_0^1 d\kappa \, D\Big[D\Big[\nabla \triangle^{m-1} \varrho_i(\boldsymbol{x} - \boldsymbol{z}_i(\kappa)) \Big] \cdot (\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i) \Big] \cdot \boldsymbol{v}(\boldsymbol{p}_i) \Big\|_{L^2_w} + \\ + \Big\| \mathbb{1}_{I_i} \boldsymbol{v}(\boldsymbol{p}_i) \int_0^1 d\kappa \, D \triangle^m \varrho_i(\boldsymbol{x} - \boldsymbol{z}_i(\kappa)) \cdot (\boldsymbol{q}_i - \widetilde{\boldsymbol{q}}_i) \Big\|_{L^2_w} \right)$$

Again we estimate the coefficients of the Jacobi matrices by K_ϱ, this time obtaining another factor $\sqrt{3}$ in the first summand such that

$$\boxed{8}_{n=2m} \leq 4\pi K_I \left(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w} \right) K_\varrho(\sqrt{3} + 1) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} \\ =: C_{16}(2m, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}. \tag{3.42}$$

The last term to be estimated for odd n is:

$$\boxed{9}_{n=2m} = 4\pi \sum_{i=1}^{N} \Big\| \nabla \triangle^m \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i) \wedge [\boldsymbol{v}(\boldsymbol{p}_i) - \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i)] \Big\|_{L^2_w} \\ \leq 4\pi \sum_{i=1}^{N} \Big\| \nabla \triangle^m \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i) \Big\|_{L^2_w} K_{vel} \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} \\ \leq 4\pi (1 + C_w \|\widetilde{\varphi}\|_{\mathcal{H}_w})^{\frac{P_w}{2}} \sum_{i=1}^{N} \Big\| \nabla \triangle^m \varrho_i \Big\|_{L^2_w} K_{vel} \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} \\ =: C_{17}(2m + 1, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}, \tag{3.43}$$

and for even n:

$$\boxed{9}_{n=2m} = 4\pi \sum_{i=1}^{N} \Big\| \nabla \left([\boldsymbol{v}(\boldsymbol{p}_i) - \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i)] \cdot \nabla \triangle^{m-1} \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i) \right) - [\boldsymbol{v}(\boldsymbol{p}_i) - \boldsymbol{v}(\widetilde{\boldsymbol{p}}_i)] \, \triangle^m \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i) \Big\|_{L^2_w}$$

$$\leq 4\pi \sum_{i=1}^{N} \left(\left(\int d^3x \, w(\boldsymbol{x}) \|D \nabla \triangle^{m-1} \varrho_i(\boldsymbol{x} - \widetilde{\boldsymbol{q}}_i)\|_{\mathbb{R}^3} \right)^{\frac{1}{2}} + \|\triangle^m \varrho_i(\cdot - \widetilde{\boldsymbol{q}}_i)\|_{L^2_w} \right) K_{vel} \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$$

$$\leq 4\pi N (1 + C_w \|\widetilde{\varphi}\|_{\mathcal{H}_w})^{\frac{P_w}{2}} \left(\left(\int d^3x \, w(\boldsymbol{x}) \|D \nabla \triangle^{m-1} \varrho_i(\boldsymbol{x})\|_{\mathbb{R}^3} \right)^{\frac{1}{2}} + \|\triangle^m \varrho_i\|_{L^2_w} \right) K_{vel} \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$$

$$=: C_{17}(2m, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}, \tag{3.44}$$

Collecting all these estimates, we finally arrive at the inequality 3.31p.33 for

$$C_4^{(n)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} := K_{vel} + C_{14}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) + \\ + C_{15}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) + C_{16}(2m + 1, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) + C_{17}(2m + 1, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})$$

which for fixed n is a continuous and non-decreasing function in the arguments $\|\varphi\|_{\mathcal{H}_w}$ and $\|\widetilde{\varphi}\|_{\mathcal{H}_w}$, and, hence $C_4^{(n)} \in$ `Bounds`.

Next we need to verify property (ii) of Definition 3.8p.21. Therefore, for $T > 0$ let $t \mapsto \varphi_t$ be a map in $C^n((-T, T), D_w(A^n))$ such that for all $k \leq n$ and $t \in (-T, T)$ it holds that $\frac{d^k}{dt^k} \varphi_t \in D_w(A^{n-k})$. We have to show that for all $j + l \leq n - 1$, $t \mapsto \frac{d^j}{dt^j} A^l J(\varphi_t)$ is continuous on $(-T, T)$ and take values in

$D_w(A^{n-1-j-l})$. By formulas (3.32p.33) and (3.33p.33) both properties are an immediate consequence of the fact that $\varrho_i \in C_c^\infty$. In fact, one finds that $t \mapsto \frac{d^j}{dt^j} A^l J(\varphi_t)$ takes values in $D_w(A^\infty)$ on $(-T, T)$.

Finally, we prove inequality (3.30p.32). In principle we could use (3.34p.33) and the estimates (3.35p.33, 3.40p.36, 3.41p.36, 3.42, 3.43, and 3.44) for $\widetilde{\varphi} = 0$ so that we only had to treat the Lorentz force. However, for this estimate we also want to work out the dependence on of J on ρ. Therefore, we regard

$$\|J(\varphi)\| \leq \sum_{1\leq i\leq N} \left[\|\boldsymbol{v}(\boldsymbol{p}_i)\|_{\mathbb{R}^3} + \left\| \sum_{j\neq i} e_{ij} \int d^3 \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i) \left(\boldsymbol{E}_j(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(\boldsymbol{x}) \right) \right\|_{\mathbb{R}^3} + \right.$$
$$\left. + \|4\pi \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i)\|_{L^2_w} \right] =: \boxed{10} + \boxed{11} + \boxed{12}$$

The first term can be treated as before, cf. (3.35p.33),

$$\boxed{10} \leq N K_{vel} \, \|\varphi\|_{\mathcal{H}_w}$$

The second term

$$\boxed{11} = \sum_{i=1}^{N} \left\| \sum_{j=1}^{N} e_{ij} \int d^3x \, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i) \left(\boldsymbol{E}_j(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{p}_i) \wedge \boldsymbol{B}_j(x) \right) \right\|_{\mathbb{R}^3} .$$

can be estimated by

$$e \sum_{i,j=1}^{N} \int d^3x \, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_i) \left(\|\boldsymbol{E}_j(\boldsymbol{x})\|_{\mathbb{R}^3} + \|\boldsymbol{B}_j(\boldsymbol{x})\|_{\mathbb{R}^3} \right).$$

Using estimate (3.37p.34) we find

$$\ldots \leq e \sum_{i=1}^{N} (1 + C_w \|\boldsymbol{q}_i\|)^{\frac{P_w}{2}} \left\| \frac{\varrho_i}{\sqrt{w}} \right\|_{L^2} \sum_{j=1} \left(\|\boldsymbol{E}_j(\boldsymbol{x})\|_{L^2_w} + \|\boldsymbol{B}_j(\boldsymbol{x})\|_{L^2_w} \right)$$
$$\leq 2Ne \sum_{i=1}^{N} \left\| \frac{\varrho_i}{\sqrt{w}} \right\|_{L^2} \sum_{i=1}^{N} (1 + C_w \|\boldsymbol{q}_i\|)^{\frac{P_w}{2}} \|\varphi\|_{\mathcal{H}_w}.$$

Finally, for the last term we obtain

$$\boxed{12} \leq 4\pi K_{vel} \sum_{i=1}^{N} \|\varrho_i(\cdot - \boldsymbol{q}_i)\|_{L^2_w} \, \|\varphi\|_{\mathcal{H}_w} \leq 4\pi K_{vel} \sum_{i=1}^{N} \|\varrho_i\|_{L^2_w} \sum_{i=1}^{N} (1 + C_w \|\boldsymbol{q}_i\|)^{P_w} \, \|\varphi\|_{\mathcal{H}_w}.$$

Hence, there is a $C_J \in \texttt{Bounds}$ for

$$C_J \left(\|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \leq i \leq N \right) := N K_{vel} + 2Ne \left\| \frac{\varrho_i}{\sqrt{w}} \right\|_{L^2} + 4\pi K_{vel} \sum_{i=1}^{N} \|\varrho_i\|_{L^2_w}.$$

This concludes the proof. □

Next we need to show the a priori bound as in (3.15p.21).

A priori Bound on the ML±SI Solutions

Lemma 3.27. *Let the map $t \mapsto \varphi_t = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\leq i\leq N}$ be a solution to*

$$\varphi_t = W_t \varphi^0 + \int_0^t W_t J(\varphi_s).$$

with $\varphi^0 = \varphi_t|_{t=0} \in D_w(A)$. Then there is a $C_{18} \in \texttt{Bounds}$ such that

$$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \leq e^{\gamma T} (1 + C_{18} T e^{C_{18} T}) \|\varphi^0\|_{\mathcal{H}_w} < \infty. \tag{3.45}$$

for $C_{18} := C_{18} \left(\|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \leq i \leq N \right)$.

Proof. By Lemma 3.23p.30 we know that

$$\|\varphi_t\|_{\mathcal{H}_w} = \|W_t\varphi^0 + \int_0^t ds\ W_{t-s}J(\varphi_s)\|_{\mathcal{H}_w} \le e^{\gamma T}\|\varphi^0\|_{\mathcal{H}_w} + \text{sign}(t)e^{\gamma T}\int_0^t ds\ \|J(\varphi_s)\|_{\mathcal{H}_w}$$

Lemma 3.26p.32 provides the bound to estimate the integrand

$$\|J(\varphi_s)\|_{\mathcal{H}_w} \le C_J \sum_{i=1}^N (1 + C_w\|\boldsymbol{q}_{i,s}\|_{\mathbb{R}^3})^{\frac{P_w}{2}}\|\varphi_s\|_{\mathcal{H}_w}$$

for every time s. Moreover, as the velocities are bounded by the speed of light we get in addition

$$\|\boldsymbol{q}_{i,s}\| = \left\|\boldsymbol{q}_i^0 + \int_0^s dr\ \boldsymbol{v}(\boldsymbol{p}_{i_r})\right\| \le \|\boldsymbol{q}_i^0\| + \text{sign}(s)\int_0^s dr\ \left\|\frac{\sigma_i \boldsymbol{p}_{i,r}}{\sqrt{m_i^2 + \boldsymbol{p}_{i,r}^2}}\right\| \le \|\varphi^0\|_{\mathcal{H}_w} + |s|.$$

Hence, for some finite $T > 0$ and $|t| \le T$ we infer the following integral inequality

$$\|\varphi_t\|_{\mathcal{H}_w} \le e^{\gamma T}\|\varphi^0\|_{\mathcal{H}_w} + \text{sign}(t)C_{19}(T)\int_0^t ds\ \|\varphi_s\|_{\mathcal{H}_w}$$

for $C_{19}(T) := e^{\gamma T}C_J N(1 + C_w(\|\varphi^0\|_{\mathcal{H}_w} + |T|))^{\frac{P_w}{2}}$, according to which by Gronwall's lemma

$$\sup_{t\in[-T,T]}\|\varphi_t\|_{\mathcal{H}_w} \le e^{\gamma T}(1 + C_{19}Te^{C_{19}T})\|\varphi^0\|_{\mathcal{H}_w} < \infty. \tag{3.46}$$

□

This proves claim (3.10p.20) for

$$C_1 := e^{\gamma T}(1 + C_{19}Te^{C_{19}T})$$

with the parameter dependence as stated in the above lemma and yields the needed bound (3.15p.21) for $C_5(T) := e^{\gamma T}(1 + C_{19}Te^{C_{19}T})$. This bound together with Lemma 3.23p.30 and Lemma 3.26p.32 fulfill all the conditions for Theorem 3.9p.21. Hence, we have shown existence and uniqueness of global solutions to (3.9p.20). To conclude the proof for part (i) and (ii) we still need to verify (3.11p.20). Let $T \ge 0$ and $\varphi, \widetilde{\varphi} : [-T,T] \to D_w(A)$ be solutions to (3.9p.20), then for $t_0, t \in [-T,T]$ we have

$$\begin{aligned}\|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} &= \left\|W_{t-t_0}(\varphi_{t_0} - \widetilde{\varphi}_{t_0}) + \int_{t_0}^t ds\ W_{t-s}(J(\varphi_s) - J(\widetilde{\varphi}_s))\right\|_{\mathcal{H}_w} \\ &\le e^{\gamma T}\|\varphi_{t_0} - \widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w} + \text{sign}(t-t_0)e^{\gamma T}\int_{t_0}^t ds\ C_4^{(1)}(\|\varphi_s\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_s\|_{\mathcal{H}_w})\ \|\varphi_s - \widetilde{\varphi}_s\|_{\mathcal{H}_w}\end{aligned}$$

by (3.31p.33). Now we use (3.45p.38) and find

$$C_{20}(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w}) := \sup_{s\in[-T,T]} C_4^{(1)}(\|\varphi_s\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_s\|_{\mathcal{H}_w}) < \infty.$$

Hence, we can apply Gronwall's lemma once again and find that (3.11p.20) holds for

$$C_2(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w}) := e^{\gamma T}(1 + C_{20}(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w})Te^{C_{20}(T,\|\varphi_{t_0}\|_{\mathcal{H}_w},\|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w})T}).$$

The Maxwell constraints

For proving part (iii) we need to study whether solutions $t \mapsto \varphi_t$ respect the constraints (3.12p.20). This, however, can be seen to be true by a short computation. Without loss of generality we may assume $t^* = 0$. Say we are given an initial value $(\boldsymbol{q}_i^0, \boldsymbol{p}_i^0, \boldsymbol{E}_i^0, \boldsymbol{B}_i^0)_{1\le i\le N} =: \varphi^0 \in D_w(A)$, then by part (i) and (ii) there exists a unique solution $t \mapsto \varphi_t$ in $C^1(\mathbb{R}, D_w(A))$ of Equation (3.14p.21). As before we use the notation $\varphi_t =: (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N}$ for $t \in \mathbb{R}$. Furthermore, let φ^0 be chosen in such a way that $\nabla \cdot \boldsymbol{E}_i^0 = 4\pi\varrho_i(\cdot - \boldsymbol{q}_i^0)$ and $\nabla \cdot \boldsymbol{B}_i^0 = 0$ hold in the distribution sense. We may write the divergence of the magnetic field of the i-th particle for each $t \in \mathbb{R}$ in the distribution sense as

$$\nabla \cdot \boldsymbol{B}_{i,t} = \nabla \cdot \left(\boldsymbol{B}_i^0 + \int_0^t \dot{\boldsymbol{B}}_{i,s}\, ds \right) = -\nabla \cdot \int_0^t ds\, \nabla \wedge \boldsymbol{E}_{i,s}$$

where we have used the equation of motion (3.14p.21) and the assumption $\nabla \cdot \boldsymbol{B}_i^0 = 0$. Since $\varphi_t \in D_w(A)$, $\nabla \wedge \boldsymbol{E}_{i,s}$ is in L^2_w. Therefore, for any $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ we find by Fubini's theorem that

$$\int d^3x\, \nabla\phi(\boldsymbol{x}) \cdot \int_0^t ds\, \nabla \wedge \boldsymbol{E}_{i,s}(\boldsymbol{x}) = \int_0^t ds \int d^3x\, \nabla\phi(\boldsymbol{x}) \cdot (\nabla \wedge \boldsymbol{E}_{i,s}(\boldsymbol{x})) = 0 \tag{3.47}$$

as for any fixed t the absolute value of the integrand is integrable as

$$\int_0^t ds \int d^3x\, |\nabla\phi(\boldsymbol{x}) \cdot (\nabla \wedge \boldsymbol{E}_{i,s}(\boldsymbol{x}))| \le \|\nabla\phi\|_{L^2_w}\, t \sup_{s\in[0,t]} \|\nabla \wedge \boldsymbol{E}_{i,s}\|_{L^2_w} \le \infty.$$

The supremum exists because of continuity. Analogously, we find for the electric fields

$$\begin{aligned}\nabla \cdot \boldsymbol{E}_{i,t} &= \nabla \cdot \left(\boldsymbol{E}_i^0 + \int_0^t ds\, \dot{\boldsymbol{E}}_{i,s} \right) \\ &= 4\pi\varrho_i(\cdot - \boldsymbol{q}_i^0) + \nabla \cdot \int_0^t ds\, \nabla \wedge \boldsymbol{B}_{i,s} - 4\pi\nabla \cdot \int_0^t ds\, \boldsymbol{v}(\boldsymbol{p}_{i,s})\varrho_i(\cdot - \boldsymbol{q}_{i,s}).\end{aligned}$$

By the same argument as in (3.47) the second term is zero. We commute the divergence with the integration since $\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}$ are continuous functions of t and $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ and find

$$\begin{aligned}\ldots &= 4\pi\varrho_i(\cdot - \boldsymbol{q}_i^0) - 4\pi \int_0^t ds\, \boldsymbol{v}(\boldsymbol{p}_{i,s}) \cdot \nabla\varrho_i(\cdot - \boldsymbol{q}_{i,s}) \\ &= 4\pi\varrho_i(\cdot - \boldsymbol{q}_i^0) + 4\pi \int_0^t \frac{d}{ds}\varrho_i(\cdot - \boldsymbol{q}_{i,s})\, ds = 4\pi\varrho_i(\cdot - \boldsymbol{q}_{i,t})\end{aligned}$$

which concludes part (iii) and the proof. □

Proof of Theorem 3.6p.20

Proof of Theorem 3.6p.20. As a last step we shall examine the regularity of the Maxwell solutions. Assume the initial value $\varphi^0 \in D_w(A^{2m})$ for some $m \in \mathbb{N}$. According to Theorem 3.5p.20, we know that there exists a unique solution $t \mapsto \varphi_t = (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N}$ which is in $C^{2m}(\mathbb{R}, D_w(A^{2m}))$ with $\varphi_t|_{t=0} = \varphi^0$. The first aim is to see whether the fields $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t}$ are more smooth than a typical function in H_w^{curl}. In order to achieve this we have in mind to apply our version of Sobolev's lemma for weighted function spaces, i.e. Lemma 3.21p.29. We know that $(\nabla\wedge)^{2l}\boldsymbol{E}_{i,t}, (\nabla\wedge)^{2l}\boldsymbol{B}_{i,t} \in H_w^{curl}$ for any $0 \le l \le m$, but then

$$\begin{aligned}(\nabla\wedge)^{2l}\boldsymbol{E}_{i,t} &= (\nabla\wedge)^{2l-2}(\nabla\wedge)^2\boldsymbol{E}_{i,t} = (\nabla\nabla\cdot - \triangle)^{l-1}\,(\nabla\wedge)^2\boldsymbol{E}_{i,t} \\ &= \sum_{k=0}^{l-1} \binom{l-1}{k} (\nabla\nabla\cdot)^k(-\triangle)^{l-1-k}(\nabla\wedge)^2\boldsymbol{E}_{i,t} = (-\triangle)^{l-1}(\nabla(\nabla\cdot\boldsymbol{E}_{i,t}) - \triangle\boldsymbol{E}_{i,t})\end{aligned}$$

in the distribution sense, where $\nabla\nabla\cdot$ denotes the gradient of the divergence. The same computation holds for $\boldsymbol{B}_{i,t}$. By inserting the constraints (3.12p.20) we find:

$$(\nabla\wedge)^{2l}\boldsymbol{E}_{i,t} = 4\pi(-1)^{l-1}\triangle^{m-1}\nabla\varrho_i(\cdot - \boldsymbol{q}_{i,t}) + (-\triangle)^l\boldsymbol{E}_{i,t} \qquad \text{and} \qquad (\nabla\wedge)^{2l}\boldsymbol{B}_{i,t} = (-\triangle)^l\boldsymbol{B}_{i,t}.$$

As $\varrho_i \in C_c^\infty$ we may conclude that for any fixed $t \in \mathbb{R}$ we have $\triangle^l \boldsymbol{E}_{i,t}, \triangle^l \boldsymbol{B}_{i,t} \in L^2_w$ for $0 \le l \le m$ and therefore $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t} \in H_w^{\triangle^m}$ which proves claim (i). In particular, for every open $O \subset\subset \mathbb{R}^3$, $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t}$ are in $H_w^{\triangle^m}(O)$ which by Theorem 3.20p.28 for $w = 1$ equals $H^{2m}(O)$. Lemma 3.15p.26 then states $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t} \in H^{2m}_{loc}$ which provides the conditions to be able to apply Theorem 3.21p.29(ii) that states: In the equivalence class of $\boldsymbol{E}_{i,t}$ and $\boldsymbol{B}_{i,t}$ there is a representative in $C^l(\mathbb{R}^3, \mathbb{R}^3)$ for $0 \le l \le 2m - 2 = n - 2$, respectively. We denote the representatives also by $\boldsymbol{E}_{i,t}$ and $\boldsymbol{B}_{i,t}$, respectively.

Moreover, we know that for any $0 \le k \le n$ the map $t \mapsto \frac{d^k}{dt^k}\varphi_t$ and hence the maps $t \mapsto \frac{d^k}{dt^k}\boldsymbol{E}_{i,t}$ and $t \mapsto \frac{d^k}{dt^k}\boldsymbol{E}_{i,t}$ are continuous. Hence, for any $\Lambda \subset\subset \mathbb{R}^4$ and for $k \le n$ the integrals

$$\int_\Lambda ds\, d^3x\, w(\boldsymbol{x}) \left\| \frac{d^k}{dt^k} \boldsymbol{E}_{i,s} \right\|^2_{\mathbb{R}^3} \qquad \text{and} \qquad \int_\Lambda ds\, d^3x\, w(\boldsymbol{x}) \left\| \partial^k_{x_j} \boldsymbol{E}_{i,s} \right\|^2_{\mathbb{R}^3} \text{ for } j = 1,2,3$$

are finite. Applying Sobolev's lemma in the form presented in [Rud73, Theorem 7.25] we yield that within the equivalence classes $\boldsymbol{E}_i$ and $\boldsymbol{B}_i$ there is a representative in $C^{n-2}(\mathbb{R}^4, \mathbb{R}^3)$, respectively, which proves claim (ii).

Assume $w \in \mathcal{W}^k$ for $k \ge 2$. Then Theorem 3.20p.28 yields that also $\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t} \in H_w^{2m=n}(\mathbb{R}^3)$, and by Theorem 3.21p.29(iii) there is a constant C such that (3.13p.20) holds for every $1 \le i \le N$ which proves claim (iii) and concludes the proof. □

The existence and uniqueness result from Theorem 3.5p.20 permits us to define a time-evolution operator induced by the Maxwell-Lorentz equations:

Definition 3.28. *We define* Maxwell-Lorentz Time-Evolution

$$M_L : \mathbb{R}^2 \times D_w(A) \to D_w(A), \quad (t, t_0, \varphi^0) \to M_L(t, t_0)[\varphi^0] = \varphi_t = W_{t-t_0}\varphi^0 + \int_{t_0}^t W_{t-s} J(\varphi_s)$$

which encodes the time-evolution of the charges as well as their electromagnetic fields from time t_0 to time t.

REMARK 3.29. *(i) By uniqueness we get for times $t_0, t_1, t \in \mathbb{R}$ that*

$$M_L(t, t_0)[\varphi^0] = M_L(t, t_1)\left[M_L(t_1, t_0)[\varphi^0]\right].$$

(ii) For the case of (ML+SIp.15), i.e. $e_{ij} = 1$ for all $1 \le i, j \le N$, and initial values $\varphi^0 \in D_w(A)$ for weights $w \in \mathcal{W}$ such that for $w(\boldsymbol{x}) = O_{\|\boldsymbol{x}\| \to \infty}(1)$ one finds by straightforward computation that the total energy defined by

$$H(t) := \sum_{i=1}^N \left[\sigma_i \sqrt{m_i^2 + \boldsymbol{p}_{i,t}^2} + \frac{1}{8\pi} \int d^3x \left(\boldsymbol{E}^2_{i,t} + \boldsymbol{B}^2_{i,t} \right) \right]$$

is a constant of motion, where we used the notation $(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{E}_{i,t})_{1 \le i \le N} = M_L(t, t_0)[\varphi^0]$ from Section 3.5p.30. However, for weights $w \in \mathcal{W}$ such that $w(\boldsymbol{x}) \to 0$ for $\|\boldsymbol{x}\| \to \infty$, the integrals in the expression of $H(t)$ diverge and the total energy is infinite.

(iii) In the case of (ML-SIp.16), i.e. $e_{ij} = 1 - \delta_{ij}$, the total energy is generically not conserved which can be understood as follows: In this case the time derivative of the electric field $\boldsymbol{E}_{i,t}$ in (3.2p.15) depends on the position $\boldsymbol{q}_{i,t}$ and velocity $\boldsymbol{v}(\boldsymbol{p}_{i,t})$ of the i-th charge which means that the charge can transfer energy by means of radiation to the field degrees of freedom. On the other hand the Lorentz force law acting on the i-th charge (3.1p.15) does not depend on the i-th field since $e_{ii} = 0$. Therefore, the i-th charge cannot be in turn decelerated whenever it radiates. This

way the charges can "pump" energy into their field degrees of freedom without "paying" by loss of kinetic energy. However, for the special initial conditions (2.15p.12) as discussed in Chapter 2p.7 the ML-SI equations inherit constants of motion from the Wheeler-Feynman equations. Unlike for ML+SI equations, it is expected that not only scattering states but also bound states like the Schild solution [Sch63] exist.

3.6 Conclusion and Outlook

Limit to Point-Like Charges. As we have discussed in the preface and in Chapter 2p.7, from a physical point of view it would be desirable to take the limit to the point particles $\varrho_i \to \delta^3$. While in the case of (ML+SIp.15) the equations of motions are ill-defined in the point particle limit, this is not the case for (ML-SIp.16) as long as the charges stay away from each other. At least for two charges of equal sign we can expect existence and uniqueness for almost all initial conditions. In such a case it is expected that the analogue of the existence and uniqueness proof for point-like charges stays essentially the same except for modifications of the norms. In a sequel we shall discuss the point particle limit of the (ML-SIp.16) case.

Thermal States. Furthermore, as pointed out in [Spo04] for thermal states at non-zero temperature, one expects the electric and magnetic fields to fluctuate without decay. So it seems natural to check if the presented treatment of the Maxwell-Lorentz equations for fields in L^2_w suffices to treat also such thermal states; recall that the weight function w has to be chosen to decay slower than exponentially.

Chapter 4

Wheeler-Feynman Equations of Motion

4.1 Chapter Overview and Results

This chapter purports the so-called *Wheeler-Feynman equations* for rigid charges. Recall the Wheeler-Feynman equations for point-like charges we discussed in Chapter 2p.7. They were given by the equations (2.16p.12) which are essentially the Lorentz force equations for the Wheeler-Feynman fields given by one half of the sum of the advanced and retarded Liénard-Wiechert fields for potentials (2.5p.8). As in the chapter before, we use non-relativistic notation in a special reference frame (cf. Section 5.1p.95) and in order to circumvent the trajectory crossing problem we smear out again the point-like charges by some smooth and compactly supported functions on $\mathbb{R}^3$. These equations then describe a system of $N \in \mathbb{N}$ classical, non-rotating, rigid charges interacting directly via the relativistic action-at-distance principle and can be written in the form of the Lorentz force law (3.1p.15):

Wheeler-Feynman eqautions for rigid charges

$$\begin{aligned} \partial_t \boldsymbol{q}_{i,t} &= \boldsymbol{v}(\boldsymbol{p}_{i,t}) := \frac{\sigma_i \boldsymbol{p}_{i,t}}{\sqrt{m_i^2 + \boldsymbol{p}_{i,t}^2}} \\ \partial_t \boldsymbol{p}_{i,t} &= \sum_{j \neq i} \int d^3x \, \varrho_i(\boldsymbol{x} - \boldsymbol{q}_{i,t}) \left(\boldsymbol{E}^{\mathrm{WF}}_{j,t}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{q}_{i,t}) \wedge \boldsymbol{B}^{\mathrm{WF}}_{j,t}(\boldsymbol{x}) \right) \end{aligned} \tag{4.1}$$

for $1 \leq i \leq N$ with a special choice of fields, the *Wheeler-Feynman fields*, given by

$$\begin{pmatrix} \boldsymbol{E}^{\mathrm{WF}}_{i,t} \\ \boldsymbol{B}^{\mathrm{WF}}_{i,t} \end{pmatrix} = \frac{1}{2} \sum_{\pm} 4\pi \int ds \int d^3y \, K^{\pm}_{t-s}(\boldsymbol{x} - \boldsymbol{y}) \begin{pmatrix} -\nabla \varrho_i(\boldsymbol{y} - \boldsymbol{q}_{i,s}) - \partial_s \left(\boldsymbol{v}(\boldsymbol{p}_{i,s}) \varrho_i(\boldsymbol{y} - \boldsymbol{q}_{i,s}) \right) \\ \nabla \wedge \left(\boldsymbol{v}(\boldsymbol{p}_{i,s}) \varrho_i(\boldsymbol{y} - \boldsymbol{q}_{i,s}) \right) \end{pmatrix}. \tag{4.2}$$

Here, $K^{\pm}_t(\boldsymbol{x}) := \triangle^{\pm}(t, \boldsymbol{x}) = \frac{\delta(\|\boldsymbol{x}\| \pm t)}{4\pi \|\boldsymbol{x}\|}$ are the advanced and retarded Green's functions of the d'Alembert operator. We shall use the same notation and terminology as for the Maxwell-Lorentz equations in Chapter 3p.15, i.e. at time t the ith charge for $1 \leq i \leq N$ is situated at position $\boldsymbol{q}_{i,t}$ in space $\mathbb{R}^3$, momentum $\boldsymbol{p}_{i,t} \in \mathbb{R}^3$ and carries the classical mass $m_i \in \mathbb{R} \setminus \{0\}$ while $\sigma_i := \mathrm{sign}(m_i)$ allows for negative masses. The geometry of the rigid charge is given by the charge distribution $\varrho_i \in C^\infty_c(\mathbb{R}^3, \mathbb{R})$ for $1 \leq i \leq N$. In the following we shall refer to the Wheeler-Feynman equations (4.1) and (4.2) as WF equations.

Guiding questions

The WF equations are more subtle than they look because $\boldsymbol{E}^{\mathrm{WF}}_{i,t}$ and $\boldsymbol{B}^{\mathrm{WF}}_{i,t}$ are functionals of the ith charge trajectory $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})$. In contrast to the Maxwell-Lorentz equations, these fields appear here only as mathematical entities, and the only dynamical degrees of freedom are the charge trajectories. In fact, the change of the state of motion of the ith charge, i.e. the left-hand side of (4.1), is given in terms of the future and the past of all $j \neq i$ charge trajectories with respect to time t. Equations of this type are commonly called *functional differential equations*. Such equations have in general very different mathematical properties compared to ordinary or partial differential equations regarding existence, uniqueness and regularity. In contrast to the intuition in usual classical mechanics, it is completely unclear whether the WF equations allow for a well-defined initial value problem for Newtonian Cauchy data, i.e. position and momentum at time zero. This also means that it is unclear how to speak about possible solutions yet and, therefore, our leading questions of this chapter are:

1. How can we speak about solutions and by means of which data can we tell solutions apart?
2. Do Wheeler-Feynman solutions exist for given Newtonian Cauchy data?

We will answer both questions partly: With regard to question 1 we shall show that for possible Wheeler-Feynman solutions with bounded accelerations and momenta it is sufficient to specify positions, momenta and Wheeler-Feynman fields at any time t_0 in order to tell them apart; the question what part of this data is also necessary remains open. Concerning question 2 we shall show further that for given Newtonian Cauchy data there exist Wheeler-Feynman solutions on finite time (though arbitrarily large) intervals; the question if there are Wheeler-Feynman solution for all times remains open.

Mathematical results in this chapter

We begin our survey with Section 4.2p.52 where we familiarize with functional differential equations and their curiosities, amongst others: non-uniqueness for Cauchy data, non-smoothing and non-existence, by means of simple examples. In order to get a feeling for the WF equations we continue the discussion with a toy model for Wheeler-Feynman electrodynamics for two repelling charges interacting only by advanced and delayed Coulomb fields in Subsection 4.2.1p.53. The remarkable feature of this toy model is that for given strips of charge trajectories solutions can be explicitly constructed in a piecewise manner. In particular, it can be seen that without demanding any regularity properties, solutions, albeit not unique, for any given Newtonian Cauchy data exist; whether more regularity could yield uniqueness remains an interesting open question. This toy model makes us confident that we do not have to fear non-existence of Wheeler-Feynman solutions for given Newtonian Cauchy data. We conclude this introductory section with Subsection 4.2.2p.55 where we discuss an idea of reformulating certain functional differential equations in terms of initial value problems, using the following strategy: We regard the functional differential equations of the following type

$$x'(t) = V(x(t), f(t,x)) \qquad \text{for} \qquad f(t,x) = \frac{1}{2}\sum_{\pm}\left[f^{\pm T} + \int_{\pm T}^{t} ds\; W(x(s))\right] \tag{4.3}$$

for given $f^{\pm T}$, functions V, W and fixed $T > 0$ and investigate the well-posedness of the initial value problem for prescribed Cauchy data $x(t)|_{t=0} = x^0$. To see the relationship with the WF equations, think of $x(t)$ as being position and momenta and $f(t)$ to represent the Maxwell fields at time t. The sum over $\pm$ is the sum over retarded as well as advanced fields. This problem can be recast into initial value problem by enlarging the phase space where we regard

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ f(t) \end{pmatrix} = \begin{pmatrix} V(x(t), f(t,x)) \\ W(x(s)) \end{pmatrix}. \tag{4.4}$$

Given initial data $(x(t), f(t))|_{t=0} = (x^0, f^0)$ there exist unique solutions $t \mapsto M_t[x^0, f^0] = (x(t), f(t))$ of this set of equations, however, in order to solve (4.3) for given x^0 we need to give the appropriate initial conditions f^0. The idea is to construct f^0 by iteration of the map

$$S[x^0, f^0] := \frac{1}{2}\sum_{\pm}\left[f^{\pm T} + \int_{\pm T}^{t} ds\ W(x(s))\right] \quad \text{for } x(s) \text{ being the first component of } M_s[x^0, f^0]. \tag{4.5}$$

For finite T the role of $f^{\pm T}$ is to fix the incoming advanced and outgoing retarded fields $f^{\pm T}$ at time $\pm T$. For $T \to \infty$ one may set $f^{\pm\infty} = 0$ in analogy with the Maxwell equations which forget their asymptotic incoming and outgoing initial fields. We discuss how for small T one can still expect unique solutions if V and W are regular enough. However, for big or infinite T, though one might still get existence of the fixed point of S, it is not clear anymore how to ensure uniqueness of solutions. Nevertheless, this method is the key idea behind our later existence proof of Wheeler-Feynman solutions on finite time intervals for given Newtonian Cauchy data.

The main mathematical results which take up the aforementioned two questions for the Wheeler-Feynman electrodynamics are then given in the two remaining sections. We start in Section 4.3p.59 where we deal with question one. As discussed in the end of the introductory Chapter 2p.7, there is an intimate connection between Wheeler-Feynman and ML-SI dynamics. The idea is to exploit this feature in the following way: If there exists any solution to the WF equations then we know that its charge trajectories $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\leq i\leq N}$ fulfill the Lorentz force law (4.1p.43) for the Wheeler-Feynman fields $t \mapsto (\boldsymbol{E}^{\text{WF}}_{i,t}, \boldsymbol{B}^{\text{WF}}_{i,t})_{1\leq i\leq N}$ given by (4.2p.43) and, moreover, the Wheeler-Feynman fields fulfill the Maxwell equations by definition of the Liénard-Wiechert fields. Hence, the map $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}^{\text{WF}}_{i,t}, \boldsymbol{B}^{\text{WF}}_{i,t})_{1\leq i\leq N}$ constitutes a solution to the ML-SI equations. Now if the Wheeler-Feynman fields are actually compatible with the set of initial values $D_w(A)$ for which our existence and uniqueness theorem of the last chapter holds (in fact, we will use $D_w(A^\infty)$ to yield strong solutions), we can conclude that this ML-SI solution is uniquely identified by specifying positions, momenta of the charge trajectories and the Wheeler-Feynman fields at one instant time t. More precisely, we shall show that for $\mathcal{T}_{\text{WF}}$ being the set of all once differentiable Wheeler-Feynman solutions $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\leq i\leq N}$ with bounded accelerations and momenta, the map

$$i_{t_0} : \mathcal{T}_{\text{WF}} \to D_w(A^\infty), \qquad \left(t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\leq i\leq N}\right) \mapsto (\boldsymbol{q}_{i,t_0}, \boldsymbol{p}_{i,t_0}, \boldsymbol{E}^{\text{WF}}_{i,t_0}, \boldsymbol{B}^{\text{WF}}_{i,t_0})_{1\leq i\leq N} \tag{4.6}$$

is well-defined and injective for all $t_0 \in \mathbb{R}$. With the preliminary work of the last chapter the only involving part in the proof is to show that the Wheeler-Feynman fields are compatible with the set of initial values $D_w(A^\infty)$ for which the existence and uniqueness results of solutions to the ML-SI equations hold. For this we solve the Maxwell equations (3.2p.15) explicitly for prescribed and sufficient regular charge trajectories $t \mapsto (\boldsymbol{q}_t, \boldsymbol{p}_t)$ and initial fields $(\boldsymbol{E}^0, \boldsymbol{B}^0)$ at some time t_0 which is the content of Subsection 4.3.1p.60. This is done by rewriting the Maxwell equations into an inhomogeneous wave equation

$$\square \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} = 4\pi \begin{pmatrix} -\nabla\rho_t - \partial_t \boldsymbol{j}_t \\ \nabla \wedge \boldsymbol{j}_t \end{pmatrix}$$

for $\rho_t := \varrho(\cdot - \boldsymbol{q}_t), \boldsymbol{j}_t := \boldsymbol{v}(\boldsymbol{p}_t)\varrho(\cdot - \boldsymbol{q}_t)$ and charge density ϱ such that any solution can then be constructed by inverting the d'Alembert operator in terms of the propagator $K_t = K_t^- - K_t^+$. We yield explicit formulas for the electric and magnetic field which we shall refer to as Kirchoff's formulas:

$$\begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \boldsymbol{E}^0 \\ \boldsymbol{B}^0 \end{pmatrix} + K_{t-t_0} * \begin{pmatrix} -4\pi\boldsymbol{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\ K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \tag{4.7}$$

and which for a charge trajectory with mass $m \neq 0$ we usually abbreviate by

$$M_{m,\varrho}[(\boldsymbol{E}^0, \boldsymbol{B}^0), (\boldsymbol{q}, \boldsymbol{p})](t, t_0). \tag{4.8}$$

The same formulas (after a partial integration) have already been found by Komech and Spohn using a slightly different technique [KS00]. Based on these Kirchoff formulas for given charge trajectories and initial fields at some time t_0 we derive the explicit form of the Liénard-Wiechert fields by a limit procedure for which t_0 is sent to $\pm\infty$ and show that the limit still fulfills the Maxwell equations (though the Liénard-Wiechert formulas are well-known for over a hundert years in the physical community we had trouble finding any reference of a rigorous derivation along with a proof that they solve the Maxwell equations in the mathematical literature). The well-known result [Jac98, Roh94, Spo04] is:

$$\begin{pmatrix} \boldsymbol{E}_t^{\pm} \\ \boldsymbol{B}_t^{\pm} \end{pmatrix} = \int d^3z\, \varrho(z) \begin{pmatrix} \boldsymbol{E}_t^{LW\pm}(\cdot - z) \\ \boldsymbol{B}_t^{LW\pm}(\cdot - z) \end{pmatrix}$$

for

$$\boldsymbol{E}_t^{LW\pm}(\boldsymbol{x}) := \left[\frac{(\boldsymbol{n} \pm \boldsymbol{v})(1 - \boldsymbol{v}^2)}{\|\boldsymbol{x} - \boldsymbol{q}\|^2 (1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} + \frac{\boldsymbol{n} \wedge [(\boldsymbol{n} \pm \boldsymbol{v}) \wedge \boldsymbol{a}]}{\|\boldsymbol{x} - \boldsymbol{q}\| (1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} \right]^{\pm}$$
$$\boldsymbol{B}_t^{LW\pm}(\boldsymbol{x}) := \mp [\boldsymbol{n} \wedge \boldsymbol{E}_t(\boldsymbol{x})]^{\pm}$$

where $\boldsymbol{n}$ is the normalized version of the vector $\boldsymbol{x} - \boldsymbol{q}$, $\boldsymbol{v}$ is the velocity and $\boldsymbol{a}$ is the acceleration while the superscript $\pm$ denotes that these entities must be evaluated at advanced and delayed time with respect to the space-time point $(\boldsymbol{x}, t)$; cf. Subsection 4.3.1p.60. As discussed in the overview of the last chapter 3.1p.15, the "worst" behaving Wheeler-Feynman trajectories we expect are the Schild solutions and those have bounded accelerations and momenta. For such charge trajectories the term, depending on the accelerations in the Liénard-Wiechert fields, does not decay fast enough for the Liénard-Wiechert fields to be square integrable. But exactly for this we have introduced the weight w in the ML-SI existence and uniqueness theorem and we show that there is a w, e.g. $w(\boldsymbol{x}) = (1 + \|\boldsymbol{x}\|^2)^{-1}$, being conform with the requirements of the theorem and in addition modulating the missing decay in the acceleration term such that $\boldsymbol{E}_t^{\pm}, \boldsymbol{B}_t^{\pm} \in L_w^2$. Coming back to the Wheeler-Feynman fields which we need to be compatible with $D_w(A^\infty)$ and which are defined by one half of the sum of the advance and retarded Liénard-Wiechert fields we yield $(\boldsymbol{q}_{i,t_0}, \boldsymbol{p}_{i,t_0}, \boldsymbol{E}_{i,t_0}^{\mathrm{WF}}, \boldsymbol{B}_{i,t_0}^{\mathrm{WF}})_{1\le i\le N} \in D_w(A^\infty)$ where we owe the regularity to the convolution representation.

In order to answer question 2 for the class of Wheeler-Feynman solutions $\mathcal{T}_{\mathrm{WF}}$, one had to determine the range of the map i_{t_0} but this task is beyond the present understanding of the WF equations. Yet we start looking for an answer to this in Section 4.4p.73 in the following way: With the characterization of Wheeler-Feynman solutions in $\mathcal{T}_{\mathrm{WF}}$ by i_{t_0} at hand it is natural to ask whether there exist Wheeler-Feynman solutions for any prescribed Newtonian Cauchy data $p = (\boldsymbol{q}_i^0, \boldsymbol{p}_i^0)_{1\le i\le N} \in \mathbb{R}^{6N}$ which are positions and momenta of the N charges at time zero postponing the question of uniqueness. For this we reformulate the Wheeler-Feynman functional differential equations (4.1p.43) and (4.2p.43) into an initial value problem as in the discussed example (4.3p.44). The analogue of equations (4.4p.44) are the ML-SI equations which we have treated in the last chapter; recall the notation of the ML-SI solutions $(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N} = M_L[p, F](t, 0)$ for Newtonian Cauchy data p and initial electromagnetic fields $F = (\boldsymbol{E}_i^0, \boldsymbol{B}_i^0)_{1\le i\le N}$ at time zero. In order to simplify the notation we denote the Liénard-Wiechert fields of a charge trajectory $t \mapsto (\boldsymbol{q}_t, \boldsymbol{p}_t)$ with mass m and charge density ϱ at time t by $M_{m,\varrho}[(\boldsymbol{q}, \boldsymbol{p})](t, \pm\infty)$. The task is now to find ML-SI equations whose charge trajectories obey the Newtonian Cauchy data

$$(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\le i\le N}\Big|_{t=0} = p \tag{4.9}$$

and whose fields at time zero are the Wheeler-Feynman fields (4.2p.43), i.e. in our new notation

$$(\boldsymbol{E}_{i,t}^{\mathrm{WF}}, \boldsymbol{B}_{i,t}^{\mathrm{WF}}) = \frac{1}{2} \sum_{\pm} M_{\varrho_i, m_i}[(\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm\infty) \tag{4.10}$$

at time $t = 0$. As in the example above, it seems natural to construct such initial fields by iteration of the map S^p (this is the analogue of (4.5p.45) for $T \to \infty$):

INPUT: $F = (\boldsymbol{E}_i^0, \boldsymbol{B}_i^0)_{1\le i\le N}$ such that $(p, F) \in D_w(A)$.

The Fixed Point Map

(i) Compute the ML-SI solution $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N} := M_L[p, F](t, 0)$.

(ii) Compute the Wheeler-Feynman fields for $1 \le i \le N$

$$(\boldsymbol{E}_{i,t}^{\mathrm{WF}}, \boldsymbol{B}_{i,t}^{\mathrm{WF}}) = \frac{1}{2} \sum_{\pm} M_{\varrho_i, m_i}[(\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm\infty).$$

OUTPUT: $S^p[F] := (\boldsymbol{E}_{i,0}^{\mathrm{WF}}, \boldsymbol{B}_{i,0}^{\mathrm{WF}})_{1\le i\le N}$.

By construction any fixed point needs to be a solution to the WF equations (4.1p.43) and (4.10p.46) for Newtonian Cauchy data (4.9p.46). Therefore, it suffices to show the existence of a fixed point of S^p. The advantages compared to other fixed point approaches are twofold: First, existence and uniqueness of solutions to functional differential equations can now be studied by the fixed point methods of nonlinear functional analysis. And second, instead of working with a norm on the space of the charge trajectories we only need to find a suitable norm on the space of initial fields at time zero. However, there are two apparent difficulties:

1. If $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N}$ as computed in step (i) has unbounded accelerations and/or momenta, then the Wheeler-Feynman Fields computed in step (ii) need not to be well-defined; cf. Theorem 4.18p.66.

2. Even if the Wheeler-Feynman Fields from step (ii) are well-defined, $(p, S^p[F])$ does not need to lie in $D_w(A)$.

That the momenta and accelerations of charge trajectories of solutions to the ML-SI equations cannot be bounded by the initial conditions in a simple way as it is the case for the ML+SI dynamics is due to the fact that we lack any kind of energy conservation on equal time hypersurfaces. These difficulties force us to regard a related problem first which, however, only yields charge trajectories which obey the Wheeler-Feynman equations on a finite but arbitrarily large time interval. In Subsection 4.5p.92 we then discuss how the applied method could also yield true Wheeler-Feynman solutions for all times.

Since the Wheeler-Feynman fields (4.10p.46) solve the Maxwell equations (as stated in Theorem 4.21p.68), we can express them for any $T > 0$ by

$$(\boldsymbol{E}_{i,t}^{\mathrm{WF}}, \boldsymbol{B}_{i,t}^{\mathrm{WF}}) = \frac{1}{2} \sum_{\pm} M_{\varrho_i, m_i}[X_{i,\pm T}^{\pm}, (\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm T) \tag{4.11}$$

for the advanced and retarded Liénard-Wiechert fields

$$X_{i,\pm T}^{\pm} := (\boldsymbol{E}_{i,\pm T}^{\mathrm{WF}}, \boldsymbol{B}_{i,\pm T}^{\mathrm{WF}}) = M_{\varrho_i, m_i}[(\boldsymbol{q}_i, \boldsymbol{p}_i)](\pm T, \pm\infty) \tag{4.12}$$

where we have been using the notation (4.8p.45). Let us then assume the fields $X_{i,\pm T}^{\pm}$ as given in terms of a function of the charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ for $1 \le i \le N$ such that they fulfill the correct Maxwell constraints at time T, respectively $-T$. In contrast to the first approach we now need to find trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N}$ that fulfill the equations (4.1p.43) and (4.11) which shall be denoted as the bWF equations (which stands for "boundary field Wheeler-Feynman equations"). However, if for $1 \le i \le N$ those fields $X_{i,+T}^{+}$, respectively $X_{i,-T}^{-}$, are for example advanced, respectively retarded, Liénard-Wiechert fields generated by well-behaving trajectories in $\mathcal{T}_{\mathrm{WF}}$, then by Theorem 4.28p.72 the corresponding ML-SI solution takes values in $D_w(A^\infty)$ for some appropriate weight function w. Furthermore, as the time-evolution of the Wheeler-Feynman fields in (4.11) goes only over a time interval $[-T, T]$ we only need to control the ML-SI solution there. For finite $T > 0$ we always get a bound on the maximal momentum and acceleration of all charge trajectories. This way the difficulties of the first approach are shifted to the existence of the boundary fields $X^{\pm}_{i,\pm T}$, $1 \le i \le N$ which we will discuss later. Again, we can formulate the question of the existence of solutions in terms of the altered fixed point map $S^{p,X^{\pm}}_{T}$:

The Fixed Point Map for given Boundary Fields

INPUT: $F = (\boldsymbol{E}^0_i, \boldsymbol{B}^0_i)_{1\le i\le N}$ for any fields such that $(p, F) \in D_w(A^\infty)$.

(i) Compute the ML-SI solution $[-T, T] \ni t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N} := M_L[p, F](t, 0)$.

(ii) Compute the fields for $1 \le i \le N$

$$(\widetilde{\boldsymbol{E}}_{i,t}, \widetilde{\boldsymbol{B}}_{i,t}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[X^{\pm}_{i,\pm T}, (\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm T) \tag{4.13}$$

where $X^{\pm}_{i,\pm T}$ are given functions of the charge trajectory $(\boldsymbol{q}_i, \boldsymbol{p}_i)$ for $1 \le i \le N$.

OUTPUT: $S^{p,X^{\pm}}_{T}[F] := (\widetilde{\boldsymbol{E}}_{i,0}, \widetilde{\boldsymbol{B}}_{i,0})_{1\le i\le N}$.

By construction any fixed point solves the bWF equations (4.1p.43) and (4.11p.47) with prescribed $(X^{\pm}_{i,\pm T})_{1\le i\le N}$ for Newtonian Cauchy data (4.9p.46). Note that any fixed point then automatically fulfills the Maxwell constraints at time zero, i.e. $\nabla \cdot \boldsymbol{E}^0_i = 4\pi\varrho_i(\cdot - \boldsymbol{q}^0_i)$ and $\nabla \cdot \boldsymbol{B}^0_i = 0$, for $1 \le i \le N$, because the boundary fields fulfill them at times $\pm T$.

This way we only get a Wheeler-Feynman interaction on a time interval within $[-T, T]$ for prescribed asymptotes whose shape for times bigger than T, respectively smaller than $-T$ is determined by the choice of $X^{\pm T}$ if one has not by chance taken the choice (4.12p.47) for the boundary fields; see Figure 4.1p.51. We shall prove existence of fixed points of $S^{p,X^{\pm}}_{T}$ for a convenient class of boundary fields (which include Liénard-Wiechert fields of charge trajectories in $\mathcal{T}_{\mathrm{WF}}$). Note that even if this class (4.12p.47) were too small to include for all Wheeler-Feynman solutions the fixed points of $S^{p,X^{\pm}}_{T}$ are still of significance because:

1. Choosing boundary fields $X^{\pm}$ which are of the form of the advanced and retarded Liénard-Wiechert, we shall show that for any $T > 0$ and T-dependent restrictions on the Newtonian Cauchy data p and charge densities ϱ_i we can always find charge trajectories that fulfill the true WF equations (4.1p.43) and (4.2p.43) for a non-zero interval within $[-L, L] \subset [-T, T]$; see Figure 4.1p.51.
2. As for large T, assuming the charge trajectories are strictly time-like and have velocity bounds smaller than one, $M_{\varrho_i,m_i}[X^{\pm}_{i,\pm T}, (\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm T)$ converges pointwise in $\mathbb{R}^3$ to $M_{\varrho_i,m_i}[(\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm\infty)$ which is independent of the boundary fields $X^{\pm}$ (cf. Theorem 4.18p.66), one can expect that a fixed point of $S^{p,X^{\pm}}_{T}$ is in some sense close to a true Wheeler-Feynman solution.

Let us sketch the idea behind the proof for the existence of fixed points of the map $S^{p,X^{\pm}}_{T}$. We aim at applying Banach's Fixed Point Theorem for small times T and Schauder's Fixed Point Theorem [Eva98, Chapter 9, Theorem 3, p.502] for all finite times T:

Schauder's Fixed Point Theorem

Theorem 4.1. *Let $\mathcal{B}$ denote a Banach space and $K \subset \mathcal{B}$ be compact and convex. Assume $S : K \to K$ is continuous, then S has a fixed point.*

We shall mostly work with the field space $\mathcal{F}_w := \bigoplus_{i=1}^N L^2_w \oplus L^2_w$ instead of the phase space of the ML-SI dynamics so that it is convenient to introduce the operators A, J which are projections of the operators A and J defined in the last chapter onto their field components. One can see immediately that A generates a γ-contraction group $(\mathsf{W}_t)_{t\in\mathbb{R}}$ on its natural domain $D_w(\mathsf{A})$ which, again, is the projection of $D_w(A)$ onto its field components. A natural choice for the Hilbert spaces of fields is

$$\mathcal{F}^n_w := \{F \in \mathcal{F}_w \mid \mathsf{A}^j F \in \mathcal{F}_w, 1 \le j \le n\} \qquad \text{with the norm} \qquad \|\cdot\|_{\mathcal{F}^n_w} := \sum_{k=0}^n \|\mathsf{A}^k \cdot\|_{\mathcal{F}_w}.$$

for $n \in \mathbb{N}$. For fixed $T > 0$ the boundary fields $X^\pm_{i,T}$ are defined such that they are functions of $(p, F) \in \mathcal{H}_w$ instead of charge trajectories as it was discussed above. The reason for that is that we can recover the charge trajectories by the ML-SI time-evolution of the initial value (p, F). With the appropriate regularity conditions on the boundary fields one can define the map $F \mapsto S^{p,X^\pm}_T[F]$ as a continuous self-mapping on $\mathcal{F}^1_w$. The most sensitive part of the proof is that the range of $S^{p,X^\pm}_T$ must be compact in order to apply Schauder's Fixed Point Theorem. This can be shown thanks to the fact that the fields generated by the charge trajectories on the time interval $[-T, T]$ can be bounded by a finite constant depending only on the Newtonian Cauchy data p and the time T. The reason that the bound is uniform in all initial fields F can be seen by rewriting (4.13p.48)

$$(\widetilde{\boldsymbol{E}}_{i,t}, \widetilde{\boldsymbol{B}}_{i,t}) = \frac{1}{2}\left(\mathsf{W}_{\mp T} X^\pm_{\pm T}[p, F] + \int_{\mp T}^0 ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s)\right) \tag{4.14}$$

where $X^\pm_{\pm T}[p, F] := (X^\pm_{i,\pm T}[p, F])_{1\le i\le N}$ and $\varphi_s := M_L[p, F](s, 0)$. The first summand is the contribution of the boundary fields which behaves as we wish. The second contribution comes from the integrated current of the charge trajectories over the interval $[-T, T]$. The estimate of the norm of the integrand in (4.14) for $\varphi_s = (\boldsymbol{q}_{i,s}, \boldsymbol{p}_{i,s}, \boldsymbol{E}_{i,s}, \boldsymbol{B}_{i,s})_{1\le i\le N}$

$$\|\mathsf{J}(\varphi_s)\|_{\mathcal{F}_w} \le \sum_{i=1}^N \|4\pi \boldsymbol{v}(\boldsymbol{p}_{i,s})\varrho_i(\cdot - \boldsymbol{q}_{i,t})\|_{L^2_w} \le \sum_{i=1}^N \|4\pi\varrho_i(\cdot - \boldsymbol{q}_{i,t})\|_{L^2_w}$$

depends only on the position of the charges at time s (since the measure $\sqrt{w} d^3x$ is not translational invariant). But as the velocities are always smaller than the speed of light, their position can be bounded by $\|p\| + s$. This observation implies that the range of $S^{p,X^\pm}_T$ is bounded. If we can in addition show that it is also compact, we can then take K to be the closed convex hull of it which is again compact, restrict $S^{p,X^\pm}_T$ to $K \to K$ so that Schauder's Fixed Point Theorem would ensure the existence of a fixed point in K. In order to show compactness we consider sequences in the range of $S^{p,X^\pm}_T$. Because of their boundedness the Banach-Alaoglu theorem then states that they have a $\mathcal{F}^1$ weakly convergent subsequence, and it is left to show that this subsequence also converges strongly. To show this, we only have to make sure that the subsequence of fields does not oscillate too wildly and that no spatially outgoing spikes are formed. By imposing further conditions on the boundary fields so that they behave as we wish we again have only to concentrate on the fields which are created by the current of the charge trajectories on the time interval $[-T, T]$. By the finite propagation speed of the Maxwell equations those created fields have support within a ball around the initial positions of the charge with radius $\|p\|_{\mathbb{R}^{6N}} + 2T + R$ for balls $B_R(0)$ of radius $R > 0$ around the origin such that $\operatorname{supp} \varrho_i \subseteq B_R(0)$. Furthermore, taking into account the Maxwell constraints, we can also find bounds on the Laplacian of those fields which depend only on T and $\|p\|_{\mathbb{R}^{6N}}$ where we use a similar technique than in the regularity proof of the Maxwell-Lorentz equations in the last chapter. Hence, there is no formation of spikes and the oscillations are mild so that we are able to show that the subsequence is also strongly convergent in $\mathcal{F}^1$. For small enough times and additional requirements on the boundary fields one can even apply Banach's fixed point theorem to ensure also uniqueness which is due to the Lipschitz continuous dependence on the initial values of the ML-SI dynamics. The exact requirements for the boundary fields are condensed in Definition 4.38p.75 where we define the classes of boundary fields. In Lemma 4.45p.79 we show that the Coulomb field is in all these classes but it is expected (though not shown) that all Liénard-Wiechert fields of charge trajectories with bounded acceleration and momentum are also included. In this sense the boundary fields can be seen as being the advanced and retarded Liénard-Wiechert fields of the asymptotes of the actual charge trajectories on time interval $[-T, T]$. In a last paragraph we then show that among these fixed point there are fields F such that the charge trajectories $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\leq i\leq N}$ of the ML-SI solution with initial value (p, F) also fulfill the true WF equations on a finite time interval. In order to see this we regard the difference of the true Wheeler-Feynman fields (4.10p.46) and fields (4.11p.47) depending on the boundary fields for the ith charge trajectory in terms of Kirchoff formulas (4.7p.45)

$$
\begin{aligned}
&M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm T) - M_{\varrho_i,m_i}[(\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm\infty) \\
&= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X^\pm_{i,\pm T} + K_{t\mp T} * \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_{i,\pm T})\varrho_i(\cdot - \boldsymbol{q}_{i,\pm T}) \\ 0 \end{pmatrix} \\
&\quad - 4\pi \int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \boldsymbol{q}_{i,s}) \\ \boldsymbol{v}(\boldsymbol{p}_{i,s})\varrho_i(\cdot - \boldsymbol{q}_{i,s}) \end{pmatrix}.
\end{aligned}
$$

Whenever this difference is zero, at least within tubes around the positions of all other $j \neq i$ charge trajectories for times t in some interval $[-L, L]$ for $L > 0$, they also solve the true Wheeler-Feynman equations on this time interval. To show that such a non-empty time interval exists we make use of the property of the Maxwell equations that Liénard-Wiechert initial fields $X^\pm_{i,\pm T}$ are cleared to zero by the first term above within the forward, respectively backward, light-cone of the space-time points $(\boldsymbol{q}_{i,\pm T}, \pm T)$ during the time-evolution to make way for the fields generated by the charge trajectory. We show this by direct computation in the case of the Coulomb fields as boundary fields using harmonic analysis. In the same region the third term, coming from the Wheeler-Feynman fields for times outside of the time interval $[-T, T]$, is naturally zero, too. Finally, the second term has support on the light-cone of space-time point $(\boldsymbol{q}_{i,\pm T}, \pm T)$ only. Hence, the above difference is zero within the intersection of the forward light-cone of $(\boldsymbol{q}_{i,-T}, -T)$ and the backward light-cone of $(\boldsymbol{q}_{i,T}, T)$.

Now it is only left to ensure that all charge trajectories stay inside this space-time region long enough so that $L > 0$. For this, if we had a uniform velocity estimate for the charge trajectories (which in the case of, for example, two charges of equal sign would be physically reasonable) we would only have to choose T large enough. However, the estimate we have for general initial conditions comes from the Gronwall estimate of the ML-SI dynamics which is T dependent. Therefore, we yield true Wheeler-Feynman solutions on a finite time interval only if the maximal distance of the initial positions is small enough. Furthermore, we have to account for the radius $R > 0$ of the extended charge which gives a restriction on the choice of the charge densities ϱ_i. However, it is strongly expected that this velocity estimate can be improved for more special initial conditions p so that all these conditions are only technicalities and this method would yield Wheeler-Feynman solutions on arbitrary large, finite time intervals.

Main results The main results of this chapter are:

1. The Kirchoff formulas for the Maxwell solutions and the derivation of the Liénard-Wiechert fields which are shown to fulfill the Maxwell equations.

2. The unique characterization of the Wheeler-Feynman solution class $\mathcal{T}_{\text{WF}}$.

3. The existence of fixed point for the map $S_T^{p,X^\pm}$ for any $T > 0$ and $p \in \mathbb{R}^{6N}$ and a class of boundary fields $X^\pm$ plus uniqueness for small enough T.

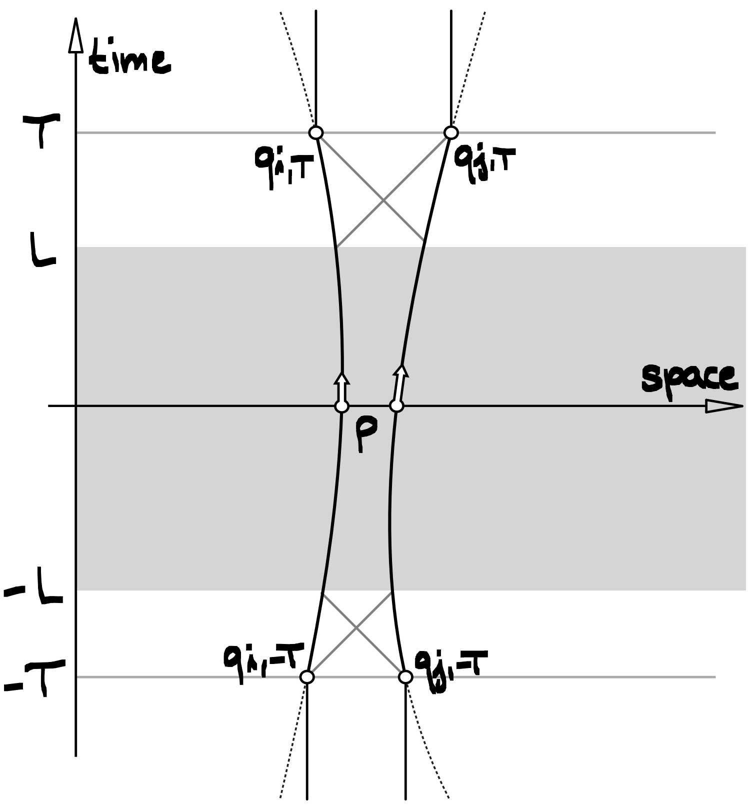


Figure 4.1: For $N = 2$ charges, some $T > 0$ and Newtonian Cauchy data p denoted by the white arrows, the charge trajectories $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1 \le i \le 2} := (\mathsf{Q} + \mathsf{P}) M_L[p, F](t, 0)$ for a hypothetical fixed point $F = S_T^{p,X^\pm}[F]$ are shown: solid black for $t \in [-T, T]$ and otherwise dashed. In this figure the boundary fields $X^\pm$ are chosen to be the Coulomb fields of charges at rest at $\boldsymbol{q}_{i,\pm T}$ for $i = 1, 2$. In this case the charge $i \neq j$ within the time interval $[-T, T]$ feels the Wheeler-Feynman fields of a trajectory being equal to the charge trajectory of charge j on time interval $[-T, T]$ and for times $t > \pm T$ being equal to the trajectory of a charge at rest at position $\boldsymbol{q}_{j,\pm T}$ (solid black straight lines). The shaded region denotes the time interval where the charge trajectories fulfill the true WF equations. The gray 45° degree lines are used to denote the intersection of the light cones of the space-time points $(T, \boldsymbol{q}_{i,\pm T})$ with the charge trajectories.

4. The existence of Wheeler-Feynman solutions on non-zero time intervals.

The Wheeler-Feynman equations appear only very sparsely in the mathematical literature. While some special solutions to the Wheeler-Feynman equations of motions were found [Sch63], general existence of solutions to these equations has only been settled in the case of restricted motion of two point particles with equal charge on a straight line in $\mathbb{R}^3$ [Bau97]. An even bigger, outstanding problem is the question how the solutions can be uniquely characterized, especially if it is possible to pose a well-defined initial value problem for WFED with Newtonian Cauchy data. Apart from the mentioned works, there exist only few discourses on WFED in the classical literature, for example special analytic solutions [Ste92], numerical approximation [DW65] and conjectures on as well as special cases of existence and uniqueness of solutions in one dimension [Dri69, Dri79]. In a recent work [Luc09] the Fokker variational principle for two charges in three dimensions is discussed mathematically, which can be used to yield Wheeler-Feynman solutions by specifying starting and ending points of the two world lines and giving in addition a part of the future of the first charge and a part of the past of the other charge. Without the Literature review restriction to the motion on the straight line, only conjectures about existence and uniqueness can be found, e.g. [WF49, DW65, And67, Syn76].

4.2 Functional Differential Equations

A functional differential equation is a differential equation that involves terms which are functionals of the solution. This way the state change given by a functional differential equation at a certain time may also depend on the past (delay equations), the future or on the entire history of the solution. The existence and uniqueness properties of this type of equations are in general very different from ordinary or partial differential equations. In order to become familiar with the concept of functional differential equations and its difficulties of an ordinary differential equation with delay, we borrow an example from [Dri77]: Let us look for solutions $x : \mathbb{R} \to \mathbb{R}$ to the differential equations

$$x'(t) = ax(t) + bx(t-r) \tag{4.15}$$

with coefficients $a, b \in \mathbb{R}$ and a constant delay $r \in \mathbb{R}$. As in the theory of ordinary differential equations commonly used we denote by $x'(t)$ the derivative $\frac{dx(t)}{dt}$. For b and r different from zero, the derivative of the solution x at time t depends not only on the solution at time t but also on the retarded time $t - r$.

Construction of solutions to a class of delay differential equations

The first question is whether (4.15) has solutions. This question can be answered constructively: Assume that we are given a once differentiable function x_0 on the interval $(-r, 0]$ (on the borders of the interval we only need the one-sided derivative) and we look for a solution x that fulfills

$$x(t) = x_0(t) \text{ on the interval } t \in (-r, 0]. \tag{4.16}$$

This initial function already specifies the solution x uniquely on the whole past $(-\infty, 0]$ as for example for $t \in (-2r, -r]$ we have

$$x(t) = \frac{x'(t+r) - ax(t+r)}{b} = \frac{x_0'(t+r) - ax_0(t+r)}{b}.$$

This construction can be continued inductively for each interval $(-nr, -(n-1)r]$ for $n \in \mathbb{N}$ to yield an almost everywhere differentiable solution x on the whole past.

Solutions do not necessarily inherit the smoothness of the initial function

However, we observe that this construction does not necessarily yield a continuous function as in general

$$\lim_{\epsilon \downarrow 0} x(-r-\epsilon) = \frac{x_0'(-\epsilon) - ax_0(-\epsilon)}{b} \neq \lim_{\epsilon \downarrow 0} x_0(-r+\epsilon) = \lim_{\epsilon \downarrow 0} x(-r+\epsilon). \tag{4.17}$$

Note that this does not only depend on the regularity of x_0 but also on (4.17). In order to give an example which we will use later, let us choose a special initial function x_0, e.g. such that for every $t \in \mathbb{R}$

$$x_0(t) := \begin{cases} 0 & \text{for } t \leq -r \\ e^{-t^{-2}} e^{-(t+r)^{-2}} & \text{for } -r < t < 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.18}$$

which is infinitely often differentiable and has the property that itself and each of its derivatives are zero at $t = 0$ and $t = -r$, i.e. $x_0^{(n)}(-r) = 0 = x_0^{(n)}(0)$ for $n \in \mathbb{N}_0$. This special choice of an initial function x_0 together with (4.16) then yields an infinitely often differentiable solution x on

$(-\infty, 0]$ since the left- and right-hand side of (4.17) are equal. For any continuous x_0 the future $[0, \infty)$ can be constructed by the *method of steps*. For $t \in (0, r]$ we have

$$x(t) = e^{at}\left(x_0(0) + b \int_0^t ds\ x_0(s-r)e^{-as}\right)$$

where the right-hand side depends only on the given function x_0. Step by step one yields x on $[0, \infty)$. This way we have constructed an infinitely often differentiable solution $x : \mathbb{R} \to \mathbb{R}$.

Uniqueness of solutions to delay differential equations

Having constructed one solution, the next question is uniqueness. Let us assume that x as well as y are continuous solutions to equation (4.15p.52) on $\mathbb{R}$, both fulfilling (4.16p.52) for a continuous x_0. By continuity the equations must fulfill the integral equation

$$\begin{aligned} x(t) - y(t) &= \int_0^t ds\ \left[a(x(s) - y(s)) + b(x(s-r) - y(s-r))\right] \\ &= a \int_0^t ds\ \left[x(s) - y(s)\right] + b \int_{-r}^{t-r} ds\ \left[x(s) - y(s)\right]. \end{aligned}$$

Using (4.16p.52) we have the estimate

$$|x(t) - y(t)| \le (|a| + |b|) \int_0^t ds\ |x(s) - y(s)|$$

for $t \ge 0$. Gronwall's lemma in differential form then yields $y(t) - x(t) = 0$ for all times $t \ge 0$. Hence, the $x_0(t) = x(t)$ for $t \in (-r, 0]$ determines the solution x uniquely for $t \ge 0$ and thus, together with the argument above, on whole $\mathbb{R}$.

Non-existence and non-uniqueness for Cauchy initial conditions

Initial conditions in form of a function $x_0 : (-r, 0] \to \mathbb{R}$ are uncommon in physical problems. Usually one hopes to tell solutions apart by giving Cauchy data at time $t = 0$. This becomes cumbersome for functional differential equations. In fact, there are counterexamples [Dri77] which prove the non-existence of solutions to linear delay equations even with constant coefficients for given Cauchy data $x(t)|_{t=0}$. And even if there is a solution for specific Cauchy initial conditions the solution will in general not be unique. As an example let us assume that $y : \mathbb{R} \to \mathbb{R}$ is a solution to (4.15p.52) fulfilling

$$y^{(n)}(t)|_{t=0} = c_n \text{ for } n \in \mathbb{N}_0 \tag{4.19}$$

and given constants $(c_n)_{n \in \mathbb{N}_0}$. Let x be the constructed solution to (4.15p.52) with initial condition (4.16p.52) for the choice (4.18p.52). Then we have

$$x^{(n)}(t) = 0 \text{ for } n \in \mathbb{N}_0$$

and therefore, by linearity, for every $c \in \mathbb{R}$, $y + cx$ is another solution fulfilling (4.19).

Note that an approximation of the delay by its Taylor series turns the delay differential equation into an ordinary differential equation and, hence, the approximate equation loses the discussed features like non-existence and non-uniqueness and therefore, the true delay character cannot be studied anymore.

4.2.1 Wheeler-Feynman Toy Model

So far we have discussed an example of a functional differential equation with delay. The next example we look at is an equation that includes also advanced effects. Let $\boldsymbol{x}, \boldsymbol{y} : \mathbb{R} \to \mathbb{R}^3$ be two trajectories of charges of equal sign which interact with their Coulomb fields only. The defining equations of the model are given by

Wheeler-Feynman toy model

$$\ddot{\boldsymbol{x}}(t) = \sum_{\pm} \frac{\boldsymbol{x}(t) - \boldsymbol{y}(t_x^\pm)}{\|\boldsymbol{x}(t) - \boldsymbol{y}(t_x^\pm)\|^3} \qquad \ddot{\boldsymbol{y}}(t) = \sum_{\pm} \frac{\boldsymbol{y}(t) - \boldsymbol{x}(t_y^\pm)}{\|\boldsymbol{y}(t) - \boldsymbol{x}(t_y^\pm)\|^3}$$
$$t_x^\pm = t \pm \|\boldsymbol{x}(t) - \boldsymbol{y}(t_x^\pm)\| \qquad t_y^\pm = t \pm \|\boldsymbol{y}(t) - \boldsymbol{x}(t_y^\pm)\| \tag{4.20}$$

In comparison to the example given in the introduction to this section, we have added two new features to the functional differential equations. First, the acceleration at time t of one trajectory does not only depend on the past but also on the future of the other trajectory. And second, the advance and delay are not static but depend on the state of the dynamical system – to be more precise on the actual position at some instant of time. Because of these two features one could regard these equations as a toy model of the Wheeler-Feynman equations.

Initial conditions

We want to get a feeling how much data of a solution is needed to identify it uniquely. The idea is to apply the same construction used for the example in the introduction of this section, where the solution to the whole past could be generated from a given piece of the solution. Let $\boldsymbol{x}_0 : D_x \to \mathbb{R}^3$ and $\boldsymbol{y}_0 : D_y \to \mathbb{R}^3$ for domains $D_x, D_y \subset \mathbb{R}$ be two time continuously differentiable functions. We assume a solution to $t \mapsto (\boldsymbol{x}(t), \boldsymbol{y}(t))$ of the equations (4.20) has initial conditions $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ if

$$\boldsymbol{x}(t) = \boldsymbol{x}_0(t), \text{ for } t \in D_x \qquad \text{and} \qquad \boldsymbol{y}(t) = \boldsymbol{y}_0(t), \text{ for } t \in D_y. \tag{4.21}$$

As the advance and delay at time $t = 0$ depends on the state of the dynamical system at $t = 0$, we cannot take any domains D_x, D_y. However, a sensible choice is apparent: $D_x, D_y \subset \mathbb{R}$ need to include the 0 and the advance and delay terms in (4.20) need to be well-defined, i.e. the equations

$$\tau_x^\pm = \pm\|\boldsymbol{x}_0(0) - \boldsymbol{y}_0(\tau_x^\pm)\| \qquad \text{and} \qquad \tau_y^\pm = \pm\|\boldsymbol{y}_0(0) - \boldsymbol{x}_0(\tau_y^\pm)\|$$

need to have solutions $\tau_x^\pm \in D_y$ and $\tau_y^\pm \in D_x$. Furthermore, as the Coulomb fields become singular whenever the two trajectories cross, we demand in addition that $\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\| > 0$ for $t \in D_x \cap D_y$. In order to avoid being over-determined we assume the special form $D_x := [\tau_y^-, \tau_y^+]$ and $D_y := [\tau_x^-, \tau_x^+]$. Finally, we need the equations (4.20) to be fulfilled for time $t = 0$. That such initial conditions exist can be seen by the following construction: Take any two trajectories $t \to \boldsymbol{x}_0(t)$ and $t \to \boldsymbol{y}_0(t)$ that do not cross and have velocities smaller than one. Compute the intersection times $\tau_x^\pm$ and $\tau_y^\pm$ of the forward and backward light cone of $\boldsymbol{x}_0(0)$, respectively $\boldsymbol{y}_0(0)$, with $t \to \boldsymbol{y}_0(t)$, respectively, $t \to \boldsymbol{x}_0(t)$. Holding these intersection points $\boldsymbol{x}_0(\tau_y^\pm)$ and $\boldsymbol{y}_0(\tau_x^\pm)$ fixed, one finally needs to correct the second derivative at time $t = 0$ in a continuous manner to be able to fulfill the equations (4.20) at $t = 0$.

Construction of solutions

In the following we give a construction yielding a unique solution $t \mapsto (\boldsymbol{x}(t), \boldsymbol{y}(t))$ of the equations (4.20) on arbitrary bounded time intervals that fulfill the initial conditions (4.21). We rearrange the equations of motion for $t \mapsto \boldsymbol{x}(t)$, cf. (4.20), into the form

$$\frac{\boldsymbol{x}(t) - \boldsymbol{y}(t_x^+)}{\|\boldsymbol{x}(t) - \boldsymbol{y}(t_x^+)\|^3} = \ddot{\boldsymbol{x}}(t) - \frac{\boldsymbol{x}(t) - \boldsymbol{y}(t_x^-)}{\|\boldsymbol{x}(t) - \boldsymbol{y}(t_x^-)\|^3}$$

The function $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}\|\boldsymbol{x}\|^{-3}$ wherever well-defined has an inverse $\boldsymbol{f}^{-1}(\boldsymbol{x}) = \boldsymbol{x}\|\boldsymbol{x}\|^{-\frac{3}{2}}$. Hence, we yield

$$\boldsymbol{y}(t_x^+) = \boldsymbol{x}(t) - \boldsymbol{f}^{-1}\left(\ddot{\boldsymbol{x}}(t) - \frac{\boldsymbol{x}(t) - \boldsymbol{y}(t_x^-)}{\|\boldsymbol{x}(t) - \boldsymbol{y}(t_x^-)\|^3}\right).$$

Now for a time $0 \leq t \leq \tau_x^+$ the right-hand side of this equation involves only known entities and we may express these equations as

$$\boldsymbol{y}(t_x^+) = \boldsymbol{x}_0(t) - \boldsymbol{f}^{-1}\left(\ddot{\boldsymbol{x}}_0(t) - \frac{\boldsymbol{x}_0(t) - \boldsymbol{y}_0(t_x^-)}{\|\boldsymbol{x}_0(t) - \boldsymbol{y}_0(t_x^-)\|^3}\right).$$

Note that this also determines the advance term

$$t_x^+ = t + \left\|\boldsymbol{f}^{-1}\left(\ddot{\boldsymbol{x}}_0(t) - \frac{\boldsymbol{x}_0(t) - \boldsymbol{y}_0(t_x^-)}{\|\boldsymbol{x}_0(t) - \boldsymbol{y}_0(t_x^-)\|^3}\right)\right\|$$

which is a simple function of t. Hence, as long as the trajectories $t \mapsto \boldsymbol{x}(t)$ and $t \mapsto \boldsymbol{y}(t)$ do not cross, the trajectory of $t \mapsto \boldsymbol{y}(t)$ which is initially defined on the domain $[\tau_x^-, \tau_x^+]$ by (4.21p.54) can be extended to the domain $[\tau_x^-, t_x^+(\tau_y^+)]$. By the same construction we can extend the trajectory $t \to \boldsymbol{x}(t)$ to the domain $[\tau_y^-, t_y^+(\tau_x^+)]$. Under the assumption that the trajectories do not cross, the maximal velocities of both trajectories are uniformly bounded and that this bound is below one, we can continue this construction to yield trajectories $t \to \boldsymbol{x}(t)$ on the domain $[\tau_y^-, \infty)$ and $t \to \boldsymbol{y}(t)$ on $[\tau_x^-, \infty)$. Note that the stated assumptions are very likely to hold at least for a lot of initial conditions as the charges are repelling and their interaction decays over their distance. A similar construction can be done for the past so that we end up with a solution $t \to (\boldsymbol{x}, \boldsymbol{y})$ of the equations (4.20p.54) on whole $\mathbb{R}$.

The above construction relies heavily on the existence of $\boldsymbol{f}^{-1}$ which enables us to solve for the future or past trajectory. For a Lorentz boosted Coulomb field this will be already not possible anymore since in this case the position and the velocity coordinates mix in the inner products and there is no unique way to tell them apart knowing only the values of the field. Whether additional conditions like smoothness and conservations laws better the situation is not known. Nevertheless, we learn from this toy model that Newtonian Cauchy data $(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{y}(t), \dot{\boldsymbol{y}}(t))|_{t=0}$ is not sufficient to identify a solution uniquely disregarding any conditions on regularity. Whether demanding smoothness for the solutions renders them unique for only Newtonian Cauchy data is an interesting question. On the other hand, we also learned that we do not have reasons to believe that we encounter non-existence of solutions to the Wheeler-Feynman equations for given Newtonian Cauchy data.

4.2.2 Reformulation in Terms of an Initial Value Problem

The type of functional differential equations we shall be looking for are of the following form:

$$x'(t) = V(x(t), f(t, x)) \tag{4.22}$$

To keep it simple let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and we take $x : \mathbb{R} \to \mathcal{X}$, $V : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and f a function of $t \in \mathbb{R}$ and a functional x taking values in $\mathcal{Y}$. To get the connection to the Wheeler-Feynman equations imagine x to encode position and momentum of all charges, f the electric and magnetic fields and V the Lorentz force. As in electrodynamics where the fields depend on the charge trajectories, f depends on the whole solution $x : \mathbb{R} \to \mathcal{X}$.

We want to study equation (4.22) in terms of Cauchy data

$$x(t)|_{t=0} = x^0 \tag{4.23}$$

for given $x^0 \in \mathcal{X}$. Therefore, the key idea of the rest of this entire chapter is to reformulate the question of the existence of solution to the functional differential equation in terms of an ordinary initial value problem. This can be done for any functional differential equation if the functional can be given in terms of a propagator, for example,

$$f(t,x) = \frac{1}{\delta_+ + \delta_-} \sum_{\pm} \delta_{\pm} \left[f^{\pm T} + \int_{\pm T}^{t} ds\ W(x(s)) \right] \text{ for all } t \in \mathbb{R} \tag{4.24}$$

for some $T \geq 0$ and initial values $f^{\pm T} \in \mathbb{R}$ and $\delta_+, \delta_- \in \{0,1\}$. In case of f encoding the electrodynamic fields of the charge trajectories specified in x, the initial values $f^{\pm T}$ will be forgotten for $T \to \infty$ which is due to the differential operator of the free Maxwell equations omitted in our simplified considerations here. For $\delta_+ = 1 = \delta_-$ we have a functional differential equation depending on the solution x in the interval $[-T, T]$, while for $\delta_+ = 0, \delta_- = 1$, respectively, $\delta_+ = 1$ and $\delta_- = 0$, it would only depend on the past $[-T, 0]$, respectively, the future $[0, T]$. Differentiation with respect to time gives

$$\frac{d}{dt} f(t,x) = W(x(t)) \tag{4.25}$$

which encodes a time-evolution of the functional f for given x. In the analogy with electrodynamics this can be viewed as the Maxwell equations with omission of the differential operator of the free Maxwell equations.

Reformulation of the initial value problem for the functional differential equation

So instead of studying (4.22p.55) and (4.23p.55) we can as well consider the equation

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ f(t) \end{pmatrix} = \begin{pmatrix} V(x(t), f(t)) \\ W(x(t)) \end{pmatrix} \tag{4.26}$$

together with the initial condition (4.23p.55). If we demand in addition

$$f(t)|_{t=0} = \frac{1}{\delta_+ + \delta_-} \sum_{\pm} \delta_{\pm} \left[f^{\pm T} + \int_{\pm T}^{0} ds\ W(x(s)) \right], \tag{4.27}$$

which by (4.25) is equivalent to (4.24), any solution x then fulfills (4.22p.55) for initial value (4.23p.55).

Let us use the notation

$$\varphi(t) = \begin{pmatrix} x(t) \\ f(t) \end{pmatrix} \qquad \text{and} \qquad J(\varphi(t)) = \begin{pmatrix} V(x(t), f(t)) \\ W(x(t)) \end{pmatrix}$$

so that (4.26) reads

$$\varphi'(t) = J(\varphi(t)). \tag{4.28}$$

Time zero Cauchy data

If J is nonlinear and Lipschitz continuous, we get existence and uniqueness of solutions to (4.26) for Cauchy data at time $t = 0$ without much effort.

Theorem 4.2. *Assume that $J : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{Y}$ is nonlinear and Lipschitz continuous with Lipschitz constant $L > 0$, i.e.*

$$\|J(\varphi - \widetilde{\varphi})\|_{\mathcal{X}\oplus\mathcal{Y}} \leq L \|\varphi - \widetilde{\varphi}\|_{\mathcal{X}\oplus\mathcal{Y}}.$$

Let $\varphi^0 \in \mathcal{X} \oplus \mathcal{Y}$, then there is a unique solution $\varphi \in C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y})$ of (4.28) and $\varphi(t)|_{t=0} = \varphi^0$.

Proof. Let us define the Banach space

$$X_\gamma := \left\{ \varphi \in C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y}) \;\middle|\; \|\varphi\|_\gamma := \sup_{t \in \mathbb{R}} e^{-\gamma|t|} \|\varphi(t)\|_{\mathcal{X}\oplus\mathcal{Y}} \right\}$$

for $\gamma \geq 0$ and the map $S : C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y}) \to C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y})$ pointwise for every $t \in \mathbb{R}$ by

$$S[\varphi](t) := \varphi^0 + \int_0^t ds\ J(\varphi(s)).$$

Let $\varphi \in X_\gamma$, then $S[\varphi]$ is in $C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y})$ and

$$\begin{aligned}
\|S[\varphi]\|_\gamma &= \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \left\| \varphi^0 + \int_0^t ds\ J(\varphi(s)) \right\|_{\mathcal{X}\oplus\mathcal{Y}} \\
&\leq \|\varphi^0\|_{\mathcal{X}\oplus\mathcal{Y}} + \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \int_{-|t|}^{|t|} ds\ \|J(\varphi(s)) - J(0) + J(0)\|_{\mathcal{X}\oplus\mathcal{Y}} \\
&\leq \|\varphi^0\|_{\mathcal{X}\oplus\mathcal{Y}} + \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \left[L \int_{-|t|}^{|t|} ds\ \|\varphi(s)\|_{\mathcal{X}\oplus\mathcal{Y}} + 2|t|\, \|J(0)\|_{\mathcal{X}\oplus\mathcal{Y}} \right] \\
&\leq \|\varphi^0\|_{\mathcal{X}\oplus\mathcal{Y}} + \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \left[L\|\varphi\|_\gamma \int_{-|t|}^{|t|} ds\ e^{\gamma|s|} + 2|t|\, \|J(0)\|_{\mathcal{X}\oplus\mathcal{Y}} \right] \\
&\leq \|\varphi^0\|_{\mathcal{X}\oplus\mathcal{Y}} + \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \left[L\|\varphi\|_\gamma 2 \frac{e^{\gamma|t|}}{\gamma} + 2|t|\, \|J(0)\|_{\mathcal{X}\oplus\mathcal{Y}} \right].
\end{aligned}$$

Now $te^{-\gamma t}$ has $t = \frac{1}{\gamma}$ as maximum so that we find

$$\|S[\varphi]\|_\gamma \leq \|\varphi^0\|_{\mathcal{X}\oplus\mathcal{Y}} + \frac{2}{\gamma} \left[L\|\varphi\|_\gamma + \|J(0)\|_{\mathcal{X}\oplus\mathcal{Y}} \right] < \infty$$

such that S is a nonlinear map $X_\gamma \to X_\gamma$ for every $\gamma \geq 0$. Furthermore, we compute

$$\begin{aligned}
\|S[\varphi] - S[\widetilde{\varphi}]\|_\gamma &= \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \left\| \int_0^t ds\ J(\varphi(s)) - J(\widetilde{\varphi}) \right\|_{\mathcal{X}\oplus\mathcal{Y}} \\
&\leq L \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \int_{-|t|}^{|t|} ds\ \|\varphi(s) - \widetilde{\varphi}(s)\|_{\mathcal{X}\oplus\mathcal{Y}} \\
&\leq L\|\varphi - \widetilde{\varphi}\|_\gamma \sup_{t\in\mathbb{R}} e^{-\gamma|t|} \int_{-|t|}^{|t|} ds\ e^{\gamma|s|} \leq \frac{2L}{\gamma} \|\varphi - \widetilde{\varphi}\|_\gamma
\end{aligned}$$

which states that S is a contraction for sufficiently large γ. Hence, by Banach's fixed point theorem we have existence and uniqueness of a fixed-point in $X_\gamma \subset C(,\mathcal{X} \oplus \mathcal{Y})$. □

Uniqueness for the functional differential equation for small times

We mainly get the uniqueness from the fact that J is nonlinear. However, this changes when we want to solve (4.28p.56) for (4.23p.55) and (4.27p.56). For small times $T \geq 0$ we can settle the issue of existence and uniqueness by a similar technique like above. Let us extend our short-hand notation with projectors on the coordinates

$$Q\varphi(t) = \begin{pmatrix} x(t) \\ 0 \end{pmatrix} \qquad \text{and} \qquad F\varphi(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

If there is no source for type errors, we shall also use $Q\varphi(t) = x(t)$ as an $\mathcal{X}$ value and $F\varphi(t) = f(t)$ as a $\mathcal{Y}$ value.

Lemma 4.3. *Let J and $x^0 \in \mathcal{X}$ be as before and let $(f^{\pm T})_{T\geq 0}$ be a family in $\mathcal{Y}$. Then there is a constant $\tau > 0$ which depends only on the Lipschitz constant L of J such that for all $0 \leq T < \tau$ there is a unique solution $\varphi \in C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y})$ of (4.28p.56) for (4.23p.55) and (4.27p.56), i.e.*

$$\varphi'(t) = J(\varphi(s)) \tag{4.29}$$

$$\varphi(t)|_{t=0} = \chi^0 + \frac{1}{\delta_+ + \delta_-} \sum_{\pm} \delta_\pm \left[\chi^{\pm T} + \int_{\pm T}^0 ds\ FJ(x(s)) \right]. \tag{4.30}$$

for $\chi^0 = \begin{pmatrix} x^0 \\ 0 \end{pmatrix}$ and $\chi^{\pm T} = \begin{pmatrix} 0 \\ f^{\pm T} \end{pmatrix}$.

Proof. From Theorem 4.2p.56 we know that for every $f^0 \in \mathcal{Y}$ there is a unique solution to (4.29) for initial value

$$\varphi(t)|_{t=0} = \begin{pmatrix} x^0 \\ f^0 \end{pmatrix}.$$

Therefore, we need only to find the one which fulfills also (4.30p.57). It seems that the natural way is to construct such initial conditions via a fixed-point map. Let us denote by $t \mapsto M_t[\varphi^0]$ the unique solution $\varphi(\cdot) \in C(\mathbb{R}, \mathcal{X} \oplus \mathcal{Y})$ with initial conditions $\varphi(t)|_{t=0} = \varphi$ for all $\varphi \in \mathcal{X} \oplus \mathcal{Y}$. Furthermore, we define a candidate for such a map by $S : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{Y}$ by

$$S^{\mathcal{X}}_T[\varphi] := x^0 + \frac{1}{\delta_+ + \delta_-} \sum_{\pm} \delta_{\pm} \left[x^{\pm T} + \int_{\pm T}^{0} ds\ FJ(M_s[\varphi]) \right] \tag{4.31}$$

for all $\varphi \in \mathcal{X} \oplus \mathcal{Y}$. For another $\widetilde{\varphi} \in \mathcal{X} \oplus \mathcal{Y}$ we find

$$\begin{aligned} \left\| S^{\mathcal{X}}_t[\varphi] - S^{\mathcal{X}}_t[\widetilde{\varphi}] \right\|_{\mathcal{X}\oplus\mathcal{Y}} &\leq \int_{-T}^{T} ds\ \|J(M_s[\varphi]) - J(M_s[\widetilde{\varphi}])\|_{\mathcal{X}\oplus\mathcal{Y}} \\ &\leq 2TL \sup_{t\in[-T,T]} \|M_t[\varphi] - M_t[\widetilde{\varphi}]\|_{\mathcal{X}\oplus\mathcal{Y}} . \end{aligned} \tag{4.32}$$

From Gronwall's lemma we get an estimate on the supremum because

$$\begin{aligned} \|M_t[\varphi] - M_t[\widetilde{\varphi}]\|_{\mathcal{X}\oplus\mathcal{Y}} &= \left\| \varphi - \widetilde{\varphi} + \int_0^t ds\ \left[J(M_s[\varphi]) - J(M_s[\widetilde{\varphi}]) \right] \right\|_{\mathcal{X}\oplus\mathcal{Y}} \\ &\leq \|\varphi - \widetilde{\varphi}\|_{\mathcal{X}\oplus\mathcal{Y}} + L \int_0^t ds\ \|M_s[\varphi] - M_s[\widetilde{\varphi}]\|_{\mathcal{X}\oplus\mathcal{Y}} \end{aligned}$$

which means that

$$\|M_t[\varphi] - M_t[\widetilde{\varphi}]\|_{\mathcal{X}\oplus\mathcal{Y}} \leq \left(1 + Lte^{Lt}\right) \|\varphi - \widetilde{\varphi}\|_{\mathcal{X}\oplus\mathcal{Y}}$$

Entering this into equations (4.32) we get

$$\left\| S^{\mathcal{X}}_t[\varphi] - S^{\mathcal{X}}_t[\widetilde{\varphi}] \right\|_{\mathcal{X}\oplus\mathcal{Y}} \leq 2TL\left(1 + LTe^{LT}\right) \|\varphi - \widetilde{\varphi}\|_{\mathcal{X}\oplus\mathcal{Y}}. \tag{4.33}$$

As $T \mapsto 2TL\left(1 + LTe^{LT}\right)$ is a continuous and strictly increasing function taking values in $[0, \infty)$, there is a $0 < \tau < \infty$ such that $2\tau L\left(1 + L\tau e^{L\tau}\right) = 1$. Hence, for any $0 \leq T < \tau$ the map $S^{\mathcal{X}}_T$ is a contraction on the Banach space $\mathcal{X} \oplus \mathcal{Y}$. The Banach's fixed point theorem ensures the existence of a unique fixed point $\varphi \in \mathcal{X} \oplus \mathcal{Y}$ and therefore

$$\varphi(t) := M_t[\varphi] := \varphi + \int_0^t ds\ J(\varphi(s))$$

is a solution to (4.29p.57) which fulfills

$$\varphi(t)|_{t=0} = \varphi = S^{\mathcal{X}}_T[\varphi] = x^0 + \frac{1}{\delta_+ + \delta_-} \sum_{\pm} \delta_{\pm} \left[x^{\pm T} + \int_{\pm T}^{0} ds\ FJ(M_s[\varphi]) \right],$$

i.e. it fulfills (4.30p.57), which concludes the proof. □

Theorem 4.4. *Let $\tau > 0$ as in the last theorem, then for all such $0 \leq T < \tau$ and functionals*

$$f(t, x) := \frac{1}{\delta_+ + \delta_-} \sum_{\pm} \delta_{\pm} \left[f^{\pm T} + \int_{\pm T}^{t} ds\ W(x(s)) \right],$$

then for every $x^0 \in \mathcal{X}$ there is a unique solution $x \in C(\mathbb{R}, \mathcal{X})$ of

$$x'(t) = V(x(t), f(t, x)) \qquad \textit{and} \qquad x(t)|_{t=0} = x^0.$$

Proof. Let φ be the unique fixed point of $S_T^{\mathcal{X}}$, i.e. (4.31), of Lemma 4.3p.57. We compute

$$F\varphi(t) = F\left(S_T^{\mathcal{X}}[\varphi] + \int_0^t ds\ J(\varphi(s))\right) = \frac{1}{\delta_+ + \delta_-}\sum_{\pm}\delta_\pm\left[\chi^{\pm T} + \int_{\pm T}^t ds\ FJ(M_s[\varphi])\right].$$

Since $M_t[\varphi] = \begin{pmatrix} x(t) \\ f(t) \end{pmatrix}$ and $J(M_s[\varphi]) = \begin{pmatrix} V(x(s), f(s)) \\ W(x(s)) \end{pmatrix}$, we yield $F\varphi(t) = f(t, x)$ which concludes the proof. □

To picture what is going on let us consider the delay case only, i.e. here $\delta_+ = 0$ and $\delta_- = 1$. This means we are looking for solutions to $\varphi'(t) = J(\varphi(t))$ for initial conditions given at different times

Why uniqueness for small times?

$$Q\varphi(t)|_{t=0} = x^0 \qquad \text{and} \qquad F\varphi(t)|_{t=-T} = f^{-T}$$

for given $x^0 \in \mathcal{X}$ and $f^{-T} \in \mathcal{Y}$. If $T > 0$ is not chosen to be too large Theorem 4.4p.58 provides a unique solution to this problem which we call $t \mapsto \varphi(t)$. In turn, Theorem 4.2p.56 states that for every prescribed $x \in \mathcal{X}$ there is a unique solution $t \mapsto \varphi^x(t) := M_{(t-T)}[(x, f^{-T})]$. By uniqueness all these trajectories do not cross in phase space $\mathcal{X} \oplus \mathcal{Y}$ unless they are the same. The mechanism which keeps them apart is the Lipschitz continuity of the vector field of the differential equation. Imagine x very close to $Q\varphi(-T)$, then by the Lipschitz continuity the vectors $J(\varphi(t))$ and $J(\varphi^x(t))$ are almost parallel for $t \in [-T, 0]$, and $Q\varphi(0)$ and $Q\varphi^x(0)$ will lie near but have no chance to become equal. The other extreme would be an x very far away from $Q\varphi(-T)$, the trajectory $t \mapsto Q\varphi^x(t)$ can then not reach x^0 anymore during the time T as the maximal velocity is bounded by the Lipschitz constant L. Hence, it is clear that T is approximately inverse proportional to the Lipschitz constant L as equation (4.33p.58) suggests.

As a concluding remark we consider the problem above for arbitrary large or even infinite T. Without more assumptions on J, it will in general not be possible to apply Banach's fixed point theorem to infer existence of solutions. In fact, for the Wheeler-Feynman equations in Subsection 4.4p.73 the existence of solutions for arbitrary but finite T can only be shown by Schauder's Fixed Point Theorem. This raises the question about uniqueness. For example, imagine we had an existence and uniqueness theorem like (4.4p.58) without the restriction that T needs to be small. Then for two trajectories $t \mapsto \varphi^x(t)$ and $t \mapsto \varphi^y(t)$ for $x, y \in \mathcal{X}$ as defined before with $x \neq y$, theorem (4.2p.56) states $\varphi^x(t) \neq \varphi^y(t)$ for all $t \in \mathbb{R}$. If at one time $t \in \mathbb{R}$ the two trajectories cross in $\mathcal{X}$ space, i.e. if we have $Q\varphi^x(t) = Q\varphi^y(t)$, theorem (4.4p.58) without the restriction on T would imply that $\varphi^x(t) = \varphi^y(t)$ for all $t \in \mathbb{R}$ which is a contradiction. Therefore, two such trajectories could never cross in $\mathcal{X}$ space. This is big restriction either on the vector fields J or on the set of initial conditions in $\mathcal{X} \oplus \mathcal{Y}$ or both. However, such a property cannot be expected for any general nonlinear and Lipschitz continuous vector field.

Why we cannot expect uniqueness for all times T

4.3 Wheeler-Feynman Initial Fields

In the following we will develop a way to characterize a class of possible Wheeler-Feynman solutions by initial values for the ML-SI dynamics. This characterization will be based on properties of the corresponding Maxwell solutions which we infer by solving the Maxwell equations explicitly in Subsection 4.3.1p.60. The Characterization of Wheeler-Feynman solutions is then done in Subsection 4.3.2p.70.

4.3.1 Solutions to the Maxwell equations

In this chapter we prove explicit representation formulas for strong solutions $t \mapsto (\boldsymbol{E}_t, \boldsymbol{B}_t)$ of the Maxwell equations given a charge trajectory or charge-current density:

Charge trajectories

Definition 4.5. *We shall call any map*

$$(\boldsymbol{q},\boldsymbol{p}) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3), \qquad t \mapsto (\boldsymbol{q}_t, \boldsymbol{p}_t)$$

a charge trajectory where $\boldsymbol{q}_t$ denotes the position and $\boldsymbol{p}_t$ the momentum of the charge with mass $m \neq 0$. We collect all time-like trajectories in the set

$$\mathcal{T}^1_\vee := \left\{ (\boldsymbol{q},\boldsymbol{p}) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3) \;\middle|\; \|\boldsymbol{v}(\boldsymbol{p}_t)\| < 1 \text{ for all } t \in \mathbb{R} \right\},$$

and all strictly time-like trajectories in the set

$$\mathcal{T}^1_\triangledown := \left\{ (\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\vee \;\middle|\; \exists v_{max} < 1 \text{ such that } \sup_{t \in \mathbb{R}} \|\boldsymbol{v}(\boldsymbol{p}_t)\| \leq v_{max} \right\}$$

where $\boldsymbol{v}(\boldsymbol{p}) := \frac{\boldsymbol{p}}{\sqrt{m^2+\boldsymbol{p}^2}}$. We shall also use the notation $\mathcal{T}_\# := \times_{i=1}^N \mathcal{T}^1_\#$ for the N-fold Cartesian product where # is a placeholder for $\vee$ or $\triangledown$. Furthermore, two charge trajectories are equal if and only if their positions and momenta are equal for all times.

Charge-current densities

Definition 4.6. *We shall call any pair of maps $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}, (t, \boldsymbol{x}) \mapsto \rho_t(\boldsymbol{x})$ and $\boldsymbol{j} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, (t, \boldsymbol{x}) \mapsto \boldsymbol{j}_t(\boldsymbol{x})$ a charge-current density whenever:*

(i) For all $\boldsymbol{x} \in \mathbb{R}^3$: $\rho_{(\cdot)}(\boldsymbol{x}) \in C^1(\mathbb{R}, \mathbb{R})$ and $\boldsymbol{j}_{(\cdot)}(\boldsymbol{x}) \in C^1(\mathbb{R}, \mathbb{R}^3)$.

(ii) For all $t \in \mathbb{R}$: $\rho_t, \partial_t \rho_t \in C^\infty(\mathbb{R}^3, \mathbb{R})$ and $\boldsymbol{j}_t, \partial_t \boldsymbol{j}_t \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

(iii) For all $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^3$: $\partial_t \rho_t(\boldsymbol{x}) + \nabla \cdot \boldsymbol{j}_t(\boldsymbol{x}) = 0$ which we call continuity equation.

We denote the set of such pairs $(\rho, \boldsymbol{j})$ by $\mathcal{D}$.

We shall also need the following connection between charge trajectories and charge-current densities:

Induced charge-current densities

Definition 4.7. *For $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ and $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\vee$ we call $(\rho, \boldsymbol{j}) \in \mathcal{D}$ defined by*

$$\rho_t(\boldsymbol{x}) := \varrho(\boldsymbol{x} - \boldsymbol{q}_t) \qquad \text{and} \qquad \boldsymbol{j}_t(\boldsymbol{x}) := \frac{\boldsymbol{p}_t}{\sqrt{m^2 + \boldsymbol{p}_t^2}} \varrho(\boldsymbol{x} - \boldsymbol{q}_t)$$

for all $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^3$ the ϱ induced charge-current density of $(\boldsymbol{q},\boldsymbol{p})$ with mass m.

The Maxwell equations including the Maxwell constraints for a given charge-current density $(\rho, \boldsymbol{j}) \in \mathcal{D}$ read:

$$\begin{aligned} \dot{\boldsymbol{E}}_t &= \nabla \wedge \boldsymbol{B}_t - 4\pi \boldsymbol{j}_t \qquad & \nabla \cdot \boldsymbol{E}_t &= 4\pi \rho_t \\ \dot{\boldsymbol{B}}_t &= -\nabla \wedge \boldsymbol{E}_t. & \nabla \cdot \boldsymbol{B}_t &= 0. \end{aligned} \tag{4.34}$$

The class of fields $(\boldsymbol{E}_t, \boldsymbol{B}_t)$ we are interested in is:

Space of the fields

Definition 4.8. $\mathcal{F}^1 := C^\infty(\mathbb{R}^3, \mathbb{R}^3) \oplus C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

The class of solutions to these Maxwell equations we want to study is characterized by:

Definition 4.9. *Let $t_0 \in \mathbb{R}$ and $F^0 \in \mathcal{F}^1$. Then any mapping $F : \mathbb{R} \to \mathcal{F}^1, t \mapsto F_t := (\boldsymbol{E}_t, \boldsymbol{B}_t)$ that solves (4.34) for initial value $F_t|_{t=t_0} = F^0$ is called a solution to the Maxwell equations with t_0 initial value F^0.*

Maxwell solutions

The explicit representation formulas are constructed with the help of:

Definition 4.10. *We set*

Green's functions of the d'Alembert

$$K_t^{\pm}(\boldsymbol{x}) := \frac{\delta(\|\boldsymbol{x}\| \pm t)}{4\pi\|\boldsymbol{x}\|}$$

where δ denotes the one-dimensional Dirac delta distribution. Furthermore, for every $f \in C^\infty(\mathbb{R}^3)$ we define

$$K_t^{\pm} * F(\boldsymbol{x}) = \begin{cases} 0 & \text{for } \pm t > 0 \\ t \fint_{\partial B_{|t|}(\boldsymbol{x})} d\sigma(y) F(\boldsymbol{y}) := t \int_{\partial B_{|t|}(\boldsymbol{x})} d\sigma(y) \frac{F(y)}{4\pi t^2} & \text{otherwise} \end{cases}$$

In the next lemma we collect useful properties of these Green's functions.

Lemma 4.11. *The distributions $K_t^{\pm}$ introduced in Definition 4.10 have the following properties:*

Green's functions properties

(i) For any $f \in C^\infty(\mathbb{R}^3)$ the mapping $(t, \boldsymbol{x}) \mapsto [K_t^{\pm} * f](\boldsymbol{x})$ is in $C^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3)$, $\square K_t^{\pm} * f = 0$ for $t \neq 0$ and for any $n \in \mathbb{N}$

$$\lim_{t \to 0\mp} \begin{pmatrix} \partial_t^{2n} K_t^{\pm} * f \\ \partial_t^{2n+1} K_t^{\pm} * f \end{pmatrix} = \begin{pmatrix} 0 \\ \mp\triangle^n f \end{pmatrix}. \tag{4.35}$$

(ii) For any $f \in C^\infty(\mathbb{R}^3)$ and $K_t = \sum_{\pm} \mp K_t^{\pm}$ the mapping $(\mathbb{R} \setminus \{0\}) \times \mathbb{R} \ni (t, \boldsymbol{x}) \mapsto [K_t^{\pm} * f](\boldsymbol{x})$ is continuously extendable to a $C^\infty(\mathbb{R} \times \mathbb{R}^3)$ function. Furthermore, $\square K_t * f = 0$ for all $t \in \mathbb{R}$.

(iii) Let $\mathbb{R}^3 \times \mathbb{R} \ni (\boldsymbol{x}, t) \mapsto f_t(\boldsymbol{x})$ be a map that is for each fixed $\boldsymbol{x} \in \mathbb{R}^3$ an once continuously differentiable function and for each fixed $t \in \mathbb{R}$ infinitely often differentiable then the following estimates hold for an $R \geq |t|$:

$$\|[K_t * f_t](\boldsymbol{x})\| \leq R \sup_{y \in \partial B_R(x)} \|f_t(\boldsymbol{y})\| \quad \text{and} \quad \|[K_t * f_t](\boldsymbol{x})\| \leq \sup_{y \in \partial B_R(x)} \left(\|f_t(\boldsymbol{y})\| + \frac{R^2}{3} \|\triangle f(\boldsymbol{y})\| \right)$$

Furthermore, for all $n \in \mathbb{N}$ it is true that

$$\lim_{t \to 0} K_t * f_t = 0 \qquad \text{and} \qquad \lim_{t \to 0} \partial_t K_t * f_t = f_0.$$

Proof. A straightforward computation(see Computation in Appendix 5.2p.97) yields

$$K_t^{\pm} * f = \mp t \fint_{\partial B_{\mp t}(0)} d\sigma(y) f(\cdot - \boldsymbol{y}) \tag{4.36}$$

$$\partial_t K_t^{\pm} * f = \mp \fint_{\partial B_{\mp t}(0)} d\sigma(y)\ f(\cdot - \boldsymbol{y}) \mp \frac{t^2}{3} \fint_{B_{\mp t}(0)} d^3y\ \triangle f(\cdot - \boldsymbol{y}) \tag{4.37}$$

$$\partial_t^2 K_t^{\pm} * f = K_t^{\pm} * \triangle f = \triangle K_t^{\pm} * f. \tag{4.38}$$

(i) Therefore, the first and second derivatives exist with respect to t, while the second derivative can be written as a spacial derivative on f. By induction one easily computes all combinations of $\boldsymbol{x}$ and t derivatives and finds that the mapping $(t,\boldsymbol{x}) \mapsto [K_t^\pm * f](\boldsymbol{x})$ is in $C^\infty\left((\mathbb{R}\setminus\{0\})\times\mathbb{R}^3\right)$. With (4.36), (4.37), (4.38) and induction in $\mathbb{N}$ together with Lebesgue's differentiation theorem one finds (4.35). (ii) With (i) we need to show that for any $f \in C^\infty(\mathbb{R}^3)$ the limits of $K_t * f$ and $\partial_t K_t * f$ from the right and from the left exist and agree at $t = 0$. The former case is clear because the limit is zero. Regarding the latter we observe

$$\lim_{t\to 0+} \partial_t K_t * f = \lim_{t\to 0+} \partial_t K_t^- * f = f = -\lim_{t\to 0-} \partial_t K_t^+ * f = \lim_{t\to 0-} \partial_t K_t * f.$$

$\lim_{t\to 0} \Box K_t * f = 0$ is a special case of the above. (iii$_{\text{p.61}}$) The estimates are the immediate consequence of (4.36$_{\text{p.61}}$) and (4.37$_{\text{p.61}}$). The limits can be computed by

$$\lim_{t\to 0} \left\|[K_t * f_t](\boldsymbol{x})\right\| \leq \lim_{t\to 0} \left\|[K_t * (f_t - f_0)](\boldsymbol{x})\right\| + \lim_{t\to 0} \left\|[K_t * f_0](\boldsymbol{x})\right\|$$

where the second term is zero by (i$_{\text{p.61}}$). For every $\boldsymbol{x} \in \mathbb{R}^3$, $f_t(\boldsymbol{x})$ is continuous in t, therefore choosing t small enough and $R > |t|$ we obtain

$$\lim_{t\to 0} \left\|[K_t * (f_t - f_0)](\boldsymbol{x})\right\| \leq R \lim_{t\to 0} \sup_{y\in B_\delta(\boldsymbol{x})} \|f_t(\boldsymbol{y}) - f_0(\boldsymbol{y})\| = 0.$$

Similarly, we find

$$\lim_{t\to 0} \left\|[\partial_t K_t * f_t](\boldsymbol{x}) - f_0(\boldsymbol{x})\right\| \leq \lim_{t\to 0} \left\|[\partial_t K_t * (f_t - f_0)](\boldsymbol{x})\right\| + \lim_{t\to 0} \left\|[\partial_t K_t * f_0](\boldsymbol{x}) - f_0(\boldsymbol{x})\right\|$$

while, again, the second term is zero by (i$_{\text{p.61}}$). The same continuity argument as above gives

$$\lim_{t\to 0} \left\|[\partial_t K_t * (f_t - f_0)](\boldsymbol{x})\right\| \leq \lim_{t\to 0} \sup_{y\in B_\delta(\boldsymbol{x})} \left(\|f_t(\boldsymbol{y}) - f_0(\boldsymbol{y})\| + \frac{R^2}{3} \|\triangle f_t(\boldsymbol{y}) - \triangle f_0(\boldsymbol{y})\| \right) = 0$$

which concludes the proof. □

REMARK 4.12. *In the future we will always denote this continuous extension by the same symbol K_t. It is often called the* propagator *of the homogeneous wave equation.*

A simply consequence of this lemma is:

Kirchoff's formula

Corollary 4.13. *A solution $t \mapsto A_t$ of the homogeneous wave equation $\Box A_t = 0$ for initial value $A_t|_{t=0} = A^0$ and $\partial_t A_t|_{t=0} = A^0$, for $A^0, \dot{A}^0 \in C^\infty(\mathbb{R}^3)$, is given by*

$$A_t = \partial_t K_t * A^0 + K_t * \dot{A}^0 \tag{4.39}$$

The next result gives explicit representation formulas of the Maxwell equations (4.34$_{\text{p.60}}$). These formulas can be constructed by the following line of thought: In the distribution sense every solution to the Maxwell equations (4.34$_{\text{p.60}}$) is also a solution to

$$\Box \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} = 4\pi \begin{pmatrix} -\nabla\rho_t - \partial_t \boldsymbol{j}_t \\ \nabla \wedge \boldsymbol{j}_t \end{pmatrix}$$

for initial values

$$(\boldsymbol{E}_t, \boldsymbol{B}_t)\big|_{t=t_0} = (\boldsymbol{E}^0, \boldsymbol{B}^0) \quad \text{as well as} \quad \partial_t(\boldsymbol{E}_t, \boldsymbol{B}_t)\big|_{t=t_0} = (\nabla \wedge \boldsymbol{B}^0 - 4\pi \boldsymbol{j}_{t_0}, -\nabla \wedge \boldsymbol{E}^0). \tag{4.40}$$

Using the abbreviation $F_t^\# = (\boldsymbol{E}_t^\#, \boldsymbol{B}_t^\#)$, using # as placeholder for future superscripts, and with the help of the Green's functions from Definition 4.10$_{\text{p.61}}$ we can easily guess the general form of any solution to these equations which is given by:

$$F_t = F_t^{hom} + \int_{-\infty}^{\infty} ds\, K_{t-t_0-s}^\pm * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \boldsymbol{j}_{t_0+s} \\ \nabla \wedge \boldsymbol{j}_{t_0+s} \end{pmatrix} \tag{4.41}$$

where any homogeneous solution F_t^{hom} fulfills $\Box F_t^{hom} = 0$. Considering the forward as well as backward time-evolution we regard two different kinds of initial value problems:

(i) Initial fields F^0 are given at some time $t_0 \in \mathbb{R} \cup \{-\infty\}$ and propagated to a time $t > t_0$.

(ii) Initial fields F^0 are given at some time $t_0 \in \mathbb{R} \cup \{+\infty\}$ and propagated to a time $t < t_0$.

The kind of initial value problem posed will then determine F_t^{hom} and the corresponding Green's function $K_t^\pm$. For (i) we shall use K_t^- and for (ii) K_t^+ which are uniquely determined by $\square K_t^\pm = \delta(t)\delta^3$ and $K_t^\pm = 0$ for $\pm t > 0$. Without a proof we note at least for time-like charge trajectories and $\mp(t - t_0) > 0$

$$\square \int_{\pm\infty}^{0} ds\; K^{\pm}_{t-t_0-s} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \boldsymbol{j}_{t_0+s} \\ \nabla \wedge \boldsymbol{j}_{t_0+s} \end{pmatrix} = \int_{\pm\infty}^{0} ds\; \square K^{\pm}_{t-t_0-s} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \boldsymbol{j}_{t_0+s} \\ \nabla \wedge \boldsymbol{j}_{t_0+s} \end{pmatrix} = 0$$

by Lemma 4.11p.61. Terms of this kind will simply be added to the homogeneous solution while here we denote this sum by the same symbol F_t^{hom}. This way we arrive at two solution formulas. One being suitable for our forwards initial value problem, i.e. $t - t_0 > 0$,

$$F_t = F_t^{hom} + 4\pi \int_0^{t-t_0} ds\; K^-_{t-t_0-s} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \boldsymbol{j}_{t_0+s} \\ \nabla \wedge \boldsymbol{j}_{t_0+s} \end{pmatrix},$$

and the other suitable for the backwards initial value problem, i.e. $t - t_0 < 0$,

$$F_t = F_t^{hom} + 4\pi \int_{t-t_0}^{0} ds\; K^+_{t-t_0-s} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \boldsymbol{j}_{t_0+s} \\ \nabla \wedge \boldsymbol{j}_{t_0+s} \end{pmatrix}.$$

As a last step one needs to identify the homogeneous solutions which satisfy the given initial conditions (4.40p.62). With Corollary 4.13p.62 we have given the explicit representation formula:

$$F_t^{hom} := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * F^0.$$

Therefore, using the definition of $K_t = \sum_\pm \mp K_t^\pm$ and a substitution in the integration variable, we finally arrive at the expression for $t \in \mathbb{R}$:

$$F_t = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * F^0 + K_{t-t_0} * \begin{pmatrix} -4\pi \boldsymbol{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\; K_{t-s} * \begin{pmatrix} -\nabla\rho_s - \partial_s \boldsymbol{j}_s \\ \nabla \wedge \boldsymbol{j}_s \end{pmatrix}.$$

Theorem 4.14. *Let $(\rho, \boldsymbol{j}) \in \mathcal{D}$ be a given charge-current density.* Maxwell solutions

(i) Given $(\boldsymbol{E}^0, \boldsymbol{B}^0) \in \mathcal{F}^1$ fulfilling the Maxwell constraints $\nabla \cdot \boldsymbol{E}^0 = 4\pi\rho_{t_0}$ and $\nabla \cdot \boldsymbol{B}_{t_0} = 0$, then for any $t_0 \in \mathbb{R}$ the mapping $t \mapsto F_t = (\boldsymbol{E}_t, \boldsymbol{B}_t)$ with

$$\begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \boldsymbol{E}^0 \\ \boldsymbol{B}^0 \end{pmatrix} + K_{t-t_0} * \begin{pmatrix} -4\pi \boldsymbol{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\; K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix}$$

*for all $t \in \mathbb{R}$ is $\mathcal{F}^1$ valued, infinitely often differentiable and a solution to the Maxwell equations (4.34**p.60**) with t_0 initial value F^0.*

(ii) Furthermore, if for fixed $t_0, t^* \in \mathbb{R}$ and $\boldsymbol{x}^* \in \mathbb{R}^3$ it holds that

$$K_{t-t_0} * \varrho_{t_0} = 0 \qquad \textit{and} \qquad K_{t-t_0} * \boldsymbol{j}_{t_0} = 0 \tag{4.42}$$

for all $t \in B_1(t^*)$ and $\boldsymbol{x} \in B_1(\boldsymbol{x}^*)$, then statement (i) restricted to such $(t, \boldsymbol{x})$ is also true for initial fields $(\boldsymbol{E}^0, \boldsymbol{B}^0) = 0$.

Proof. The regularity for the first two terms is given by Lemma 4.11p.61. The third term is well-defined by Definition 4.6p.60. Lemma 4.11p.61 states that its integrand is infinitely often differentiable in t and $\boldsymbol{x}$. As the integral goes over a compact set it inherits the regularity from the integrand. In the following we treat both cases (i) and (ii) together. We shall frequently commute spatial differential operators with integrals which is justified because the integrals go over compact sets and the integrand is continuously differentiable. It is convenient to make partial integrations in the third term first to yield:

$$K_{t-t_0} * \begin{pmatrix} -4\pi \boldsymbol{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\ K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} = 4\pi \int_{t_0}^{t} ds\ \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix}.$$

The spatial partial integrations hold by Definition 4.10p.61. The partial integration in s holds as, according to Lemma 4.11p.61, the boundary terms give $4\pi[K_{t-s} * \boldsymbol{j}_s]_{s=t_0}^{s=t} = -4\pi K_{t-t_0} * \boldsymbol{j}_{t_0}$. Next we verify the Maxwell constraints. At first for the electric field:

$$\nabla \cdot \boldsymbol{E}_t = \partial_t K_{t-t_0} * \nabla \cdot \boldsymbol{E}^0 + 4\pi \int_{t_0}^{t} ds\ \left[-\triangle K_{t-s} * \rho_s - \partial_t K_{t-s} * \nabla \cdot \boldsymbol{j}_s \right].$$

Applying the continuity equation, cf. 4.6p.60, in the last term we get

$$\ldots = \partial_t K_{t-t_0} * \nabla \cdot \boldsymbol{E}^0 + 4\pi \int_{t_0}^{t} ds\ \left[-\triangle K_{t-s} * \rho_s + \partial_t K_{t-s} * \partial_s \rho_s \right]$$

After a partial integration in the last term we find

$$\ldots = \partial_t K_{t-t_0} * \nabla \cdot \boldsymbol{E}^0 + 4\pi \left[\partial_s K_{t-s} * \rho_s \right]_{s=t_0}^{s=t} + 4\pi \int_{t_0}^{t} ds\ \Box K_{t-s} * \rho_s.$$

Lemma 4.11p.61 identifies the middle term $4\pi \left[\partial_s K_{t-s} * \rho_s \right]_{s=t_0}^{s=t} = 4\pi\rho_t - 4\pi\partial_t K_{t-t_0} * \rho_{t_0}$ and states that the last term is zero. Therefore,

$$\ldots = \partial_t K_{t-t_0} * \nabla \cdot \boldsymbol{E}^0 - 4\pi\partial_t K_{t-t_0} * \rho_{t_0} + 4\pi\rho_t.$$

In the case (i) we have $\nabla \cdot \boldsymbol{E}^0 = 4\pi\varrho_{t_0}$ and the first two terms cancel each other. In the case (ii) these two terms are identically zero because of (4.42p.63). Hence, we get for both cases $\nabla \cdot \boldsymbol{E}_t = 4\pi\rho_t$. Second, for the magnetic field we immediately get $\nabla \cdot \boldsymbol{B}_t = \partial_t K_{t-t_0} * \nabla \cdot \boldsymbol{B}_0 = 0$ because in the case (i) $\nabla \cdot \boldsymbol{B}_0 = 0$ and in the case (ii) $\boldsymbol{B}_0 = 0$. Therefore, the Maxwell constraints are fulfilled in both cases. Next we verify the rest of the Maxwell equations:

$$\begin{aligned} \boxed{13} &:= \left(\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix} \right) \begin{pmatrix} \boldsymbol{E}_t \\ \boldsymbol{B}_t \end{pmatrix} = \begin{pmatrix} \triangle + \nabla\wedge(\nabla\wedge\cdot) & 0 \\ 0 & \triangle + \nabla\wedge(\nabla\wedge\cdot) \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \boldsymbol{E}^0 \\ \boldsymbol{B}^0 \end{pmatrix} \\ &+ 4\pi\partial_t \int_{t_0}^{t} ds\ \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} - 4\pi \int_{t_0}^{t} ds\ \begin{pmatrix} 0 & \nabla\wedge(\nabla\wedge\cdot) \\ 0 & \partial_t\nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \\ &=: \boxed{14} + \boxed{15} + \boxed{16} \end{aligned}$$

where we have used Equation (4.38p.61) from Lemma 4.11p.61 in the first term, which together with $\nabla \cdot \boldsymbol{B}^0 = 0$ further reduces to

$$\boxed{14} = \nabla K_{t-t_0} * \begin{pmatrix} \nabla \cdot \boldsymbol{E}^0 \\ 0 \end{pmatrix}.$$

The time derivative in the second term gives

$$\begin{aligned} \boxed{15} = 4\pi\partial_t \int_{t_0}^{t} ds\ \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} &= 4\pi \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \Bigg|_{s\to t} \\ &+ 4\pi \int_{t_0}^{t} ds\ \begin{pmatrix} -\partial_t\nabla & -\partial_t^2 \\ 0 & \partial_t\nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \end{aligned}$$

where Lemma 4.11p.61 states that the first term on the right-hand side equals $-4\pi\begin{pmatrix}\boldsymbol{j}_t\\0\end{pmatrix}$. Therefore, with $\nabla\wedge(\nabla\wedge\cdot)=\nabla(\nabla\cdot(\cdot))-\triangle$ we yield

$$\boxed{13}=\nabla K_{t-t_0}*\begin{pmatrix}\nabla\cdot\boldsymbol{E}^0\\0\end{pmatrix}+\begin{pmatrix}-4\pi\boldsymbol{j}_t\\0\end{pmatrix}+4\pi\int_{t_0}^t ds\begin{pmatrix}-\partial_t\nabla & -\square-\nabla(\nabla\cdot)\\0&0\end{pmatrix}K_{t-s}*\begin{pmatrix}\rho_s\\\boldsymbol{j}_s\end{pmatrix}.$$

According to Lemma 4.11p.61, the term involving the $\square$ is zero. Inserting the continuity equation for the current, i.e. $\nabla\cdot\boldsymbol{j}_t=-\partial_t\rho_t$, together with another partial integration in the last term, the electric (first) component of this vector equals

$$\ldots=-4\pi\boldsymbol{j}_t+\left(\nabla K_{t-t_0}*\nabla\cdot\boldsymbol{E}^0+4\pi\left[K_{t-s}*\nabla\rho_s\right]_{s=t_0}^{s=t}\right).$$

Again, by Lemma 4.11p.61 the braket yields

$$K_{t-t_0}*\nabla\cdot\boldsymbol{E}^0-4\pi K_{t-t_0}*\nabla\rho_{t_0}$$

In the case (i) $\nabla\cdot\boldsymbol{E}^0=4\pi\rho_{t_0}$ so that both terms cancel while in case (ii) both terms are identically zero by $\boldsymbol{E}^0=0$ and (4.42p.63). Hence,

$$\boxed{13}=\begin{pmatrix}-4\pi\boldsymbol{j}_t\\0\end{pmatrix},$$

and, thus, $t\to(\boldsymbol{E}_t,\boldsymbol{B}_t)$ solves the Maxwell equations (4.34p.60). The initial values can be computed with Lemma 4.11p.61

$$\begin{pmatrix}\boldsymbol{E}_t\\\boldsymbol{B}_t\end{pmatrix}\Bigg|_{t=t_0}=\lim_{t\to t_0}\begin{pmatrix}\partial_t&\nabla\wedge\\-\nabla\wedge&\partial_t\end{pmatrix}K_{t-t_0}*\begin{pmatrix}\boldsymbol{E}^0\\\boldsymbol{B}^0\end{pmatrix}=\begin{pmatrix}\boldsymbol{E}^0\\\boldsymbol{B}^0\end{pmatrix}.$$

□

REMARK 4.15. *Clearly one needs less regularity of the initial values in order to get a strong solution. However, in our context we will only need initial values in $\mathcal{F}^1$. The formula of the solutions, after the additional partial integration as noted in the beginning of the proof, agrees with the one in [KS00][(A.24),(A.25)]*[1] *which was derived with the help of the Fourier transform. For the purposes in this work, this direct approach is, however, more convenient for our regularity requirements.*

Theorem 4.14p.63 gives rise to the following definition:

Definition 4.16. *Let $(\rho,\boldsymbol{j})$ be the $\varrho\in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ induced charge-current density of a given a charge trajectory $(\boldsymbol{q},\boldsymbol{p})\in\mathcal{T}_\vee^1$ with mass $m\neq 0$, cf. Definition 4.7p.60. Then denote the solution $t\mapsto F_t$ of the Maxwell equations given by Theorem 4.14p.63 corresponding to $(\rho,\boldsymbol{j})$ and t_0 initial values $F^0=(\boldsymbol{E}_0,\boldsymbol{B}_0)\in\mathcal{F}^1$ by*

Maxwell time-evolution

$$t\mapsto M_{\varrho,m}[F^0,(\boldsymbol{q},\boldsymbol{p})](t,t_0):=F_t.$$

The second result of this section puts the well-known Liénard-Wiechert field formulas of time-like charge trajectories on mathematical rigorous grounds.

Definition 4.17. *Let $(\boldsymbol{q},\boldsymbol{p})\in\mathcal{T}_\nabla^1$ be a strictly time-like charge trajectory and $(\rho,\boldsymbol{j})$ the $\varrho\in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ induced charge-current density for some mass $m\neq 0$, cf. Definitions 4.5p.60 and 4.7p.60. Then we define*

Liénard-Wiechert fields

$$t\mapsto M_{\varrho,m}[(\boldsymbol{q},\boldsymbol{p})](t,\pm\infty):=4\pi\int_{\pm\infty}^t ds\,K_{t-s}*\begin{pmatrix}-\nabla&-\partial_s\\0&\nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s\\\boldsymbol{j}_s\end{pmatrix}$$

which we call the advanced and retarded Liénard-Wiechert fields of the charge trajectory $(\boldsymbol{q},\boldsymbol{p})$.

[1]There seems to be a misprint in equation [KS00][(A.24)]. However, (A.20) from which it is derived is correct.

That this definition makes sense for charge trajectories in $\mathcal{T}^1_\nabla$ is part of the content of the next theorem:

Liénard-Wiechert fields

Theorem 4.18. *Let $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\nabla$ be a strictly time-like charge trajectory and $(\rho,\boldsymbol{j})$ the $\varrho \in C^\infty_c(\mathbb{R}^3,\mathbb{R})$ induced charge-current density for some mass $m \neq 0$, cf. Definitions 4.5*p.60 *and 4.7*p.60*. Furthermore, let $F^0 = (\boldsymbol{E}^0,\boldsymbol{B}^0) \in \mathcal{F}^1$ be fields which fulfill the Maxwell constraints $\nabla\cdot\boldsymbol{E}^0 = 4\pi\rho_{t_0}$ and $\nabla\cdot\boldsymbol{B}_{t_0} = 0$ as well as*

$$\|\boldsymbol{E}^0(\boldsymbol{x})\| + \|\boldsymbol{B}^0(\boldsymbol{x})\| + \|\boldsymbol{x}\|\sum_{i=1}^{3}\left(\|\partial_{\boldsymbol{x}_i}\boldsymbol{E}^0(\boldsymbol{x})\| + \|\partial_{\boldsymbol{x}_i}\boldsymbol{B}^0(\boldsymbol{x})\|\right) = \underset{\|\boldsymbol{x}\|\to\infty}{\mathrm{O}}(\|\boldsymbol{x}\|^{-\epsilon}) \tag{4.43}$$

for some $\epsilon > 0$. Then for all $t \in \mathbb{R}$

$$\begin{aligned} M_{\varrho,m}[(\boldsymbol{q},\boldsymbol{p})](t,\pm\infty) &= \text{pw-lim}_{t_0\to\pm\infty} M_{\varrho,m}[F^0,(\boldsymbol{q},\boldsymbol{p})](t,t_0) \\ &= 4\pi\int_{\pm\infty}^{t} ds\,\left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s\\ \boldsymbol{j}_s\end{pmatrix}\right] = \int d^3z\,\varrho(z)\begin{pmatrix}\boldsymbol{E}^{LW\pm}_t(\cdot - z)\\ \boldsymbol{B}^{LW\pm}_t(\cdot - z)\end{pmatrix} \end{aligned} \tag{4.44}$$

is in $\mathcal{F}^1$ for

$$\boldsymbol{E}^{LW\pm}_t(\boldsymbol{x}-z) := \left[\frac{(\boldsymbol{n}\pm\boldsymbol{v})(1-\boldsymbol{v}^2)}{\|\boldsymbol{x}-z-\boldsymbol{q}\|^2(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^3} + \frac{\boldsymbol{n}\wedge[(\boldsymbol{n}\pm\boldsymbol{v})\wedge\boldsymbol{a}]}{\|\boldsymbol{x}-z-\boldsymbol{q}\|(1\pm\boldsymbol{n}\cdot\boldsymbol{v})^3}\right]^{\pm} \tag{4.45}$$

$$\boldsymbol{B}^{LW\pm}_t(\boldsymbol{x}-z) := \mp[\boldsymbol{n}\wedge\boldsymbol{E}_t(\boldsymbol{x}-z)]^{\pm} \tag{4.46}$$

and

$$\begin{array}{llll} \boldsymbol{q}^\pm := \boldsymbol{q}_{t^\pm} & \boldsymbol{v}^\pm := \boldsymbol{v}(\boldsymbol{p}_{t^\pm}) & \boldsymbol{a}^\pm := \dot{\boldsymbol{v}}^\pm \\ \boldsymbol{n}^\pm := \frac{\boldsymbol{x}-z-\boldsymbol{q}^\pm}{\|\boldsymbol{x}-z-\boldsymbol{q}^\pm\|} & t^\pm = t\pm\|\boldsymbol{x}-z-\boldsymbol{q}^\pm\|. \end{array} \tag{4.47}$$

In this context pw-lim *denotes the point-wise limit in $\mathbb{R}^3$.*

For the proof we need the following lemma:

Lemma 4.19. *Given a strictly time-like charge trajectory $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\nabla$ and a function f on $\mathbb{R}^3$ with* $\operatorname{supp} f \subseteq B_R(0)$ *for some $R > 0$ and $\boldsymbol{x}^* \in \mathbb{R}^3$ there exists a $T_{max} > 1$ so that*

$$K_r * f(\cdot - \boldsymbol{q}_{t\pm r}) = 0$$

for all $\boldsymbol{x} \in B_1(\boldsymbol{x}^*)$ and $|r| > T_{max}$.

Proof. Since

$$K_r * f(\cdot - \boldsymbol{q}_{t\pm r}) = r \fint_{\partial B_{|r|}(\boldsymbol{x})} d\sigma(y)\, f(\boldsymbol{y}-\boldsymbol{q}_{t\pm r})$$

this expression is zero if $\partial B_{|r|}(\boldsymbol{x}) \cap B_R(\boldsymbol{q}_{t\pm r}) = \emptyset$. On the one hand, for $\boldsymbol{x} \in B_1(\boldsymbol{x}^*)$, $\boldsymbol{y} \in \partial B_{|r|}(\boldsymbol{x})$ gives

$$\|\boldsymbol{x}-\boldsymbol{y}\| \geq \|\boldsymbol{x}^*-\boldsymbol{y}\| - \|\boldsymbol{x}-\boldsymbol{x}^*\| < |r| - 1. \tag{4.48}$$

In the following we consider $|r| > 1$ such that the right-hand side above is positive. On the other hand, if $\boldsymbol{y} \in B_R(\boldsymbol{q}_{t\pm r})$, we have

$$\|\boldsymbol{x}-\boldsymbol{y}\| \leq \|\boldsymbol{x}^*-\boldsymbol{q}_{t\pm r}\| + 1 + \|\boldsymbol{q}_{t\pm r}-\boldsymbol{y}\| \leq \|\boldsymbol{x}^*-\boldsymbol{q}_{t\pm r}\| + 1 + R \leq \|\boldsymbol{x}^*-\boldsymbol{q}_t\| + 1 + v_{max}|r| + R.$$

The last estimate is due to the strictly time-like nature of the charge trajectory; cf. Definition 4.5p.60. Combining this estimate with (4.48) we get $\partial B_{|r|}(\boldsymbol{x}) \cap B_R(\boldsymbol{q}_{t\pm r}) = \emptyset$ whenever

$$|r| > \max\left\{1, \frac{\|\boldsymbol{x}^*-\boldsymbol{q}_t\| + 2 + R}{1-v_{max}}\right\} =: T_{max}.$$

□

Proof of Theorem 4.18

Proof of Theorem 4.18. Fix $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^3$. By Theorem 4.14p.63 for every $t_0, t \in \mathbb{R}$

$$M_{\varrho,m}[F^0,(\boldsymbol{q},\boldsymbol{p})](t,t_0) := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \boldsymbol{E}^0 \\ \boldsymbol{B}^0 \end{pmatrix} + K_{t-t_0} * \begin{pmatrix} -4\pi \boldsymbol{j}_{t_0} \\ 0 \end{pmatrix}$$
$$+ 4\pi \int_{t_0}^{t} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} =: \boxed{17} + \boxed{18} + \boxed{19}$$

is in $\mathcal{F}^1$. At first we show that for $t_0 \to \pm\infty$ the terms $\boxed{17}$ and $\boxed{18}$ vanish with the help of (4.43p.66), which ensures that there is a constant $1 \le C_{21} < \infty$ such that for $\|\boldsymbol{x}\|$ large enough

$$\left(\|\boldsymbol{E}^0(\boldsymbol{x})\| + \|\boldsymbol{B}^0(\boldsymbol{x})\| + \|\boldsymbol{x}\| \sum_{i=1}^{3} \left(\|\partial_{\boldsymbol{x}_i}\boldsymbol{E}^0(\boldsymbol{x})\| + \|\partial_{\boldsymbol{x}_i}\boldsymbol{B}^0(\boldsymbol{x})\| \right) \right) \|\boldsymbol{x}\|^{\epsilon} \le C_{21}.$$

By Definition 4.10p.61 and for large enough t_0 we get:

$$\left\| [\nabla \wedge K_{t-t_0} * \boldsymbol{E}^0](\boldsymbol{x}) \right\| \le |t-t_0| \fint_{\partial B_{|t-t_0|}(\boldsymbol{x})} d\sigma(y) \frac{\left\|\nabla \wedge \boldsymbol{E}^0(\boldsymbol{y})\right\| \left\|\boldsymbol{y}\right\|^{1+\epsilon}}{\left\|\boldsymbol{y}\right\|^{1+\epsilon}}$$
$$\le |t-t_0| \fint_{\partial B_1(0)} d\sigma(y) \frac{C_{21}}{\left\|\boldsymbol{x} - |t-t_0|\boldsymbol{y}\right\|^{-(1+\epsilon)}} \le \frac{C_{21}|t-t_0|}{(|t-t_0| - \|\boldsymbol{x}\|)^{1+\epsilon}} \xrightarrow[t_0\to\pm\infty]{\mathbb{R}} 0$$

where the constant $C_{21} < \infty$ is given by (4.43p.66). By Equation (4.37p.61) we have

$$\left\| [\partial_t K_{t-t_0} * \boldsymbol{E}^0](\boldsymbol{x}) \right\| \le \fint_{\partial B_{|t-t_0|}(\boldsymbol{x})} d\sigma(y)\, \|\boldsymbol{E}^0(\boldsymbol{y})\| + |t-t_0| \fint_{\partial B_1(0)} d\sigma(y)\, \|\boldsymbol{y} \cdot \nabla \boldsymbol{E}^0(\boldsymbol{x} - |t-t_0|\boldsymbol{y})\|.$$

Let again t_0 be sufficiently large. The first term on the right-hand side equals

$$\fint_{\partial B_{|t-t_0|}(\boldsymbol{x})} d\sigma(y) \frac{\|\boldsymbol{E}^0(\boldsymbol{y})\|\, \|\boldsymbol{y}\|^{\epsilon}}{\|\boldsymbol{y}\|^{\epsilon}} \le \frac{C_{21}}{(|t-t_0| - \|\boldsymbol{x}\|)^{\epsilon}} \xrightarrow[t_0\to\pm\infty]{\mathbb{R}} 0,$$

while the second term is smaller or equals

$$|t-t_0| \fint_{\partial B_1(0)} d\sigma(y) \frac{\sum_{i=1}^{3} \|\partial_{\boldsymbol{x}_i}\boldsymbol{E}^0(\boldsymbol{x} - |t-t_0|\boldsymbol{y})\|\, \|\boldsymbol{x} - |t-t_0|\boldsymbol{y}\|^{1+\epsilon}}{\|\boldsymbol{x} - |t-t_0|\boldsymbol{y}\|^{1+\epsilon}}$$
$$\le \frac{C_{21}|t-t_0|}{(|t-t_0| - \|\boldsymbol{x}\|)^{1+\epsilon}} \xrightarrow[t_0\to\pm\infty]{\mathbb{R}} 0.$$

Next we show that in the limit $t_0 \to \pm\infty$ the term $\boxed{18}$ also vanishes. As $(\boldsymbol{q},\boldsymbol{p})$ is a time-like charge trajectory we can apply Lemma 4.19p.66 for $r = t - t_0$ which yields

$$\|[K_{t-t_0} * \boldsymbol{j}_{t_0}](\boldsymbol{x})\| = 0$$

for large enough $|t_0|$. Therefore, we can conclude that term $\boxed{18}$ is zero for t_0 large enough. The same holds with $\boldsymbol{E}^0$ replaced by $\boldsymbol{B}^0$, and therefore we find

$$\lim_{t_0\to\pm\infty} \boxed{19} = 4\pi \int_{\pm\infty}^{t} ds \left[K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \right](\boldsymbol{x}) = \left(M_{\varrho,m}[(\boldsymbol{q},\boldsymbol{p})](t,\pm\infty) \right)(\boldsymbol{x})$$
$$= 4\pi \int_0^{\infty} dr \left[K_r * \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_{t\pm r} \\ \boldsymbol{j}_{t\pm r} \end{pmatrix} \right](\boldsymbol{x}) = \int d^3y \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \frac{\rho(\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{q}_{t\pm\|\boldsymbol{y}\|})}{\|\boldsymbol{y}\|} \begin{pmatrix} 1 \\ \boldsymbol{v}_{t\pm r} \end{pmatrix} \quad (4.49)$$
$$=: \begin{pmatrix} \boldsymbol{E}_t^{\pm}(\boldsymbol{x}) \\ \boldsymbol{B}_t^{\pm}(\boldsymbol{x}) \end{pmatrix}.$$

Let us first compute the electric fields

$$E_t^\pm(x) = \int d^3y \left[\frac{-\nabla\rho(x - y - q_{t\pm\|y\|})}{\|y\|} + \frac{v_{t\pm\|y\|} \cdot \nabla\rho(x - y - q_{t\pm\|y\|})\, v_{t\pm\|y\|}}{\|y\|} - \frac{\rho(x - y - q_{t\pm\|y\|})\, a_{t\pm\|y\|}}{\|y\|} \right].$$

In order to simplify this expression we make a transformation of the integration variable:

$$y \to z(y) := x - y - q_{t\pm\|y\|} \tag{4.50}$$

Here, we use that $(q,p) \in \mathcal{T}^1_\nabla$ is a strictly time-like charge trajectory. We observe that $z(\cdot)$ is a diffeomorphism because, first, it is bijective since for $\sup_{t\in\mathbb{R}} \|v_t\| \le v_{max} < 1$ the equation $y(z) = x - z - q_{t\pm\|y(z)\|}$ has a unique solution $y(z)$ for all $z \in \mathbb{R}^3$ which is given by $\{q^\pm\} = \bigcup_{r\ge0}(\partial B_r(x - z) \cap \{q_{t\pm r}\})$, i.e. the intersection of the charge trajectory and the forward, respectively backward, light cone of $x - z$. And second, $z(\cdot)$ is continuously differentiable with $(\partial_{y_i} z_j(y))_{1\le i,j\le3} = -\delta_{ij} \pm v_{j,t\pm\|y\|}\frac{y_i}{\|y\|}$ such that it has a non-zero determinant which equals $(-1 \pm v_{t\pm\|y\|} \cdot \frac{y}{\|y\|})$, again because $\sup_{t\in\mathbb{R}} \|v_t\| \le v_{max} < 1$, and therefore the inverse of $z(\cdot)$ is also continuously differentiable. In order to make the notation more readable we shall use the abbreviations (4.47p.66). We then get

$$\begin{aligned} E_t^\pm(x) &= \int d^3z\, \frac{-\nabla\rho(z) + v^\pm \cdot \nabla\rho(z)\, v^\pm - \rho(z)\, a^\pm}{\|x - z - q^\pm\|(1 \pm n^\pm \cdot v^\pm)} \\ &= \int d^3z\, \rho(z)\Bigg[\nabla_z \frac{1}{\|x - z - q^\pm\|(1 \pm n^\pm \cdot v^\pm)} - \sum_{k=1}^{3} \partial_{z_k} \frac{v_k^\pm\, v^\pm}{\|x - z - q^\pm\|(1 \pm n^\pm \cdot v^\pm)} \\ &\quad - \frac{a^\pm}{\|x - z - q^\pm\|(1 \pm n^\pm \cdot v^\pm)}\Bigg]. \end{aligned} \tag{4.51}$$

after a partial integration. Note that for this we only need almost everywhere differentiability. Doing the same for the magnetic field yields

$$B_t^\pm(x) = \int d^3z\, \rho(z)\left[-\nabla \wedge \frac{v^\pm}{\|x - z - q^\pm\|(1 \pm n^\pm \cdot v^\pm)}\right] \tag{4.52}$$

After a tedious but not really interesting computation(see Computation in Appendix 5.3p.98) one finds that Equation (4.44p.66) holds. Since we can represent the Maxwell solution by a convolution with a $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ function it is immediate that $F_t^\pm \in \mathcal{F}^1$. This concludes the proof. □

REMARK 4.20. *Condition (4.43p.66) guarantees that in the limit $t_0 \to \pm\infty$ the initial value F^0 are forgotten by the time-evolution of the Maxwell equations. Note that in order to compute the Liénard-Wiechert fields the strictly time-like nature of the charge trajectory is sufficient for the limit to exists $t_0 \to \pm\infty$. This condition could be softened into an integrability condition for more general ρ and j, e.g. one must only demand that the right-hand side of (4.49p.67) is finite. However, the Liénard-Wiechert fields for time-like charge trajectories would then in general not be given by (4.44p.66) since (4.50) does not have to be bijective anymore. This fact is indicated by the blow up of the factors $(1 \pm n \cdot v)^{-3}$ in Equation (4.45p.66) for $v \to 1$.*

Liénard-Wiechert Fields Solve the Maxwell equations

Theorem 4.21. *Let $(q,p) \in \mathcal{T}^1_\nabla$ be a strictly time-like charge trajectory and (ρ, j) the $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ induced charge-current density for some mass $m \neq 0$, cf. Definitions 4.5p.60 and 4.7p.60. Then the Liénard-Wiechert fields $M_{\varrho,m}[(q,p)](t, \pm\infty)$ are a solution to the Maxwell equations (4.34p.60) including the Maxwell constraints for all $t \in \mathbb{R}$.*

Proof. $(\rho,\boldsymbol{j})$ is the $\varrho \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ induced charge-current density of the strictly time-like charge trajectory $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\nabla$. Hence, for any $t \in \mathbb{R}$

$$\rho_t = \varrho(\cdot - \boldsymbol{q}_t) \qquad \text{and} \qquad \boldsymbol{j}_t = \boldsymbol{v}(\boldsymbol{p}_t)\varrho(\cdot - \boldsymbol{q}_t).$$

Therefore, Lemma 4.19p.66, for the choice $r = t - s$, states that for all $t^* \in \mathbb{R}$ and $\boldsymbol{x}^* \in \mathbb{R}^3$ there exists a constant $1 < T_{max} < \infty$ such that: For all $t \in B_1(t^*)$ and $\boldsymbol{x} \in B_1(\boldsymbol{x}^*)$

$$\left[K_{t-s} * \begin{pmatrix}\rho_s\\ \boldsymbol{j}_s\end{pmatrix}\right](\boldsymbol{x}) = 0 \text{ if } |s| > T := T_{max} + |t^*| + 1.$$

This allows for any $t \in B_1(t^*)$ and $\boldsymbol{x} \in B_1(\boldsymbol{x}^*)$ to rewrite Equation (4.49p.67) into

$$\left(M_{\varrho,m}[(\boldsymbol{q},\boldsymbol{p})](t,\pm\infty)\right)(\boldsymbol{x}) = \begin{pmatrix}\boldsymbol{E}_t^\pm(\boldsymbol{x})\\ \boldsymbol{B}_t^\pm(\boldsymbol{x})\end{pmatrix} = 4\pi\int_{\pm\infty}^t ds\, \left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s\\ \boldsymbol{j}_s\end{pmatrix}\right](\boldsymbol{x}) \tag{4.53}$$

$$= 4\pi\int_{\pm T}^t ds\, \left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s\\ \boldsymbol{j}_s\end{pmatrix}\right](\boldsymbol{x}). \tag{4.54}$$

So, for $t_0 = \pm T$, the right-hand side of (4.53) equals

$$K_{t-t_0} * \begin{pmatrix}-4\pi\boldsymbol{j}_{t_0}\\ 0\end{pmatrix} + 4\pi\int_{t_0}^t ds\, \left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s\\ \boldsymbol{j}_s\end{pmatrix}\right](\boldsymbol{x})$$

which by Theorem 4.14p.63(ii) solves the Maxwell Equation including the Maxwell constraints 4.34p.60 for all $t \in B_1(t^*)$ and $\boldsymbol{x} \in B_1(\boldsymbol{x}^*)$. Since $t^* \in \mathbb{R}$ and $\boldsymbol{x}^* \in \mathbb{R}^3$ are arbitrary, the Maxwell Equation including the Maxwell constraints are fulfilled for all $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^3$ which concludes the proof. □

From their explicit expressions we immediately get a simple bound on the Liénard-Wiechert fields:

Liénard-Wiechert estimate

Corollary 4.22. *Let $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\nabla$ be a strictly time-like charge trajectory and $(\rho,\boldsymbol{j})$ the $\varrho \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ induced charge-current density for some mass $m \neq 0$, cf. Definitions 4.5p.60 and 4.7p.60. Furthermore, assume there exists an $a_{max} < \infty$ such that $\sup_{t\in\mathbb{R}} \|\partial_t \boldsymbol{v}(\boldsymbol{p}_t)\| \leq a_{max}$, we then get a simple estimate for the Liénard-Wiechert fields for all $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$ and multi-index $\alpha \in \mathbb{N}^3$:*

$$\|D^\alpha \boldsymbol{E}_t^\pm(\boldsymbol{x})\| + \|D^\alpha \boldsymbol{B}_t^\pm(\boldsymbol{x})\| \leq \frac{C_{22}{}^{(\alpha)}}{(1-v_{max})^3}\left(\frac{1}{1+\|\boldsymbol{x}-\boldsymbol{q}_t\|^2} + \frac{a_{max}}{1+\|\boldsymbol{x}-\boldsymbol{q}_t\|}\right)$$

for

$$\begin{pmatrix}\boldsymbol{E}_i^\pm\\ \boldsymbol{B}_i^\pm\end{pmatrix} := M_{\varrho,m}[(\boldsymbol{q},\boldsymbol{p})](t,\pm\infty),$$

a family of finite constants $(C_{22}{}^{(\alpha)})_{\alpha\in\mathbb{N}^3}$ and v_{max} as defined in Definition 4.5p.60.

Proof. From Theorem 4.18p.66 we know that for this sub-light charge trajectory the Liénard-Wiechert fields take the form

$$\begin{pmatrix}\boldsymbol{E}_i^\pm(\boldsymbol{x})\\ \boldsymbol{B}_i^\pm(\boldsymbol{x})\end{pmatrix} = \int d^3z\, \varrho(\boldsymbol{x}-z)\begin{pmatrix}\boldsymbol{E}_i^{LW\pm}(z)\\ \boldsymbol{B}_i^{LW\pm}(z)\end{pmatrix}. \tag{4.55}$$

As the integrand is infinitely often differentiable in $\boldsymbol{x}$ and has compact support, the derivatives for any multi-index $\alpha \in \mathbb{N}^3$ are given by

$$D^\alpha F_t^\pm(\boldsymbol{x}) = \int d^3z\, D^\alpha \varrho_i(\boldsymbol{x}-z)\begin{pmatrix}\boldsymbol{E}_i^\pm(z)\\ \boldsymbol{B}_i^\pm(z)\end{pmatrix}.$$

First, we take a look at (4.45p.66) and (4.46p.66) for given $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. As we have a strictly time-like charge trajectory $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{T}^1_\nabla$, $\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}^\pm\|$ is the smallest if we assume the worst case, i.e. that from time t on the rigid charge moves into the future (respectively into the past) with the speed of light towards the point $\boldsymbol{x}-\boldsymbol{z}$. Therefore, $\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}^\pm\| \leq \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}_t\|$ and, hence,

$$\|\boldsymbol{B}_t^{LW\pm}(\boldsymbol{x}-\boldsymbol{z})\| + \|\boldsymbol{E}_t^{LW\pm}(\boldsymbol{x}-\boldsymbol{z})\| \leq \frac{2}{(1-v_{max})^3}\left[\frac{1}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}_t\|^2} + \frac{a_{max}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}_t\|}\right]^\pm \tag{4.56}$$

because $\sup_{t\in\mathbb{R}}\|\boldsymbol{v}_t\| \leq v_{max} < 1$. The rest is straightforward computation (see Computation in Appendix 5.4p.99). □

4.3.2 Unique Identification of Wheeler-Feynman Solutions

Using the results of Section 4.3.1p.60 we can give a sensible definition of what we mean by solutions to the Wheeler-Feynman equations (4.1p.43) and (4.2p.43). We restrict the class of possible Wheeler-Feynman solutions to:

Class of Wheeler-Feynman solutions

Definition 4.23. *Let $\mathcal{T}_{\mathrm{WF}}$ denote the set of strictly time-like charge trajectories $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N} \in \mathcal{T}_\nabla$ with masses $m_i \neq 0$, $1 \leq i \leq N$ and with the properties:*

(i) There exists an $a_{max} < \infty$ such that $\sup_{t\in\mathbb{R}}\|\partial_t \boldsymbol{v}(\boldsymbol{p}_{i,t})\| \leq a_{max}$, i.e. the accelerations of the charges are bounded.

(ii) for all times $t \in \mathbb{R}$ solve the Wheeler-Feynman equations (4.1p.43) and (4.2p.43).

REMARK 4.24. *(i) Note that this definition is sensible because with $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N} \in \mathcal{T}_\nabla$, equations (4.2p.43) for $1 \leq i \leq N$ can by Definition 4.17p.65 be rewritten as:*

$$(\boldsymbol{E}_{i,t}^{\mathrm{WF}}, \boldsymbol{B}_{i,t}^{\mathrm{WF}}) = \frac{1}{2}\sum_\pm M_{\varrho_i,m_i}[(\boldsymbol{q}_i,\boldsymbol{p}_i)](t,\pm\infty).$$

Theorem 4.18p.66 guarantees that the right-hand side is well-defined. Furthermore, charge trajectories in $\mathcal{T}^1_\nabla$ are once continuously differentiable so that the left-hand side of (4.1p.43) is also well-defined. The bound on the acceleration will give us a bound on the Wheeler-Feynman fields in the $\mathcal{H}_w$ norm; see Lemma 4.26p.71.

(ii) Furthermore, it is highly expected that $\mathcal{T}_{\mathrm{WF}}$ is non-empty for two reasons: 1. In the point particle case there are explicit solutions to the Wheeler-Feynman equations known, i.e. the Schild solutions [Sch63] and the solutions of Bauer's existence theorem [Bau97], which yield strictly time-like charge trajectories with bounded accelerations. 2. Physically, one would expect that in general scattering solutions have accelerations that decay at $t \to \pm\infty$.

For the main theorem of this section we need the following lemmas. First, we give an example of a suitable weight w in $\mathcal{W}^\infty$.

Explicit expression for the weight w

Lemma 4.25. *For $\boldsymbol{x} \mapsto w(\boldsymbol{x}) := (1-\|\boldsymbol{x}\|^2)^{-1}$ it holds $w \in \mathcal{W}^\infty$; cf. Equation (3.6p.18).*

Proof. By Equation (3.8p.19) w is in $\mathcal{W}$. Thus, it is left to show that this w is also in $\mathcal{W}^k$ for any $k \in \mathbb{N}$. To see this let us consider

$$\begin{aligned}
0 &= D^\alpha\left(w(\boldsymbol{x})(1+\|\boldsymbol{x}\|^2)\right)\\
&= \sum_{k_1,k_2,k_3=0}^{\alpha_1,\alpha_2,\alpha_3}\binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\partial_1^{\alpha_1-k_1}\partial_2^{\alpha_2-k_2}\partial_3^{\alpha_3-k_3}w(\boldsymbol{x})\partial_1^{k_1}\partial_2^{k_2}\partial_3^{k_3}(1+\|\boldsymbol{x}\|^2)\\
&= (D^\alpha w(\boldsymbol{x}))(1+\|\boldsymbol{x}\|^2) + \sum_{i=1}^3 \alpha_i\left(\partial_i^{\alpha_i-1}w(\boldsymbol{x})\right)2x_i + \sum_{i=1}^3\frac{1}{2}\alpha_i(\alpha_i-1)\left(\partial_i^{\alpha_i-2}w(\boldsymbol{x})\right)2
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ is a multi-index. This leads to the recursive estimate

$$\left|D^\alpha w(\boldsymbol{x})\right| \le w(\boldsymbol{x}) \left(\sum_{i=1}^{3} 2\alpha_i \left|\partial_i^{\alpha_i - 1} w(\boldsymbol{x})\right| |x_i| + \sum_{i=1}^{3} \alpha_i(\alpha_i - 1) \left|\partial_i^{\alpha_i - 2} w(\boldsymbol{x})\right| \right)$$

in the sense that terms involving ∂^l for negative l equal zero. Hence, the left-hand side can be bounded by lower derivatives, and therefore, by induction over the multi-index α, we get constants $C^\alpha < \infty$ such that $|D^\alpha w(\boldsymbol{x})| \le C^\alpha w(\boldsymbol{x})$. Furthermore, from the computation

$$D^\alpha w(\boldsymbol{x}) = D^\alpha \left(\sqrt{w(\boldsymbol{x})}\, \sqrt{w(\boldsymbol{x})} \right) = \sum_{k_1,k_2,k_3=0}^{\alpha_1,\alpha_2,\alpha_3} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \partial_1^{\alpha_1-k_1} \partial_2^{\alpha_2-k_2} \partial_3^{\alpha_3-k_3} \sqrt{w(\boldsymbol{x})} \partial_1^{k_1} \partial_2^{k_2} \partial_3^{k_3} \sqrt{w(\boldsymbol{x})}$$

and with $I_\alpha := \{k \in \mathbb{N}^3 \mid 0 \le k_i \le \alpha_i, i = 1, 2, 3\} \setminus \{(0, 0, 0), \alpha\}$ we get the recursive formula

$$\left|D^\alpha \sqrt{w(\boldsymbol{x})}\right| \le \frac{1}{2} \Bigg[C^\alpha \sqrt{w(\boldsymbol{x})} + \frac{1}{\sqrt{w(\boldsymbol{x})}} \sum_{(k_1,k_2,k_3)\in I_\alpha} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \times \\ \times \left|\partial_1^{\alpha_1-k_1} \partial_2^{\alpha_2-k_2} \partial_3^{\alpha_3-k_3} \sqrt{w(\boldsymbol{x})}\right| \left|\partial_1^{k_1} \partial_2^{k_2} \partial_3^{k_3} \sqrt{w(\boldsymbol{x})}\right| \Bigg].$$

where we have used the above established estimate $|D^\alpha w(\boldsymbol{x})| \le C^\alpha w(\boldsymbol{x})$. Again, the left-hand side can be bounded by lower derivatives, and therefore, by induction over the multi-index α, we yield finite constants C_α such that also $|D^\alpha \sqrt{w}| \le C_\alpha w$. Therefore, $w \in \mathcal{W}^k$ for any $k \in \mathbb{N}$ and, thus, $w \in \mathcal{W}^\infty$. □

Second, we show that this weight w decays quickly enough such that all Liénard-Wiechert fields of strictly time-like charge trajectories in $\mathcal{T}^1_\triangledown$ with bounded accelerations lie in $D_w(A^\infty)$.

Lemma 4.26. *Let $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N} \in \mathcal{T}^1_\triangledown$ with masses $m_i \neq 0$, $1 \le i \le N$, and assume there exists an $a_{max} < \infty$ such that $\sup_{t\in\mathbb{R}} \|\partial_t \boldsymbol{v}(\boldsymbol{p}_{i,t})\| \le a_{max}$. Define $t \mapsto (\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t}) := M[(\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm\infty)$. Then there exists a $w \in \mathcal{W}^\infty$ such that for any $\boldsymbol{q}_i, \boldsymbol{p}_i \in \mathbb{R}^3$, $1 \le i \le N$, it is true that*

Regularity of the Liénard-Wiechert fields

$$(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N} \in D_w(A^\infty), \ \textit{for all } t \in \mathbb{R}.$$

Proof. The charge trajectories are in $\mathcal{T}^1_\triangledown$ and therefore strictly time-like. Furthermore, they have bounded accelerations. Therefore, by Corollary 4.22p.69, for $1 \le i \le N$ and each multi-index $\alpha \in \mathbb{N}^3$ there exists a constant $C_{22}{}^{(\alpha)} < \infty$ such that

$$\|D^\alpha \boldsymbol{E}^\pm_{i,t}(\boldsymbol{x})\| + \|D^\alpha \boldsymbol{B}^\pm_{i,t}(\boldsymbol{x})\| \le \frac{C_{22}}{(1 - v_{max})^3} \left(\frac{1}{1 + \|\boldsymbol{x} - \boldsymbol{q}_t\|^2} + \frac{a_{max}}{1 + \|\boldsymbol{x} - \boldsymbol{q}_t\|} \right).$$

Hence, for $w(\boldsymbol{x}) = \frac{1}{1+\|\boldsymbol{x}\|^2}$ we get

$$\left\|A^n(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}^\pm_{i,t}, \boldsymbol{B}^\pm_{i,t})\right\|_{\mathcal{H}_w} \le \sum_{i=1}^{N} \sum_{|\alpha|\le n} \left(\|\boldsymbol{q}_{i,t}\| + \|\boldsymbol{p}_{i,t}\| + \int d^3x\, w(\boldsymbol{x}) \left(\|D^\alpha \boldsymbol{E}^\pm_{i,t}(\boldsymbol{x})\|^2 + \|D^\alpha \boldsymbol{B}^\pm_{i,t}(\boldsymbol{x})\|^2 \right) \right)$$

which is finite, so that for any $t \in \mathbb{R}$ we have $\varphi_t \in D_w(A^\infty)$. □

Third, we show that the charge trajectories in $\mathcal{T}_{\mathrm{WF}}$ and their Liénard-Wiechert fields give rise to a ML-SI solution.

Wheeler-Feynman trajectories give rise to ML-SI solutions

Lemma 4.27. *Let* $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N} \in \mathcal{T}_{\mathrm{WF}}$ *and* $(\boldsymbol{E}^{\mathrm{WF}}_{i,t},\boldsymbol{B}^{\mathrm{WF}}_{i,t}) := \frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[(\boldsymbol{q}_i,\boldsymbol{p}_i)](t,\pm\infty)$ *for all* $t\in\mathbb{R}$. *Define*

$$t\mapsto \varphi_t := (\boldsymbol{q}_{i,t},\boldsymbol{p}_{i,t},\boldsymbol{E}^{\mathrm{WF}}_{i,t},\boldsymbol{B}^{\mathrm{WF}}_{i,t})_{1\leq i\leq N}.$$

Then for the case (ML-SI$_{\text{p.16}}$*), i.e.* $e_{ij} = 1-\delta_{ij}$, $1\leq i,j\leq N$, *and any* $t_0, t\in\mathbb{R}$ *it holds*

$$\varphi_t = M_L[\varphi_{t_0}](t,t_0);$$

cf. Definition 3.28$_{\text{p.41}}$.

Proof. First, as the charge trajectories fulfill the Wheeler-Feynman equations (4.1$_{\text{p.43}}$), they also fulfill the Lorentz force law (3.1$_{\text{p.15}}$) because we set $e_{ij} = 1-\delta_{ij}$, $1\leq i,j\leq N$. Second, by Theorem 4.21$_{\text{p.68}}$ the fields $(\boldsymbol{E}^{\pm}_{i,t},\boldsymbol{B}^{\pm}_{i,t})$ solve the Maxwell equations including the Maxwell constraints, both given in the set of equations (3.2$_{\text{p.15}}$). Therefore, $t\mapsto\varphi_t$ is a solution to the ML-SI equations, i.e. the coupled set of equations (3.1$_{\text{p.15}}$) plus (3.2$_{\text{p.15}}$) for $e_{ij}=1-\delta_{ij}$, $1\leq i,j\leq N$. By Lemma 4.26$_{\text{p.71}}$ for any t_0 we yield $\varphi_{t_0}\in D_w(A^\infty)$ so that the existence assertion of Theorem 3.5$_{\text{p.20}}$ states that there is a solution $t\mapsto\widetilde{\varphi}_t$ of the ML-SI equations with $\varphi_{t_0}=\widetilde{\varphi}_{t_0}$ while the uniqueness assertion of that theorem states that if $\varphi_{t_0}=\widetilde{\varphi}_{t_0}$ for any $t_0\in\mathbb{R}$, we have $\varphi_t=\widetilde{\varphi}_t$ for all $t\in\mathbb{R}$. Therefore, we conclude $\varphi_t=\widetilde{\varphi}_t = M_L[\varphi_{t_0}](t,t_0)$ for all $t\in\mathbb{R}$. □

From these lemmas and the uniqueness of ML-SI solutions it follows that all Wheeler-Feynman solutions in $\mathcal{T}_{\mathrm{WF}}$ can be identified uniquely by specifying their positions, momenta and Wheeler-Feynman fields at a certain time $t_0\in\mathbb{R}$:

Sufficient Wheeler-Feynman initial conditions

Theorem 4.28. *There exists a* $w\in\mathcal{W}^\infty$ *such that for each* $t_0\in\mathbb{R}$ *the following map is injective:*

$$i_{t_0}:\mathcal{T}_{\mathrm{WF}}\to D_w(A^\infty), \qquad (\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N}\mapsto(\boldsymbol{q}_{i,t_0},\boldsymbol{p}_{i,t_0},\boldsymbol{E}^{\mathrm{WF}}_{i,t_0},\boldsymbol{B}^{\mathrm{WF}}_{i,t_0})_{1\leq i\leq N}$$

where $(\boldsymbol{E}^{\mathrm{WF}}_{i,t_0},\boldsymbol{B}^{\mathrm{WF}}_{i,t_0}) := \frac{1}{2}\sum_{\pm}M_{\varrho_i,m_i}[(\boldsymbol{q}_i,\boldsymbol{p}_i)](t_0,\pm\infty)$.

Proof. Let $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N}, (\widetilde{\boldsymbol{q}}_i,\widetilde{\boldsymbol{p}}_i)_{1\leq i\leq N}\in\mathcal{T}_{\mathrm{WF}}$ and $t_0\in\mathbb{R}$. Define $\varphi_t := (\boldsymbol{q}_{i,t},\boldsymbol{p}_{i,t},\boldsymbol{E}^{\mathrm{WF}}_{i,t},\boldsymbol{B}^{\mathrm{WF}}_{i,t})_{1\leq i\leq N}$ and $\widetilde{\varphi}_t := (\widetilde{\boldsymbol{q}}_{i,t},\widetilde{\boldsymbol{p}}_{i,t},\widetilde{\boldsymbol{E}}^{\mathrm{WF}}_{i,t},\widetilde{\boldsymbol{B}}^{\mathrm{WF}}_{i,t})_{1\leq i\leq N}$ for all $t\in\mathbb{R}$ as in Lemma 4.27. By Lemma 4.26$_{\text{p.71}}$ there is a $w\in\mathcal{W}^\infty$ such that $\varphi_{t_0},\widetilde{\varphi}_{t_0}\in D_w(A^\infty)$ and therefore the range of i_{t_0} is a subset of $D_w(A^\infty)$. From Lemma 4.27 we know in addition that for all $t\in\mathbb{R}$, $\varphi_t = M_L[\varphi_{t_0}](t,t_0)$ and $\widetilde{\varphi}_t = M_L[\widetilde{\varphi}_{t_0}](t,t_0)$. Assume $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N}\neq(\widetilde{\boldsymbol{q}}_i,\widetilde{\boldsymbol{p}}_i)_{1\leq i\leq N}$, i.e. there exist $t\in\mathbb{R}$ such that we have $(\boldsymbol{q}_{i,t},\boldsymbol{p}_{i,t})_{1\leq i\leq N}\neq(\widetilde{\boldsymbol{q}}_{i,t},\widetilde{\boldsymbol{p}}_{i,t})_{1\leq i\leq N}$. For such t we have then $M_L[\varphi_{t_0}](t,t_0)=\varphi_t\neq\widetilde{\varphi}_t = M_L[\widetilde{\varphi}_{t_0}](t,t_0)$. The uniqueness assertion of Theorem 3.5$_{\text{p.20}}$ then states $\varphi_{t_0}\neq\varphi_{t_0}$. By construction $\varphi_{t_0} = i_{t_0}((\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\leq i\leq N}$ and $\widetilde{\varphi}_{t_0} = i_{t_0}(\widetilde{\boldsymbol{q}}_i,\widetilde{\boldsymbol{p}}_i)_{1\leq i\leq N}$. Hence $i_{t_0}:\mathcal{T}_{\mathrm{WF}}\to D_w(A^\infty)$ is injective. □

REMARK 4.29. *Note that the weight function* w *could be chosen to decay faster than the choice in Lemma 4.25*$_{\text{p.70}}$. *This freedom allows to generalize Theorem 4.28 also for possible Wheeler-Feynman solutions whose acceleration is not bounded but may grow with* $t\to\pm\infty$. *This is due to the fact that growth of the acceleration* $\boldsymbol{a}$ *in equations (4.44*$_{\text{p.66}}$*) can be pushed down by the weight* w. *However, since the weight* w *must be at least in* $\mathcal{W}^1$, *which then ensures by Lemma 3.23*$_{\text{p.30}}$ *that the group* $(W_t)_{t\in\mathbb{R}}$ *exists, one can only allow the acceleration* $\boldsymbol{a}$ *to grow slower than exponentially.*

4.4 Existence of Wheeler-Feynman Initial Fields

We shall now come to the question of existence of Wheeler-Feynman solutions. Once and for all we fix the parameters:

Definition 4.30. *To the very end of this chapter we fix the charge distributions* $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ *such that* $\operatorname{supp} \varrho_i \subset B_R(0) \subset \mathbb{R}^3$ *for one finite* $R > 0$ *and the masses* $m_i \neq 0$, $1 \leq i \leq N$. *Furthermore, we shall use the (ML-SIp.16) choice, i.e.* $e_{ij} = 1 - \delta_{ij}$, $1 \leq i, j \leq N$, *whenever we use the notation* $M_L[\cdot]$ *from Definition 3.28p.41 or the Maxwell-Lorentz equations which we refer to as the ML-SI equations. Furthermore, we choose a weight* $w \in \mathcal{W}^\infty$ *for which Theorem 4.28 holds.*

Global definition of w, ϱ_i, m_i and e_{ij}

In subsection 4.4.2p.75 we shall formalize the map $S_T^{p,X^\pm}$ and prove the existence of a fixed point. The proof will rely on the explicit expressions for the Maxwell fields of the ML-SI dynamics of chapter 3p.15 in terms of the Kirchoff's formulas given in section 4.3.1p.60. Therefore, we inserted a small intermediate subsection before the main proof which will provide all necessary formulas.

4.4.1 The Maxwell Fields of the Maxwell-Lorentz Dynamics

This intermediate subsection is supposed to bring quickly together the solution theories of the Maxwell-Lorentz equations (chapter 3p.15) on $D_w(A)$ and the Maxwell equations (subsection 4.3.1p.60) on $\mathcal{F}^1$. In particular, it will provide explicit formulas for the Maxwell solutions expressed by $(W_t)_{t\in\mathbb{R}}$ and J on a suitable domain. We recall the Newtonian phase space $\mathcal{P} = \mathbb{R}^{6N}$, the space of weighted square integrable fields $\mathcal{F}_w$, the phase space $\mathcal{H}_w = \mathcal{P} \oplus \mathcal{F}_w$ of the Maxwell-Lorentz equations, cf. Definition 3.2p.19, the definition of the operator A on $D_w(A) \subset \mathcal{H}_w$, cf. Definition 3.3p.19, as well as the one of the operator J on $\mathcal{H}_w$, cf. Definition 3.4p.19. In order not to blow up the notation we use the following:

Notation 4.31. *For any* $\varphi = (\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N} \in \mathcal{H}_w$ *we define the projectors* $\mathsf{Q}, \mathsf{P}, \mathsf{F}$ *by*

Projectors $\mathsf{P}, \mathsf{Q}, \mathsf{F}$

$$\mathsf{Q}\varphi = (\boldsymbol{q}_i, 0, 0, 0)_{1\leq i\leq N}, \qquad \mathsf{P}\varphi = (0, \boldsymbol{p}_i, 0, 0)_{1\leq i\leq N}, \qquad \mathsf{F}\varphi = (0, 0, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\leq i\leq N}.$$

Wherever formal type errors do not lead to ambiguities we sometimes forget about or add the zero components and write, e.g.,

$$(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\leq i\leq N} = (\mathsf{Q} + \mathsf{P})\varphi \qquad \text{or} \qquad (\boldsymbol{q}_i, \boldsymbol{p}_i, 0, 0)_{1\leq i\leq N} = (\mathsf{Q} + \mathsf{P})(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\leq i\leq N}.$$

As we now treat N fields simultaneously, we need to extend $\mathcal{F}^1$, cf. Definition 4.8p.60, according to:

Definition 4.32. $\mathcal{F} := \bigoplus_{i=1}^N C^\infty(\mathbb{R}^3, \mathbb{R}^3) \oplus C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

Space of N smooth fields

Furthermore, we recall that A is the generator of a γ contractive group $(W_t)_{t\in\mathbb{R}}$ on $D_w(A)$ which was the content of Definition 3.24p.32 and its preceding lemma. Since we shall mainly work in field spaces, we need the projections of the operators A, W_t and J onto field space $\mathcal{F}_w$:

Definition 4.33. *For all* $t \in \mathbb{R}$ *and* $\varphi \in \mathcal{H}_w$ *we define*

Projection of A, W_t, J to field space $\mathcal{F}_w$

$$\mathsf{A} := \mathsf{F}A\mathsf{F}, \qquad \mathsf{W}_t := \mathsf{F}W_t\mathsf{F} \qquad \text{and} \qquad \mathsf{J} := \mathsf{F}J(\varphi).$$

The natural domain of A, W_t *is given by* $D_w(\mathsf{A}) := \mathsf{F}D_w(A) \subset \mathcal{F}_w$. *We shall also need* $D_w(\mathsf{A}^n) := \mathsf{F}D_w(A^n) \subset \mathcal{F}_w$ *for every* $n \in \mathbb{N} \cup \{\infty\}$. *Clearly, the operator* A *on* $D_w(\mathsf{A})$ *is also closed and inherits also the resolvent properties from* A *on* $D_w(A)$. *Furthermore, this implies* $(\mathsf{Q} + \mathsf{P})W_t = \mathrm{id}_{\mathcal{P}}$ *and* $\mathsf{F}W_t = \mathsf{W}_t$ *so that* $(\mathsf{W}_t)_{t\in\mathbb{R}}$ *is also a* γ *contractive group on the smaller space* $D_w(\mathsf{A})$. *Finally, note also that by the definition of* J *we have* $\mathsf{J}(\varphi) = \mathsf{J}((\mathsf{Q} + \mathsf{P})\varphi)$ *for all* $\varphi \in \mathcal{H}_w$, *i.e.* J *does not depend on the field components* $\mathsf{F}\varphi$.

The following corollary translates the explicit Kirchoff formulas for free Maxwell solutions computed in subsection 4.3.1p.60 into the language of the group $(\mathsf{W}_t)_{t\in\mathbb{R}}$. We have used Kirchoff's formulas for initial fields in $\mathcal{F}$ while the group $(\mathsf{W}_t)_{t\in\mathbb{R}}$ operates on $D_w(\mathsf{A})$. Therefore, by uniqueness, we expect to be able to express free Maxwell solution generated by the group by Kirchoff's formulas as long as the initial conditions come from $\mathcal{F} \cap D_w(\mathsf{A})$.

Kirchoff's formulas in terms of $(W_t)_{t\in\mathbb{R}}$

Corollary 4.34. *Let $w \in \mathcal{W}^1$, $F \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ for some $n \in \mathbb{N}$, and*

$$(\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\leq i\leq N} := \mathsf{W}_t F, \text{ for all } t \in \mathbb{R}.$$

Then

$$\begin{pmatrix} \widetilde{\boldsymbol{E}}_{i,t} \\ \widetilde{\boldsymbol{B}}_{i,t} \end{pmatrix} = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \begin{pmatrix} \boldsymbol{E}_{i,0} \\ \boldsymbol{B}_{i,0} \end{pmatrix} - \int_0^t ds\, K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\boldsymbol{E}_{i,0} \\ \nabla\nabla\cdot\boldsymbol{B}_{i,0} \end{pmatrix}.$$

fulfill $\boldsymbol{E}_{i,t} = \widetilde{\boldsymbol{E}}_{i,t}$ and $\boldsymbol{B}_{i,t} = \widetilde{\boldsymbol{B}}_{i,t}$ for all $t \in \mathbb{R}$ and $1 \leq i \leq N$ in the L^2_w sense. Furthermore, for all $t \in \mathbb{R}$ it holds also that $(\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\leq i\leq N} \in D_w(\mathsf{A}^n) \cap \mathcal{F}$.

Proof. By the group properties $\mathsf{W}_t F \in D_w(\mathsf{A}^n)$ and by Definition 4.33p.73 and 3.3p.19

$$\partial_t \mathsf{W}_t F = \mathsf{A}\mathsf{W}_t F = (0, 0, \nabla\wedge\boldsymbol{B}_{i,t}, -\nabla\wedge\boldsymbol{E}_{i,t})_{1\leq i\leq N}.$$

Since $(\boldsymbol{E}_{i,0}, \boldsymbol{B}_{i,0}) \in \mathcal{F}$, a straight-forward computation together with the properties of K_t from Lemma 4.11p.61 yields

$$\begin{aligned} \boxed{20} &= \left(\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix}\right) \begin{pmatrix} \boldsymbol{E}_{i,t} \\ \boldsymbol{B}_{i,t} \end{pmatrix} \\ &= -\partial_t \int_{t_0}^t ds\, K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\boldsymbol{E}_{i,0} \\ \nabla\nabla\cdot\boldsymbol{B}_{i,0} \end{pmatrix} + \begin{pmatrix} \partial_t^2 + \nabla\wedge(\nabla\wedge\cdot) & 0 \\ 0 & \partial_t^2 + \nabla\wedge(\nabla\wedge\cdot) \end{pmatrix} K_t * \begin{pmatrix} \boldsymbol{E}_{i,0} \\ \boldsymbol{B}_{i,0} \end{pmatrix} \end{aligned}$$

Applying $\nabla\wedge(\nabla\wedge\cdot) = \nabla(\nabla\cdot) - \triangle$ and Lemma 4.11p.61 again gives

$$(\partial_t^2 - \triangle) K_t * \begin{pmatrix} \boldsymbol{E}_{i,0} \\ \boldsymbol{B}_{i,0} \end{pmatrix} = 0$$

and

$$\begin{aligned} \partial_t \int_0^t ds\, K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\boldsymbol{E}_{i,0} \\ \nabla\nabla\cdot\boldsymbol{B}_{i,0} \end{pmatrix} &= K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\boldsymbol{E}_{i,0} \\ \nabla\nabla\cdot\boldsymbol{B}_{i,0} \end{pmatrix}\Bigg|_{s\to t} - \left[K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\boldsymbol{E}_{i,0} \\ \nabla\nabla\cdot\boldsymbol{B}_{i,0} \end{pmatrix}\right]_{s\to 0}^{s\to t} \\ &= K_t * \begin{pmatrix} \nabla\nabla\cdot\boldsymbol{E}_{i,0} \\ \nabla\nabla\cdot\boldsymbol{B}_{i,0} \end{pmatrix}. \end{aligned}$$

Hence, we get $\boxed{20} = 0$ and, therefore, for $\widetilde{F}_t := (\widetilde{\boldsymbol{E}}_{i,t}, \widetilde{\boldsymbol{B}}_{i,t})_{1\leq i\leq N}$ it is true that $\partial_t \widetilde{F}_t = \mathsf{A}\widetilde{F}_t$ in the strong sense. By the group properties W_t and A commute on $D_w(\mathsf{A})$ which implies

$$\partial_t \left(\mathsf{W}_{-t}\widetilde{F}_t\right) = -\mathsf{A}\mathsf{W}_{-t}\widetilde{F}_t + \mathsf{W}_{-t}\mathsf{A}\widetilde{F}_t = 0.$$

Therefore, $\widetilde{F}_t = \mathsf{W}_t\widetilde{F}_0 = \mathsf{W}_t F_0 = \chi_t$ as by definition $F_0 = F = \widetilde{F}_0$. This means in particular that $\boldsymbol{E}_{i,t} = \widetilde{\boldsymbol{E}}_{i,t}$ and $\boldsymbol{B}_{i,t} = \widetilde{\boldsymbol{B}}_{i,t}$ for all $t \in \mathbb{R}$ and $1 \leq i \leq N$ in the L^2_w sense. Furthermore, as $F \in D_w(\mathsf{A}^n) \cap \mathcal{F}$, Lemma 4.11p.61 states that $\widetilde{F}_t \in \mathcal{F}$, and by the group properties of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ we also have $F_t \in D_w(\mathsf{A}^n)$ for all $t \in \mathbb{R}$. Hence, $F_t = \widetilde{F}_t \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ for all $t \in \mathbb{R}$ which concludes the proof. □

A ready application of this corollary is the following lemma which allows to express the smooth inhomogeneous Maxwell solutions of subsection 4.3.1p.60 in terms of $(\mathsf{W}_t)_{t\in\mathbb{R}}$.

Lemma 4.35. *Let $t, t_0 \in \mathbb{R}$ be given times, $F = (F_i)_{1\le i\le N} \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ for some $n \in \mathbb{N}$ be given initial fields and $(\boldsymbol{q}_i, \boldsymbol{p}_i) \in \mathcal{T}^1_\vee$ time-like charge trajectories for $1 \le i \le N$. If in addition the initial fields $F_i = (\boldsymbol{E}_i, \boldsymbol{B}_i)$ fulfill the Maxwell constraints*

The Maxwell solutions in terms of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ and J

$$\nabla \cdot \boldsymbol{E}_i = 4\pi\varrho_i(\cdot - \boldsymbol{q}_{i,t_0}) \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B}_i = 0$$

for $1 \le i \le N$, then for all $t \in \mathbb{R}$

$$F_t := \mathsf{W}_{t-t_0}\chi + \int_{t_0}^t ds\, \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}^n) = (M_{\varrho_i,m_i}[F_i, (\boldsymbol{q}_i, \boldsymbol{p}_i)](t, t_0))_{1\le i\le N}.$$

in the L^2_w sense where $\varphi_s := (\mathsf{Q}+\mathsf{P})(\boldsymbol{q}_{i,s}, \boldsymbol{p}_{i,s})_{1\le i\le N}$ for $s \in \mathbb{R}$. Furthermore, $F_t \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ for all $t \in \mathbb{R}$.

Proof. This can be computed by applying Corollary 4.34p.74 twice and using one partial integration. □

4.4.2 Newtonian Cauchy Data: Wheeler-Feynman Interaction for Finite Times

The strategy will be to use Banach's and Schauder's fixed point theorem to prove the existence of a fixed point of S_T. The following normed spaces will prove to be suitable for this problem:

Definition 4.36. *For $n \in \mathbb{N}$ let $\mathcal{F}^n_w$ be the linear space of functions $F \in D_w(\mathsf{A}^n)$ with*

Hilbert Spaces for the fixed point theorem

$$\|F\|_{\mathcal{F}^n_w(B)} := \left(\sum_{k=0}^n \|\mathsf{A}^k F\|_{\mathcal{F}_w}\right)^{\frac{1}{2}}.$$

with $B = \mathbb{R}^3$ in which case we simply write $\|\cdot\|_{\mathcal{F}^n_w}$ instead of $\|\cdot\|_{\mathcal{F}^n_w(\mathbb{R})}$. For other $B \subset \mathbb{R}^3$ we shall use this notation to split up integration domains. We shall use this notation also for $\mathcal{F}_w = \mathcal{F}^0_w$.

Lemma 4.37. *For $n \in \mathbb{N}$, $\mathcal{F}^n_w$ is a Hilbert space.*

Proof. This is an immediate consequence of Theorem 3.14p.26 which relies on the fact that A is closed on $D_w(\mathsf{A})$. □

Next we specify the class of boundary fields $(X^\pm_{i,\pm T})_{1\le i\le N}$ which we want to allow.

Definition 4.38. *For weight $w \in \mathcal{W}$ and $n \in \mathbb{N}$ let $\mathcal{A}^n_w$ be the set of maps*

The class of boundary fields $\mathcal{A}^n_w$, $\widetilde{\mathcal{A}}^n_w$ and $\mathcal{A}^{\mathrm{Lip}}_w$

$$X : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty) \cap \mathcal{F}, \qquad (T, \varphi) \mapsto X_T[\varphi]$$

which have the following properties for all $p \in \mathcal{P}$ and $T \in \mathbb{R}$:

(i) There is a function $C_{23} \in$ `Bounds` such that for all $\varphi \in D_w(A)$ with $(\mathsf{Q}+\mathsf{P})\varphi = p$ it is true that $\|X_T[\varphi]\|_{\mathcal{F}^n_w} \le C_{23}{}^{(n)}(|T|, \|p\|)$.

(ii) The map $F \mapsto X_T[p, F]$ as $\mathcal{F}^1_w \to \mathcal{F}^1_w$ is continuous.

(iii) For $(\boldsymbol{E}_{i,T}, \boldsymbol{B}_{i,T})_{1\le i\le N} := X_T[\varphi]$ and $(\boldsymbol{q}_{i,T}, \boldsymbol{p}_{i,T})_{1\le i\le N} := (\mathsf{Q}+\mathsf{P})M_L[\varphi](T, 0)$ one has $\nabla\cdot\boldsymbol{E}_{i,T} = 4\pi\varrho_i(\cdot - \boldsymbol{q}_{i,T})$ and $\nabla\cdot\boldsymbol{B}_{i,T} = 0$.

Let the subset $\widetilde{\mathcal{A}}^n_w \subset \mathcal{A}^n_w$ comprise such maps X that fulfill:

(iv) For balls $B_\tau := B_\tau(0) \subset \mathbb{R}^3$ *with radius* $\tau > 0$ *around the origin and any bounded set* $M \subset D_w(\mathsf{A})$ *it holds that* $\lim_{\tau\to\infty} \sup_{F\in M} \|X_T[p,F]\|_{\mathcal{F}^n_w(B^c_\tau)} = 0$.

Furthermore, let the subset $\mathcal{A}^{\mathrm{Lip}}_w \subset \mathcal{A}^1_w$ *comprise such maps X that fulfill:*

(v) There is a function $C_{24} \in \mathtt{Bounds}$ *such that for all* $\varphi, \widetilde{\varphi} \in D_w(A)$ *with* $(\mathtt{Q}+\mathtt{P})\varphi = p = (\mathtt{Q}+\mathtt{P})\widetilde{\varphi}$ *it is true that* $\|X_T[\varphi] - X_T[\widetilde{\varphi}]\|_{\mathcal{F}^1_w} \leq |T| C_{24}(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) \, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$.

REMARK 4.39. *The boundary fields needed are now encoded via* $(X^\pm_{i,\pm T})_{1\le i\le N} := X^\pm_{\pm T}[\varphi]$ *for two elements* $X^\pm \in \mathcal{A}^n_w$ *and some* $\varphi \in D_w(A)$. *The dependence of* $X^\pm_{\pm T}$ *on a* $\varphi \in D_w(A)$ *instead of the charge trajectories* $(\boldsymbol{q}_i, \boldsymbol{p}_i)$, $1 \le i \le N$, *in* $\mathcal{T}^1_\nabla$ *makes sense as* φ *carries the whole information about the charge trajectories by* $t \mapsto (\mathtt{Q}+\mathtt{P})M_L[\varphi](t,0)$ *which are the charge trajectories of the ML-SI solutions. As we shall discuss after showing that these classes are not empty, one can imagine their elements to be the Liénard-Wiechert fields of any charge trajectories in* $\mathcal{T}^1_\nabla$ *which continue the ML-SI charge trajectories on either the time interval* $(-\infty, -T]$ *or* $[T, \infty)$ *for the given* $T \in \mathbb{R}$. *Finally, the reason why we define three classes* $\mathcal{A}^n_w$, $\widetilde{\mathcal{A}}^n_w$ *and* $\mathcal{A}^{\mathrm{Lip}}$ *is to distinguish clearly the properties needed, first, to define what we mean by a bWF solution, second, to show existence of bWF solutions, and third, to show uniqueness of the bWF solution for small enough* T. *Note also that* $\mathcal{A}^{n+1}_w \subset \mathcal{A}^n_w$ *as well as* $\widetilde{\mathcal{A}}^{n+1}_w \subset \widetilde{\mathcal{A}}^n_w$ *for* $n \in \mathbb{N}$.

Having this we can formalize what we mean by a solution to the bWF equations for Newtonian Cauchy data and boundary fields.

bWF solutions for Newtonian Cauchy Data and boundary fields

Definition 4.40. *Let* $T > 0$, *Newtonian Cauchy data* $p \in \mathcal{P}$ *and two boundary fields* $X^\pm \in \mathcal{A}^1_w$ *be given. We define* $\mathcal{T}^{p,X^\pm}_T$ *to be the set of time-like charge trajectories in* $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N} \in \mathcal{T}_\vee$ *which solve the bWF equations in the form (4.1p.43) and*

$$(\boldsymbol{E}^{\mathrm{WF}}_{i,t}, \boldsymbol{B}^{\mathrm{WF}}_{i,t}) = \left(\frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[X^\pm_{\pm T}[p,F], (\boldsymbol{q}_i,\boldsymbol{p}_i)](0,\pm T)\right) \quad \text{for} \quad F = (\boldsymbol{E}^{\mathrm{WF}}_{i,t}, \boldsymbol{B}^{\mathrm{WF}}_{i,t})_{1\le i\le N}\Big|_{t=0} \tag{4.57}$$

and initial conditions (4.9p.46). We shall call every element of $\mathcal{T}^{p,X^\pm}_T$ *a bWF solution for initial value p and boundary fields* $X^\pm$ *at time* T.

REMARK 4.41. *Equations (4.11p.47) which where used in the introduction were replaced by (4.57) because it turns out to be more convenient to have the boundary fields depending only on ML-SI initial data* (p, F) *which encodes the trajectory. By Definition 4.38p.75(iv) the boundary fields fulfill the Maxwell constraints at time* $\pm T$. *This is important as our formulas for the Maxwell solutions of subsection 4.3.1p.60 are only valid if the Maxwell constraints are fulfilled. Though this requirement could be loosened by refining the formulas for the Maxwell fields it is natural to stick with it because the fields of true Wheeler-Feynman solution fulfills the Maxwell equations including the constraints, and the final goal is to find solutions for* $T \to \infty$.

Now we can define a convenient fixed point map whose fixed points are the bWF solutions.

The fixed point map S_T

Definition 4.42. *For finite time* $T > 0$, *Newtonian Cauchy data* $p \in \mathcal{P}$ *and boundary fields* $X^\pm \in \mathcal{A}^1_w$, *we define*

$$S^{p,X^\pm}_T : D_w(\mathsf{A}) \to D_w(\mathsf{A}^\infty), \qquad F \mapsto S^{p,X^\pm}_T[F]$$

for

$$S^{p,X^\pm}_T[F] := \frac{1}{2}\sum_{\pm}\left[\mathsf{W}_{\mp T} X^\pm_{\pm T}[p,F] + \int_{\pm T}^{t} ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s[p,F])\right]$$

where $\varphi_s[p,F] := M_L[p,F](s,0)$ *for* $s \in \mathbb{R}$ *is the ML-SI solution, cf. Definition 3.28p.41, for initial value* $(p,F) \in D_w(A)$.

We got to make sure that the fixed point map is well-defined and that its possible fixed points have the desired properties, i.e. their corresponding charge trajectories are in $\mathcal{T}_T^{p,X^\pm}$.

Theorem 4.43. *For finite time $T > 0$, Newtonian Cauchy data $p \in \mathcal{P}$ and boundary fields $X^\pm \in \mathcal{A}_w^1$ the following is true:*

The Map S_T and its fixed points

*(i) The map $S_T^{p,X^\pm}$ introduced in Definition 4.42*p.76 *is well-defined.*

(ii) For $F \in D_w(\mathsf{A})$, setting $(X_{i,\pm T}^\pm)_{1\le i\le N} := X_{\pm T}^\pm[p,F]$ and denoting the ML-SI charge trajectories

$$t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{q}_{i,t})_{1\le i\le N} := (\mathsf{Q}+\mathsf{P})M_L[p,F](t,0)$$

by $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N}$ it holds that

$$S_T^{p,X^\pm}[F] = \frac{1}{2}\sum_\pm \left(M_{\varrho_i,m_i}[X_{i,\pm T}^\pm, (\boldsymbol{q}_i,\boldsymbol{p}_i)](0,\pm T)\right)_{1\le i\le N} \in D_w(\mathsf{A}^\infty) \cap \mathcal{F}.$$

(iii) For any $F = S_T^{p,X^\pm}[F]$ it is true that and that the corresponding charge trajectories $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N}$ as defined in (ii) are in $\mathcal{T}_T^{p,X^\pm}$.

Proof. (i) Let $F \in D_w(\mathsf{A})$, then $(p,F) \in D_w(A)$ and, hence, by the ML±SI existence and uniqueness Theorem 3.5p.20 $t \mapsto \varphi_t := M_L[\varphi](t,0)$ is a once continuously differentiable map $\mathbb{R} \to D_w(A) \subset \mathcal{H}_w$. By properties of J stated in lemma (3.26p.32) we know that $A^k J : \mathcal{H}_w \to D_w(A^\infty) \subset \mathcal{H}_w$ is locally Lipschitz continuous for any $k \in \mathbb{N}$. By projecting onto field space $\mathcal{F}_w$, cf. Definition4.33p.73, we yield that also $\mathsf{A}^k\mathsf{J} : \mathcal{H}_w \to D_w(\mathsf{A}^\infty) \subset \mathcal{F}_w$ is locally Lipschitz continuous. Hence, by the group properties of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ we know that $s \mapsto \mathsf{W}_{-s}\mathsf{A}^k\mathsf{J}(\varphi_s)$ for any $k \in \mathbb{N}$ is continuous. Therefore, we may apply Corollary 5.6p.100 which states that

$$\mathsf{A}^k \int_{\pm T}^0 ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s) = \int_{\pm T}^0 ds\, \mathsf{W}_{-s}\mathsf{A}^k\mathsf{J}(\varphi_s)$$

while the integral on the right-hand side exists because the integrand is continuous and the integral goes over a compact set. As this holds for any $k \in \mathbb{N}$, $\int_{\pm T}^0 ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}^\infty)$. Furthermore, by Definition 4.38p.75 the term $X_{\pm T}^\pm[p,F]$ is in $D_w(\mathsf{A}^\infty)$ and therefore $\mathsf{W}_{\mp T}X_{\pm T}^\pm[p,F] \in D_w(\mathsf{A}^\infty)$ by the group properties. Hence, the map $S_T^{\phi,X^\pm}$ is well-defined as a map $D_w(A) \to D_w(\mathsf{A}^\infty)$.

(ii) For $F \in D_w(\mathsf{A})$ let $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N}$ denote the charge trajectories $t \mapsto (\boldsymbol{q}_{i,t},\boldsymbol{p}_{i,t})_{1\le i\le N} = (\mathsf{Q}+\mathsf{P})\varphi_t$ of $t \mapsto \varphi_t := M_L[p,F](t,0)$. Since $(p,F) \in D_w(A)$, we know again by Theorem 3.5p.20 that these charge trajectories are once continuously differentiable as $\mathbb{R} \to D_w(A) \subset \mathcal{H}_w$. As the absolute value of the velocity is given by $\|\boldsymbol{v}(\boldsymbol{p}_{i,t})\| = \frac{\|\boldsymbol{p}_{i,t}\|}{\sqrt{m_i^2+\boldsymbol{p}_{i,t}^2}} < 1$, we conclude that $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N}$ are once continuously differentiable and time-like and therefore in $\mathcal{T}_\vee$, cf. Definition 4.5p.60. Furthermore, the boundary fields $X_{\pm T}^\pm[p,F]$ are in $D_w(\mathsf{A}^\infty) \cap \mathcal{F}$ and obey the Maxwell constraints by the definition of $\mathcal{A}_w^n$. So we can apply Lemma 4.35p.75 which states for $(X_{i,\pm T}^\pm)_{1\le i\le N} := X_{\pm T}^\pm[p,F]$ that

$$(M_{\varrho_i,m_i}[X_i,(\boldsymbol{q}_i,\boldsymbol{p}_i](t,\pm T))_{1\le i\le N} = \mathsf{W}_{t\mp T}X_\pm^\pm[p,F] + \int_{\pm T}^t ds\, \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}) \cap \mathcal{F}. \quad (4.58)$$

For $t = 0$ this proves claim (ii).

(iii) Finally, assume there is an $F \in \mathcal{F}_w$ such that $F = S_T^{p,X^\pm}[F]$. By (ii) this implies $F \in D_w(\mathsf{A}^\infty) \cap \mathcal{F}$. Let $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N}$ and $t \mapsto \varphi_t$ be as defined in the proof of (ii) which now is infinitely often differentiable as $\mathbb{R} \to \mathcal{H}_w$ since $(p, F) \in D_w(A^\infty)$. We shall show that the following integral equality holds

$$\varphi_t = (p, 0) + \int_0^t ds\ (\mathsf{Q} + \mathsf{P})J(\varphi_s) + \frac{1}{2}\sum_\pm \left[W_{t\mp T}(0, X^\pm_{\pm T}[p, F]) + \int_{\pm T}^t ds\ W_{t-s}\mathsf{F}J(\varphi_s) \right] \quad (4.59)$$

for all $t \in \mathbb{R}$; keep in mind that $t \mapsto \varphi_t$ depends also on (p, F). Then differentiation with respect to time t of the phase space components of $(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t}, \boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t})_{1\le i\le N} := \varphi_t$ yields $\partial_t(\mathsf{Q} + \mathsf{P})\varphi_t = (\mathsf{Q} + \mathsf{P})J(\varphi_t)$ which by definition of J yields

$$\begin{aligned} \partial_t \boldsymbol{q}_{i,t} &= \boldsymbol{v}(\boldsymbol{p}_{i,t}) := \frac{\sigma_i \boldsymbol{p}_{i,t}}{\sqrt{m_i^2 + \boldsymbol{p}_{i,t}^2}} \\ \partial_t \boldsymbol{p}_{i,t} &= \sum_{j\neq i} \int d^3x\ \varrho_i(\boldsymbol{x} - \boldsymbol{q}_{i,t}) \left(\boldsymbol{E}_{j,t}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{q}_{i,t}) \wedge \boldsymbol{B}_{j,t}(\boldsymbol{x}) \right). \end{aligned} \quad (4.60)$$

Furthermore, the field components fulfill

$$\begin{aligned} \mathsf{F}\varphi_t &= \mathsf{F}\frac{1}{2}\sum_\pm \left[W_{t\mp T}(0, X^\pm_{\pm T}[\varphi]) + \int_{\pm T}^t ds\ W_{t-s}\mathsf{F}J(\varphi_s) \right] \\ &= \frac{1}{2}\sum_\pm \left[\mathsf{W}_{t\mp T} X^\pm_{\pm T}[p, F] + \int_{\pm T}^t ds\ \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \right] \end{aligned}$$

where we only used the definition of the projectors, cf. Definition 4.33p.73. Hence, by (4.58p.77) we know

$$(\boldsymbol{E}_{i,t}, \boldsymbol{B}_{i,t}) = \frac{1}{2}\sum_\pm M_{\varrho_i, m_i}[F_i, (\boldsymbol{q}_i, \boldsymbol{p}_i](t, \pm T). \quad (4.61)$$

Finally, we have

$$(\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\le i\le N}\Big|_{t=0} = p = (\boldsymbol{q}_i^0, \boldsymbol{p}_i^0)_{1\le i\le N}. \quad (4.62)$$

Equations (4.60), (4.61) and (4.62) are exactly the bWF equations (4.1p.43) and (4.57p.76) for Newtonian Cauchy data (4.9p.46). Hence, since in (ii) we proved that $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N}$ are in $\mathcal{T}_\vee$, we conclude that they are also in $\mathcal{T}_T^{p,X^\pm}$, cf. Definition 4.40p.76.

Now it is only left to prove that the integral equation (4.59) holds. By Definition 3.28p.41, φ_t fulfills

$$\varphi_t = W_t(p, F) + \int_0^t ds\ W_{t-s}J(\varphi_s)$$

for all $t \in \mathbb{R}$. Inserting the fixed point equation $F = S_T^{p,X^\pm}[F]$, i.e.

$$F = \mathsf{W}_{t\mp T} X^\pm_\pm[p, F] + \int_{\pm T}^t ds\ \mathsf{W}_{t-s}\mathsf{J}(\varphi_s),$$

we find

$$\varphi_t = (p, 0) + \frac{1}{2}\sum_\pm W_{t\mp T}(0, X^\pm_{\pm T}[p, F]) + \frac{1}{2}\sum_\pm W_t \int_{\pm T}^0 ds\ W_{-s}(0, \mathsf{J}(\varphi_s)) + \int_0^t ds\ W_{t-s}J(\varphi_s).$$

Using the continuity of the integrands we may apply Lemma 5.5p.100 to commute W_t with the integral. This together with $J = (\mathsf{Q} + \mathsf{P})J + \mathsf{F}J$ and that $(\mathsf{Q} + \mathsf{P})W_t = \mathrm{id}_{\mathcal{P}}$ yields the desired result (4.59) for all $t \in \mathbb{R}$ which concludes the proof. □

In the next Lemma we discuss a simple element $C \in \mathcal{A}_w^n$ and thereby show that the classes of boundary fields $\widetilde{\mathcal{A}}_w^n$ and $\mathcal{A}_w^{\mathrm{Lip}}$ are not empty.

Definition 4.44. *For $n \in \mathbb{N}$ define $C : \mathbb{R} \times \mathcal{H}_w^n \to D_w(\mathsf{A}^n)$, $(T, \varphi) \mapsto C_T[\varphi]$ to be*

Coulomb boundary field

$$C_T[\varphi] := \left(\boldsymbol{E}_i^C(\cdot - \boldsymbol{q}_{i,T}), 0\right)_{1 \le i \le N}$$

where $(\boldsymbol{q}_{i,T})_{1\le i\le N} := \mathsf{Q}M_L[\varphi](T,0)$ and the Coulomb field

$$(\boldsymbol{E}_i^C, 0) := M_{\varrho_i, m_i}[t \mapsto (0,0)](0, -\infty) = \left(\int d^3z\, \varrho_i(\cdot - z)\frac{z}{\|z\|^3}, 0\right).$$

*Note that the last equality holds by Theorem 4.18**p.66**.*

Lemma 4.45. *For any $n \in \mathbb{N}$ and any $w \in \mathcal{W}$ the set $C \in \mathcal{A}_w^n \cap \mathcal{A}_w^{\mathrm{Lip}}$.*

The class of boundary fields is non-empty

Proof. We need to show the properties given in Definition 4.38p.75. Fix $T > 0$ and $p \in \mathcal{P}$. Recall the definition of C_T as introduced in Definition 4.44. Let $\varphi \in D_w(A)$ and $F = \mathsf{F}\varphi$ for $(\mathsf{Q} + \mathsf{P})\varphi = p$. Furthermore, we define $(\boldsymbol{q}_{i,T})_{1\le i\le N} := QM_L[\varphi](T, 0)$. Since $\boldsymbol{E}^C$ is a Liénard-Wiechert field of the charge trajectory $t \mapsto (\boldsymbol{q}_{i,T}, 0)$ in $\mathcal{T}_\nabla^1$, we can apply Corollary 4.22p.69 to yield the following estimate for any multi-index $\alpha \in \mathbb{N}^3$ and $\boldsymbol{x} \in \mathbb{R}^3$

$$\left\|D^\alpha \boldsymbol{E}^C(\boldsymbol{x})\right\|_{\mathbb{R}^3} \le \frac{C_{25}{}^{(\alpha)}}{1 + \|\boldsymbol{x}\|^2}. \tag{4.63}$$

which allows to define the finite constants $C_{26}{}^{(\alpha)} := \|D^\alpha \boldsymbol{E}_C\|_{L^2_w}$. Using the properties of the weight $w \in \mathcal{W}$ we find

$$\|C_T[\varphi]\|^2_{\mathcal{F}_w^n} \le \sum_{k=0}^n \|\mathsf{A}^k C_T[\varphi]\|_{\mathcal{F}_w} \le \sum_{k=0}^n \sum_{i=1}^N \left\|(\nabla\wedge)^k \boldsymbol{E}_i^C(\cdot - \boldsymbol{q}_{i,T})\right\|_{L^2_w} \le \sum_{k=0}^n \sum_{|\alpha|\le k} \sum_{i=1}^N \left\|D^\alpha \boldsymbol{E}_i^C\right\|_{L^2_w}$$
$$\le \sum_{k=0}^n \sum_{|\alpha|\le k} \sum_{i=1}^N \left(1 + C_w \left\|\boldsymbol{q}_{i,T}\right\|\right)^{\frac{P_w}{2}} \left\|D^\alpha \boldsymbol{E}_i^C\right\|_{L^2_w} \le \sum_{k=0}^n \sum_{|\alpha|\le k} \sum_{i=1}^N \left(1 + C_w \left\|\boldsymbol{q}_{i,T}\right\|\right)^{\frac{P_w}{2}} C_{26}{}^{(\alpha)} < \infty.$$

This implies $C_T \in D_w(A^\infty) \cap \mathcal{F}$ and that $C : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty) \cap \mathcal{F}$ is well-defined.

Note that the right-hand side depends only on $\left\|\boldsymbol{q}_{i,T}\right\|$ which is bounded by $\|p\| + |T|$ since the maximal velocity is below one. Hence, property (i) holds for

$$C_{23}{}^{(n)}(|T|, \|p\|) := \sum_{k=0}^n \sum_{|\alpha|\le k} \sum_{i=1}^N (1 + C_w (\|p\| + |T|))^{\frac{P_w}{2}} C_{26}{}^{(\alpha)}.$$

Instead of showing property (ii), we prove the stronger property (v). For this let $\widetilde{\varphi} \in D_w(A)$ such that $(\mathsf{Q} + \mathsf{P})\varphi = (\mathsf{Q} + \mathsf{P})\widetilde{\varphi}$, $(\widetilde{\boldsymbol{q}}_{i,T})_{1\le i\le N} := \mathsf{Q}M_L[\widetilde{\varphi}](T, 0)$. Starting with

$$\|C_T[\varphi] - C_T[\widetilde{\varphi}]\|_{\mathcal{F}_w^1} \le \sum_{i=1}^N \sum_{|\alpha|\le 1} \left\|D^\alpha \left(\boldsymbol{E}^C(\cdot - \boldsymbol{q}_{i,T}) - \boldsymbol{E}^C(\cdot - \widetilde{\boldsymbol{q}}_{i,T})\right)\right\|_{L^2_w}$$

we compute

$$\left\|D^\alpha \left(\boldsymbol{E}^C(\cdot - \boldsymbol{q}_{i,T}) - \boldsymbol{E}^C(\cdot - \widetilde{\boldsymbol{q}}_{i,T})\right)\right\|_{L^2_w} = \left\|\int_0^1 d\lambda\, (\widetilde{\boldsymbol{q}}_{i,T} - \boldsymbol{q}_{i,t}) \cdot \nabla D^\alpha \boldsymbol{E}_{i,T}^C(\cdot - \widetilde{\boldsymbol{q}}_{i,T} + \lambda(\widetilde{\boldsymbol{q}}_{i,T} - \boldsymbol{q}_{i,t}))\right\|_{L^2_w}$$
$$\le \int_0^1 d\lambda\, \left\|(\boldsymbol{q}_{i,t} - \widetilde{\boldsymbol{q}}_{i,T}) \cdot \nabla D^\alpha \boldsymbol{E}^C(\cdot - \widetilde{\boldsymbol{q}}_{i,T} + \lambda(\widetilde{\boldsymbol{q}}_{i,T} - \boldsymbol{q}_{i,t}))\right\|_{L^2_w}$$

where in the last step we have used Minkowski's inequality. Therefore, for all $|\alpha| \leq 1$ we get

$$\sum_{|\alpha|\leq 1} \left\| D^\alpha \left(\boldsymbol{E}^C(\cdot - \boldsymbol{q}_{i,T}) - \boldsymbol{E}^C(\cdot - \widetilde{\boldsymbol{q}}_{i,T}) \right) \right\|_{L^2_w}$$
$$\leq \|\boldsymbol{q}_{i,T} - \widetilde{\boldsymbol{q}}_{i,T}\|_{\mathbb{R}^3} \sup_{0\leq\lambda\leq 1} \sum_{|\beta|\leq 2} \left\| D^\beta \boldsymbol{E}^C(\cdot + \lambda(\boldsymbol{q}_{i,T} - \widetilde{\boldsymbol{q}}_{i,T})) \right\|_{L^2_w}.$$

The estimate ($4.63_{\text{p.79}}$) and the properties of $w \in \mathcal{W}$ yield

$$\left\| D^\beta \boldsymbol{E}^C(\cdot - \widetilde{\boldsymbol{q}}_{i,T} + \lambda(\widetilde{\boldsymbol{q}}_{i,T} - \boldsymbol{q}_{i,t})) \right\|_{L^2_w} \leq \left(1 + C_w \lambda \left\| \widetilde{\boldsymbol{q}}_{i,T} + \lambda(\widetilde{\boldsymbol{q}}_{i,T} - \boldsymbol{q}_{i,t}) \right\|_{\mathbb{R}^3}\right)^{\frac{P_w}{2}} \left\| D^\beta \boldsymbol{E}^C \right\|_{L^2_w}$$
$$\leq (1 + C_w(\|\boldsymbol{q}_i\|_{\mathbb{R}^3} + \|\widetilde{\boldsymbol{q}}_i\|_{\mathbb{R}^3} + 2|T|)^{\frac{P_w}{2}} C_{26}{}^{(\beta)}$$

Furthermore, since the maximal velocity is smaller than one, property (v) holds for

$$C_{24}(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) := N \sum_{|\beta|\leq 2} (1 + C_w(\|\mathsf{Q}\varphi\|_{\mathbb{R}^3} + \|\mathsf{Q}\widetilde{\varphi}\|_{\mathbb{R}^3} + 2|T|)^{\frac{P_w}{2}} C_{26}{}^{(\beta)}.$$

(iii) holds by Theorem $4.14_{\text{p.63}}$.

(iv) Let $B_\tau(0) \subset \mathbb{R}^3$ be a ball of radius $\tau > 0$ around the origin. For any $F \in D_w(\mathsf{A})$ we define $(\boldsymbol{q}_{i,T})_{1\leq i\leq N} := QM_L[\varphi](T, 0)$ and yield

$$\|C^T[p, F]\|_{\mathcal{F}^n_w(B^c_\tau(0)} \leq \sum_{i=1}^N \sum_{|\alpha|\leq n} \left\| D^\alpha \boldsymbol{E}^C(\cdot - \boldsymbol{q}_{i,T}) \right\|_{L^2_w(B^c_\tau(0))}$$
$$\leq \sum_{i=1}^N \sum_{|\alpha|\leq n} \left(1 + C_w \|\boldsymbol{q}_{i,T}\|\right)^{\frac{P_w}{2}} \left\| D^\alpha \boldsymbol{E}^C \right\|_{L^2_w(B^c_\tau(\boldsymbol{q}_{i,T}))}.$$

Note again that the maximal velocity is smaller than one so that $\|\boldsymbol{q}_{i,T} \leq \|\boldsymbol{q}^0_i\| + T$. Hence, for $\tau > \|\boldsymbol{q}^0_i\| + T$ define $r(\tau) := \tau - \|\boldsymbol{q}^0_i\| + T$ such that it holds

$$\sup_{F\in D_w(\mathsf{A})} \|C^T[p, F]\|_{\mathcal{F}^n_w(B^c_\tau(0)} \leq \sum_{i=1}^N \sum_{|\alpha|\leq n} \left(1 + C_w \|\boldsymbol{q}_{i,T}\|\right)^{\frac{P_w}{2}} \left\| D^\alpha \boldsymbol{E}^C \right\|_{L^2_w(B^c_{r(\tau)}(0))} \xrightarrow[\tau\to\infty]{} 0$$

To summarize we have shown that for all $n \in \mathbb{N}$ the map C as introduced in Definition $4.44_{\text{p.79}}$ is an element of $\widetilde{\mathcal{A}}^n_w \cap \mathcal{A}^{\text{Lip}}$ which is a subset of $\mathcal{A}^n_w$. □

REMARK 4.46. *In view of ($4.13_{\text{p.48}}$) the boundary fields are a guess of how the charge trajectories $(\boldsymbol{q}^0_i, \boldsymbol{p}_i)_{1\leq i\leq N}$ continue on the intervals $(-\infty, -T]$ and $[T, \infty)$. Instead of the Coulomb fields of a charge at rest we could have also taken the Liénard-Wiechert fields of a charge trajectory which starts at $\boldsymbol{q}_{i,T}$ and has constant momentum $\boldsymbol{p}_{i,T}$ for $(\boldsymbol{q}_{i,T}, \boldsymbol{p}_{i,T})_{1\leq i\leq N} := (\mathsf{Q} + \mathsf{P})M_L[\varphi](T, 0)$ with only minor modification (the result would be a Lorentz boosted Coulomb field). Such boundary fields would also be in $\mathcal{A}^{\text{Lip}}_w$ as for $(\boldsymbol{p}_{i,T})_{1\leq i\leq N} := \mathsf{P}M_L[\widetilde{\varphi}](T, 0)$ we have*

$$\|\boldsymbol{p}_{i,T} - \widetilde{\boldsymbol{p}}_{i,T}\| \leq \int_0^T ds\, \|\dot{\boldsymbol{p}}_{i,s} - \dot{\widetilde{\boldsymbol{p}}}_{i,s}\| \leq T \sup_{s\in[0,T]} \|\dot{\boldsymbol{p}}_{i,s} - \dot{\widetilde{\boldsymbol{p}}}_{i,s}\|$$

while the supremum exists because the charge trajectories are smooth thanks to $\varphi, \widetilde{\varphi} \in D_w(A)$. Only if one wanted to continue the charge trajectories $(\boldsymbol{q}^0_i, \boldsymbol{p}_i)_{1\leq i\leq N}$ in ($4.13_{\text{p.48}}$) more smoothly, for example also continuously in the acceleration, the resulting boundary fields would not lie in $\mathcal{A}^{\text{Lip}}_w$ anymore but rather in $\widetilde{\mathcal{A}}^1_w$ since in general different initial value for the ML-SI equations yield different accelerations at time zero.

Now we are able to state the main theorem of this section:

Theorem 4.47. *Let Newtonian Cauchy data $p \in \mathcal{P}$ be given. For the maps $S_T^{p,X^\pm}$ for finite $T > 0$ as defined in Definition 4.42**p.76* *the following is true:*

$S_T^{p,X^\pm}$ has a fixed point

(i) For any boundary fields $X^\pm \in \mathcal{A}_w^{\mathrm{Lip}}$ and T sufficiently small, $S_T^{p,X^\pm}$ has a unique fixed point.

(ii) For any boundary fields $X^\pm \in \mathcal{A}_w^3$ and finite $T > 0$, the map $S_T^{p,X^\pm}$ has a fixed point.

The strategy for the proof is to use Banach's and Schauder's fixed point theorem. Before we give a proof of Theorem 4.47 we collect the needed estimates and properties of $S_T^{p,X^\pm}$ in a series of lemmas.

Lemma 4.48. *For $n \in \mathbb{N}_0$ the following is true:*

Estimates on $\mathcal{F}_w^n$

(i) For all $t \in \mathbb{R}$ and $F \in D_w(\mathsf{A}^n)$ it holds that $\|\mathsf{W}_t F\|_{\mathcal{F}_w^n} \le e^{\gamma|t|}\|F\|_{\mathcal{F}_w^n}$.

(ii) For all $\varphi \in \mathcal{H}_w$ there is a constant $C_{27}^{(n)} \in \mathtt{Bounds}$ such that

$$\|\mathsf{J}(\varphi)\|_{\mathcal{F}_w^n} \le C_{27}^{(n)}(\|\mathsf{Q}\varphi\|_{\mathcal{H}_w}).$$

(iii) For all $\varphi, \widetilde{\varphi} \in \mathcal{H}_w$ there is a $C_{28}^{(n)} \in \mathtt{Bounds}$ such that

$$\|\mathsf{J}(\varphi) - \mathsf{J}(\widetilde{\varphi})\|_{\mathcal{F}_w^n} \le C_{28}^{(n)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}.$$

Proof. (i) By Definition 4.33p.73 and Lemma 3.10p.22, $(\mathsf{W}_t)_{t\in\mathbb{R}}$ is a γ contractive group generated by A on $D_w(\mathsf{A})$. Hence, A and W_t commute for any $t \in \mathbb{R}$ which implies for all $F \in D_w(\mathsf{A}^n)$ that

$$\|\mathsf{W}_t F\|_{\mathcal{F}_w^n}^2 = \sum_{k=0}^n \|\mathsf{A}^k \mathsf{W}_t F\|_{\mathcal{F}_w}^2 = \sum_{k=0}^n \|\mathsf{W}_t \mathsf{A}^k F\|_{\mathcal{F}_w}^2 \le e^{\gamma|t|}\sum_{k=0}^n \|\mathsf{A}^k F\|_{\mathcal{F}_w}^2 = e^{\gamma|t|}\|F\|_{\mathcal{F}_w^n}.$$

For (ii) let $(\boldsymbol{q}_i, \boldsymbol{p}_i, \boldsymbol{E}_i, \boldsymbol{B}_i)_{1\le i\le N} = \varphi \in \mathcal{H}_w$. Using then the definitions of J, cf. Definition 4.33p.73 and 3.4p.19, we find

$$\|\mathsf{J}(\varphi)\|_{\mathcal{F}_w^n} \le \sum_{k=0}^n \|(\nabla\wedge)^k \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i)\|_{L_w^2}.$$

By applying the triangular inequality one finds a constant C_{29}, e.g. $C_{29} = 4\sqrt{6}$, for which

$$\|(\nabla\wedge)^k \boldsymbol{v}(\boldsymbol{p}_i)\varrho_i(\cdot - \boldsymbol{q}_i)\|_{L_w^2} \le (C_{29})^n \sum_{|\alpha|\le n} \|\boldsymbol{v}(\boldsymbol{p}_i) D^\alpha \varrho_i(\cdot - \boldsymbol{q}_i)\|_{L_w^2} \le (C_{29})^n \sum_{|\alpha|\le n} \|D^\alpha \varrho_i(\cdot - \boldsymbol{q}_i)\|_{L_w^2}$$

whereas in the last step we used the fact that the maximal velocity is smaller than one. Using the properties of the weight function $w \in \mathcal{W}$, cf. Definition 3.1p.18, we conclude

$$\|D^\alpha \varrho_i(\cdot - \boldsymbol{q}_i)\|_{L_w^2} \le (1 + C_w\|\boldsymbol{q}_i\|)^{\frac{P_w}{2}} \|D^\alpha \varrho_i\|_{L_w^2}.$$

Collecting these estimates we yield that claim (ii) holds for

$$C_{27}^{(n)}(\|\mathsf{Q}\varphi\|_{\mathcal{H}_w}) := (C_{29})^n \sum_{i=1}^N (1 + C_w\|\boldsymbol{q}_i\|)^{\frac{P_w}{2}} \sum_{|\alpha|\le n} \|D^\alpha \varrho_i\|.$$

Claim (iii) is shown by repetitively applying estimate (3.31p.33) of Lemma 3.26p.32 on the right-hand side of

$$\|\mathsf{J}(\varphi) - \mathsf{J}(\widetilde{\varphi})\|_{\mathcal{F}_w^n} \le \sum_{k=0}^n \|A^k[J(\varphi) - J(\widetilde{\varphi})]\|_{\mathcal{H}_w}$$

which yields a constant $C_{28}^{(n)} := \sum_{k=0}^n C_4^{(k)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})$ where $C_4 \in \mathtt{Bounds}$ was defined in Lemma 3.26p.32. This concludes the proof. □

Properties of $S_T^{p,X^\pm}$

Lemma 4.49. *Let $T > 0$, $p \in \mathcal{P}$ and $X^\pm \in \mathcal{A}_w^n$ for $n \in \mathbb{N}$. Then it holds:*

(i) There is a function $C_{30} \in \texttt{Bounds}$ such that for all $F \in \mathcal{F}_w^1$ we have

$$\|S_T^{p,X^\pm}[p,F]\|_{\mathcal{F}_w^n} \leq C_{30}{}^{(n)}(T,\|p\|).$$

(ii) $F \mapsto S_T^{p,X^\pm}[F]$ as $\mathcal{F}_w^1 \to \mathcal{F}_w^1$ is continuous.

If $X^\pm \in \mathcal{A}_w^{\mathrm{Lip}}$, it is also true that:

(iii) There is a function $C_{31} \in \texttt{Bounds}$ such that for all $F, \widetilde{F} \in \mathcal{F}_w^1$ we have

$$\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^1} \leq T C_{31}(T,\|p\|,\|F\|_{\mathcal{F}_w},\|\widetilde{F}\|_{\mathcal{F}_w})\|F-\widetilde{F}\|_{\mathcal{F}_w}.$$

Proof. Fix $T > 0$, $p \in \mathcal{P}$, $X^\pm \in \mathcal{A}_w^n$ for $n \in \mathbb{N}$. Before we prove the claims we preliminarily recall the relevant estimates of the ML-SI dynamics. Throughout the proof and for for any $F, \widetilde{F} \in \mathcal{F}_w^n$ we define $D_w(A^n) \ni \varphi = (p,F)$ and $D_w(A^n) \ni \widetilde{\varphi} = (p,\widetilde{F})$ and furthermore the ML-SI solutions $\varphi_t := M_L[\varphi](t,0)$ and $\widetilde{\varphi}_t := M_L[\widetilde{\varphi}](t,0)$ for any $t \in \mathbb{R}$. Recall the estimate (3.11p.20) from the ML±SI existence and uniqueness Theorem 3.5p.20 which gives the following T dependent upper bounds on these ML-SI solutions:

$$\sup_{t\in[-T,T]} \|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_2(T,\|\varphi\|_{\mathcal{H}_w},\|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi-\widetilde{\varphi}\|_{\mathcal{H}_w}, \tag{4.64}$$

$$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \leq C_2(T,\|\varphi\|_{\mathcal{H}_w},0)\|\varphi\|_{\mathcal{H}_w} \quad \text{and} \quad \sup_{t\in[-T,T]} \|\widetilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_2(T,\|\widetilde{\varphi}\|_{\mathcal{H}_w},0)\|\widetilde{\varphi}\|_{\mathcal{H}_w}. \tag{4.65}$$

To prove claim (i) we estimate

$$\|S_T^{p,X^\pm}[F]\|_{\mathcal{F}_w^n} \leq \left\| \frac{1}{2}\sum_\pm \mathsf{W}_{\mp T} X^\pm_{\pm T}[p,F] \right\|_{\mathcal{F}_w^n} + \left\| \frac{1}{2}\sum_\pm \int_{\pm T}^0 ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s) \right\|_{\mathcal{F}_w^n} =: \boxed{21} + \boxed{22},$$

cf. Definition 4.42p.76 where $S_T^{p,X^\pm}$ was defined. By the estimate given in Lemma 4.48p.81(i) and the property given in Definition 4.38p.75(i) of the boundary fields we find

$$\boxed{21} \leq \frac{1}{2}\sum_\pm \|\mathsf{W}_{\mp T} X^\pm_{\pm T}[p,F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T}\|X^\pm_{\pm T}[p,F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} C_{23}{}^{(n)}(T,\|\phi\|_{\mathcal{H}_w}).$$

Furthermore, using in addition the estimates given in Lemma 4.48p.81(i-ii) we get a bound for the next term by

$$\boxed{22} \leq T e^{\gamma T} \sup_{s\in[-T,T]} \|\mathsf{J}(\varphi_s)\|_{\mathcal{F}_w^n} \leq T e^{\gamma T} \sup_{s\in[-T,T]} C_{27}(\|\mathsf{Q}\varphi_s\|_{\mathcal{H}_w}) \leq T e^{\gamma T} C_{27}(\|p\|+T)$$

whereas the last step is implied by the fact that the maximal velocity is below one. These estimates prove claim (i) for

$$C_{30}{}^{(n)}(T,\|\phi\|_{\mathcal{H}_w^n}) := e^{\gamma T}\left(C_{23}{}^{(n)}(T,\|p\|) + T C_{27}(\|p\|+T)\right).$$

Next we prove claim (ii). Therefore, we regard

$$\begin{aligned}\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^n} &\leq e^{\gamma T}\|X^\pm_{\pm T}[\varphi] - X^\pm_{\pm T}[\widetilde{\varphi}]\|_{\mathcal{F}_w^n} + T e^{\gamma T} \sup_{s\in[-T,T]} \|\mathsf{J}(\varphi_s) - \mathsf{J}(\widetilde{\varphi}_s)\|_{\mathcal{F}_w^n} \\ &=: \boxed{23} + \boxed{24}\end{aligned}$$

where we have already applied Lemma 4.48p.81(i). Next we use Lemma 4.48p.81(iii) on $\boxed{24}$ and yield

$$\boxed{24} \leq Te^{\gamma T} \sup_{s\in[-T,T]} C_{28}{}^{(n)}(\|\varphi_s\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_s\|_{\mathcal{H}_w})\|\varphi_s - \widetilde{\varphi}_s\|_{\mathcal{H}_w}$$

Finally, by the ML-SI estimates (4.64) and (4.65) we yield

$$\boxed{24} \leq TC_{32}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n})\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} \tag{4.66}$$

for

$$C_{32}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n}) := e^{\gamma T} C_{28}{}^{(n)}\Big(C_2(T, \|\varphi\|_{\mathcal{H}_w}, 0)\|\varphi\|_{\mathcal{H}_w}, C_2(T, 0, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi\|_{\mathcal{H}_w}\Big)\times$$
$$\times\, C_2(T, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}).$$

For $\widetilde{F} \to F$ in $\mathcal{F}_w^1$ these estimates imply $S_T^{p,X^\pm}[\widetilde{F}] \to S_T^{p,X^\pm}[F]$ in $\mathcal{F}_w^1$ since here $\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} = \|F - \widetilde{F}\|_{\mathcal{F}_w}$ which proves claim (ii).

(iii) Let now $X^\pm \in \mathcal{A}_w^{\text{Lip}}$. Term $\boxed{23}$ then behaves by Definition 4.38p.75 as

$$\boxed{23} \leq TC_{24}{}^{(n)}(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\; \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$$

Together with the estimate (4.66) this proves claim (ii) for

$$C_{31}{}^{(n)}(T, \|p\|, \|F\|_{\mathcal{F}_w}, \|\widetilde{F}\|_{\mathcal{F}_w}) := C_{24}{}^{(n)}(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) + C_{32}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n})$$

since in our case $\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} = \|F - \widetilde{F}\|_{\mathcal{F}_w}$. □

Before we proof the main theorem of this section we need a last lemma which gives a criterion for precompactness of sequences in L_w^2.

Criterion for precompactness

Lemma 4.50. *Let $(\boldsymbol{F}_n)_{n\in\mathbb{N}}$ be a sequence in $L_w^2(\mathbb{R}^3, \mathbb{R}^3)$ such that*

(i) The sequence $(\boldsymbol{F}_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{H}_w^\triangle$.

(ii) $\lim_{\tau\to\infty} \sup_{n\in\mathbb{N}} \|\boldsymbol{F}_n\|_{L_w^2(B_\tau^c(0))} = 0.$

Then the sequence $(F_n)_{n\in\mathbb{N}}$ is precompact, i.e. it contains a convergent subsequence.

Proof. (see Proof in Appendix 5.4p.101) The idea for the proof is based on [Lie01, Chapter 8, Proof of Theorem 8.6, p.208]. □

REMARK 4.51. *Of course one only needs to control solely the gradient, however, the Laplace turns out to be more convenient for the later application of the lemma.*

Now we can prove the first main theorem of this subsection.

Proof of Theorem 4.47p.81

Proof of Theorem 4.47p.81. Fix $p \in \mathcal{P}$.

(i) Let $X^\pm \in \mathcal{A}_w^{\text{Lip}} \subset \mathcal{A}_w^1$, then Lemma 4.49p.82(i) states

$$\|S_T^{p,X^\pm}[p, F]\|_{\mathcal{F}_w^1} \leq C_{30}{}^{(1)}(T, \|p\|) =: r.$$

Hence, the map $S_T^{p,X^\pm}$ restricted to the ball $B_r(0) \subset \mathcal{F}_w^1$ with radius r around the origin is a nonlinear self-mapping. Lemma 4.49p.82(iii) states for all $T > 0$ and $F, \widetilde{F} \in B_r(0) \subset D_w(\mathsf{A})$ that

$$\begin{aligned}\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^1} &\le TC_{31}(T, \|p\|, \|F\|_{\mathcal{F}_w}, \|\widetilde{F}\|_{\mathcal{F}_w})\|F - \widetilde{F}\|_{\mathcal{F}_w} \\ &\le TC_{31}(T, \|p\|, r, r)\|F - \widetilde{F}\|_{\mathcal{F}_w}.\end{aligned}$$

where we have also used that $C_{31} \in$ Bounds is a continuous and strictly increasing function of its arguments. Hence, for T sufficiently small we have $TC_{31}(T, \|p\|, r, r) < 1$ such that $S_T^{p,X^\pm}$ is a contraction on $B_r(0) \subset \mathcal{F}_w^1$. By Banach's fixed point theorem $S_T^{p,X^\pm}$ has a unique fixed point in $B_r(0) \subset \mathcal{F}_w^1$.

(ii) Given a finite $T > 0$, $p \in \mathcal{P}$ and $X^\pm \in \widetilde{\mathcal{A}}_w^3$ Lemma 4.43p.77(i) states for all $F \in \mathcal{F}_w^1$

$$\|S_T^{p,X^\pm}[p, F]\|_{\mathcal{F}_w^1} \le \|S_T^{p,X^\pm}[p, F]\|_{\mathcal{F}_w^3} \le C_{30}{}^{(3)}(T, \|p\|) =: r. \tag{4.67}$$

Let K be the closed convex hull of $M := \{S_T^{p,X^\pm}[F] \mid F \in \mathcal{F}_w^1\} \subset B_r(0) \subset \mathcal{F}_w^1$. By Lemma 4.43p.77(ii) we know that the map $S_T^{p,X^\pm} : K \to K$ is continuous as a map $\mathcal{F}_w^1 \to \mathcal{F}_w^1$. If M is compact, it implies that K is compact, and hence, Schauder's Fixed Point Theorem 4.1p.48 ensures the existence of a fixed point.

It is left to show that M is compact. Therefore, let $(G_m)_{m\in\mathbb{N}}$ be a sequence in M. We need to show that it contains an $\mathcal{F}_w^1$ convergent subsequence. To show this we intend to use Lemma 4.50p.83. By definition there is a sequence $(F_m)_{m\in\mathbb{N}}$ in $B_r(0) \subset \mathcal{F}_w^1$ such that $G_m := S_T^{p,X^\pm}[F_m]$, $m \in \mathbb{N}$. We define for $m \in \mathbb{M}$

$$(\boldsymbol{E}_i^{(m)}, \boldsymbol{B}_i^{(m)})_{1\le i\le N} := S_T^{p,X^\pm}[F_m].$$

Recall the definition of the norm of $\mathcal{F}_w^n$, cf. Definition 4.36p.75, for some $(\boldsymbol{E}_i, \boldsymbol{B}_i)_{1\le i\le N} = F \in \mathcal{F}_w^n$ and $n \in \mathbb{N}$

$$\|F\|_{\mathcal{F}_w^n}^2 = \sum_{k=0}^n \|\mathsf{A}^k F\|_{\mathcal{F}_w}^2 = \sum_{k=0}^n \sum_{i=1}^N \left(\|(\nabla\wedge)^k \boldsymbol{E}_i\|_{L_w^2}^2 + \|(\nabla\wedge)^k \boldsymbol{B}_i\|_{L_w^2}^2\right). \tag{4.68}$$

Therefore, since A on $D_w(\mathsf{A})$ is closed, $(G_m)_{m\in\mathbb{N}}$ has an $\mathcal{F}_w^1$ convergent subsequence if and only if all the sequences $((\nabla\wedge)^k \boldsymbol{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k \boldsymbol{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le n$ have a common convergent subsequence in L_w^2.

To show this we first provide the bounds needed for Lemma 4.50p.83(i). Estimate (4.67) implies that

$$\sum_{k=0}^3 \sum_{i=1}^N \left(\|(\nabla\wedge)^k \boldsymbol{E}_i^{(m)}\|_{L_w^2}^2 + \|(\nabla\wedge)^k \boldsymbol{B}_i^{(m)}\|_{L_w^2}^2\right) = \|G_m\|_{\mathcal{F}_w^3}^2 \le r^2 \tag{4.69}$$

for all $m \in \mathbb{N}$. Furthermore, by Lemma 4.43p.77(ii) the fields $(\boldsymbol{E}_i^{(m)}, \boldsymbol{B}_i^{(m)})_{1\le i\le N}$ are a solution to the Maxwell equations at time zero and hence, by Theorem 4.14p.63 fulfill the Maxwell constraints for $(\boldsymbol{q}_i^0, \boldsymbol{p}_i^0)_{1\le i\le N} := p$ which read

$$\nabla \cdot \boldsymbol{E}^{(m)} = 4\pi\varrho_i(\cdot - \boldsymbol{q}_i^0) \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B}_i^{(m)} = 0.$$

Also by Theorem 4.14p.63, G_m is in $\mathcal{F}$ so that for every $k \in \mathbb{N}_0$

$$(\nabla\wedge)^{k+2}\boldsymbol{E}_i^{(m)} = 4\pi\delta_{k0}\nabla\varrho_i(\cdot - \boldsymbol{q}_i^0) - \triangle(\nabla\wedge)^k \boldsymbol{E}_i^{(m)} \qquad \text{and} \qquad (\nabla\wedge)^{k+2}\boldsymbol{B}_i^{(m)} = -\triangle(\nabla\wedge)^k \boldsymbol{B}_i^{(m)}$$

where δ_{k0} is the Kronecker delta which is zero except for $k = 0$. Estimate (4.69) implies for all $m \in \mathbb{N}$ that

$$\begin{aligned}
&\sum_{k=0}^{1}\sum_{i=1}^{N}\left(\|\triangle(\nabla\wedge)^k \boldsymbol{E}_i^{(m)}\|_{L^2_w}^2 + \|\triangle(\nabla\wedge)^k \boldsymbol{B}_i^{(m)}\|_{L^2_w}^2\right)\\
&\leq 2\sum_{k=0}^{1}\sum_{i=1}^{N}\left(\|(\nabla\wedge)^{k+2} \boldsymbol{E}_i^{(m)}\|_{L^2_w}^2 + \|(\nabla\wedge)^{k+2} \boldsymbol{B}_i^{(m)}\|_{L^2_w}^2\right) + 2\sum_{i=1}^{N}\|4\pi\nabla\varrho_i(\cdot - \boldsymbol{q}_i^0)\|_{L^2_w}\\
&\leq 2r^2 + 8\pi\sum_{i=1}^{N}\left(1 + C_w\left\|\boldsymbol{q}_i^0\right\|\right)^{P_w}\|\nabla\varrho_i\|_{L^2_w}^2
\end{aligned}$$

where we made use of the properties of the weight $w \in \mathcal{W}$. Note that the right-hand does not depend on m. Therefore, all the sequences $((\nabla\wedge)^k\boldsymbol{E}_i^{(m)})_{m\in\mathbb{N}}$, $(\triangle(\nabla\wedge)^k\boldsymbol{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\boldsymbol{B}_i^{(m)})_{m\in\mathbb{N}}$, $(\triangle(\nabla\wedge)^k\boldsymbol{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \leq i \leq N$ are uniformly bounded in L^2_w.

Second, we need to show that all the sequences $((\nabla\wedge)^k\boldsymbol{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\boldsymbol{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \leq i \leq N$ decay uniformly at infinity to meet condition (ii) of Lemma 4.50p.83. Define $(\boldsymbol{E}_{i,\pm T}^{(m),\pm}, \boldsymbol{B}_{i,\pm T}^{(m),\pm})_{1\leq i\leq N} := X^{\pm}_{\pm T}[p, F_m]$ for $m \in \mathbb{N}$ and denote the ith charge trajectory $t \mapsto (\boldsymbol{q}_{i,t}^{(m)}, \boldsymbol{p}_{i,t}^{(m)}) := (\mathsf{Q}+\mathsf{P})M_L[p, F_m](t, 0)$ by $(\boldsymbol{q}_i^{(m)}, \boldsymbol{p}_i^{(m)})$, $1 \leq i \leq N$. Using Lemma 4.43p.77(ii) and afterwards Lemma 4.14p.63 we can write the fields as

$$\begin{aligned}
\begin{pmatrix}\boldsymbol{E}_i^{(m)}\\ \boldsymbol{B}_i^{(m)}\end{pmatrix} &= \frac{1}{2}\sum_{\pm} M_{\varrho,m_i}[(\boldsymbol{E}^{\pm}_{i,\pm T}, \boldsymbol{E}^{\pm}_{i,\pm T}), (\boldsymbol{q}_i^{(m)}, \boldsymbol{p}_i^{(m)})](0, \pm T)\\
&= \frac{1}{2}\sum_{\pm}\Bigg[\begin{pmatrix}\partial_t & \nabla\wedge\\ -\nabla\wedge & \partial_t\end{pmatrix}K_{t\mp T} * \begin{pmatrix}\boldsymbol{E}_{i,\pm T}^{(m),\pm}\\ \boldsymbol{B}_{i,\pm T}^{(m),\pm}\end{pmatrix} + K_{t\mp T} * \begin{pmatrix}-4\pi\boldsymbol{j}_{i,\pm T}^{(m)}\\ 0\end{pmatrix}\\
&\quad + 4\pi\int_{\pm T}^{t} ds\, K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_{i,s}^{(m)}\\ \boldsymbol{j}_{i,s}^{(m)}\end{pmatrix}\Bigg]\Bigg|_{t=0} =: \boxed{25} + \boxed{26} + \boxed{27}
\end{aligned}$$

where $\rho_{i,t}^{(m)} := \varrho_i(\cdot - \boldsymbol{q}_{i,t}^{(m)})$ and $\boldsymbol{j}_{i,t}^{(m)} := \boldsymbol{v}(\boldsymbol{p}_{i,t}^{(m)})\rho_{i,t}$ for all $t \in \mathbb{R}$.

We shall show that there is a $\tau^* > 0$ such that for all $m \in \mathbb{N}$ the terms $\boxed{26}$ and $\boxed{27}$ are point-wise zero on $B^c_{\tau^*}(0) \subset \mathbb{R}^3$. Recalling the computation rules for K_t from Lemma 4.11p.61 we calculate

$$\|4\pi[K_{\mp T} * \boldsymbol{j}_{\pm T}^{(m)}](\boldsymbol{x})\|_{\mathbb{R}^3} \leq 4\pi T \fint_{B_T(\boldsymbol{x})} d\sigma(y)\, \varrho_i(\boldsymbol{y} - \boldsymbol{q}_{\pm T}^{(m)}).$$

The right-hand side is zero for all $\boldsymbol{x} \in \mathbb{R}^3$ such that $\partial B_T(\boldsymbol{x}) \cap \operatorname{supp}\varrho_i(\cdot - \boldsymbol{q}_{\pm T}) = \emptyset$. Because the charge distributions have compact support there is a $R > 0$ such that $\operatorname{supp}\varrho_i \subseteq B_R(0)$ for all $1 \leq i \leq N$. Now for any $1 \leq i \leq N$ and $m \in \mathbb{N}$ we have

$$\operatorname{supp}\varrho_i(\cdot - \boldsymbol{q}_{i,\pm T}^{(m)}) \subseteq B_R(\boldsymbol{q}_{i,\pm T}^{(m)}) \subseteq B_{R+T}(\boldsymbol{q}_i^0)$$

since the supremum of the velocities of the charge $\sup_{t\in[-T,T], m\in\mathbb{N}}\|\boldsymbol{v}(\boldsymbol{p}_{i,t}^{(m)}\|$ is smaller or equal one. Hence, $\partial B_T(\boldsymbol{x}) \cap B_{R+T}(\boldsymbol{q}_i^0) = \emptyset$ for all $\boldsymbol{x} \in B^c_\tau(0)$ with $\tau > \|p\| + R + 2T$.

Considering $\boxed{27}$ we have

$$\left\|4\pi\int_{\pm T}^{0} ds\left[K_{-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_{i,s}^{(m)}\\ \boldsymbol{j}_{i,s}^{(m)}\end{pmatrix}\right](\boldsymbol{x})\right\|_{\mathbb{R}^3\oplus\mathbb{R}^2} \leq 4\pi\int_{\pm T}^{0} ds\, s \fint_{\partial B_{|s|}(\boldsymbol{x})} d\sigma(y)\, \|\boldsymbol{G}(\boldsymbol{y} - \boldsymbol{q}_s^{(m)})\|_{\mathbb{R}^3\oplus\mathbb{R}^3} \tag{4.70}$$

where we used the abbreviation

$$\boldsymbol{G} := \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_{i,s}^{(m)} \\ \boldsymbol{j}_{i,s}^{(m)} \end{pmatrix}$$

and the computation rules for K_t given in Lemma 4.11p.61. As $\operatorname{supp} \boldsymbol{G} \subseteq \operatorname{supp} \varrho_i \subseteq B_R(0)$, the right-hand side of (4.70p.85) is zero for all $\boldsymbol{x} \in \mathbb{R}$ such that

$$\bigcup_{s\in[-T,T]} \left[\partial B_{|s|}(\boldsymbol{x}) \cap B_R(\boldsymbol{q}_{i,s}^{(m)})\right] = \emptyset.$$

Now the left-hand side is subset equal

$$\cup_{s\in[-T,T]} \partial B_{|s|}(\boldsymbol{x}) \bigcap \cup_{s\in[-T,T]} B_R(\boldsymbol{q}_{i,s}^{(m)}) \qquad \subseteq \qquad B_T(\boldsymbol{x}) \cap B_{R+T}(\boldsymbol{q}_i^0)$$

which is equal the empty set for all $\boldsymbol{x} \in B_\tau^c(0)$ with $\tau > \|p\| + R + 2T$.

Hence, setting $\tau^* := \|p\| + R + 2T$ we conclude that for all $\tau > \tau^*$ the terms $\boxed{26}$ and $\boxed{27}$ and all their derivatives are zero on $B_\tau^c(0) \subset \mathbb{R}^3$. That means in order to show that all the sequences $((\nabla\wedge)^k \boldsymbol{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k \boldsymbol{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$ decay uniformly at spatial infinity, it suffices to show

$$\lim_{\tau\to\infty} \sup_{m\in\mathbb{N}} \sum_{k=0}^{n} \sum_{i=1}^{N} \left(\|(\nabla\wedge)^k \boldsymbol{e}_i^{(m)}\|_{L_w^2(B_\tau^c(0))} + \|(\nabla\wedge)^k \boldsymbol{b}_i^{(m)}\|_{L_w^2(B_\tau^c(0))} \right) = 0. \tag{4.71}$$

for

$$\begin{pmatrix} \boldsymbol{e}_i^{(m)} \\ \boldsymbol{b}_i^{(m)} \end{pmatrix} := \boxed{25} = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * \begin{pmatrix} \boldsymbol{E}_{i,\pm T}^{(m),\pm} \\ \boldsymbol{B}_{i,\pm T}^{(m),\pm} \end{pmatrix} \Bigg|_{t=0}$$

for $1 \le i \le N$. Let $\boldsymbol{F} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $\tau > 0$. By computation rules for K_t given in Lemma 4.11p.61 we then yield

$$\begin{aligned}
&\|\nabla \wedge K_{\mp T} * \boldsymbol{F}\|_{L_w^2(B_{\tau+T}^c(0))} = \|K_{\mp T} * \nabla \wedge \boldsymbol{F}\|_{L_w^2(B_{\tau+T}^c(0))} \le \left\| T \fint_{\partial B_T(0)} d\sigma(y)\, \nabla \wedge \boldsymbol{F}(\cdot - \boldsymbol{y}) \right\|_{L_w^2(B_{\tau+T}^c(0))} \\
&\le T \fint_{\partial B_T(0)} d\sigma(y)\, \|\nabla \wedge \boldsymbol{F}(\cdot - \boldsymbol{y})\|_{L_w^2(B_{\tau+T}^c(0))} \le T \sup_{\boldsymbol{y}\in\partial B_T(0)} \|\nabla \wedge \boldsymbol{F}(\cdot - \boldsymbol{y})\|_{L_w^2(B_{\tau+T}^c(0))} \\
&\le T \sup_{\boldsymbol{y}\in\partial B_T(0)} (1 + C_w\|\boldsymbol{y}\|)^{\frac{P_w}{2}} \|\nabla \wedge \boldsymbol{F}(\cdot - \boldsymbol{y})\|_{L_w^2(B_{\tau+T}^c(0))} \le T(1 + C_w T)^{\frac{P_w}{2}} \|\nabla \wedge \boldsymbol{F}\|_{L_w^2(B_\tau^c(0))}.
\end{aligned}$$

We also estimate using the computation rules for K_t given in Lemma 4.11p.61 the term

$$\begin{aligned}
&\|\partial_t K_{t\mp T}|_{t=0} * \boldsymbol{F}\|_{L_w^2(B_{\tau+T}^c(0))} = \left\| \fint_{\partial B_T(0)} d\sigma(y)\, \boldsymbol{F}(\cdot - \boldsymbol{y}) + \frac{T^2}{3} \fint_{B_T(0)} d^3y\, \triangle \boldsymbol{F}(\cdot - \boldsymbol{y}) \right\|_{L_w^2(B_{\tau+T}^c(0))} \\
&\le \fint_{\partial B_T(0)} d\sigma(y)\, \|\boldsymbol{F}(\cdot - \boldsymbol{y})\|_{L_w^2(B_{\tau+T}^c(0))} + \frac{T^2}{3} \fint_{B_T(0)} d^3y\, \|\triangle \boldsymbol{F}(\cdot - \boldsymbol{y})\|_{L_w^2(B_{\tau+T}^c(0))} \\
&\le (1 + C_w T)^{\frac{P_w}{2}} \|\boldsymbol{F}\|_{L_w^2(B_\tau^c(0))} + \frac{T^2}{3} (1 + C_w T)^{\frac{P_w}{2}} \|\triangle \boldsymbol{F}\|_{L_w^2(B_\tau^c(0))}.
\end{aligned}$$

Substituting $\boldsymbol{F}$ with $(\nabla\wedge)^k\boldsymbol{E}^{(m),\pm}_{i,\pm T}$ and $(\nabla\wedge)^k\boldsymbol{B}^{(m),\pm}_{i,\pm T}$ for $k=0,1$ and $1\leq i\leq N$ in the two estimates above yields

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{i=1}^{N}\left(\|(\nabla\wedge)^k\boldsymbol{e}^{(m)}_i\|_{L^2_w(B^c_{\tau+T}(0))}+\|(\nabla\wedge)^k\boldsymbol{b}^{(m)}_i\|_{L^2_w(B^c_{\tau+T}(0))}\right)\\
&\leq(1+C_wT)^{\frac{P_w}{2}}\Bigg(\|(\nabla\wedge)^k\boldsymbol{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}+\|(\nabla\wedge)^k\boldsymbol{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}+\\
&\qquad+\frac{T^2}{3}\left(\|(\nabla\wedge)^k\triangle\boldsymbol{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}+\|(\nabla\wedge)^k\triangle\boldsymbol{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}\right)+\\
&\qquad+T\left(\|(\nabla\wedge)^{k+1}\boldsymbol{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}+\|(\nabla\wedge)^{k+1}\boldsymbol{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}\right)\Bigg).
\end{aligned}\tag{4.72}$$

Now $X^\pm$ lie in $\widetilde{\mathcal{A}}^3_w\subset\mathcal{A}^3_w$ which means that the fields $\boldsymbol{E}^{(m),\pm}_{i,\pm T}$ and $\boldsymbol{B}^{(m),\pm}_{i,\pm T}$ for $1\leq i\leq N$ fulfill the Maxwell constraints so that

$$\|(\nabla\wedge)^k\triangle\boldsymbol{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}=\|(\nabla\wedge)^{k+2}\boldsymbol{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}+4\pi\|(\nabla\wedge)^k\nabla\varrho_i(\cdot-\boldsymbol{q}^{(m)}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}$$

and

$$\|(\nabla\wedge)^k\triangle\boldsymbol{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}=\|(\nabla\wedge)^{k+2}\boldsymbol{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}.$$

Applying Definition 4.38p.75(iv) yields

$$\lim_{\tau\to\infty}\sup_{m\in\mathbb{N}}\sum_{j=0}^{3}\sum_{i=1}^{N}\left\|(\nabla\wedge)^j\boldsymbol{E}^{(m),\pm}_{i,\pm T}\|^2_{L^2_w(B^c_\tau(0))}+\|(\nabla\wedge)^j\boldsymbol{B}^{(m),\pm}_{i,\pm T}\|^2_{L^2_w(B^c_\tau(0))}\right)\leq\lim_{\tau\to\infty}\sup_{m\in\mathbb{N}}\|\chi^\pm_{\pm T}[p,F_m]\|^2_{\mathcal{H}^n_w}=0$$

because $F_m\in B_r(0)\subset\mathcal{F}^1_w$ for all $m\in\mathbb{N}$, which implies (4.71p.86) by the above estimates. By the above estimate (4.72) we conclude that equation (4.71p.86) holds which we proved to be sufficient to show the uniform decay at spatial infinity of all the sequences $((\nabla\wedge)^k\boldsymbol{E}^{(m)}_i)_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\boldsymbol{B}^{(m)}_i)_{m\in\mathbb{N}}$ for $k=0,1$ and $1\leq i\leq N$.

Let us summarize using the abbreviations $\boldsymbol{E}^{(m,k)}_i:=(\nabla\wedge)^k\boldsymbol{E}^{(m)}_i$ and $\boldsymbol{B}^{(m,k)}_i:=(\nabla\wedge)^k\boldsymbol{B}^{(m)}_i$ for $k=0,1$, $1\leq i\leq N$ and $m\in\mathbb{N}$: First, we have shown that the sequences $(\boldsymbol{E}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\boldsymbol{B}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\triangle\boldsymbol{E}^{(m,k)}_i)_{m\in\mathbb{N}}$ and $(\triangle\boldsymbol{B}^{(m,k)}_i)_{m\in\mathbb{N}}$ are all uniformly bounded in L^2_w. Second, we have shown that the sequences $(\boldsymbol{E}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\boldsymbol{B}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\triangle\boldsymbol{E}^{(m,k)}_i)_{m\in\mathbb{N}}$ decay uniformly at spatial infinity.

Having this we can now successively apply Lemma 4.50p.83 to yield the common $\mathcal{F}^1_w$ convergent subsequence: Fix $1\leq i\leq N$. Let $(\boldsymbol{E}^{(m^0_l,0)}_i)_{l\in\mathbb{N}}$ be the L^2_w convergent subsequence of $(\boldsymbol{E}^{(m,0)}_i)_{m\in\mathbb{N}}$ and $(\boldsymbol{E}^{(m^1_l,1)}_i)_{l\in\mathbb{N}}$ the L^2_w convergent subsequence of $(\boldsymbol{E}^{(m^0_l,1)}_i)_{l\in\mathbb{N}}$. In the same way we proceed with the other indices $1\leq i\leq N$ and the magnetic fields, every time choosing a further subsequence of the previous one. Let us denote the final subsequence by $(m_l)_{l\in\mathbb{N}}\subset\mathbb{N}$. Then we have constructed sequences $(G_{m_l})_{l\in\mathbb{N}}$ as well as $(\mathsf{A}G_{m_l})_{l\in\mathbb{N}}$ which are convergent in $\mathcal{F}_w$. However, A on $D_w(\mathsf{A})$ is closed so that this implies convergence of $(G_{m_l})_{l\in\mathbb{N}}$ in $\mathcal{F}^1_w$. As $(G_m)_{m\in\mathbb{N}}$ was arbitrary, we conclude that every sequence in M has an $\mathcal{F}^1_w$ convergent subsequence and therefore M is compact which had to be shown. □

Having established the existence of a fixed point F for all times $T>0$, Newtonian Cauchy data $p\in\mathcal{P}$ and boundary fields $(X^\pm_{i,\pm T})_{1\leq i\leq N}=X^\pm\in\widetilde{\mathcal{A}}^3_w$, Theorem 4.43p.77(iii) states that the charge trajectories $t\mapsto(\boldsymbol{q}_{i,t},\boldsymbol{p}_{i,t})_{1\leq i\leq N}:=(\mathsf{Q}+\mathsf{P})M_L[p,F](t,0)$ are in $\mathcal{T}^{p,X^\pm}_T$. This means they are time-like charge trajectories that solve the bWF equations for all times $t\in\mathbb{R}$, which are given by the Lorentz force law (4.1p.43) together with the equation for the fields (4.57p.76)

$$(\boldsymbol{E}^{\mathrm{WF}}_{i,t},\boldsymbol{B}^{\mathrm{WF}}_{i,t})=\frac{1}{2}\sum_{\pm}M_{\varrho_i,m_i}[X^\pm_{i,\pm T},(\boldsymbol{q}_i,\boldsymbol{p}_i)](t,\pm T),\tag{4.73}$$

$1 \le i \le N$, and initial conditions (4.9p.46).

However, how much do such fixed points tell us about the true Wheeler-Feynman solution? With regard to this question we shall show in the following that under conditions on the Newtonian Cauchy data p and on the charge densities ϱ_i, $1 \le i \le N$, one always finds a fixed point F whose corresponding charge trajectories fulfill the Wheeler-Feynman equations, i.e. the Lorentz force law (4.1p.43) together with the fields (4.10p.46)

$$(\boldsymbol{E}^{\mathrm{WF}}_{i,t}, \boldsymbol{B}^{\mathrm{WF}}_{i,t}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[(\boldsymbol{q}_i,\boldsymbol{p}_i)](t,\pm\infty), \tag{4.74}$$

$1 \le i \le N$, for t in a time interval $[-L, L]$ for some $L > 0$. Note that for this to be true the right-hand side of equation (4.73p.87) does not have to agree with the right-hand side of (4.74) everywhere on $\mathbb{R}^3$ but only within the tubes of radius R around the position of the $j \neq i$ charge trajectories as only these values enter the Lorentz force law (4.1p.43).

Partial Wheeler-Feynman solutions for Newtonian Cauchy data

Definition 4.52. *For Newtonian Cauchy data $p \in \mathcal{P}$ we define $\mathcal{T}^L_{\mathrm{WF}}$ to be the set of time-like charge trajectories in $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N} \in \mathcal{T}_\vee$ which solve the Wheeler-Feynman equations in the form (4.1p.43) and (4.74) for time $t \in [-L, L]$ and initial conditions (4.9p.46). We shall call every element of $\mathcal{T}^L_{\mathrm{WF}}$ a partial Wheeler-Feynman solution for initial value p.*

In order to see that a bWF solution $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N} \in \mathcal{T}^{p,X^\pm}_T$ is also a partial Wheeler-Feynman solution we have to regard the difference

$$\begin{aligned} &M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\boldsymbol{q}_i,\boldsymbol{p}_i)](t,\pm T) - M_{\varrho_i,m_i}[(\boldsymbol{q}_i,\boldsymbol{p}_i)](t,\pm\infty) \\ &= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X^\pm_{i,\pm T} + K_{t\mp T} * \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_{i,\pm T})\varrho_i(\cdot - \boldsymbol{q}_{i,\pm T}) \\ 0 \end{pmatrix} \\ &\quad - 4\pi \int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \boldsymbol{q}_{i,s}) \\ \boldsymbol{v}(\boldsymbol{p}_{i,s})\varrho_i(\cdot - \boldsymbol{q}_{i,s}) \end{pmatrix}. \end{aligned} \tag{4.75}$$

where we used Definition 4.16p.65 with Theorem 4.14p.63 as well as Definition 4.17p.65. Whenever the difference is zero everywhere within the tubes around the positions of the $j \neq i$ charge trajectories for $t \in [-L, L]$, the charge trajectories $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N}$ are in $\mathcal{T}^L_{\mathrm{WF}}$. This is certainly not true for all boundary fields $X^\pm \in \widetilde{\mathcal{H}}^3_w$. However, it is the case for the advanced, respectively retarded, Liénard-Wiechert fields of any charge trajectories which continue $(\boldsymbol{q}_i,\boldsymbol{p}_i)_{1\le i\le N}$ on the time interval $[T,\infty)$, respectively $(-\infty,-T]$, and we shall show this in the particular case of the Coulomb boundary fields C, cf. Definition 4.44p.79.

In fact, for the Coulomb boundary fields $C \in \widetilde{\mathcal{H}}^3_w \cap \mathcal{H}^{\mathrm{Lip}}_w$ we find that the difference discussed above is for "+" zero everywhere on the backward light-cone of the space-time point $(T, \boldsymbol{q}_{i,T})$ as well as for "−" everywhere on the forward light-cone of $(-T, \boldsymbol{q}_{i,-T})$.

Shadows of the boundary fields and Wheeler-Feynman fields

Lemma 4.53. *Let $\boldsymbol{q}, \boldsymbol{v} \in \mathbb{R}^3$, $\varrho \in C^\infty_c(\mathbb{R}^3,\mathbb{R})$ such that $\operatorname{supp}\varrho \subseteq B_R(0)$ for some finite $R > 0$. Furthermore, let $\boldsymbol{E}^C$ be the Coulomb field of a charge at rest at the origin*

$$\boldsymbol{E}^C := \int d^3z\, \varrho(\cdot - z)\frac{z}{\|z\|^3}$$

Then for $T > R$

$$\left[\begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * \begin{pmatrix} \boldsymbol{E}^C(\cdot - \boldsymbol{q}) \\ 0 \end{pmatrix} + K_{t\mp T} * \begin{pmatrix} -4\pi \boldsymbol{v}\varrho(\cdot - \boldsymbol{q}) \\ 0 \end{pmatrix}\right](\boldsymbol{x}) = 0 \tag{4.76}$$

and

$$\int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \boldsymbol{q}_{i,s}) \\ \boldsymbol{v}(\boldsymbol{p}_{i,s})\varrho_i(\cdot - \boldsymbol{q}_{i,s}) \end{pmatrix} (\boldsymbol{x}) = 0 \tag{4.77}$$

for $t \in (-T + R, T - R)$ and $\boldsymbol{x} \in B_{|t\mp T|-R}(\boldsymbol{q})$.

Proof. Let $t \in [-T+R, T-R]$. With regard to the second term we compute

$$\left\|-4\pi \boldsymbol{v}\left[K_{t\mp T} * \varrho(\cdot - \boldsymbol{q})\right](\boldsymbol{x})\right\| = 4\pi\|\boldsymbol{v}\|\left|(t\mp T) \fint_{\partial B_{|t\mp t|}(0)} d\sigma(y)\, \varrho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})\right|$$
$$\leq 4\pi\|\boldsymbol{v}\||t\mp T| \sup|\varrho| \fint_{\partial B_{|t\mp t|}(\boldsymbol{q})} d\sigma(y)\, \mathbb{1}_{B_R(\boldsymbol{x})}(\boldsymbol{y})$$

where we used Definition 4.10p.61 for $K_{t\mp T}$. Now $\boldsymbol{x} \in B_{|t\mp T|-R}(\boldsymbol{q})$ implies $\partial B_{|t\mp T|}(\boldsymbol{q}) \cap B_R(\boldsymbol{x}) = \emptyset$ and hence that the term above is zero.

With regard to the first term we note that the only non-zero contribution is $\partial_t K_{t\mp T} * \boldsymbol{E}_i^C$ since $\nabla \wedge \boldsymbol{E}^C = 0$. We shall need the computation rules for K_t as given in Lemma 4.11p.61 and in particular equation (4.37p.61) which in our case reads

$$\left[\partial_t K_{t\mp T} * \boldsymbol{E}^C(\cdot - \boldsymbol{q})\right](\boldsymbol{x}) = \fint_{\partial B_{|t\mp T|}(0)} d\sigma(y)\, \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}) + (t\mp T)\partial_t \fint_{\partial B_{|t\mp T|}(0)} d\sigma(y)\, \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}) \tag{4.78}$$

$$= \fint_{\partial B_{|t\mp T|}(0)} d\sigma(y)\, \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}) + \frac{(t\mp T)^2}{3} \fint_{B_{|t\mp T|}(0)} d^3y\, \triangle \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{y}) =: \boxed{28} + \boxed{29}. \tag{4.79}$$

Using Lebesgue's theorem we start with

$$\boxed{28} = \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{q}) + \int_0^{|t\mp T|} ds\, \partial_s \fint_{\partial B_s(0)} d\sigma(y)\, \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})$$
$$= \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{q}) + \int_0^{|t\mp T|} dr\, \frac{r}{3} \fint_{B_r(0)} d^3y\, \triangle \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}).$$

Furthermore, we know that $0 = (\nabla\wedge)^2 \boldsymbol{E}^C = \nabla(\nabla\cdot\boldsymbol{E}^C) - \triangle\boldsymbol{E}^C$ and $\nabla\cdot\boldsymbol{E}^C = 4\pi\varrho$. So we continue the computation with

$$\boxed{28} = \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{q}) + \int_0^{|t\mp T|} dr\, \frac{r}{3} \fint_{B_r(0)} d^3y\, 4\pi\nabla\varrho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})$$
$$= \boldsymbol{E}^C(\boldsymbol{x}-\boldsymbol{q}) - \int_0^{|t\mp T|} dr\, \frac{1}{r^2} \int_{\partial B_r(0)} d\sigma(y)\, \frac{\boldsymbol{y}}{r}\varrho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})$$

where we have used (4.78) to evaluate the derivative and in addition used Stoke's Theorem. Note that the minus sign in the last line is due to the fact that ∇ acts on $\boldsymbol{x}$ and not $\boldsymbol{y}$. Inserting the definition of the Coulomb field $\boldsymbol{E}^C$ we finally get

$$\boxed{28} = \int_{B^c_{|t\mp T|}(0)} d^3y\, \varrho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q})\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|^3}.$$

This integral is zero if, for example, $B^c_{|t\mp T|}(\boldsymbol{q}) \cap B_R(\boldsymbol{x}) = \emptyset$ and this is the case for $\boldsymbol{x} \in B_{|t\mp T|-R}(\boldsymbol{q})$. So it remains to show that $\boxed{29}$ also vanishes. Therefore, using $\triangle\boldsymbol{E}^C = 4\pi\nabla\varrho$ as before, we get

$$\boxed{29} = -\int_{\partial B_{|t\mp T|}(0)} d\sigma(y)\, \frac{\boldsymbol{y}}{(t\mp T)^2}\varrho(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{q}).$$

This expression is zero, for example, when $\partial B_{|t\mp T|}(\boldsymbol{q}) \cap B_R(\boldsymbol{x}) = \emptyset$ which is true for $\boldsymbol{x} \in B_{|t\mp T|-R}(\boldsymbol{q})$. Hence, we have shown that for $t \in (-T+R, T-R)$ and $\boldsymbol{x} \in B_{|t\mp T|}(\boldsymbol{q})$ the term (4.76p.88) is zero.

Looking at the support of the integrand and the integration domain in term (4.77p.88) we find that for all $t \in (-T+R, T-R)$ it is zero for all $\boldsymbol{x} \in \mathbb{R}^3$ such that

$$\bigcup_{|s|>T} (\partial B_{|t-s|}(\boldsymbol{x}) \cap B_R(\boldsymbol{q}_s)) = \emptyset. \tag{4.80}$$

Hence, for $t \in (-T+R, T-R)$ and $\boldsymbol{x} \in B_{|t\mp T|}(\boldsymbol{q})$ the term (4.77p.88) is also zero which concludes the proof. □

REMARK 4.54. *This lemma directly applies to the difference (4.75p.88) we were discussing before. By looking at the explicit formulas for the Maxwell solutions given in Theorem 4.14p.63 we recognize that this difference term is in some sense the free time-evolution of the initial fields. This time-evolution makes sure that the initial fields coming from a charge at rest have to make way for the new fields generated by the current of the charge during the time interval* $[-T, T]$. *This will certainly hold for all boundary fields which are Liénard-Wiechert fields of given charge trajectories on the intervals* $(-\infty, T]$ *and* $[T, \infty)$ *not only for the case of a charge at rest.*

Now that we know a big region where the difference (4.75p.88) is zero we have to make sure that the charge trajectories spend the time interval $[-L, L]$ there. For this we need a uniform momentum estimate:

Uniform velocity bound

Lemma 4.55. *For finite* $T > 0$ *and* $r > 0$ *there is a continuous and strictly increasing map* $v^{a,b} : \mathbb{R}^+ \to [0,1)$, $T \mapsto v_T^{a,b}$ *such that*

$$\sup\left\{ \|\boldsymbol{v}(\boldsymbol{p}_{i,t})\|_{\mathbb{R}^3} \,\middle|\, t \in [-T,T], \|p\| \le a, F \in \operatorname{Ran} S_T^{p,C}, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N \right\} \le v_T^{a,b}.$$

for $(\boldsymbol{p}_{i,t})_{1\le i\le N} := \mathrm{P} M_L[p,F](t,0)$ *for all* $t \in \mathbb{R}$.

Proof. Recall the estimate (3.10p.20) from the ML±SI existence and uniqueness Theorem 3.5p.20 which gives the following T dependent upper bounds on these ML-SI solutions for all $\varphi \in D_w(A)$:

$$\sup_{t\in[-T,T]} \|M_L[\varphi](t,0)\|_{\mathcal{H}_w} \le C_1\left(T, \|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \le i \le N\right) \|\varphi\|_{\mathcal{H}_w}. \tag{4.81}$$

Note further that by Lemma 4.49p.82 since $C \in \mathcal{A}^1_w$, there is a $C_{30}{}^{(1)} \in$ `Bounds` such that fields $F \in \operatorname{Ran} S_T^{p,C} \in D_w(\mathsf{A}^\infty)$ fulfill

$$\|F\|_{\mathcal{F}_w} \le C_{30}{}^{(1)}(T, \|p\|) \le C_{30}{}^{(1)}(T, a).$$

Therefore, setting $c := a + C_{30}{}^{(1)}(T,a)$ we estimate the maximal momentum of the charges by

$$\begin{aligned}
&\sup\left\{ \|\boldsymbol{v}(\boldsymbol{p}_{i,t})\|_{\mathbb{R}^3} \,\middle|\, t \in [-T,T], \|p\| \le a, F \in \operatorname{Ran} S_T^{p,C}, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N \right\}\\
&\le \sup\left\{ \|\boldsymbol{v}(\boldsymbol{p}_{i,t})\|_{\mathbb{R}^3} \,\middle|\, t \in [-T,T], \varphi \in D_w(A), \|\varphi\|_{\mathcal{H}_w} \le c, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N \right\}\\
&\le C_1(T, b, b,)\, c =: p_T^{a,b} < \infty.
\end{aligned}$$

Now, since C_2 as well as $C_{30}{}^{(1)}$ are in `Bounds` the map $T \mapsto p_T^{a,b}$ as $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous and strictly increasing. We conclude that claim is fulfilled for the choice

$$v_T^{a,b} := \frac{p_T^{a,b}}{\sqrt{m^2 + (p_T^{a,b})^2}}$$

and $m := \min_{1\le i\le N} |m_i|$. □

With this we can formulate our last result.

Existence of partial Wheeler-Feynman solutions

Theorem 4.56. *For* $(\boldsymbol{q}_i^0, \boldsymbol{p}_i^0)_{1\le i\le N} = p \in \mathcal{P}$ *define*

$$\triangle q_{max}(p) := \max_{1\le i,j\le N} \|\boldsymbol{q}_i^0 - \boldsymbol{q}_j^0\|.$$

Choose $T > 0$, $a > 0$ *and* $b > 0$. *Furthermore, let* $R > 0$ *be a radius small enough such that the charge densities fulfill* $\operatorname{supp}\varrho_i \subseteq B_R(0)$, $\|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b$, $(1 - v_T^{a,b})T - 2R > 0$ *and* $T > 2R$. *Then for Newtonian Cauchy data*

$$p \in \left\{ p \in \mathcal{P} \,\middle|\, \|p\| \le a, \triangle q_{max}(p) < (1 - v_T^{a,b})T - 2R \right\}$$

the charge trajectories in $\mathcal{T}_T^{p,C} \subset \mathcal{T}_{\mathrm{WF}}^L$ *for* $L := \frac{(1-v_T^{a,b})T - \triangle q_{max} - 2R}{1+v_T^{a,b}} > 0$.

Proof. Let F be a fixed point $F = S_T^{p,C}[F]$ which exists by Theorem 4.47p.81. Define the charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N}$ by $t \mapsto (\boldsymbol{q}_{i,t}, \boldsymbol{p}_{i,t})_{1\le i\le N} := (\mathsf{Q}+\mathsf{P})M_L[p,F](t,0)$. By the fixed point properties of F we know that these trajectories are in $\mathcal{T}_T^{p,C}$ and therefore solve the bWF equations, i.e. the Lorentz force law (4.1p.43) and the equations for the fields (4.73p.87) for Newtonian Cauchy data (4.9p.46). In order to show that the charge trajectories $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{1\le i\le N}$ are also in $\mathcal{T}_{\mathrm{WF}}^L$ for the given L we need to show that the difference (4.75p.88)

$$\begin{aligned} &M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm T) - M_{\varrho_i,m_i}[(\boldsymbol{q}_i, \boldsymbol{p}_i)](t, \pm\infty) \\ &= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X^\pm_{i,\pm T} + K_{t\mp T} * \begin{pmatrix} -4\pi \boldsymbol{v}(\boldsymbol{p}_{i,\pm T})\varrho_i(\cdot - \boldsymbol{q}_{i,\pm T}) \\ 0 \end{pmatrix} \\ &\quad - 4\pi \int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \boldsymbol{q}_{i,s}) \\ \boldsymbol{v}(\boldsymbol{p}_{i,s})\varrho_i(\cdot - \boldsymbol{q}_{i,s}) \end{pmatrix}. \end{aligned}$$

is zero for times $t \in [-L, L]$ at least for all points $\boldsymbol{x}$ in a tube around the position of the $j \neq i$ charge trajectories. Lemma 4.53p.88 states that this expression is zero for all $t \in [-T+R, T-R]$ and $\boldsymbol{x} \in B_{|t\mp T|-R}(\boldsymbol{q}_{i,\pm T})$. So it is sufficient to show that the charge trajectories spend the time interval $[-L, L]$ in this particular space-time region. Clearly, the position $\boldsymbol{q}_i^0$ at time zero is in $B_{T-R}(\boldsymbol{q}_{i,\pm T})$. Hence, we need to compute the earliest exit time L of this space-time region of a charge trajectory j in the worst case. The exit time L is the time when the jth charge trajectory leaves the region $B_{|L\mp T|-R}(\boldsymbol{q}_{i,\pm T})$. By Lemma 4.55p.90 the charges can in the worst case move apart from each other with velocity $v_T^{a,b}$ during the time interval $[-T, T]$. Putting the origin at $\boldsymbol{q}_i^0$ we can compute the exit time L by

$$-v_T^{a,b} T = \|\boldsymbol{q}_j^0 - \boldsymbol{q}_i^0\| + 2R + v_T^{a,b} L - (T - L)$$

This gives $L := \frac{(1-v_T^{a,b})T - \triangle q_{max} - 2R}{1+v_T^{a,b}} > 0$ as long as $\triangle q_{max} < (1 - v_T^{a,b})T$ wich is the case. □

REMARK 4.57. *The intention behind this theorem is to only show that at least on finite intervals it is possible to find Wheeler-Feynman solutions for Newtonian Cauchy data. The conditions are very restrictive, however, only technical. For example, if we consider the special case of two charges of the same sign and positive masses one can expect to get a uniform bound* $v_T^{a,b} \le v^{a,b} < 1$ *on the maximal velocity for all times. In this case all the restrictions disappear because for any choice of* $p \in \mathcal{P}$ *we can take* T *to be large enough to ensure* $\triangle q_{max}(p) < (1 - v^{a,b})T - 2R$.

4.5 Conclusion and Outlook

As discussed in the introductory Section 4.1p.43, the question whether there exist true Wheeler-Feynman solutions for all times for given Newtonian Cauchy data remains open. There are at least two ways to proceed from here:

1) Fixed Point Theorems for $T \to \infty$. The problem we have already addressed towards the end of Section 4.1p.43 is that we are missing uniform bounds for the accelerations and velocities of the charges. Note that since the S^p needs to be a self-mapping, we cannot easily put such conditions in by hand but rather have to work them out given the form of the ML-SI and WF equations. There are two fixed point theorems which seem most convenient for this case:

First, one could try to apply Krasnosel'skii's Fixed Point Theorem in the following way:

Theorem 4.58 (Krasnosel'skii's Fixed Point Theorem [Sma74] Chapter 4.4 Theorem 4.4.1)**.** *Let $\mathcal{B}$ be a Banach space, and R, S maps on $M \subseteq \mathcal{B}$ with the following properties:*

(i) $\operatorname{Ran}(S+R) \subseteq M$,

(ii) *S is continuous and* $\operatorname{Ran} S$ *is compact,*

(iii) *R is a contraction.*

Then the map $S + R$ has a fixed point on M.

For any Newtonian Cauchy data $p \in \mathcal{P}$, boundary fields $X^\pm \in \mathcal{A}^1_w$ and field $F \in D_w(\mathsf{A}^\infty) \cap \mathcal{F}$ let $(S^{p,X^\pm}_{i,T}[F])_{1\leq i\leq N} := S^{p,X^\pm}_T[F]$ and $(X^{pm}_{i,\pm T}[p,F])_{1\leq i\leq N} := X^{pm}_{\pm T}[p,F]$. Using the Kirchoff formulas for the Maxwell equations we have for all $1 \leq i \leq N$

$$\begin{aligned} S^{p,X^\pm}_{i,T} &= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * X^\pm_{i,\pm T}[p,F] + K_{t-t_0} * \begin{pmatrix} -4\pi \boldsymbol{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^t ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \boldsymbol{j}_s \end{pmatrix} \\ &=: R^p_{i,T}[F] + S^p_{i,T}[F]. \end{aligned}$$

Define $R^p_T := (R^p_{i,T})_{1\leq i\leq N}$ and $S^p_T := (S^p_{i,T})_{1\leq i\leq N}$. We have shown that for all $T > 0$ and given boundary fields $X^\pm$ the map $S^{p,X^\pm}_T$ is continuous and has compact range if the boundary fields decay uniformly at spatial infinity and depend continuously on F. Therefore, this holds also for zero boundary fields and one yields that S^p_T is continuous and $\operatorname{Ran} S^p_T$ is compact. Now replace the boundary fields $X^\pm_{i,T}$ in the definition of R^p_T by the true boundary fields of a possible Wheeler-Feynman solution (4.12p.47) as discussed in the overview Section 4.1p.43, i.e.

$$X^\pm_{i,\pm T} := M_{\varrho_i,m_i}[(\boldsymbol{q}_i,\boldsymbol{p}_i)](\pm T, \pm\infty). \tag{4.12}$$

So S^p_T are the fields generated by the charge trajectories in the time interval $[-T,T]$. Similarly R^p_T are the fields generated in the time interval $[-\infty,-T] \cup [T,\infty]$ which are *morally* transported from $\pm T$ to time zero with the help of the free Maxwell time-evolutions. The hope is that for large T one can get good enough bounds on the accelerations and velocities of the charge trajectories such that R^p_T becomes a contraction. The hope is based on the facts that, first, the Maxwell time-evolution forgets its initial fields at large times which gives additional decay, and second, at least for scattering states one should be able to get bounds on the accelerations and velocities for large T. Note that we can choose T to be as large as we want which does not change the needed properties of S^p_T.

Another but apparently more difficult way is to try to apply Schäfer's Fixed Point Theorem:

Theorem 4.59 (Schäfer's Fixed Point Theorem [Eva98]). *Let $\mathcal{B}$ be a Banach space. If the map $S : \mathcal{B} \to \mathcal{B}$ is continuous and compact (i.e. it maps bounded sequences to precompact ones) and the set*

$$\{F \in \mathcal{B} \mid F = \lambda S[F], 0 \leq \lambda \leq 1\} \tag{4.82}$$

is bounded, then S has a fixed point.

Using our results concerning $S_T^{p,X^\pm}$, one can show that even for $T \to \infty$ it remains continuous and compact if the ML-SI dynamics comply with the following physically reasonable a priori bound: For each ball B of initial conditions in $D_w(A^\infty)$ there is a constant $K < \infty$ such that

$$\sup_{t\in\mathbb{R}} \sup_{\varphi\in B} \left(\|\mathrm{P}\varphi_t\|_{\mathcal{H}_w} + \|\mathrm{P}J(\varphi_t)\|_{\mathcal{H}_w} \right) \leq K$$

for $\varphi_t := M_L[\varphi](t,0)$ with $\varphi \in B$. This estimate implies that the acceleration and momenta of the charge trajectories of ML-SI solutions having initial value in B are bounded by a finite constant depending only on B. Even if we assume such a bound for the ML-SI dynamics as given, we need to show the a priori bound (4.82). The case $\lambda = 1$ for which we need to get an a priori bound for possible Wheeler-Feynman solutions is difficult enough. For $\lambda < 1$ the chance of getting appropriate bounds is even smaller because we would be looking at altered Wheeler-Feynman equations whose fields would not even satisfy the Maxwell constraints. It seems that the appearance of the parameter λ, which comes from a topological argument of the Leray-Schauder degree theory, does not have any connection with the resulting dynamics. For this reason it seems rather difficult to conduct such a program.

2) Continuation of Fixed Points for Finite T. In general it appears that fixed point methods for $T \to \infty$ rely on uniform bounds of the accelerations and momenta of the charge trajectories. At the present stage it is not clear how such bounds could be computed – having in mind that on equal time hypersurfaces we do not even have constants of motion for the ML-SI dynamics. This situation seems rather bad but it is indeed not surprising as even for ordinary differential equations one is usually not able to give uniform bounds for all times but only for compact time intervals. As we proceeded in the proof of existence of solutions for the ML-SI equations one usually computes local solutions and then concatenates them which yields solutions on any finite but arbitrary large time interval. For this purpose, bounds being uniform on only compact time intervals like the ones we have for the ML-SI dynamics are sufficient. The problem with functional differential equations is that we do not know what the fixed points of $S_T^{p,X^\pm}$ and $S_{\widetilde{T}}^{p,\widetilde{X}^\pm}$ for two times $T < \widetilde{T}$ and boundary field $X^\pm, \widetilde{X}^\pm$ have in common. For example, one could imagine that in the worst case the maps have not even the same number of fixed points so that even for arbitrarily small time distances $\widetilde{T} - T$ the fixed points of both maps do not even have to be close in the Banach space. However, as soon as we have found a Wheeler-Feynman solutions on a finite interval the situation changes. First, we get conservation laws from the Wheeler-Feynman action integral by Noether's theorem, and second, we get consistency conditions similar to those we found for the toy model in Section 4.2.1p.53 which were sufficient to construct the whole solution. This approach we regard to be most promising and we shall focus in our future work on:

- Getting a better velocity estimate than Lemma 4.55p.90 in order to yield Wheeler-Feynman solutions for Newtonian Cauchy data $p \in \mathbb{R}^{6N}$ on arbitrary large time intervals without extra conditions on p and charge densities ϱ_i.
- Studying how the conservation laws and consistency conditions for Wheeler-Feynman trajectories strips can be used for the concatenation of different Wheeler-Feynman solution on finite intervals.

- Because the strips of world lines which by the Wheeler-Feynman equations are only connected by space-time points on world lines within time interval $[-L, L]$ are independent on the boundary fields $X^{\pm}$, we have reason to believe that these strips are already the unique Wheeler-Feynman trajectories corresponding to Newtonian Cauchy data p.

Chapter 5

Appendix for Part I

5.1 Equations of Motion in a Special Reference Frame

In classical electromagnetism a theory about time-like world lines in Minkowski space $\mathbb{M} := (\mathbb{R}^4, g)$ for the metric tensor g is given by

$$(g^{\mu}{}_{\nu})_{\mu,\nu=0,1,2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the following we use the standard four-vector notation and Einstein's summation convention, e.g. for $x, y \in \mathbb{M}$ we write

$$x_\mu y^\mu = \sum_{\mu,\nu=0}^{3} g^{\mu}{}_{\nu} x^\nu y^\mu.$$

A time-like world line is a set $\{x^\mu(s) \mid s \in \mathbb{R}\} \subset \mathbb{M}$ parametrized by a differential function $x : \mathbb{R} \to \mathbb{M}$ such that the corresponding velocity fulfills $\dot{x}_\mu(s)\dot{x}^\mu(s) > 0$ for all $s \in \mathbb{R}$. For time $t : \mathbb{R} \to \mathbb{R}$ and position $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^3$ in a special reference frame we have

$$x^\mu(s) = \begin{pmatrix} t(s) \\ \boldsymbol{q}(t(s)) \end{pmatrix}^{\mu} \quad \text{for all } s \in \mathbb{R}.$$

and define the velocity by

$$u^\mu(s) = \frac{d}{ds} x^\mu(s) = \frac{dt(s)}{ds} \begin{pmatrix} 1 \\ \boldsymbol{v}(t(s)) = \frac{d\boldsymbol{q}(t)}{dt}\big|_{t=t(s)} \end{pmatrix} \quad \text{for all } s \in \mathbb{R}. \tag{5.1}$$

Since the world line is time-like we know that

$$0 < u_\mu(s) u^\mu(s) = \left(\frac{dt(s)}{ds}\right)^2 \left(1 - \boldsymbol{v}(t(s))^2\right)$$

so that $\frac{dt(s)}{ds} \neq 0$ and $0 \leq \|\boldsymbol{v}(s)\| < 1$ for all $s \in \mathbb{R}$. A natural choice is

$$\frac{dt(s)}{ds} = \frac{1}{\sqrt{1 - \boldsymbol{v}(t(s))^2}} =: \gamma(\boldsymbol{v}(t(s))) \tag{5.2}$$

because then $u_\mu(s)u^\mu(s) = 1$ and u^μ transforms as a Lorentz four-vector for all $s \in \mathbb{R}$. This way formulas involving terms dependent on u^μ will be Lorentz invariant. This also implies that

$$0 = \frac{d}{ds}\left(u_\mu(s)u^\mu(s)\right) = 2u_\mu(s)\frac{du^\mu(s)}{ds}, \tag{5.3}$$

i.e. that the acceleration $\frac{du^\mu(s)}{ds}$ is always Minkowski-orthogonal to the velocity at $s \in \mathbb{R}$. A simple Lorentz invariant force law for a time-like world line parameterized by x is then given by

$$\frac{d}{ds}\begin{pmatrix} x^\mu(s) \\ mu^\mu(s) \end{pmatrix} = \begin{pmatrix} u^\mu(s) \\ u_\nu(s)F^{\mu\nu}(x(s)) \end{pmatrix} \tag{5.4}$$

where F is an anti-symmetric tensor field on $\mathbb{M}$ and $m \in \mathbb{R} \setminus \{0\}$ is the mass of the particle. The anti-symmetry of F ensures that the world line is time-like because

$$mu_\mu(s)\frac{du^\mu}{ds} = u_\mu F^{\mu\nu}(x(s))u_\nu(s) = -u_\mu F^{\nu\mu}(x(s))u_\nu(s) = u_\nu F^{\nu\mu}(x(s))u_\mu(s)$$

holds only if the right-hand side equals zero.

At some places we need to express this relativistic notation by choosing coordinates in a special reference frame where we use position $\boldsymbol{q}$ and momentum $\boldsymbol{p}$ as in Hamilton mechanics which depend on the actual time t to describe the world line. The upper equation (5.4) is only the definition (5.1p.95) with (5.2p.95), i.e.

$$\frac{d}{ds}t(s) = \gamma(\boldsymbol{v}(t(s))) \qquad \text{and} \qquad \frac{d}{ds}\boldsymbol{q}(t(s)) = \gamma(\boldsymbol{v}(t(s)))\boldsymbol{v}(t(s)) \tag{5.5}$$

for $\boldsymbol{v}(t) = \frac{d\boldsymbol{q}(t)}{dt}$ and all $s \in \mathbb{R}$. From the lower equation of (5.4) we yield

$$m\frac{d}{ds}\gamma(\boldsymbol{v}(t(s))) = u_\nu(s)F^{0\nu}(x(s)) \qquad \text{and} \qquad \frac{d}{ds}\boldsymbol{p}(t(s)) = \left(u_\nu(s)F^{i\nu}(x(s))\right)_{i=1,2,3}$$

for $\boldsymbol{p}(t) = m\gamma(\boldsymbol{v}(t))\boldsymbol{v}(t)$. Using the notation $\boldsymbol{v} = (v_1, v_2, v_3)$ the second equation gives

$$\frac{d}{dt}\boldsymbol{p}(t) = \left(F^{i0}(t, \boldsymbol{q}(t)) - \sum_{j=1}^{3} v_j(t)F^{ij}(t, \boldsymbol{q}(t))\right)_{i=1,2,3}. \tag{5.6}$$

The anti-symmetry of F yields $F^{00} = 0$ and

$$m\frac{d}{dt}\gamma(\boldsymbol{v}(t)) = -\sum_{j=1}^{3} v_j(t)\cdot F^{0j}(t, \boldsymbol{q}(t)) = \sum_{j=1}^{3} v_j(t)F^{j0}(t, \boldsymbol{q}(t)).$$

But (5.6) gives

$$\sum_{j=1}^{3} v_j(t)F^{j0}(t, \boldsymbol{q}(t)) = \boldsymbol{v}(t)\cdot\frac{d\boldsymbol{p}(t)}{dt} + \sum_{i,j=1}^{3} v_i(t)v_j(t)F^{ij}(t, \boldsymbol{q}(t)) = \boldsymbol{v}(t)\cdot\frac{d\boldsymbol{p}(t)}{dt}$$

so that we find

$$m\frac{d}{dt}\gamma(\boldsymbol{v}(t)) = \boldsymbol{v}(t)\cdot\frac{d\boldsymbol{p}(t)}{dt}.$$

However, this is trivially fulfilled because it basically expresses the Minkowski orthogonality property (5.3) of the four-vector velocity and acceleration:

$$0 = mu_\mu(s)\frac{d}{ds}u^\mu(s) = \gamma(\boldsymbol{v}(t))\begin{pmatrix}1\\ \boldsymbol{v}(t)\end{pmatrix}_\mu \frac{d}{ds}\begin{pmatrix}m\gamma(\boldsymbol{v}(t))\\ \boldsymbol{p}(t)\end{pmatrix}^\mu \Bigg|_{t=t(s)}$$

$$= \gamma^2(\boldsymbol{v}(t))\left(m\frac{d}{dt}\gamma(\boldsymbol{v}(t)) - \boldsymbol{v}(t)\cdot\frac{d\boldsymbol{p}(t)}{dt}\right)\Bigg|_{t=t(s)}$$

as $\gamma^2(\boldsymbol{v}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{R}^3$. Collecting the non-trivial equations (5.6) and (5.5) we can rewrite (5.4) in a special reference frame by

$$\frac{d}{dt}\begin{pmatrix}\boldsymbol{q}(t)\\ \boldsymbol{p}(t)\end{pmatrix} = \begin{pmatrix}\boldsymbol{v}(t) = \frac{\boldsymbol{p}(t)}{\sqrt{m^2+\boldsymbol{p}^2(t)}}\\ \left(F^{i0}(t,\boldsymbol{q}(t)) - \sum_{j=1}^3 v_j(t)F^{ij}(t,\boldsymbol{q}(t))\right)_{i=1,2,3}\end{pmatrix}.$$

5.2 Missing Proofs and Computations for Section 3.2

Proof of Theorem 3.12$_{p.25}$. Let $\boldsymbol{f} \in L^2_w$, then by definition $\sqrt{w}\boldsymbol{f} \in L^2$. Furthermore, C_c^∞ is dense in L^2. Therefore, one finds a sequence $(\boldsymbol{g}_n)_{n\in\mathbb{N}}$ in C_c^∞ and, hence, a sequence $\boldsymbol{f}_n := \frac{\boldsymbol{g}_n}{\sqrt{w}}$, $n \in \mathbb{N}$, in $C_c^\infty \cap L^2_w$, since $w \in C^\infty(\mathbb{R}^3, \mathbb{R}^+\setminus\{0\})$ by definition of $\mathcal{W}$, such that $\|\boldsymbol{f}-\boldsymbol{f}_n\|_{L^2_w} = \|\sqrt{w}\boldsymbol{f}-\boldsymbol{g}_n\|_{L^2} \to 0$ for $n \to \infty$. □ Proof of Theorem 3.12p.25

Computation 5.1. *Let $(\varphi_{n,(\cdot)})_{n\in\mathbb{N}}$ be a Cauchy sequence in $X_{T,n}$, then $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N}$ such that $\|A^j\varphi_{k,t} - A^j\varphi_{l,t}\|_{\mathcal{B}} \leq \|\varphi_k - \varphi_l\|_{X_{T,n}} < \epsilon$ for all $k,l > N$, $t \in [-T,T]$, $j \leq n$. Hence, each $(A^j\varphi_{n,t})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}$ which converges to some $\varphi_t^j \in \mathcal{B}$. Since A is closed, we know $\varphi_t^j \in D(A^{n-j})$ and $\varphi_t^j = A^j\varphi_t^0$. Moreover, the convergence is uniform on $[-T,T]$:* Computation 5.1

$$\|\varphi_k - \varphi^0\|_{X_{T,n}} = \sup_{t\in[-T,T]}\sum_{j=0}^n \|A^j(\varphi_{k,t} - \varphi_t^0)\|_{\mathcal{B}} = \sup_{t\in[-T,T]}\lim_{l\to\infty}\sum_{j=0}^n \|A^j(\varphi_{k,t} - \varphi_{l,t})\|_{\mathcal{B}}$$

$$\leq \sup_{t\in[-T,T]}\sup_{l>N}\sum_{j=0}^n \|A^j(\varphi_{k,t} - \varphi_{l,t})\|_{\mathcal{B}} = \sup_{l>N}\|\varphi_{k,(\cdot)} - \varphi_{l,(\cdot)}\|_{X_{T,n}} < \epsilon$$

We still need to show that $t \mapsto A^j\varphi_t^0$ is continuous on $(-T,T)$. Let $\epsilon > 0$ and pick a $k \in \mathbb{N}$ such that $\|A^j\varphi_{k,t} - A^j\varphi_t^0\|_{\mathcal{B}} < \epsilon/3$. The mappings $t \mapsto A^j\varphi_{k,t}$ are continuous on $(-T,T)$, so for each $j \leq n$, $t \in (-T,T)$ we find a $\delta > 0$ such that $|t-s| < \delta$ implies $\|A^j\varphi_{k,t} - A^j\varphi_{k,s}\|_{\mathcal{B}} < \epsilon/3$. For $|t-s| < \delta$ we find

$$\|A^j\varphi_t^0 - A^j\varphi_s^0\|_{\mathcal{B}} \leq \|A^j\varphi_t^0 - A^j\varphi_{k,t}\|_{\mathcal{B}} + \|A^j\varphi_{k,t} - A^j\varphi_{s,k}\|_{\mathcal{B}} + \|A^j\varphi_{s,k}^0 - A^j\varphi_s^0\|_{\mathcal{B}} < \epsilon$$

which proves the continuity of $t \mapsto A^j\varphi_t^0$.

5.3 Missing Computation for Section 4.3.1

Computation 5.2. *For $\mp t > 0$ and $f \in C^\infty(\mathbb{R}^3)$*

$$K_t^\pm * f = \int d^3y\ K_t^\pm(\boldsymbol{y})f(\cdot - \boldsymbol{y}) = \int_0^\infty dr \int_{\partial B_r(0)} d\sigma(y)\frac{\delta(r \pm t)}{4\pi r}f(\cdot - \boldsymbol{y})$$

$$= \mp t \fint_{\partial B_{\mp t}(0)} d\sigma(y) f(\cdot - \boldsymbol{y})$$

is in $C^\infty(\mathbb{R}^3)$ as a function of $\boldsymbol{x}$ and is continuous in t except at $t = 0$. For $\pm t > 0$ this expression is zero by definition. Next we compute the first derivative with respect to time $\mp t > 0$.

$$\begin{aligned}\partial_t K_t^\pm * f &= \mp \fint_{\partial B_{\mp t}(0)} d\sigma(y)\, f(\cdot - \boldsymbol{y}) \mp t\partial_t \fint_{\partial B_{\mp t}(0)} d\sigma(y)\, f(\cdot - \boldsymbol{y})\\ &= \mp \fint_{\partial B_{\mp t}(0)} d\sigma(y)\, f(\cdot - \boldsymbol{y}) \mp \frac{t^2}{3} \fint_{B_{\mp t}(0)} d^3y\, \triangle f(\cdot - \boldsymbol{y})\end{aligned}$$

As before we have used the Gauß-Green theorem.

$$\begin{aligned}&\partial_t \fint_{\partial B_{\mp t}(0)} d^3y\, f(\cdot - \boldsymbol{y}) = \partial_t \fint_{\partial B_1(0)} d\sigma(y)\, f(\cdot - (\mp t)\boldsymbol{y}) = \fint_{\partial B_1(0)} d\sigma(y)\, (\pm \boldsymbol{y}) \cdot \nabla f(\cdot - (\mp t)\boldsymbol{y})\\ &= -\frac{1}{4\pi t^2} \int_{\partial B_{\mp t}(0)} d\sigma(y)\, \frac{\boldsymbol{y}}{t} \cdot \nabla f(\cdot - \boldsymbol{y}) = \mp \frac{1}{4\pi t^2} \int_{\partial B_{\mp t}(0)} d\sigma(y)\, \frac{\boldsymbol{y}}{\mp t} \cdot \nabla_{\boldsymbol{y}} f(\cdot - \boldsymbol{y})\\ &= \mp \frac{1}{4\pi t^2} \int_{B_{\mp t}(0)} d^3y\, \nabla_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{y}} f(\cdot - \boldsymbol{y}) = \frac{t}{3} \fint_{B_{\mp t}(0)} d^3y\, \triangle f(\cdot - \boldsymbol{y})\end{aligned}$$

The second derivative is then given by

$$\begin{aligned}\partial_t^2 K_t^\pm * f &= \partial_t \left(\mp \fint_{\partial B_{\mp t}(0)} d\sigma(y)\, f(\cdot - \boldsymbol{y}) \mp \frac{t^2}{3} \fint_{B_{\mp t}(0)} d^3y\, \triangle f(\cdot - \boldsymbol{y}) \right)\\ &= \mp \frac{t}{3} \fint_{B_{\mp t}(0)} d^3y\, \triangle f(\cdot - \boldsymbol{y}) + \partial_t \left(\frac{1}{4\pi t} \right) \int_{B_{\mp t}(0)} d^3y\, \triangle f(\cdot - \boldsymbol{y})\\ &\quad + \frac{1}{4\pi t} \partial_t \int_0^{\mp t} dr \int_{\partial B_r(0)} d\sigma(y)\, \triangle f(\cdot - \boldsymbol{y}) = \mp t \fint_{\partial B_{\mp t}(0)} d\sigma(y)\, \triangle f(\cdot - \boldsymbol{y})\\ &= K_t^\pm * \triangle f = \triangle K_t^\pm * f\end{aligned}$$

Computation 5.3. *Here we compute the differentiation which was not performed in Theorem 4.18$_{p.66}$, Equation (4.51$_{p.68}$). At first we compute the derivative of $t^\pm$ defined in (4.47$_{p.66}$). Recall that all entities with a superscript $\pm$ depend on $t^\pm$. For any $k = 1, 2, 3$*

$$\partial_{z_k} t^\pm = \pm \partial_{z_k} \|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}^\pm\|.$$

Now

$$\partial_{z_k} \|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}^\pm\| = \frac{x_j - z_j - q_j^\pm}{\|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}^\pm\|} (-\delta_{kj} - \partial_{z_k} q_j^\pm) = -n_k^\pm - n_j^\pm\, \partial_{z_k} q_j^\pm$$

where $(\delta_{ij})_{1 \le i,j \le 3}$ is the Kronecker delta, i.e. the identity on the space of $\mathbb{R}^{3x3}$ matrices, and we have used Einstein's summation convention (we sum over double indices). On the other hand $\partial_{z_k} q^\pm = v^\pm\, \partial_{z_k} t^\pm$, such that we can plug all of these equations together and find

$$\partial_{z_k} t^\pm = \frac{\mp n_k^\pm}{1 \pm \boldsymbol{n}^\pm \cdot \boldsymbol{v}^\pm} \qquad \text{and in return} \qquad \partial_{z_k} \frac{1}{\|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}^\pm\|} = \frac{n_k^\pm}{\|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}^\pm\|^2 (1 \pm \boldsymbol{n}^\pm \cdot \boldsymbol{v}^\pm)}.$$

With these formulas at hand it is straightforward to compute the rest. Let us drop the superscript $\pm$ in order to make the following formulas more readable. We find

$$\partial_{z_k} \frac{1}{1 \pm \boldsymbol{n} \cdot \boldsymbol{v}} = \frac{\pm v_k + \boldsymbol{n} \cdot \boldsymbol{v}\, v_k - \boldsymbol{v}^2\, n_k \mp \boldsymbol{n} \cdot \boldsymbol{v}\, n_k}{\|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}\| (1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3} + \frac{\boldsymbol{n} \cdot \boldsymbol{a}\, n_k}{(1 \pm \boldsymbol{n} \cdot \boldsymbol{v})^3}.$$

Let us denote the three integrands on the right-hand side of Equation (4.51p.68) by $\boxed{30}$, $\boxed{31}$ *and* $\boxed{32}$. *Plugging in the above equations we find*

$$\boxed{30} = \frac{\boldsymbol{n}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2} + \frac{\pm\boldsymbol{v} + \boldsymbol{n}\cdot\boldsymbol{v}\,\boldsymbol{v} - \boldsymbol{v}^2\,\boldsymbol{n} \mp \boldsymbol{n}\cdot\boldsymbol{v}\,\boldsymbol{n}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3} + \frac{\boldsymbol{n}\cdot\boldsymbol{a}\,\boldsymbol{n}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3},$$

$$\begin{aligned}\boxed{31} =& \frac{-\boldsymbol{n}\cdot\boldsymbol{v}\,\boldsymbol{v}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2} + \frac{\mp\boldsymbol{v}^2\,\boldsymbol{v} \pm (\boldsymbol{n}\cdot\boldsymbol{v})^2\,\boldsymbol{v}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3}\\ &+ \frac{-\boldsymbol{n}\cdot\boldsymbol{a}\,\boldsymbol{n}\cdot\boldsymbol{v}\,\boldsymbol{v}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3} + \frac{\pm\boldsymbol{n}\cdot\boldsymbol{a}\,\boldsymbol{v} \pm \boldsymbol{n}\cdot\boldsymbol{v}\,\boldsymbol{a}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2}\end{aligned}$$

and

$$\boxed{32} = \frac{-\boldsymbol{a}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})}.$$

These three terms add up to the right-hand side of (4.45p.66). Furthermore, let us denote the integrand of the right-hand side of Equation (4.52p.68) by $\boxed{33}$, *then*

$$\begin{aligned}\boxed{33} =& \frac{-\boldsymbol{n}\wedge\boldsymbol{v}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2} + \frac{\boldsymbol{v}^2\,\boldsymbol{n}\wedge\boldsymbol{v} \pm \boldsymbol{n}\cdot\boldsymbol{v}\,\boldsymbol{n}\wedge\boldsymbol{v}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|^2(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3}\\ &+ \frac{-\boldsymbol{n}\cdot\boldsymbol{a}\,\boldsymbol{n}\wedge\boldsymbol{v}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^3} + \frac{\pm\boldsymbol{n}\wedge\boldsymbol{a}}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}\|(1 \pm \boldsymbol{n}\cdot\boldsymbol{v})^2}\end{aligned}$$

which after appropriate insertion of factors of the form $\boldsymbol{n}\wedge\boldsymbol{n} = 0$ *gives the right-hand side of (4.46p.66).*

Computation 5.4. *We only consider the case for* $\varrho \in C_c^\infty$. *Substitution of* ϱ *by* $D^\alpha\varrho \in C_c^\infty$ *for any multi-index* $\alpha \in \mathbb{N}^3$ *yields the desired estimates for the general case for which only the constants* C_{25} *change according to Equation (5.8). It suffices to show that for* $n \leq 2$ *there exist positive constants* ${C_{33}}^{(n)} < \infty$ *such that*

$$\left|\int d^3z\,\frac{\varrho(\boldsymbol{z})}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}_t\|^n}\right| \leq \frac{{C_{33}}^{(n)}}{1+\|\boldsymbol{x}-\boldsymbol{q}_t\|^n}. \tag{5.7}$$

Since $\varrho \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ *there exists a* $R < \infty$ *such that* $\operatorname{supp}\varrho \subseteq B_R(0)$. *So for some* $\epsilon > 0$ *we have*

$$\left|\int d^3z\,\frac{\varrho(\boldsymbol{z})}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}_t\|^n}\right| \leq \sup_{\boldsymbol{y}\in\mathbb{R}^3}|\varrho(\boldsymbol{y})|\left[\int_{B_\epsilon^c(0)\cap B_R(\boldsymbol{x}-\boldsymbol{q}_t)} d^3y\,\frac{1}{\|\boldsymbol{y}\|^n} + \int_{B_\epsilon(0)\cap B_R(\boldsymbol{x}-\boldsymbol{q}_t)} d^3y\,\frac{1}{\|\boldsymbol{y}\|^n}\right] =: \boxed{34} + \boxed{35}$$

which involved a substitution in the integration variable, and we have used the notation $B_\epsilon^c(0) := \mathbb{R}^3 \setminus B_\epsilon(0)$. *For* $\boldsymbol{x} \in B_{R+\epsilon}^c(\boldsymbol{q}_t)$ *the term* $\boxed{35}$ *is zero and*

$$\boxed{34} \leq \frac{\sup_{\boldsymbol{y}\in\mathbb{R}^3}|\varrho(\boldsymbol{y})|\,\frac{4}{3}\pi R^3}{(\|\boldsymbol{x}-\boldsymbol{q}_t\|-R)^n} =: \frac{C_{34}}{(\|\boldsymbol{x}-\boldsymbol{q}_t\|-R)^n}. \tag{5.8}$$

On the other hand for $\boldsymbol{x} \in B_{R+\epsilon}(\boldsymbol{q}_t)$ *and* $\epsilon < R$ *we find*

$$\boxed{34} \leq \frac{C_{34}}{\epsilon^n} \text{and} \qquad \boxed{35} \leq 4\pi\int_0^\epsilon dr\,r^{2-n} =: {C_{35}}^{(n)}.$$

Plugging these estimates in the left-hand side of (5.7) we find

$$\left|\int d^3z\,\frac{\varrho(\boldsymbol{z})}{\|\boldsymbol{x}-\boldsymbol{z}-\boldsymbol{q}_t\|^n}\right| \leq \begin{cases}\frac{C_{34}}{\epsilon^n} + {C_{36}}^{(n)} & \text{for } \boldsymbol{x} \in B_{R+\epsilon}(\boldsymbol{q}_t)\\ \frac{C_{34}}{(\|\boldsymbol{x}-\boldsymbol{q}_t\|-R)^n} & \text{otherwise.}\end{cases}$$

Clearly one finds an appropriate constant ${C_{37}}^{(n)} < \infty$ *such that*

$$\left| \int d^3z \, \frac{\varrho(z)}{\|\boldsymbol{x} - \boldsymbol{z} - \boldsymbol{q}_t\|^n} \right| \leq \frac{{C_{37}}^{(n)}}{1 + \|\boldsymbol{x} - \boldsymbol{q}_t\|^n}.$$

This together with (4.56_{p.70}) *gives* $C_{25} := 2({C_{37}}^{(n=2)} + {C_{37}}^{(n=1)})$.

5.4 Missing Lemmas and Proofs for Section 4.4.2

Lemma 5.5. *Let A be the operator defined in Definitions 3.3p.19. Furthermore, for some $n \geq 1$ let $t \mapsto A^k \varphi_t$ be a continuous map $\mathbb{R} \to D_w(A^{n-k}) \subset \mathcal{H}_w$ for $0 \leq k \leq n$. Then it is true that:*

$$A^k \int_0^t ds \; \varphi_s = \int_0^t ds \; A^k \varphi_s \qquad \text{and} \qquad W_r \int_0^t ds \; \varphi_s = \int_0^t ds \; W_r \varphi_s$$

for all $t, r \in \mathbb{R}$.

Proof. First, we show the equality on the left-hand side of the claim. Since the integrands are continuous, we can define the integrals as $\mathcal{H}_w$ limits $N \to \infty$ of the Riemann sums for all $t \in \mathbb{R}$

$$\sigma_N^k = \frac{t}{N} \sum_{j=1}^{N} A^k \varphi_{\frac{t}{N} j}$$

for $k \leq n$. By Lemma 3.23p.30 the operator A is closed on $D_w(A)$ so that A^k is closed on $D_w(A^k)$. Since $(\sigma_N^k)_{N \in \mathbb{N}}$ converge to, say, σ^k in $\mathcal{H}_w$, we get $\sigma^0 \in D_w(A^k)$ and $A^k \sigma^0 = \sigma^k$ which is exactly the equality on the left-hand side of the claim.

Second, we show the equality on the right-hand side. Therefore, for any $r, t \in \mathbb{R}$ we get

$$\frac{d}{dr} W_{-r} \int_0^t ds \; W_r \varphi_s = -A W_{-r} \int_0^t ds \; W_r \varphi_s + W_{-r} \int_0^t ds \; A W_r \varphi_s = 0$$

by the equality on the left-hand side of the claim. Hence,

$$W_{-r} \int_0^t ds \; W_r \varphi_s = \int_0^t ds \; \varphi_s \qquad \text{or} \qquad W_r \int_0^t ds \; \varphi_s = \int_0^t ds \; W_r \varphi_s.$$

This proves the right-hand side of the claim and concludes the proof. □

Corollary 5.6. *Let A and J be the operators defined in Definition 4.33p.73, i.e. the projection of A and J to field space. Furthermore, for some $n \geq 1$ let $t \mapsto \mathsf{A}^k F_t$ be a continuous map $\mathbb{R} \to D_w(\mathsf{A}^{n-k}) \subset \mathcal{F}_w$ for $0 \leq k \leq n$. Then it is also true that:*

$$\mathsf{A}^k \int_0^t ds \; F_s = \int_0^t ds \; \mathsf{A}^k F_s \qquad \text{and} \qquad \mathsf{W}_r \int_0^t ds \; F_s = \int_0^t ds \; \mathsf{W}_r F_s$$

for all $t, r \in \mathbb{R}$.

Proof. By Definition 3.3p.19 we have $A = (0, \mathsf{A})$ on $D_w(A)$ so that $W_t = (\mathrm{id}_{\mathcal{P}}, \mathsf{W}_t)$ on $D_w(A)$ for all $t \in \mathbb{R}$. Apply Lemma 5.5 and $t \mapsto (0, F_t)$ and project to field space $\mathcal{F}_w$ to yield the claim. □

*Proof of Lemma 4.50*p.83. Since by (i) the sequence $(\boldsymbol{F}_n)_{n\in\mathbb{N}}$ is uniformly bounded in the Hilbert space $\mathcal{H}_w^\triangle$ the Banach-Alaoglu Theorem states that it has a weakly convergent subsequence in $H_w^\triangle$ which we denote by $(\boldsymbol{G}_n)_{n\in\mathbb{N}}$. Let the convergence point be denoted by $\boldsymbol{F} \in H_w^\triangle$. We have to show that under the assumptions this subsequence is also strongly convergent in L_w^2. The idea is the following: Far away from the origin (ii) makes sure that the formation of spikes is suppressed while oscillations can be controlled by the Laplace which behave nicely by (i). So let $\epsilon > 0$ and divide the integration domain for $\tau > 0$

$$\|\boldsymbol{F} - \boldsymbol{G}_n\|_{L_w^2} \leq \|\boldsymbol{F} - \boldsymbol{G}_n\|_{L_w^2(B_\tau(0))} + \|\boldsymbol{F} - \boldsymbol{G}_n\|_{L_w^2(B_\tau^c(0))}.$$

Now by assumption (ii) we know for τ large enough it holds for all $n \in \mathbb{N}$ that

$$\|\boldsymbol{F} - \boldsymbol{G}_n\|_{L_w^2(B_\tau^c(0))} < \epsilon.$$

By Lemma 3.15p.26 the norm on $L_w^2(B_\tau(0))$ is equivalent to the one on $L^2(B_\tau(0))$ so that it suffices to show that there is an $N \in \mathbb{N}$ such that

$$\|\boldsymbol{F} - \boldsymbol{G}_n\|_{L^2(B_\tau(0))} < \epsilon \tag{5.9}$$

for all $n > N$. Before we do this let us introduce a tool to control possible oscillations. We define for any $\boldsymbol{H} \in L_{loc}^1$ the heat kernel

$$(e^{\triangle t}\boldsymbol{H})(\boldsymbol{x}) = h_t * \boldsymbol{G} := \frac{1}{(4\pi t)^{\frac{3}{2}}} \int d^3y \, \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4t}\right) \boldsymbol{H}(\boldsymbol{y}).$$

Denoting the Fourier transformation $\widehat{\cdot}$ and using Plancherel's Theorem we find

$$\begin{aligned} \|(1 - e^{\triangle t})\boldsymbol{H}\|_{L^2}^2 &= \|(1 - \widehat{h_t})\widehat{\boldsymbol{H}}\|_{L_w^2}^2 = \int d^3k \, \|\widehat{\boldsymbol{H}}(\boldsymbol{k})\|^2 \left(1 - \exp(-\boldsymbol{k}^2 t)\right)^2 \\ &\leq |t| \, \|k^2 \widehat{\boldsymbol{H}}\|_{L_w^2}^2 = |t| \, \|\triangle \widehat{\boldsymbol{H}}\|_{L_w^2}^2. \end{aligned} \tag{5.10}$$

Hence, we expand by triangle inequality

$$\begin{aligned} \|\boldsymbol{F} - \boldsymbol{G}_n\|_{L^2(B_\tau(0))} &\leq \|(1 - e^{\triangle t})\boldsymbol{G}_n\|_{L^2(B_\tau(0))} + \|(1 - e^{\triangle t})\boldsymbol{F}\|_{L^2(B_\tau(0))} + \|(1 - e^{\triangle t})(\boldsymbol{F} - \boldsymbol{G}_n)\|_{L^2(B_\tau(0))} \\ &=: \boxed{36} + \boxed{37} + \boxed{38}. \end{aligned}$$

We start with the first term. Using the estimate (5.10) for small enough $t > 0$ yields

$$\boxed{36} \leq \sqrt{t} \, \|\triangle G_n\|_{L^2(B_\tau(0))} < \frac{\epsilon}{3}$$

because $(\triangle G_n)_{n\in\mathbb{N}}$ is uniformly bounded in L_w^2 by (i). The same procedure for the second term yield

$$\boxed{36} \leq \sqrt{t} \, \|\triangle G_n\|_{L^2(B_\tau(0))} \leq \sqrt{t} \, \liminf_{n\to\infty} \|\triangle G_n\|_{L^2(B_\tau(0))} < \frac{\epsilon}{3}$$

where we use the lower semi-continuity of the norm and again (i). By weak convergence in L_w^2 we get the pointwise convergence for all $\boldsymbol{x} \in \mathbb{R}^3$ that

$$\left\| \mathbb{1}_{B_\tau(0)}(\boldsymbol{x}) \left[e^{\triangle t}(\boldsymbol{F} - \boldsymbol{G}_n) \right](\boldsymbol{x}) \right\|_{\mathbb{R}^3} \xrightarrow[n\to\infty]{} 0.$$

Furthermore, by Schwarz's inequality we get the estimate

$$\left\| \mathbb{1}_{B_\tau(0)}(\boldsymbol{x}) \left[e^{\triangle t}\boldsymbol{G}_n \right](\boldsymbol{x}) \right\|_{\mathbb{R}^3} \leq \mathbb{1}_{B_\tau(0)} \|h_t\|_{L^2(B_\tau(0))} \|\boldsymbol{G}_n\|_{L^2(B_\tau(0))}.$$

Again the right-hand side is uniformly bounded by (i). Hence, by dominated convergence $(e^{\triangle t}\boldsymbol{G}_n)_{n\in\mathbb{N}}$ converges in $L^2(B_\tau(0))$ to $e^{\triangle t}\boldsymbol{F}$. Therefore, for an $N \in \mathbb{N}$ large enough we have

$$\boxed{39} = \|(1 - e^{\triangle t})(\boldsymbol{F} - \boldsymbol{G}_n)\|_{L^2(B_\tau(0))} \leq \frac{\epsilon}{3}.$$

The estimate for the three terms prove claim (5.9). Thus, we conclude that $(G_n)_{n\in\mathbb{N}}$ is a strongly convergent subsequence of $(F_n)_{n\in\mathbb{N}}$ in L_w^2. □

Part II

Pair Creation

Chapter 6

Absorber Subsystems and Effective Fields

Let us consider a universe of N charges and N fields as described in Chapter 2p.7 which are ruled by the ML-SI equations for initial conditions, i.e. position and momenta of the charges and initial fields $F_i(x)|_{x^0=0}$ at time $x^0 = 0$ such that the absorber assumption (2.9p.9) holds. As explained, we may borrow the argument of Wheeler and Feynman to deduce an effective force (2.12p.10) for the ith charge, i.e.

$$m_i \dddot{z}_i^{\mu}(\tau) = e_i \sum_{j \neq i} F_j^{\mu\nu}(z_i(\tau))\dot{z}_{i,\nu}(\tau) = e_i \left[\sum_{j \neq i} \left(F_{j,0} + F_{j,-} \right) + \frac{1}{2} \left(F_{i,-} - F_{i,+} \right) \right]^{\mu\nu} (z_i(\tau))\, \dot{z}_{i,\nu}(\tau). \tag{2.12}$$

Subsystems of the omnipresent absorber medium

Now, the computation of the world line of the ith charge would require the knowledge of the initial conditions of the universe. Keep in mind that the absorber medium is made out of a large number of N charges – as we have argued we understand the absorber assumption to be arising from a thermodynamical argument. Furthermore, one probably may not allow for big regions of charge-free space within the absorber medium. In this sense, the absorber medium must be omnipresent and it becomes difficult to talk about subsystems of only a few charges in the universe as they cannot be completely isolated in general from the rest of the absorber medium. This raises the question how we could possibly treat subsystems (e.g. an charge in the Millikan experiment) of the universe with a small amount of charges with respect to N.

Definition of an equilibrium state

Up to now we cannot deduce a satisfactory answer by a physical argument using first principles, but based on empiricism and induction we are able to explain the mechanism which leads to an effective description of subsystems. The first observation is that despite the omnipresent absorber medium there are vast regions of space in which there are apparently neither charges nor free fields. In order to explain this observation we assume that the absorber medium is close to an *equilibrium state* which is defined through the condition that for every $1 \leq i \leq N$

$$\sum_{j \neq i} \left(F_{j,0} + F_{j,-} \right) \approx 0 \tag{6.1}$$

Small deviations from equilibrium and the equilibrium assumption

near the ith world line. Again "≈" stands for equality in the thermodynamic limit $N \to \infty$ which for simplicity we replace by "=". It is important to note that condition (6.1) would not render any implications on the absorber assumption (2.9p.9) which in contrast does not need to hold near the ith world line but only in some distance to all the N charges. Given such an equilibrium state one charge, e.g. the electron in the receptor in one of our eyes, does not "see" any other charges apart from the radiation reaction term. The small portions of charged matter and free fields we actually do "see" in our universe can be considered a small deviation from this equilibrium state (based on cosmologic models one approximates the amount of matter we can see to be only 4% of all matter in our universe). So it seems natural to rather not describe the motion of the N charges and N fields of the whole universe but to describe only the motion of these small portions of charged matter and free fields which we deviate from this equilibrium state while we neglect the rest. We shall call the assumption that our universe is close to an equilibrium state defined by (6.1) the *equilibrium assumption*.

Effective description of subsystems

In order to become more concrete let us consider an experiment involving $n \ll N$ charges that deviated from equilibrium which is placed inside a Faraday cage (or is done under conditions like in the Millikan experiment) – imagine any typical experiment in which the n charges constitute the apparatuses like capacitors, solenoids, wires, etc. as well as the charges to be examined (loosely speaking any charge we can "see"). Let us give these n charges labels $i \in I$ for some index set $I \subset \{1, \ldots, N\}$ with $|I| = n$. Regarding the n charges as a small deviation from the equilibrium state we may introduce an effective field

$$F_i^{\text{eff}} := \sum_{j \neq i} \left(F_{j,0} + F_{j,-} \right) = \sum_{j \in I \setminus \{i\}} \left(F_{j,0} + F_{j,-} \right) \tag{6.2}$$

so that in order to compute the world line of the charge $i \in I$ by

$$m_i \ddot{z}_i^\mu(\tau) = e_i \sum_{j \neq i} F_j^{\mu\nu}(z_i(\tau)) \dot{z}_{i,\nu}(\tau) = e_i \left[F_i^{\text{eff}} + \frac{1}{2} \left(F_{i,-} - F_{i,+} \right) \right]^{\mu\nu} (z_i(\tau))\, \dot{z}_{i,\nu}(\tau)$$

we only need to account for the $I \setminus \{i\}$ other charges. By the assumption that only the n charges deviate from an equilibrium state, the portions of the rest of the absorber medium, which happen to be also in the interior of the Faraday cage, can also be neglected. In this way the equilibrium assumption $(6.1_{\text{p.105}})$ gives rise to effective equations of motion for the n degrees of freedom. A justification of such an effective field hypothesis and its regime of validity, however, must at some point be given by statistical mechanics.

Creation processes as result of the effective description

In order for such an effective description to hold over bigger space-time regions we have to allow the effective field to change in time, not only because the fields on the right-hand side of (6.2) change in time but also because the rest of the absorber medium changes in time. To understand this better let us consider a more artificial example: Let Λ_i denote the space-time neighborhood of the ith world line during some time interval $[t_0, t_1]$. We imagine a configuration of world lines of the N charges such that the effective field fulfills the equilibrium assumption

$$F_i^{\text{eff}}(x) = \sum_{j \neq i} \left(F_{j,0} + F_{j,-} \right)(x) = 0, \text{ for } x \in \Lambda_i. \tag{6.3}$$

There the ith charge effectively obeys the equation

$$m_i \ddot{z}_i^\mu(\tau) = e_i \frac{1}{2} \left[F_{i,-} - F_{i,+} \right]^{\mu\nu} (z_i(\tau))\, \dot{z}_{i,\nu}(\tau) = \frac{2}{3} e_i^2 \left(\dddot{z}_i^\mu(\tau) \dot{z}_i^\nu(\tau) - \dddot{z}_i^\nu(\tau) \dot{z}_i^\mu(\tau) \right) \dot{z}_{i,\nu}(\tau)$$

in space-time region Λ_i which is known as the Lorentz-Dirac equation. So apart from the radiation damping term, the ith charge as a spectator does not "see" any other charges when passing through the space-time region Λ_i. Let furthermore $\widetilde{\Lambda}_i$ denote the space-time neighborhood of the ith world line during the later time interval $[t_1, t_2]$, and imagine further that only one world line $\tau \mapsto z_k(\tau)$ of the rest of the absorber medium deviates slightly from the world line that would render (6.3) to hold also in $\widetilde{\Lambda}_i$. Hence, the kth world line induces a small deviation from the equilibrium state; see Figure $6.1_{\text{p.107}}$. Instead of falling back to the ML-SI equations of N

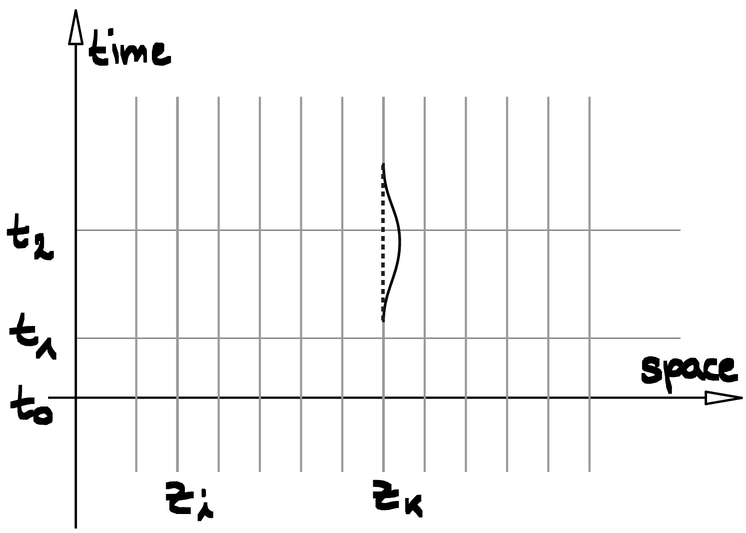


Figure 6.1: World lines of charges in the discussed example. At t_0 the charges are in an equilibrium state. On the time interval $[t_1, t_2]$ some external force drives the kth world line away from its equilibrium world line $\widetilde{z}_k$ (dashed line). The ith charge does only "see" the disturbance (black dashed as well as solid line) since the net forces from the gray world lines vanish (where we assume that the disturbance has a negligible effect on the other world lines). After time t_2 the kth charge relaxes back to its equilibrium position.

charges and fields and forgetting about the equilibrium assumption, it is sensible to introduce the retarded field $\widetilde{F}_{k,-}$ of an imaginary world line $\tau \mapsto \widetilde{z}_k$ which would satisfy the condition

$$\sum_{j\neq i, j\neq k} \left(F_{j,0} + F_{j,-}\right)(x) + \widetilde{F}_{k,-}(x) = 0, \text{ for } x \in \widetilde{\Lambda}_i.$$

By introducing this imaginary field, the effective field hypothesis then reads

$$F_i^{\text{eff}}(x) = \sum_{j\neq i} \left(F_{j,0} + F_{j,-}\right)(x) = F_{k,0} + F_{k,-} - \widetilde{F}_{k,-}, \text{ for } x \in \widetilde{\Lambda}_i \tag{6.4}$$

in which case the ith charge effectively obeys

$$m_i \ddot{z}_i^{\mu}(\tau) = e_i \sum_{j\neq i} F_j^{\mu\nu}(z_i(\tau))\dot{z}_{i,\nu}(\tau) = e_i \left[F_{k,0} + F_{k,-} - \widetilde{F}_{k,-} + \frac{1}{2}\left(F_{i,-} - F_{i,+}\right)\right]^{\mu\nu} (z_i(\tau))\, \dot{z}_{i,\nu}(\tau)$$

in the space-time region $\widetilde{\Lambda}_i$. The three new fields are the free field of the kth charge, the retarded field of the kth charge and another one due to an imaginary charge with opposite sign on the world line $\tau \mapsto \widetilde{z}_k(\tau)$. Therefore, a good hypothesis for the effective field valid in both space-time regions $\Lambda_i \cup \widetilde{\Lambda}_i$ has to allow for creation (and vice versa the annihilation) of fields to agree with the actual value of the sum of retarded fields $\sum_{j\neq i}\left(F_{j,0} + F_{j,-}\right)$ appearing in the fundamental ML-SI equations (2.12p.105). Note that the left-hand side of (6.4) is a field which has the N world lines as charge sources where the right-hand side has only the imaginary world-line $\tau \mapsto \widetilde{z}_k(\tau)$ as a charge source. This, however, implies no contradiction because the equality with the right-hand side holds only in a neighborhood of the ith world line which is distant to the $j \neq i$ sources.

Such an creation or annihilation process is an artefact of the effective description of the many degrees of freedom through a lot fewer dynamical degrees of freedom by means of the equilibrium assumption. The fundamental theory, however, is always about N charges with their N

fields obeying the ML-SI equations. Nothing is ever created or annihilated, only the effective description was adapted. Moreover, the described creation process is a subjective perception of the ith charge only (therefore F_i^{eff} depends on the index i). Let us summarize:

1. A universe governed by the ML-SI equations which is close to an equilibrium state in the sense of ($6.1_{\text{p.105}}$) allows for the introduction of an effective external field in order to describe subsystems of the universe.
2. Effectively only the small deviations from an equilibrium state ($6.1_{\text{p.105}}$) need to be described while the rest of the absorber medium can be neglected.
3. Under certain circumstances such an effective description of the absorber medium is forced to involve creation, respectively, annihilation processes.

Relaxation to an equilibrium state and the quantum analogue

A small disturbance as introduced by means of the deviation of the kth world line in the above classical example from its equilibrium position would be expected to quickly relax back to an equilibrium position. Therefore, in order to observe such a creation process one would need to separate the kth charge from the world line $\tau \mapsto \widetilde{z}_k(\tau)$ significantly by external forces. The relaxation back to an equilibrium state is more complicated in quantum theory. As an analogue to the classical example, let us consider the quantum theoretic description of N electrons which obey the N particle Dirac time-evolution. As it is well-known, the spectrum of the free one-particle Dirac operator $H^0 = -i\alpha\cdot\nabla+\beta m$ is $(-\infty,-m]\cup[+m,+\infty)$, and thus, the two components of the spectrum give rise to a splitting of the one-particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3,\mathbb{C}^4)$, i.e. the space of square integrable $\mathbb{C}^4$ valued functions on $\mathbb{R}^3$, into two spectral subspaces $\mathcal{H} = \mathcal{H}_-\oplus\mathcal{H}_+$. The absorber medium of N electrons is then represented by a wave function in the antisymmetric N-fold tensor product of $\mathcal{H}$.

One point needs to be clarified before we continue: For a finite number of electrons one expects dynamical instabilities of the electrons when coupling them to an additional dynamical degree of freedom such as the electrodynamic field. Instabilities may arise since by continuous emission of radiation the finitely many electrons could cascade deeper and deeper to more negative energy energy states which are unoccupied. However, for a direct electron-electron interaction, e.g. a Coulomb pair potential, instable dynamics are not to be expected. This is also the case for the Wheeler-Feynman-like action-at-a-distance we have in mind and which is later sketched in Chapter $8_{\text{p.159}}$.

Although we do not yet have a fully interacting quantum theory at hand, energetic considerations suggest to assume that initially at time t_0 an equilibrium state, i.e. a state for which the net interaction between the N electrons vanishes (cf. in the classical example ($6.3_{\text{p.106}}$)), is described by a N particle wave function built from tensor components in $\mathcal{H}_-$. Whenever we deviate only slightly from such an equilibrium state we may as a first approximation forget about the electron-electron interactions. In order to model a disturbance which deviates, for example, one electron out of its equilibrium position we introduce an external four-vector potential $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_0, -\vec{\mathsf{A}})$ being non-zero only during the time interval $[t_1,t_2]$; compare the classical example above. Owing to the Dirac time-evolution subject to this external potential, transitions of tensor components of the N electron wave function between the negative and the positive spectral subspace are allowed only if the external potential is non-zero. Let us assume that the external potential causes such a transition and that when it is switched off at time t_2 a tensor component of the N particle wave function is still in $\mathcal{H}_+$. Since the free Dirac time-evolution forbids further transitions between the spectral subspaces, this tensor component will remain in $\mathcal{H}_+$ forever, thus, leaving a defect (or hole) in the initial equilibrium state of N electrons. As in the classical example, it is now convenient instead of falling back to describing all N electrons to only describe the small deviation with respect to the initial equilibrium state; analogous to

($6.4_{p.107}$). In fact, the equilibrium assumption, i.e. that the deviations from equilibrium are small, is the only means to justify the negligence of electron-electron interaction (after all we intend to describe an electron gas!). Such a description then naturally leads to *pair creation* and vice versa to *pair annihilation* by the same mechanism as explained in the classical example. If we had also introduced an interaction between the electrons analogous to the classical ML-SI equations, a decay to the negative spectral subspace would then again be possible, however, in comparison to the classical case the relaxation back to the initial equilibrium state is more complex. By the Pauli exclusion principle a transition from one tensor component in $\mathcal{H}_+$ of the N particle wave function back to $\mathcal{H}_-$ only occurs if a compatible state in $\mathcal{H}_-$ is unoccupied, and this becomes more improbable the larger N is. Finally, as in the classical example, it is important to stress that the number of pairs present at some time t is in general a subjective impression of the spectator charge and has no meaning as an absolute value. Only under the assumption that the state at time t_0 as well as the state at time t_2 (when the external field is again zero) are small deviations of the same equilibrium state, the number of pairs with respect to this equilibrium state gets an objective meaning. Then any spectator charge of the rest of the absorber medium would agree on the same amount of pairs.

To our understanding this effective description was Dirac original idea:

> Admettons que dans l'Univers tel que nous le connaissons, les états d'energie négative soient presque tous occupés par des électrons, et que la distribution ainsi obtenue ne soit pas accessible à notre observation à cause de son uniformité dans toute l'etendue de l'espace. Dans ces conditions, tout état d'energie négative non occupé représentant une rupture de cette uniformité, doit se révévler à observation comme une sorte de lacune. Il es possible d'admettre que ces lacunes constituent les positrons.
>
> P.A.M. Dirac, Théorie du Positron (1934), in Selected Papers on Quantum Electrodynamics, Ed. J. Schwinger, Dover Pub. (1958)

Since we have not found a professional translation, we try to give an unprofessional one to the best of our knowledge: "Let us assume that in the universe as we know it the negative energy states are almost all occupied with electrons and that the corresponding distribution [of the electrons] is not accessible through our observation because of its uniformity in the vastness of space. Under such conditions the whole non-occupied negative energy state representing a disturbance of this uniformity reveals itself as a kind of hole to our observation. It is possible to assume that these holes constitute the positrons."

We modeled the uniformness in the classical example from above by the equilibrium assumption ($6.3_{p.106}$) while in the quantum analogue we simply omitted any interaction between the electrons. Furthermore, as the net interaction is zero the state of the absorber medium of the N electrons with their fields are *physically inaccessible* to our observation. In his book he writes earlier:

> The exclusion principle will operate to prevent a positive-energy electron ordinarily from making transitions to states of negative energy. It will be possible, however, for such an electron to drop into an unoccupied state of negative energy. In this case we should have an electron and positron disappearing simultaneously, their energy being emitted in the form of radiation. The converse process would consist in the creation of an electron and a positron from electromagnetic radiation.
>
> P.A.M. Dirac , Theory of the positron, in: The Principles of Quantum Mechanics, Oxford (1930)

By this reasoning the relaxation back to initial equilibrium is more complex than in the classical case. In fact, Dirac's theory predicted the existence and properties of positrons, pair creation and pair annihilation, which were verified by Anderson [And33] a short time later. We summarize:

1. Under the equilibrium assumption the effective description gives rise to creation and annihilation processes in classical as well as quantum theory.
2. The relaxation back to equilibrium in quantum theory is more complex than in the classical analogue.
3. Only under the equilibrium assumption the quantum theoretic description of the Dirac sea which neglects the electron-electron interaction makes sense as an approximation.

Overview of the mathematical results

As we have addressed the modeling of the absorber medium as well as the need of an effective description of the universe of N charges and N fields close to an equilibrium state indicates that we need to regard a thermodynamic limit $N \to \infty$ as it is common in statistical mechanics. This way we also make sure that a state with finitely many pairs is really a small deviation from equilibrium as there are always infinitely many electrons at their equilibrium positions. For finite N this border could not be drawn so sharply. Such mathematical idealizations are typical for thermodynamic limits in statistical mechanics, however, they should always be read as: N extremely large but finite. This is where our mathematical work starts in Chapter 7p.111: The main goals are the construction of an absorber medium consisting of infinitely many electrons, the implementation of a quantum theoretic time-evolution subject to an external four-vector potential and the computation of the induced pair creation rates. The construction of the absorber medium is presented in Section 7.2p.116 together with the mathematical framework needed to construct a time-evolution. We give the time-evolution in Section 7.3p.130. This part ends with the computation of pair-creation rates in Section 7.4p.151 and with a conclusion and outlook of the next important open questions concerning this topic.

Chapter 7

Time-Evolution of Dirac Seas in an External Field

7.1 Chapter Overview and Results

Terminology: dirac sea = absorber medium, equilibrium state = vacuum state

Though we have in mind an effective description of the absorber medium by means of the discussed equilibrium assumption for $N \to \infty$ many electrons, we shall use the common terminology and write instead of the absorber medium: *Dirac sea*, and furthermore, refer to a Dirac sea in equilibrium, defined by the condition that all net interactions between the electrons vanish as described in the introductory chapter 6p.105, as the *vacuum state* or *vacuum vector*. We want to emphasize from the very beginning that the equilibrium condition does not single out one particular vacuum state. Even in the classical analogue many different configurations of charges and fields satisfy (6.1p.105). Furthermore, as Dirac points out, by the vanishing of the net interactions of the electrons due to their uniform distribution the vacuum is inaccessible to our observation. Hence, we have to formulate the time-evolution of the Dirac sea without singling out a specific vacuum vector. This idea stands in contrast to standard quantum field theory.

State of the Art and the Problem. In the language of quantum field theory the Dirac sea is represented in the so-called second quantization procedure by the "vacuum vector" on which two types of creation operators act. Those creating electrons and those creating positrons. This way one implements Dirac's idea that one only considers the "net description of particles: electrons and positrons" and neglects what is going on "deep down in the sea", assuming that nothing physically relevant happens in there. This is close to the idea of the introductory Chapter 6p.105 that we should effectively only describe small deviations away from the equilibrium states which we shall discuss later in detail. The Hilbert space for this infinitely many particle system is the *Fock space* built by successive applications of creation operators on the vacuum.

In the case of zero external fields based on energetic considerations we argued that the vacuum state should be formed by a wedge product of one-particle wave functions in $\mathcal{H}_-$. However, in the presence of an external field the choice of $\mathcal{H}_-$ is not obvious at all. Furthermore, Dirac's invention and likewise quantum field theory are plagued by a serious problem: As soon as an electromagnetic field $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_0, -\vec{\mathsf{A}})$ enters the Dirac equation, i.e. as soon as "interaction is turned on", one has generically transitions of negative energy wave functions to positive energy wave functions, i.e. pair creation and pair annihilation – for a mathematical proof of pair creation in the adiabatic regime see [PD08b, PD08a]. While pair creation and annihilation is an observed phenomenon, it nevertheless has mathematically a devastating side effect. Figuratively speaking, the negative energy states are "rotated" by the external field and thus develop components in the positive energy subspace. Thus, the Dirac sea containing infinitely many particles generically produces under the influence of an external field infinitely many transitions between $\mathcal{H}_-$ and $\mathcal{H}_+$ as soon as the field acts. The resulting state does not anymore belong to the Fock space and there is no reason to hope that in general a lift of the one-particle Dirac time-evolution to this Fock space exists.

Shale-Stinespring condition explained

In the mathematical language this problem is rephrased as follows. The one-particle Dirac time evolution $U^{\mathsf{A}}(t_1,t_0) : L^2(\mathbb{R}^3,\mathbb{C}^4) \to L^2(\mathbb{R}^3,\mathbb{C}^4)$ from time t_0 to time t_1 induced by the Dirac hamiltonian $H^{\mathsf{A}(t)} = -i\alpha\cdot(\nabla - \vec{\mathsf{A}}(t)) + \beta m + \mathsf{A}(t)^0$ for an external field A can be lifted (however, non uniquely) if and only if the two non-diagonal parts $U^{\mathsf{A}}(t_1,t_0)_{\pm\mp} := P_\pm U^{\mathsf{A}}(t_1,t_0)P_\mp$ are Hilbert-Schmidt operators (the elementary charge e is included in the four-vector field A). Here $P_\pm$ are the spectral projectors to the spectral subspaces $\mathcal{H}_\pm$, and the Hilbert-Schmidt property means that $U^{\mathsf{A}}(t_1,t_0)_{\pm\mp}$ has a kernel which is square integrable.

This condition can be intuitively understood as follows: Let us regard the time-evolution $U^{\mathsf{A}}(t_1,t_0)$ as a map from $\mathcal{H}_-\oplus\mathcal{H}_+$ to $\mathcal{H}_-\oplus\mathcal{H}_+$ and write it in matrix form:

$$U^{\mathsf{A}}(t_1,t_0) = \begin{pmatrix} U^{\mathsf{A}}_{++}(t_1,t_0) & U^{\mathsf{A}}_{+-}(t_1,t_0) \\ U^{\mathsf{A}}_{-+}(t_1,t_0) & U^{\mathsf{A}}_{--}(t_1,t_0) \end{pmatrix}. \tag{7.1}$$

The non-diagonal terms describe pair creation and annihilation. In leading order, neglecting multiple pair creation, the squared Hilbert-Schmidt norm

$$\|U^{\mathsf{A}}_{+-}(t_1,t_0)\|^2_{I_2} = \sum_{n\in\mathbb{N}} \|U^{\mathsf{A}}_{+-}(t_1,t_0)\varphi_n\|^2_{\mathcal{H}} \tag{7.2}$$

may be interpreted as the probability to create a pair from the Dirac sea; here $(\varphi_n)_{n\in\mathbb{N}}$ denotes any orthonormal basis of $\mathcal{H}_-$.

The above condition appears in quite general settings as Shale-Stinespring condition [SS65]; see also [PS70]. It is fulfilled if and only if the magnetic part $\vec{\mathsf{A}}$ of the four-vector field A vanishes so that generically the Shale-Stinespring condition does not hold and second quantization of the Dirac equation following the usual quantization rules is impossible [Rui77, Rui77]. The catastrophe of infinitely many particle creations happens as long as the field is acting. The situation is better in scattering theory. Consider a scattering situation where the external field has compact space and time support. Generically, incoming wave functions of the sea are rotated out of the sea at intermediate times. However, “most” of them are essentially rotated back into the sea again when the field vanishes. In other words the vacuum is “more or less” restored, so that one expects “ingoing” states in Fock space to be transformed to “outgoing” states in Fock space. From this, one computes transition amplitudes between “in” and “out” states, describing e.g. pair creation. The creation of infinitely many pairs during the action of the field is often referred to as creation of so-called “virtual pairs”. The non-diagonal of the scattering matrix $P_\pm S^{\mathsf{A}} P_\mp$ consists of Hilbert-Schmidt operators and therefore, by Shale-Stinespring, can be lifted to Fock space [Bel75, Bel76] – however with the caveat that the lifting is only unique up to a phase.

In physics one computes finite transition amplitudes and expectation values by a way of formal perturbation series and renormalization. Within these series one rules out divergent terms coming from the virtual pair creation by physical principles like continuity equation, Lorentz and gauge invariance and in the end one needs in addition to renormalize the charge of the particles [Fey49, Sch51]; see [Dys06] for an extraordinarily nice and comprehensive exposition. In the present paper we arrive at finite transition amplitudes induced by U^{A} without renormalization which in our setting is replaced by an operation from the right introduced in equation (7.7p.115).

The implementation of a second quantized time-evolution in the presence of an external field between time-varying Fock spaces has been envisaged the first time in [FS79] and one version was given concretely in [LM96]. For three reasons we need to reconsider the construction of the time-evolution again:

1. We want to give a construction of the absorber medium, i.e. of the Dirac sea of infinitely many electrons, which is obscured in the standard Fock space construction or the abstract representation of the commutator algebras.
2. This explicit construction of the absorber medium will provide a simple picture in which the implementation of the time-evolution can be understood, and furthermore, means to naturally circumvent the addressed problem of the so-called virtual pair creation and its ill-defined renormalization program.
3. This construction provides a convenient way to identify the arbitrariness of the corresponding implementation of the time-evolution which has not been studied in [LM96] or anywhere else.

Heuristics. Let us give a heuristic description of how we construct the second quantized time-evolution of the Dirac Hamiltonian in the presence of a time-dependent, external field. Our work in this field of quantum electrodyanmics was mainly inspired by Dirac's original idea [Dir34], the work of Fierz and Scharf [FS79], Scharf's book [Sch95], and also by Pressley and Segal's book [PS88] as well as the work of Mickelsson [LM96, Mic98]. What is described in this subsection will be rigorously introduced and proven in Sections 7.2p.116 and 7.3p.130. The definitions and assertions will later be formulated in a general form.

Recall the ideas from the introductory Chapter 6p.105: A description of the time-evolution of Dirac seas subject to only an external field makes only sense under the discussed equilibrium assumption which allows to neglect all electron-electron interactions. The equilibrium assumption furthermore gives rise to a description of states which are close to a vacuum state (i.e. equilibrium state) which was defined by the condition that the electron-electron interaction vanishes – which clearly does not uniquely specify a vacuum state. Moreover, following [Dir34] we have indicated that one could represent a vacuum state as an infinite wedge product of one-particle wave functions in $\mathcal{H}_-$. The choice of $\mathcal{H}_-$ was justified by energy considerations in the case an absent external field. For non-zero external field this choice is neither intuitive nor canonical. To generalize this we shall construct vacuum states as infinite wedge products corresponding to any *polarization*, i.e. a closed subspace of $\mathcal{H}$ with infinite dimension and codimension like $\mathcal{H}_-$. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis that spans a polarization V. The wedge product of all φ_n, $n \in \mathbb{N}$, is then supposed to represent a Dirac sea whose electrons have the one-particle wave functions φ_n, $n \in \mathbb{N}$. We introduce an equivalence class $\mathcal{S} = \mathcal{S}(\varphi)$ of other representatives, namely of all sequences $\psi = (\psi_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that the $\mathbb{N} \times \mathbb{N}$-matrices

$$(\langle\psi_n, \psi_m\rangle)_{n,m\in\mathbb{N}} \qquad \text{and} \qquad (\langle\psi_n, \varphi_m\rangle)_{n,m\in\mathbb{N}}, \tag{7.3}$$

$\langle\cdot,\cdot\rangle$ denoting the inner product on $\mathcal{H}$, differing from the unity matrix only by a matrix in the trace class and thus have a determinant. In this case we write $\psi \sim \phi$. We then define the following bracket:

$$\langle\psi,\chi\rangle := \det(\langle\psi_n,\chi_m\rangle)_{n,m\in\mathbb{N}} = \lim_{k\to\infty} \det(\langle\psi_n,\chi_m\rangle)_{n,m=1,\dots,k}, \quad \psi,\chi \in \mathcal{S}. \tag{7.4}$$

With this at hand one constructs a Hilbert space $\mathcal{F}_\mathcal{S} = \mathcal{F}_{\mathcal{S}(\varphi)}$, where the bracket gives rise to the inner product. We refer to $\mathcal{F}_\mathcal{S}$ as an *infinite wedge space*. By this construction, see Definition 7.17p.124 (Infinite wedge spaces), a sequence $\psi \in \mathcal{S}$ is mapped to the wedge product $\Lambda\psi = \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \ldots \in \mathcal{F}_\mathcal{S}$. This space comprises all Dirac seas which are only small deviations from the vacuum state $\Lambda\varphi$. However, from this construction the distinction of a vacuum state among Dirac seas blurs. In fact, if we take any $\psi \sim \varphi$ we end up with $S = S(\varphi) = S(\psi)$ and the very same Fock space $\mathcal{F}_S$ which we could interpret as comprising all Dirac seas which are small deviations of the vacuum $\Lambda\psi$. This is because whether a Dirac sea is a vacuum state or not has nothing to do with the polarization V or the choice of basis φ. A vacuum state was defined by the condition that the net interaction between the electrons vanishes – and as we do not know how to model the quantum interaction between electrons yet, we have to introduce vacuum states a posteriori by hand, a point which we shall come back to when we compute pair creation with this external field model. Figuratively speaking, the $\sim$ equivalence relation or better the finiteness of the inner product only relates Dirac seas which are equal "deep down in the sea". Which Dirac seas relate to each other in this sense depends on the polarization V and on the choice of basis φ.

As addressed a fixed polarization like $\mathcal{H}_-$ causes problems in the presence of an external field. Therefore, the idea is to adapt the polarization according to the external field. Consider therefore H^A, the Dirac operator for a fixed, time-independent field A. The spectrum is in general not anymore as simple as in the free case and there is no canonical way of defining a splitting into subspaces $\mathcal{H} = \mathcal{H}_-^A \oplus \mathcal{H}_+^A$. The question how to split into subspaces becomes in particular interesting when the external field is time-dependent, in which case we denote the field by the sans serif letter A. So suppose that at time t_0 the field is zero and at a later time the field is switched on. As a first guess, one could choose $U^{\mathsf{A}}(t, t_0)\mathcal{H}_-$ as the polarization at time t. This choice, however, depends not only on the field A at time t but also on the whole history $(\mathsf{A}(s))_{s\leq t}$. Consider another field $\widetilde{\mathsf{A}}$ with $\widetilde{\mathsf{A}}(t_0) = \mathsf{A}(t_0)$ and $\widetilde{\mathsf{A}}(t) = \mathsf{A}(t)$. We shall show that the orthogonal projectors onto $U^{\mathsf{A}}(t, t_0)\mathcal{H}_-$ and $U^{\widetilde{\mathsf{A}}}(t, t_0)\mathcal{H}_-$ differ only by a Hilbert-Schmidt operator. This motivates to consider only classes of polarizations instead of polarizations proper. The equivalence relation "$\approx$" between polarizations is given by the condition that the difference of the corresponding orthogonal projectors is a Hilbert-Schmidt operator. In the polarization classes associated with $\mathsf{A}(t)$ there exists a mathematically simple representative, namely $\mathcal{H}_-^{\mathsf{A}(t)} := e^{Q^{\mathsf{A}(t)}}\mathcal{H}_-$; the operator $Q^{\mathsf{A}(t)}$ will quite naturally appear as the key object in the variant of the Born series of U^{A} that we use in Subsection 7.3.1p.130. This means that the only physical input we have is the class of all polarizations equivalent to $e^{Q^{\mathsf{A}(t)}}\mathcal{H}_-$ while single polarizations are only man-made and may merely serve as a coordinate within the polarization class. However, physically relevant expressions do not depend on coordinates.

Setting now

$$P_\pm^{\mathsf{A}(t)} := e^{Q^{\mathsf{A}(t)}} P_\pm e^{-Q^{\mathsf{A}(t)}} \tag{7.5}$$

one can expect that the non-diagonal operators $P_\pm^{A(t)} U^{\mathsf{A}}(t, t_0) P_\mp$ are both Hilbert-Schmidt operators which we prove in Subsection 7.3.1p.130. Therefore, by letting the polarization class vary appropriately with time, we obtain a time-evolution operator which fulfills the Shale-Stinespring condition. This gives rise to consider time-varying Fock spaces in order to lift the one-particle Dirac time-evolution; see also [FS79]. A related but different approach to obtain a time-evolution is given in [LM96, Mic98].

We shall later refine the equivalence relation $\approx$ in the following sense: For two polarizations V, W we define $V \approx_0 W$ to mean $V \approx W$ and that V and W have the same "relative charge". Intuitively the "relative charge" has the following meaning: Consider two states $\Lambda\varphi$ and $\Lambda\psi$ where φ and ψ are orthonormal bases of V and W, respectively. Then the relative charge is the difference of the electric charges of the physical sates represented by $\Lambda\varphi$ and $\Lambda\psi$, respectively. Mathematically the relative charge is defined in terms of Fredholm indices in Definition 7.3p.118. The use of the Fredholm index to describe the relative charge is quite frequent in the literature; see e.g. [PS88, LM96, HLS05]. The relation $\approx_0$ is also an equivalence relation on the set of polarizations, and one finds an intimate connection between this equivalence relation $\approx_0$ on the set of polarizations and the equivalence relation $\sim$ on the set the Dirac seas: Two equivalent Dirac seas span two equivalent polarizations and for every two polarizations $W \approx_0 V$ such that φ spans V there is a Dirac sea $\Lambda\psi \in \mathcal{F}_{\mathcal{S}(\varphi)}$ such that $\psi \sim \varphi$ and ψ spans W. Consequently, every wedge space can be associated with one polarization class with respect to $\approx_0$. Details are given in Section 7.2.1p.116.

On the other hand, assuming φ spans V, not all Dirac seas $\Lambda\psi$ such that ψ spans $W \approx_0 V$ are in $\mathcal{F}_{\mathcal{S}(\varphi)}$ because one can obviously find an orthonormal basis ψ of W for which $(\langle\psi_n, \varphi_m\rangle)_{n,m\in\mathbb{N}}$ differs from the identity by more than a trace class operator. Because of this we below consider operations (the operations from the right) that mediate between all wedge spaces belonging to the same polarization class with respect to $\approx_0$. These operations are needed to define later the physically relevant transition amplitudes.

On any element of $\mathcal{S}$ the action of any unitary map U on $\mathcal{H}$ is then naturally defined by having it act on each factor of the wedge product. Consequently we have a (left) operation on any $\mathcal{F}_{\mathcal{S}}$, namely $\mathcal{L}_U : \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{U\mathcal{S}}$, such that

$$\mathcal{L}_U\,(\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \ldots) = U\psi_1 \wedge U\psi_2 \wedge U\psi_3 \wedge \ldots, \quad \psi \in \mathcal{S}, \tag{7.6}$$

which then incorporates a "lift" of U as a unitary map from one wedge space to another. Now, this can of course also be done for the one-particle time-evolution $U = U^{\mathsf{A}}(t, t_0)$ for fixed times t_0 and t. However, we need to find a way to relate the Dirac seas in $\mathcal{F}_{\mathcal{S}}$ to the ones in $\mathcal{F}_{U\mathcal{S}}$ by considering the "net balance" between them. As we discussed already the physical input given are the polarization classes at times t_0 and t. We choose any orthonormal basis $\varphi(t_0)$ of $\mathcal{H}_-^{\mathsf{A}(t_0)}$ and likewise $\varphi(t)$ of $\mathcal{H}_-^{\mathsf{A}(t)}$ and denote their equivalence classes with respect to $\sim$ by $\mathcal{S}(\varphi(t_0))$ and $\mathcal{S}(\varphi(t))$. This way physical "in" and "out" states can be described by elements in $\mathcal{F}_{\mathcal{S}(\varphi(t_0))}$ and $\mathcal{F}_{\mathcal{S}(\varphi(t))}$, respectively. But in general $U\mathcal{S}(\varphi(t_0))$ will not be equal $\mathcal{S}(\varphi(t))$ so that $\mathcal{L}_U\psi^{in}$ and ψ^{out} are likely to lie in different wedge spaces. We show that the polarization classes of $\mathcal{F}_{U\mathcal{S}(\varphi(t_0))}$ and $\mathcal{F}_{\mathcal{S}(\varphi(t))}$ are the same. Therefore, the only difference between those two spaces may come from our specific choice of bases $\varphi(t_0)$ and $\varphi(t)$. In order to make them compatible we introduce another operation (from the right): For all unitary $\mathbb{N}\times\mathbb{N}$-matrices $R = (R_{nm})_{n,m\in\mathbb{N}}$, we define the operation from the right as follows. For $\psi \in \mathcal{S}$, let $\psi R := (\sum_{n\in\mathbb{N}} \psi_n R_{nm})_{m\in\mathbb{N}}$. In this way, every unitary R gives rise to a unitary map $\mathcal{R}_R : \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{\mathcal{S}R}$, such that

$$\mathcal{R}_R \Lambda\psi = \Lambda(\psi R), \quad \psi \in \mathcal{S}. \tag{7.7}$$

By construction the left operations and the operations from the right commute. We show that two such unitary matrices R, R' yield $\mathcal{F}_{\mathcal{S}R} = \mathcal{F}_{\mathcal{S}R'}$ if and only if $R^{-1}R'$ has a determinant. We show in Subsection 7.3.1p.130 that there always exists a unitary matrix R for which $U\mathcal{S}(t_0)R = \mathcal{S}(t)$. Now we have all we need to compute the transition amplitudes:

$$|\left\langle\psi^{\text{out}}, \mathcal{R}_R \circ \mathcal{L}_U\psi^{\text{in}}\right\rangle|^2. \tag{7.8}$$

The matrix R, however, is not unique because for any R' with $\det R' = 1$ one has $U\mathcal{S}(t_0)R = U\mathcal{S}(t_0)RR' = \mathcal{S}(t)$ so that the arbitrariness in R' gives rise to a phase which, however, has no effect on the transition amplitudes. The operations from the left and from the right are introduced in Subsection 7.2.2p.125 while in Subsection 7.2.3p.127 we identify the conditions under which R exists.

So far we have constructed Dirac seas, implemented their time-evolutions up to the arbitrariness of a phase and gave the corresponding transition amplitudes which are finite and unique. These transition amplitudes can now be used to compute pair creation rates if one introduces vacuum states a posteriori as we have discussed. Along the ideas of Chapter 6p.105 we compute and interpret the pair creation rates in Section 7.4p.151.

Main results

What is new? This work essentially adds three new results to the topic of second quantized Dirac time-evolution:

1. With the help of the operator $Q^{\mathsf{A}(t)}$ we show in Theorem 7.31p.132, Subsection 7.3.1p.130, that the set of all polarizations $\approx U^{\mathsf{A}}(t,t_0)\mathcal{H}_-$ equals the set of polarizations $\approx e^{Q^{\mathsf{A}(t)}}\mathcal{H}_-$. Since $Q^{\mathsf{A}(t)}$ is given by an explicit expression in contrary to $U^{\mathsf{A}}(t,t_0)$, most relevant computations simplify significantly.

2. We show in Theorem 7.38p.145, Subsection 7.3.2p.144, that polarization classes are uniquely identified by the magnetic components $\vec{\mathsf{A}}(t)$ of the field A at time t. This generalizes the case of $\vec{\mathsf{A}}(t) = 0$ regarded in [Rui77] to general $\vec{\mathsf{A}}(t)$ and will together with the wedge space construction allow to identify the arbitrariness of the implementation of the time-evolution.

3. Along the lines of Dirac's original idea in [Dir34] we construct Fock spaces with the help of infinite wedge products corresponding to polarization classes in Definition 7.17p.124, Subsection 7.2.1. As suggested by Fierz and Scharf in [FS79] we show that the Dirac time-evolution can be implemented as unitary maps between corresponding wedge spaces, i.e. between time-varying Fock spaces, in Theorem 7.41p.150, Subsection 7.3.3p.150. We conclude this work with a brief discussion of gauge transformations of the external field; see Theorem 7.44p.154, Subsection 7.5p.154.

7.2 Explicit Construction of the Absorber

7.2.1 Construction

In this section we give a rigorous construction of infinite wedge products which we described in the introduction. Throughout this work the notion *Hilbert space* stands for *separable, infinite dimensional, complex Hilbert space*. Let $\mathcal{H}$ and ℓ be Hilbert spaces with corresponding scalar products $\langle\cdot,\cdot\rangle$. For a typical example think of $\mathcal{H} = L^2(\mathbb{R}^3,\mathbb{C}^4)$ and $\ell = \ell_2(\mathbb{N})$, the space of square summable sequences in $\mathbb{C}$. The space ℓ will only play a role of an index space which enumerates vectors in bases of $\mathcal{H}$. We refer to $\mathcal{H}$ as the *one-particle* Hilbert space. Furthermore, we denote the space of so-called *trace class* operators on ℓ, i.e. bounded operators T on ℓ for which $\|T\|_{\mathrm{I}_1} := \operatorname{tr}\sqrt{T^*T}$ is finite, by $\mathrm{I}_1(\ell)$, the superscript $*$ denoting the Hilbert space adjoint. We say a bounded linear operator T on a Hilbert space ℓ *has a determinant* $\det T := \lim_{n\to\infty}\det(T_{i,j})_{i,j\leq n}$ if it differs from the identity operator id_ℓ on ℓ only by a trace class operator, i.e. $T - \mathrm{id}_\ell \in \mathrm{I}_1(\ell)$; see [GGK90]. We also need the space of Hilbert-Schmidt operators, i.e. the space of bounded operators $T : \ell \to \mathcal{H}$ such that the Hilbert-Schmidt norm $\|T\|_{\mathrm{I}_2} := \sqrt{\operatorname{tr} T^*T}$ is finite. The space of Hilbert-Schmidt operators is denoted by $\mathrm{I}_2 = \mathrm{I}_2(\ell,\mathcal{H})$ and write $\mathrm{I}_2(\mathcal{H}) = \mathrm{I}_2(\mathcal{H},\mathcal{H})$.

At first let us define the notions: polarizations, polarization classes and the set of Dirac seas from the introduction.

Polarizations and polarization classes

Definition 7.1.

1. *Let* $\mathtt{Pol}(\mathcal{H})$ *denote the set of all closed subspaces* $V \subset \mathcal{H}$ *such that* V *and* $V^\perp$ *are both infinite dimensional. Any* $V \in \mathtt{Pol}(\mathcal{H})$ *is called a* polarization *of* $\mathcal{H}$. *For* $V \in \mathtt{Pol}(\mathcal{H})$, *let* $P_V : \mathcal{H} \to V$ *denote the orthogonal projection of* $\mathcal{H}$ *onto* V.

2. *For* $V, W \in \mathtt{Pol}(\mathcal{H})$, $V \approx W$ *means* $P_V - P_W \in \mathrm{I}_2(\mathcal{H})$.

The space $\mathtt{Pol}(\mathcal{H})$ is a kind of Grassmann space of all infinite dimensional closed subspaces with infinite dimensional complement. Obviously, the relation $\approx$ is an equivalence relation on $\mathtt{Pol}(\mathcal{H})$. Its equivalence classes $C \in \mathtt{Pol}(\mathcal{H})/\approx$ are called *p*olarization classes. Its basic properties are collected in the following lemma. We denote by $|_{X\to Y}$ the restriction to the map $X \to Y$.

Lemma 7.2. *For $V, W \in \mathtt{Pol}(\mathcal{H})$, the following are equivalent:* Properties of $\approx$

(a) $V \approx W$

(b) $P_{W^\perp}P_V \in \mathrm{I}_2(\mathcal{H})$ *and* $P_W P_{V^\perp} \in \mathrm{I}_2(\mathcal{H})$

(c) The operators $P_V P_W P_V|_{V\to V}$ *and* $P_W P_V P_W|_{W\to W}$ *both have determinants.*

(d) The operators $P_V P_W P_V|_{V\to V}$ *and* $P_{V^\perp}P_{W^\perp}P_{V^\perp}|_{V^\perp\to V^\perp}$ *both have determinants.*

(e) $P_W|_{V\to W}$ *is a Fredholm operator and* $P_{W^\perp}|_{V\to W^\perp} \in \mathrm{I}_2(V)$.

Proof.

(a)⇒(b): Let $V, W \in \mathtt{Pol}(\mathcal{H})$ fulfill $P_V - P_W \in \mathrm{I}_2(\mathcal{H})$. We conclude that

$$P_{W^\perp}P_V = (\mathrm{id}_{\mathcal{H}} - P_W)P_V = (P_V - P_W)P_V \in \mathrm{I}_2(\mathcal{H}) \text{ and} \tag{7.9}$$
$$P_W P_{V^\perp} = P_W(\mathrm{id}_{\mathcal{H}} - P_V) = -P_W(P_V - P_W) \in \mathrm{I}_2(\mathcal{H}). \tag{7.10}$$

(b)⇒(c): Assuming (b), we conclude

$$P_V - P_V P_W P_V = P_V P_{W^\perp} P_V = (P_{W^\perp}P_V)^*(P_{W^\perp}P_V) \in \mathrm{I}_1(\mathcal{H}) \text{ and} \tag{7.11}$$
$$P_W - P_W P_V P_W = P_W P_{V^\perp} P_W = (P_W P_{V^\perp})^*(P_W P_{V^\perp}) \in \mathrm{I}_1(\mathcal{H}). \tag{7.12}$$

This implies $(P_V - P_V P_W P_V)|_{V\to V} \in \mathrm{I}_1(V)$ and $(P_W - P_W P_V P_W)|_{W\to W} \in \mathrm{I}_1(W)$ and thus the claim (c).

(c)⇒(d): Assuming (c), we need to show that $P_{V^\perp}P_{W^\perp}P_{V^\perp}|_{V^\perp\to V^\perp}$ has a determinant. Indeed: As $P_W P_V P_W|_{W\to W}$ has a determinant, we know that

$$(P_{V^\perp}P_W)^*(P_{V^\perp}P_W) = P_W P_{V^\perp} P_W = P_W - P_W P_V P_W \in \mathrm{I}_1(\mathcal{H}) \tag{7.13}$$

and thus $P_{V^\perp}P_W \in \mathrm{I}_2(\mathcal{H})$. This implies

$$P_{V^\perp} - P_{V^\perp}P_{W^\perp}P_{V^\perp} = P_{V^\perp}P_W P_{V^\perp} = (P_{V^\perp}P_W)(P_{V^\perp}P_W)^* \in \mathrm{I}_1(\mathcal{H}). \tag{7.14}$$

The claim $(P_{V^\perp}P_{W^\perp}P_{V^\perp}|_{V^\perp\to V^\perp}) \in \mathrm{id}_{V^\perp} + \mathrm{I}_1(V^\perp)$ follows.

(d)⇒(b): Assuming (d), we know

$$(P_W P_{V^\perp})^*(P_W P_{V^\perp}) = P_{V^\perp}P_W P_{V^\perp} = P_{V^\perp} - P_{V^\perp}P_{W^\perp}P_{V^\perp} \in \mathrm{I}_1(\mathcal{H}) \text{ and} \tag{7.15}$$
$$(P_{W^\perp}P_V)^*(P_{W^\perp}P_V) = P_V P_{W^\perp} P_V = P_V - P_V P_W P_V \in \mathrm{I}_1(\mathcal{H}). \tag{7.16}$$

This implies the claim (b).

(b)⇒(a): Assuming $P_{W^\perp}P_V \in \mathrm{I}_2(\mathcal{H})$ and $P_W P_{V^\perp} \in \mathrm{I}_2(\mathcal{H})$, we conclude that

$$\begin{aligned} P_V - P_W &= (P_V - P_W P_V) - (P_W - P_W P_V) \\ &= P_{W^\perp}P_V - P_W P_{V^\perp} \in \mathrm{I}_2(\mathcal{H}). \end{aligned} \tag{7.17}$$

(b)⇒(e): We write the identity on $\mathcal{H}$ in matrix form

$$\mathrm{id}_{\mathcal{H}} : V \oplus V^{\perp} \to W \oplus W^{\perp}, \qquad (x,y) \mapsto \begin{pmatrix} P_W|_{V\to W} & P_W|_{V^\perp \to W} \\ P_{W^\perp}|_{V\to W^\perp} & P_{W^\perp}|_{V^\perp\to W^\perp} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7.18}$$

Assuming (b) we know that the non-diagonal operators $P_{W^\perp}|_{V\to W^\perp}$ and $P_W|_{V^\perp\to W}$ are Hilbert-Schmidt operators. Subtracting the non-diagonal from the identity we get a new map

$$Q : V \oplus V^{\perp} \to W \oplus W^{\perp}, \qquad (x,y) \mapsto \begin{pmatrix} P_W|_{V\to W} & 0 \\ 0 & P_{W^\perp}|_{V^\perp\to W^\perp} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7.19}$$

which is by construction a perturbation of the identity by a compact operator and, thus, a Fredholm operator. However, this holds if and only if both $P_W|_{V\to W}$ and $P_{W^\perp}|_{V^\perp\to W^\perp}$ are Fredholm operators which implies (e).

(e)⇒(b): Assuming (e) we compute

$$\begin{aligned} 0 &= P_{V^\perp}|_V = P_{V^\perp}(P_W + P_{W^\perp})|_V \\ &= P_{V^\perp}|_W \; P_W|_{V\to W} + P_{V^\perp}|_{W^\perp} \; P_{W^\perp}|_{V\to W^\perp} \end{aligned} \tag{7.20}$$

from which follows that $P_{V^\perp}|_W \;\; P_W|_V$ is a Hilbert-Schmidt operator since $P_{V^\perp}|_{W^\perp}$ is a bounded operator and by assumption we have $P_{W^\perp}|_V \in \mathrm{I}_2(V)$. Furthermore, by assumption $P_W|_{V\to W}$ is a Fredholm operator so that $P_{V^\perp}|_W \; P_W|_{V\to W} \in \mathrm{I}_2(V)$ yields $P_{V^\perp}|_W \in \mathrm{I}_2(W)$. Finally, we have $P_{W^\perp}|_V = P_{W^\perp}P_V|_V$ and $P_{V^\perp}|_W = P_{V^\perp}P_W|_W$ so that $P_{W^\perp}P_V, P_{V^\perp}P_W \in \mathrm{I}_2(\mathcal{H})$ which implies the claim (b).

□

Note that in general $P_{W^\perp}P_V \in \mathrm{I}_2(\mathcal{H})$ is not equivalent to $P_W P_{V^\perp} \in \mathrm{I}_2(\mathcal{H})$. As an example, take V and W such that $V \subset W$ and V has infinite codimension in W; compare with condition (e) of the above lemma. Condition (e) appears in Chapter 7 of [PS88] where an equivalence class $C \in \mathtt{Pol}(\mathcal{H})/\!\approx$ is endowed with the structure of a complex manifold modeled on infinite dimensional separable Hilbert spaces. Consequently, the space $\mathtt{Pol}(\mathcal{H})$ is a complex manifold – the Grassmann manifold of $\mathcal{H}$ – which decomposes into the equivalence classes $C \in \mathtt{Pol}(\mathcal{H})/\!\approx$ as open and closed submanifolds.

Where exactly the spectrum is cut into parts by a choice of a polarization in a polarization class will determine the relative charge between two Dirac seas. Within one polarization class the charge may only differ by a finite number from one chosen polarization to another. Given $V, W \in \mathtt{Pol}(\mathcal{H})$ with $V \approx W$, we know from Lemma 7.2p.117(e) that $P_W|_{V\to W}$ and $P_V|_{W\to V}$ are Fredholm operators. So we are led to the definition of the *relative charge*:

Relative charge

Definition 7.3. *For $V, W \in \mathtt{Pol}(\mathcal{H})$ with $V \approx W$, we define the* relative charge *of V, W to be the Fredholm index of $P_W|_{V\to W}$:*

$$\begin{aligned} \mathrm{charge}(V,W) &:= \mathrm{ind}(P_W|_{V\to W}) = \dim\ker(P_W|_{V\to W}) - \dim\ker(P_W|_{V\to W})^* \\ &= \dim\ker(P_W|_{V\to W}) - \dim\mathrm{coker}(P_W|_{V\to W}). \end{aligned} \tag{7.21}$$

Let $\mathrm{U}(\mathcal{H},\mathcal{H}')$ be the set of unitary operators $U : \mathcal{H} \to \mathcal{H}'$. We collect some basic properties of the relative charge:

Relative charge properties

Lemma 7.4. *Let $C \in \mathtt{Pol}(\mathcal{H})/\!\approx$ be a polarization class and $V, W, X \in C$. Then the following holds:*

1. $\operatorname{charge}(V, W) = -\operatorname{charge}(W, V)$
2. $\operatorname{charge}(V, W) + \operatorname{charge}(W, X) = \operatorname{charge}(V, X)$
3. *Let $\mathcal{H}'$ be another Hilbert space and $U \in \mathrm{U}(\mathcal{H}, \mathcal{H}')$.*
   Then $\operatorname{charge}(V, W) = \operatorname{charge}(UV, UW)$.
4. *Let $U \in \mathrm{U}(\mathcal{H}, \mathcal{H})$ such that $UC = C$. Then* $\operatorname{charge}(V, UV) = \operatorname{charge}(W, UW)$.

Proof. (a) $P_W|_{V\to W}$ and $P_V|_{W\to V}$ are Fredholm operators with

$$\begin{aligned}&\operatorname{charge}(W, V) + \operatorname{charge}(V, W) = \operatorname{ind}(P_V|_{W\to V}) + \operatorname{ind}(P_W|_{V\to W})\\ &= \operatorname{ind}(P_V P_W|_{V\to V}) = \operatorname{ind}(P_V P_W P_V|_{V\to V}) = 0,\end{aligned} \tag{7.22}$$

as $P_V P_W P_V|_{V\to V}$ is a perturbation of the identity map on V by a compact operator.

(b) As P_W and P_X differ only by a compact operator, we get

$$\begin{aligned}&\operatorname{charge}(V, W) + \operatorname{charge}(W, X) = \operatorname{ind}(P_W|_{V\to W}) + \operatorname{ind}(P_X|_{W\to X})\\ &= \operatorname{ind}(P_X P_W|_{V\to X}) = \operatorname{ind}(P_X P_X|_{V\to X}) = \operatorname{charge}(V, X).\end{aligned} \tag{7.23}$$

(c) This follows immediately since unitary transformations do not change the Fredholm index.

(d) We know $UV \approx V \approx W \approx UW$ by assumption. Using parts (a), (b) and (c) of the lemma, this implies

$$\begin{aligned}&\operatorname{charge}(V, UV) = \operatorname{charge}(V, W) + \operatorname{charge}(W, UW) + \operatorname{charge}(UW, UV)\\ &= \operatorname{charge}(V, W) + \operatorname{charge}(W, UW) + \operatorname{charge}(W, V) = \operatorname{charge}(W, UW).\end{aligned} \tag{7.24}$$

□

With the notion of relative charge we refine the polarization classes further into classes of polarizations of equal relative charge:

Definition 7.5. *For $V, W \in \mathtt{Pol}(\mathcal{H})$, $V \approx_0 W$ means $V \approx W$ and* $\operatorname{charge}(V, W) = 0$.

Equal charge classes

By Lemma 7.4p.118 (Relative charge properties) the relation $\approx_0$ is an equivalence relation on $\mathtt{Pol}(\mathcal{H})$. This finer relation is better adapted for the lift of unitary one-particle operators like the Dirac time-evolution which conserve the charge.

Next we introduce the mathematical representation of the Dirac seas (they are, however, not the physical states yet because these objects still miss the antisymmetry and can not be linearly combined):

Definition 7.6.

Dirac seas

(a) Let $\mathtt{Seas}(\mathcal{H}) = \mathtt{Seas}_\ell(\mathcal{H})$ be the set of all linear, bounded operators $\Phi : \ell \to \mathcal{H}$ such that range $\Phi \in \mathtt{Pol}(\mathcal{H})$ *and $\Phi^*\Phi : \ell \to \ell$ has a determinant, i.e. $\Phi^*\Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$.*

(b) Let $\mathtt{Seas}^\perp(\mathcal{H}) = \mathtt{Seas}^\perp_\ell(\mathcal{H})$ denote the set of all linear isometries $\Phi : \ell \to \mathcal{H}$ in $\mathtt{Seas}_\ell(\mathcal{H})$.

(c) For any $C \in \mathtt{Pol}(\mathcal{H})/\approx_0$ let $\mathtt{Ocean}(C) = \mathtt{Ocean}_\ell(C)$ be the set of all $\Phi \in \mathtt{Seas}^\perp_\ell(\mathcal{H})$ such that range $\Phi \in C$.

Thus (as in geography) an ocean consists of a collection of related seas; see also Figure 7.1p.130 at the end of this section. To connect to the introduction in Subsection 7.1p.113 consider the following example: In the case of $\ell = \ell_2(\mathbb{N})$ we encode this map in an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of V such that for the canonical basis $(e_n)_{n\in\mathbb{N}}$ in ℓ^2 one has $\Phi e_n = \varphi_n$ for all $n \in \mathbb{N}$.

The set $\mathtt{Seas}(\mathcal{H})$ can naturally be structured by the relation introduced now:

Relation between Dirac seas

Definition 7.7. *For* $\Phi, \Psi \in \mathtt{Seas}(\mathcal{H})$*,* $\Phi \sim \Psi$ *means* $\Phi^*\Psi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$*, i.e.* $\Phi^*\Psi$ *has a determinant.*

In the forthcoming Corollary 7.9 we show that $\sim$ is an equivalence relation. For its proof we need the following lemma, which will also be frequently used later because it allows us to work for most purposes with Dirac seas in $\mathtt{Seas}^\perp(\mathcal{H})$ instead of $\mathtt{Seas}(\mathcal{H})$:

Isometries are good enough

Lemma 7.8. *For every* $\Psi \in \mathtt{Seas}(\mathcal{H})$ *there exist* $\Upsilon \in \mathtt{Seas}^\perp(\mathcal{H})$ *and* $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ *which fulfill* $\Psi = \Upsilon R$*,* $\Upsilon^*\Psi = R \geq 0$*,* $\Upsilon \sim \Psi$*, and* $R^2 = \Psi^*\Psi$*.*

Proof. Let $\Psi \in \mathtt{Seas}(\mathcal{H})$. The operator $\Psi^*\Psi : \ell \to \ell$ has a determinant and is hence a Fredholm operator. In particular, $\ker(\Psi^*\Psi) = \ker\Psi$ is finite dimensional. Let $\Psi = VR$ be the polar decomposition of Ψ, with $R = \sqrt{\Psi^*\Psi}$ and $V : \ell \to \mathcal{H}$ being a partial isometry with $\ker V = (\mathrm{range}\, R)^\perp = \ker\Psi$. Then V and Ψ have the same range, and this range has infinite codimension in $\mathcal{H}$. Since $\ker V$ has finite dimension, we can extend the restriction of V to $(\ker V)^\perp$ to an isometry $\Upsilon : \ell \to \mathcal{H}$. We get: $\Upsilon^*\Psi = V^*\Psi = V^*VR = R \geq 0$ and $\Upsilon R = VR = \Psi$. Now as $R^2 = \Psi^*\Psi$ has a determinant, its square root R has also a determinant. This implies $\Upsilon \sim \Psi$. □

Corollary 7.9. *The relation* $\sim$ *is an equivalence relation on* $\mathtt{Seas}(\mathcal{H})$*.*

Proof. By definition of $\mathtt{Seas}(\mathcal{H})$, the relation $\sim$ is reflexive. To show symmetry, take $\Phi, \Psi \in \mathtt{Seas}(\mathcal{H})$ with $\Phi \sim \Psi$. We conclude $\Psi^*\Phi - \mathrm{id}_\ell = (\Phi^*\Psi - \mathrm{id}_\ell)^* \in \mathrm{I}_1(\ell)$ and thus $\Psi \sim \Phi$. To show transitivity, let $\Phi, \Psi, \Gamma \in \mathtt{Seas}(\mathcal{H})$ with $\Phi \sim \Psi$ and $\Psi \sim \Gamma$. By Lemma 7.8 (Isometries are good enough), take $\Upsilon \in \mathtt{Seas}^\perp(\mathcal{H})$ and $R \geq 0$ corresponding to Ψ. Let $P : \mathcal{H} \to \mathcal{H}$ denote the orthogonal projection having the same range as Υ, and let $P^c = \mathrm{id}_\mathcal{H} - P$ denote the complementary projection. In particular, one has $P = \Upsilon\Upsilon^*$. Then

$$\Phi^*\Gamma = \Phi^*P\Gamma + \Phi P^c\Gamma = (\Phi^*\Upsilon)(\Gamma^*\Upsilon)^* + \Phi^*P^c\Gamma. \tag{7.25}$$

Now since $\Phi \sim \Psi$ we know that $\Phi^*\Psi = \Phi^*\Upsilon R$ has a determinant. Since R has also a determinant, we conclude that $\Phi^*\Upsilon$ has a determinant, too. Using $\Psi \sim \Gamma$, the same argument shows that $\Gamma^*\Upsilon$ has a determinant, and thus $(\Phi^*\Upsilon)(\Gamma^*\Upsilon)^*$ has a determinant. Next we show that $P^c\Gamma$ is a Hilbert-Schmidt operator. Indeed, $(P^c\Gamma)^*(P^c\Gamma) = \Gamma^*P^c\Gamma = \Gamma^*\Gamma - \Gamma^*P\Gamma = \Gamma^*\Gamma - (\Gamma^*\Upsilon)(\Gamma^*\Upsilon)^*$ is a difference of two operators having a determinant, since $\Gamma^*\Gamma$ and $\Gamma^*\Upsilon$ both have determinants. Hence, $(P^c\Gamma)^*(P^c\Gamma) \in \mathrm{I}_1(\ell)$, which implies $P^c\Gamma \in \mathrm{I}_2(\ell)$. The same argument, applied to Φ instead of Γ, shows that $P^c\Phi \in \mathrm{I}_2(\ell)$. We conclude $\Phi^*P^c\Gamma = (P^c\Phi)^*(P^c\Gamma) \in \mathrm{I}_1(\ell)$. Using (7.25) this yields that $\Phi^*\Gamma$ has a determinant since it is given as the sum of an operator that has a determinant plus an operator which is trace class. This proves that $\Phi \sim \Gamma$. □

For $\Phi \in \mathtt{Seas}(\mathcal{H})$, the equivalence class of Φ with respect to $\sim$ turns out to form an affine space. The following definition and lemma characterize these equivalence classes. These properties will later be used to show that the wedge spaces to be constructed (in forthcoming Definition 7.17p.124 (Infinite wedge spaces)) are separable spaces.

Dirac sea classes

Definition 7.10. *Let* $\Phi \in \mathtt{Seas}(\mathcal{H})$*.*

1. *Let* $\mathcal{S}(\Phi) \subset \mathtt{Seas}(\mathcal{H})$ *denote the equivalence class of* Φ *with respect to* $\sim$*.*

2. *For bounded linear operators $L : \ell \to \mathcal{H}$, we define $\|L\|_\Phi := \|\Phi^* L\|_{I_1} + \|L\|_{I_2}$ and vector space*

$$\mathcal{V}(\Phi) := \{L : \ell \to \mathcal{H} \mid L \text{ is linear and bounded with } \|L\|_\Phi < \infty\}.$$

Lemma 7.11. *Let $\Phi \in \texttt{Seas}(\mathcal{H})$.*

Dirac sea class properties

(a) It holds that $\mathcal{S}(\Phi) = \Phi + \mathcal{V}(\Phi)$.

(b) For $\Psi \in \texttt{Seas}(\mathcal{H})$ with $\Phi \sim \Psi$, one has $\mathcal{V}(\Phi) = \mathcal{V}(\Psi)$, and the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Psi$ are equivalent.

Proof. (a) Take $\Psi \sim \Phi$. By definition, $\Phi^*\Psi \in \mathrm{id}_\ell + I_1(\ell)$ and $\Phi^*\Phi \in \mathrm{id}_\ell + I_1(\ell)$. The difference yields $\Phi^*(\Psi - \Phi) \in I_1(\ell)$. Similarly, $\Psi^*\Psi \in \mathrm{id}_\ell + I_1(\ell)$ and $\Psi^*\Phi \in \mathrm{id}_\ell + I_1(\ell)$. Combining all this, we get $(\Psi-\Phi)^*(\Psi-\Phi) \in I_1(\ell)$, and hence $\Psi - \Phi \in I_2(\ell, \mathcal{H})$. This shows $\Psi - \Phi \in \mathcal{V}(\Phi)$.

Conversely, take $B \in \mathcal{V}(\Phi)$. We set $\Psi = \Phi + B$. First we show that $\operatorname{range}\Psi \in \texttt{Pol}(\mathcal{H})$, i.e. that it is closed and has infinite dimension and codimension. In order to do this we use the following general fact: A Fredholm operator between two Hilbert spaces maps closed, infinite dimensional and infinite codimensional subspaces, respectively, to closed, infinite dimensional, and infinite codimensional subspaces, respectively. Consider

$$\widetilde{\Phi} : \ell \oplus \operatorname{range}\Phi^\perp \to \mathcal{H}, \qquad (x, y) \mapsto \Phi x + y \tag{7.26}$$

$$\widetilde{\Psi} : \ell \oplus \operatorname{range}\Phi^\perp \to \mathcal{H}, \qquad (x, y) \mapsto \Phi x + Bx + y \tag{7.27}$$

with the direct sum is understood as orthogonal direct sum. Since $\operatorname{range}\Phi$ is in $\texttt{Pol}(\mathcal{H})$ and therefore closed, the map $\widetilde{\Phi}$ is onto. Furthermore, $\Phi^*\Phi \in \mathrm{id}_\mathcal{H} + I_1(\mathcal{H})$ is a perturbation of the identity by a compact operator and therefore a Fredholm operator. In particular, this implies $\dim\ker\Phi = \dim\ker\Phi^*\Phi < \infty$. Thus, $\widetilde{\Phi}$ is also a Fredholm operator. Now $\widetilde{\Psi}$ is a perturbation of $\widetilde{\Phi}$ by the compact operator $(x, y) \mapsto Bx$ and therefore it is Fredholm operator too. Since $\ell \oplus 0$ is closed, infinite dimensional, infinite and codimensional, it holds $\operatorname{range}\Psi = \widetilde{\Psi}(\ell \oplus 0)$.

Using $\Phi \in \texttt{Seas}(\mathcal{H})$ and the definition of $\mathcal{V}(\Phi)$, we get $\Psi^*\Psi = \Phi^*\Phi + \Phi^*B + (\Phi^*B)^* + B^*B \in (\mathrm{id}_\ell + I_1) + I_1 + I_1 + I_2 I_2 = \mathrm{id}_\ell + I_1(\ell)$. This shows $\Psi \in \texttt{Seas}(\mathcal{H})$. Furthermore, $\Phi^*\Psi = \Phi^*\Phi + \Phi^*B \in (\mathrm{id}_\ell + I_1) + I_1 = \mathrm{id}_\ell + I_1(\ell)$ holds. This yields $\Psi \sim \Phi$.

(b) Since $\Phi \sim \Psi$, there is a $L \in \mathcal{V}(\Phi)$ such that $\Phi = \Psi + L$. Let $M \in \mathcal{V}(\Psi)$. Using the triangle inequality in $I_1(\ell)$ and $\|L^*M\|_{I_1} \le \|L\|_{I_2}\|M\|_{I_2}$, we get

$$\begin{aligned}\|M\|_\Phi &= \|\Phi^*M\|_{I_1} + \|M\|_{I_2} \le \|\Psi^*M\|_{I_1} + \|L^*M\|_{I_1} + \|M\|_{I_2} \\ &\le (1 + \|L\|_{I_2})(\|\Psi^*M\|_{I_1} + \|M\|_{I_2}) = (1 + \|L\|_{I_2})\|M\|_\Psi.\end{aligned} \tag{7.28}$$

In the same way, we get $\|M\|_\Psi \le (1 + \|L\|_{I_2})\|M\|_\Phi$.

□

The equivalence classes of $\texttt{Seas}(\mathcal{H})/\sim$ and $\texttt{Pol}(\mathcal{H})/\approx_0$ go hand in hand quite naturally as the following lemma shows.

Lemma 7.12. *Let $C \in \texttt{Pol}(\mathcal{H})/\approx_0$ and $\Phi \in \texttt{Ocean}(C)$, then we have*

Connection between $\sim$ and $\approx_0$

$$C = \{\operatorname{range}\Psi \mid \Psi \in \texttt{Seas}^\perp(\mathcal{H}) \text{ such that } \Psi \sim \Phi\}.$$

Proof. Let $C' := \{\operatorname{range}\Psi \mid \Psi \in \mathtt{Seas}^\perp(\mathcal{H}) \text{ such that } \Psi \sim \Phi\}$ and $V := \operatorname{range}\Phi$.

$C' \subseteq C$: Let $W \in C'$, then there is a $\Psi \sim \Phi$ such that $\operatorname{range}\Psi = W$. One has

$$P_V P_W P_V|_{V\to V} = \Phi(\Phi^*\Psi\Psi^*\Phi)\Phi^*|_{V\to V}. \tag{7.29}$$

But $\Phi^*\Psi, \Psi^*\Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$, hence, $\Phi^*\Psi\Psi^*\Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ and $\Phi^*|_V$ is unitary, so we conclude that $P_V P_W P_V|_{V\to V} \in \mathrm{id}_V + \mathrm{I}_1(V)$ and has a determinant. Analogously, we get that $P_W P_V P_W|_{W\to W} = \Psi(\Psi^*\Phi\Phi^*\Psi)\Psi^*$ has a determinant because, again, $\Psi^*|_W$ is unitary. Lemma 7.2p.117 (Properties of $\approx$) then states $V \approx W$. We still need to show $V \approx_0 W$. Therefore, consider $\operatorname{charge}(W, V) = \operatorname{ind}(P_V|_{W\to V})$ and $P_V|_{W\to V} = \Phi\Phi^*\Psi\Psi^*|_{W\to V}$. Since $\Phi^*\Psi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ and $\Psi^*|_W$ is unitary, $\Psi^* P_V|_{W\to V}\Psi = \Psi^*\Phi\Phi^*\Psi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ which is a perturbation of the identity by a compact operator. Therefore, $\operatorname{ind}(P_V|_{W\to V}) = 0$. Hence, we have shown that $V \approx_0 W$ and therefore $W \in C$.

$C' \supseteq C$: Let $W \in C$, then $W \approx V$ and $\operatorname{charge}(V, W) = 0$. We need to find an isometry $\Psi \sim \Phi$ such that $\operatorname{range}\Psi = W$. We make a polar decomposition of $P_W\Phi$. By Lemma 7.2p.117(e) we know that $\operatorname{range} P_W|_V$ is closed. There is a partial isometry $U : \ell \to \operatorname{range} P_W\Phi = \operatorname{range} P_W|_V = \overline{\operatorname{range} P_W\Phi} \subset \mathcal{H}$ with $\ker U = \ker P_W\Phi$ such that $P_W\Phi = U|P_W\Phi|$ where $|P_W\Phi|$ is given by the square root of the positive semi-definite operator $(P_W\Phi)^*(P_W\Phi)$. Furthermore, $(P_W\Phi)^*(P_W\Phi) = \Phi^* P_V P_W P_V \Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ by Lemma 7.2p.117 and, hence, a Fredholm operator. That means also that $\ker((P_W\Phi)^*(P_W\Phi)) = \ker P_W\Phi$ is finite dimensional. Moreover, $0 = \operatorname{charge}(V, W) = \operatorname{ind}(P_W|_{V\to W})$ implies that $\dim\ker P_W\Phi = \dim W/(\operatorname{range} P_W|_V) = \dim W \cap (\operatorname{range} P_W|_V)^\perp$. Thus, there is another partial isometry $\widetilde{U} : \ell \to \mathcal{H}$ of finite rank such that $\widetilde{U}|_{\ker P_W\Phi}$ maps $\ker P_W\Phi$ unitarily onto $W \cap (\operatorname{range} P_W\Phi)^\perp$ and $\widetilde{U}|_{(\ker P_W\Phi)^\perp} = 0$. We set $\Psi := U + \widetilde{U} : \ell \to \mathcal{H}$, and get $\Psi^*\Psi = U^*U + \widetilde{U}^*\widetilde{U} = 1$ and therefore $\Psi \in \mathtt{Seas}^\perp(\mathcal{H})$. By construction, $\operatorname{range}\Psi = W$ holds. Furthermore,

$$U^*\Phi = U^* P_W\Phi = U^* U|P_W\Phi| = |P_W\Phi| \tag{7.30}$$

which has a determinant since $|P_W\Phi| \geq 0$ and

$$|P_W\Phi|^2 = \Phi^* P_V P_W P_V \Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell) \tag{7.31}$$

as $P_V P_W P_V|_{V\to V} \in \mathrm{id}(V) + \mathrm{I}_1(V)$ by Lemma 7.2p.117. On the other hand, $\widetilde{U}^*\Phi$ has finite rank since $\widetilde{U}$ does. Hence, $\Psi^*\Phi = U^*\Phi + \widetilde{U}^*\Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$, i.e. $\Psi \sim \Phi$, which means that $W \in C'$.

□

Now we begin with the construction of the infinite wedge spaces for each equivalence class of Dirac seas $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$. We follow the standard linear algebra method: First, we construct with the elements of $\mathcal{S}$ a space of formal linear combinations $\mathbb{C}^{(\mathcal{S})}$ which we equip with a semi-definite sesquilinear form that in turn induces a semi-norm. Completion with respect to this semi-norm yields the infinite wedge space of $\mathcal{S}$.

Formal linear combinations

Construction 7.13.

1. *For any set $\mathcal{S}$, let $\mathbb{C}^{(\mathcal{S})}$ denote the set of all maps $\alpha : \mathcal{S} \to \mathbb{C}$ for which $\{\Phi \in \mathcal{S} \mid \alpha(\Phi) \neq 0\}$ is finite. For $\Phi \in \mathcal{S}$, we define $[\Phi] \in \mathbb{C}^{(\mathcal{S})}$ to be the map fulfilling $[\Phi](\Phi) = 1$ and $[\Phi](\Psi) = 0$ for $\Phi \neq \Psi \in \mathcal{S}$. Thus, $\mathbb{C}^{(\mathcal{S})}$ consists of all finite formal linear combinations $\alpha = \sum_{\Psi\in\mathcal{S}} \alpha(\Psi)[\Psi]$ of elements of $\mathcal{S}$ with coefficients in $\mathbb{C}$.*

2. *Now let* $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$ *as in Definition 7.10**p.120* *(Dirac sea classes). We define the map* $\langle\cdot,\cdot\rangle : \mathcal{S}\times\mathcal{S} \to \mathbb{C}$, $(\Phi,\Psi) \to \langle\Phi,\Psi\rangle := \det(\Phi^*\Psi)$. *Note that this is well defined since for* $\Phi,\Psi \in \mathcal{S}$ *the fact* $\Phi \sim \Psi$ *implies that* $\Phi^*\Psi$ *has a determinant.*

3. *Taking* $\mathcal{S}$ *as before, let* $\langle\cdot,\cdot\rangle : \mathbb{C}^{(\mathcal{S})} \times \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}$ *denote the sesquilinear extension of* $\langle\cdot,\cdot\rangle : \mathcal{S}\times\mathcal{S} \to \mathbb{C}$, *defined as follows: For* $\alpha,\beta \in \mathbb{C}^{(\mathcal{S})}$,

$$\langle\alpha,\beta\rangle = \sum_{\Phi\in\mathcal{S}}\sum_{\Psi\in\mathcal{S}} \overline{\alpha(\Phi)}\beta(\Psi)\det(\Phi^*\Psi). \tag{7.32}$$

The bar denotes the complex conjugate. Note that the sums consist of at most finitely many nonzero summands. In particular, we have $\langle[\Phi],[\Psi]\rangle = \langle\Phi,\Psi\rangle$ *for* $\Phi,\Psi \in \mathcal{S}$.

Lemma 7.14. *The sesquilinear form* $\langle\cdot,\cdot\rangle : \mathbb{C}^{(\mathcal{S})} \times \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}$ *is hermitian and positive semi-definite, i.e.* $\langle\alpha,\beta\rangle = \overline{\langle\beta,\alpha\rangle}$ *and* $\langle\alpha,\alpha\rangle \geq 0$ *hold for all* $\alpha,\beta \in \mathbb{C}^{(\mathcal{S})}$.

Proof. For $\Phi,\Psi \in \mathcal{S}$ we have

$$\langle\Phi,\Psi\rangle = \det(\Phi^*\Psi) = \overline{\det(\Psi^*\Phi)} = \overline{\langle\Psi,\Phi\rangle} \tag{7.33}$$

This implies that $\langle\cdot,\cdot\rangle : \mathbb{C}^{(\mathcal{S})} \times \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}$ is hermitian. Let $\alpha \in \mathcal{S}$. We get

$$\langle\alpha,\alpha\rangle = \sum_{\Phi\in\mathcal{S}}\sum_{\Psi\in\mathcal{S}} \overline{\alpha(\Phi)}\alpha(\Psi)\det(\Phi^*\Psi). \tag{7.34}$$

Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis in ℓ. In the following we abbreviate $\mathbb{N}_m = \{1,\ldots,m\}$. Fredholm determinants are approximated by finite-dimensional determinants (see Section VII.3, Theorem 3.2 in [GGK90]), therefore

$$\det(\Phi^*\Psi) = \lim_{m\to\infty} \det(\left\langle e_i, \Phi^*\Psi e_j\right\rangle)_{i,j\in\mathbb{N}_m}. \tag{7.35}$$

Let $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. For every $i,j \in \mathbb{N}$, we get

$$\left\langle e_i, \Phi^*\Psi e_j\right\rangle = \lim_{n\to\infty}\sum_{k=1}^{n} \langle\Phi e_i, f_k\rangle \left\langle f_k, \Psi e_j\right\rangle, \tag{7.36}$$

and, hence, for every $m \in \mathbb{N}$

$$\begin{aligned}\det(\left\langle e_i, \Phi^*\Psi e_j\right\rangle)_{i,j\in\mathbb{N}_m} &= \lim_{n\to\infty}\det\left(\sum_{k=1}^{n}\langle\Phi e_i, f_k\rangle\left\langle f_k,\Psi e_j\right\rangle\right)_{i,j\in\mathbb{N}_m} \\ &= \lim_{n\to\infty}\sum_{\substack{I\subseteq\mathbb{N}_n\\|I|=m}} \det(\langle\Phi e_i, f_k\rangle)_{\substack{k\in I\\ ,i\in\mathbb{N}_m}} \det(\left\langle f_k,\Psi e_j\right\rangle)_{\substack{k\in I,\\ j\in\mathbb{N}_m}}\end{aligned} \tag{7.37}$$

Substituting this in (7.34) and (7.35) we conclude

$$\begin{aligned}\langle\alpha,\alpha\rangle &= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\Phi\in\mathcal{S}}\sum_{\Psi\in\mathcal{S}}\sum_{\substack{I\subseteq\mathbb{N}_n\\|I|=m}} \overline{\alpha(\Phi)}\alpha(\Psi)\overline{\det(\langle f_k,\Phi e_i\rangle)_{\substack{k\in I,\\ i\in\mathbb{N}_m}}}\det(\left\langle f_k,\Psi e_j\right\rangle)_{\substack{k\in I,\\ j\in\mathbb{N}_m}} \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{\substack{I\subseteq\mathbb{N}_n\\|I|=m}}\left|\sum_{\Phi\in\mathcal{S}}\alpha(\Phi)\det(\langle f_k,\Phi e_i\rangle)_{\substack{k\in I,\\ i\in\mathbb{N}_m}}\right|^2 \geq 0.\end{aligned} \tag{7.38}$$

□

Definition 7.15. *Let* $\|\cdot\| : \mathbb{C}^{(\mathcal{S})} \to \mathbb{R}$, $\alpha \mapsto \|\alpha\| = \sqrt{\langle\alpha,\alpha\rangle}$ *denote the semi-norm associated to* $\langle\cdot,\cdot\rangle$, *and* $N_{\mathcal{S}} = \{\alpha \in \mathbb{C}^{(\mathcal{S})} \mid \langle\alpha,\alpha\rangle = 0\}$ *denote the null space of* $\mathbb{C}^{(\mathcal{S})}$ *with respect to* $\|\cdot\|$.

This null space $N_{\mathcal{S}}$ is quite large. The following lemma identifies a few elements of this null space and is also the key ingredient to Corollary 7.18p.124 (Null space) and therewith to Lemma 7.23p.127 (Uniqueness up to a phase) of the subsection.

Lemma 7.16. *For $\Phi \in \mathcal{S}$ and $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$, one has $\Phi R \in \mathcal{S}$ and $[\Phi R] - (\det R)[\Phi] \in N_{\mathcal{S}}$.*

Proof. First, we observe that $(\operatorname{range}(\Phi R))^\perp \supseteq (\operatorname{range}\Phi)^\perp$ is infinite-dimensional. Since $\Phi^*\Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ and $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$, we have $\Phi^*(\Phi R) \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ and $(\Phi R)^*(\Phi R) = R^*(\Phi^*\Phi)R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$. This shows $\Phi R \in \mathtt{Seas}(\mathcal{H})$ and $\Phi \sim \Phi R$, and thus $\Phi R \in \mathcal{S}$. We calculate:

$$\begin{aligned} \|[\Phi R] - (\det R)[\Phi]\|^2 =& \det((\Phi R)^*(\Phi R)) - (\det R)\det((\Phi R)^*\Phi) \\ &- \overline{\det R}\det(\Phi^*\Phi R) + |\det R|^2 \det(\Phi^*\Phi) \\ =& 2|\det R|^2\det(\Phi^*\Phi) - 2|\det R|^2\det(\Phi^*\Phi) = 0. \end{aligned} \tag{7.39}$$

□

Now we have everything needed to define the most important objects in this work: The *infinite wedge spaces*. These spaces shall make up the playground for the second quantized Dirac time-evolution:

Infinite wedge spaces

Definition 7.17. *Let $\mathcal{F}_{\mathcal{S}}$ be the completion of $\mathbb{C}^{(\mathcal{S})}$ with respect to the semi-norm $\|\cdot\|$. We refer to $\mathcal{F}_{\mathcal{S}}$ as infinite wedge space over $\mathcal{S}$. Let $\iota : \mathbb{C}^{(\mathcal{S})} \to \mathcal{F}_{\mathcal{S}}$ denote the canonical map. The sesquilinear form $\langle\cdot,\cdot\rangle : \mathbb{C}^{(\mathcal{S})} \times \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}$ induces a scalar product $\langle\cdot,\cdot\rangle : \mathcal{F}_{\mathcal{S}} \times \mathcal{F}_{\mathcal{S}} \to \mathbb{C}$. Let $\Lambda : \mathcal{S} \to \mathcal{F}_{\mathcal{S}}$ denote the canonical map $\Lambda\Phi = \iota([\Phi])$, $\Phi \in \mathcal{S}$.*

Note that $\iota[N_{\mathcal{S}}] = \{0\}$. Hence, the null space is automatically factored out during the completion procedure. In fact, the null space of the canonical map $\iota : \mathbb{C}^{(\mathcal{S})} \to \mathcal{F}_{\mathcal{S}}$ equals $\ker\iota = N_{\mathcal{S}}$. Thus we can rewrite Lemma 7.16 in the following way:

Null space

Corollary 7.18. *For $\Phi \in \mathcal{S}$ and $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$, one has $\Lambda(\Phi R) = (\det R)\Lambda\Phi$.*

Combining the above Corollary 7.18 with Lemma 7.8p.120 (Isometries are good enough), we get the following: For every $\Phi \in \mathcal{S}$ there are $\Upsilon \in \mathcal{S} \cap \mathtt{Seas}^\perp(\mathcal{H})$ and $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ with $r = \det R \in \mathbb{R}_0^+$ such that $\Lambda\Phi = r\Lambda\Upsilon$. As a consequence, $\{\Lambda\Psi \mid \Psi \in \mathcal{S} \cap \mathtt{Seas}^\perp(\mathcal{H})\}$ spans a dense subspace of $\mathcal{F}_{\mathcal{S}}$. The scalar product $\langle\cdot,\cdot\rangle$ gives $\mathcal{F}_{\mathcal{S}}$ the structure of a separable Hilbert space:

Separability

Lemma 7.19. *The inner product space $(\mathcal{F}_{\mathcal{S}}, \langle\cdot,\cdot\rangle)$ is separable.*

Proof. It suffices to show that there exists a countable dense subset of $\Lambda\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathcal{F}_{\mathcal{S}}}$ in $\mathcal{F}_{\mathcal{S}}$. Choose $\Phi \in \mathcal{S}$. By Lemma 7.11p.121 (Dirac sea class properties) we then know that $\mathcal{S} = \Phi + \mathcal{V}(\Phi)$. Now the set of operators of finite rank is dense and separable in $(\mathcal{V}(\Phi), \|\cdot\|_\Phi)$. Hence, we can choose a countable, dense subset D in $(\mathcal{V}(\Phi), \|\cdot\|_\Phi)$. We show now that $\Lambda(\Phi + D)$ is dense in $\Lambda\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathcal{F}_{\mathcal{S}}}$. Let $\Psi = \Phi + L \in \mathcal{S}$ with $L \in \mathcal{V}(\Phi)$. We find a sequence $(L_n)_{n\in\mathbb{N}}$ in D with $\|L_n - L\|_\Phi \to 0$ for $n \to \infty$ and define $\Psi_n := \Phi + L_n$. One then obtains the following estimate for all large n:

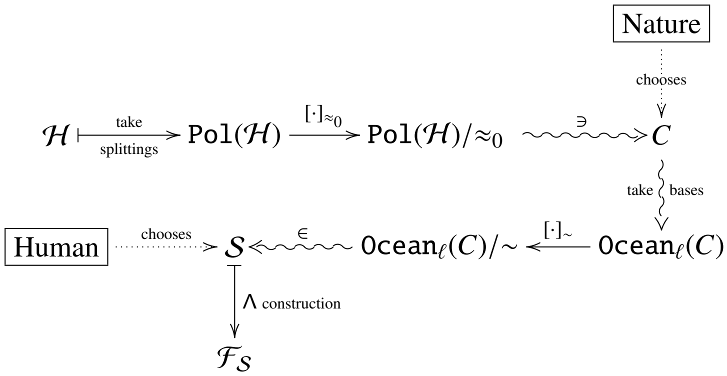
$$\begin{aligned} \|\Lambda\Psi - \Lambda\Psi_n\|^2_{\mathcal{F}_{\mathcal{S}}} &= \langle \Lambda\Psi - \Lambda\Psi_n, \Lambda\Psi - \Lambda\Psi_n\rangle_{\mathcal{F}_{\mathcal{S}}} \\ &= \det(\Psi^*\Psi) - \det(\Psi^*\Psi_n) - \det(\Psi_n^*\Psi) + \det(\Psi_n^*\Psi_n) \\ &\le C_{38}(\Psi)\left(\|\Psi^*(\Psi - \Psi_n)\|_{\mathrm{I}_1} + \|\Psi_n^*(\Psi - \Psi_n)\|_{\mathrm{I}_1}\right) \end{aligned} \tag{7.40}$$

by local Lipschitz continuity of the Fredholm determinant with respect to the norm in $I_1(\ell)$; see [Sim05, Theorem 3.4 p. 34]. The constant $C_{38}(\Psi) < \infty$ depends only on Ψ. Next the triangle inequality applied to the second term gives

$$\begin{aligned} \ldots &\le C_{38}(\Psi)\left(2\|\Psi^*(\Psi-\Psi_n)\|_{I_1} + \|(\Psi-\Psi_n)^*(\Psi-\Psi_n)\|_{I_1}\right) \\ &\le 2C_{38}(\Psi)\|\Psi-\Psi_n\|_\Psi = 2C_{38}(\Psi)\|L-L_n\|_\Psi \\ &\le C_{39}(\Psi,\Phi)\|L-L_n\|_\Phi \xrightarrow{n\to\infty} 0 \end{aligned} \tag{7.41}$$

for some constant $C_{39}(\Psi,\Phi) < \infty$ depending only on Ψ and Φ since the norms $\|\cdot\|_\Psi$ and $\|\cdot\|_\Phi$ are equivalent by Lemma 7.11p.121 (Dirac sea class properties). This shows that $\Lambda(\Phi + D)$ is a countable, dense subset of $\Lambda(\mathcal{S})$. □

The following diagram summarizes the setup:



Note that, by Lemma 7.12p.121 (Connection between $\sim$ and $\approx_0$), $\mathcal{F}_\mathcal{S}$ carries the whole information of the polarization class $C \in \mathtt{Pol}(\mathcal{H})/\approx_0$. However, it depends on a choice of basis. In this sense we say that the wedge space $\mathcal{F}_\mathcal{S}$ belongs to polarization class C.

7.2.2 Operations from the Left and from the Right

Having constructed the infinite wedge spaces $\mathcal{F}_\mathcal{S}$ for each $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$ we now introduce two types of operations on them which are the tools needed in the next subsection. In the following let $\mathcal{H}'$, ℓ' be also two Hilbert spaces.

Construction 7.20.

The left operation

1. *The following operation from the left is well-defined:*

$$\mathrm{U}(\mathcal{H},\mathcal{H}') \times \mathtt{Seas}_\ell(\mathcal{H}) \to \mathtt{Seas}_\ell(\mathcal{H}'), \quad (U,\Phi) \mapsto U\Phi.$$

2. *This operation from the left is compatible with the equivalence relation $\sim$ in the following sense: For $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$ and $\Phi,\Psi \in \mathtt{Seas}_\ell(\mathcal{H})$, one has $\Phi \sim \Psi$ if and only if $U\Phi \sim U\Psi$ in $\mathtt{Seas}_\ell(\mathcal{H}')$. Thus, the action of U on $\mathtt{Seas}_\ell(\mathcal{H})$ from the left induces also an operation from the left on equivalence classes modulo $\sim$ as follows. For $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\sim$ and $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$,*

$$U\mathcal{S} = \{U\Phi \mid \Phi \in \mathcal{S}\} \in \mathtt{Seas}_\ell(\mathcal{H}')/\sim.$$

3. *For $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$ and $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\!\sim$, the induced operation $\mathcal{L}_U : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(U\mathcal{S})}$, given by*

$$\mathcal{L}_U\left(\sum_{\Phi\in\mathcal{S}} \alpha(\Phi)[\Phi]\right) = \sum_{\Phi\in\mathcal{S}} \alpha(\Phi)[U\Phi],$$

is an isometry with respect to the hermitian forms $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{(\mathcal{S})}$ and on $\mathbb{C}^{(U\mathcal{S})}$. In particular one has $\mathcal{L}_U[N_\mathcal{S}] \subseteq N_{U\mathcal{S}}$.

4. *For every $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$, the operation from the left $\mathcal{L}_U : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(U\mathcal{S})}$ induces a unitary map $\mathcal{L}_U : \mathcal{F}_\mathcal{S} \to \mathcal{F}_{U\mathcal{S}}$, characterized by $\mathcal{L}_U(\Lambda\Phi) = \Lambda(U\Phi)$ for $\Phi \in \mathcal{S}$. This operation is functorial in the following sense. Let $\mathcal{H}''$ be another Hilbert space. For $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$, $V \in \mathrm{U}(\mathcal{H}',\mathcal{H}'')$ and $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\!\sim$, one has $\mathcal{L}_U\mathcal{L}_V = \mathcal{L}_{UV} : \mathcal{F}_\mathcal{S} \to \mathcal{F}_{UV\mathcal{S}}$ and $\mathcal{L}_{\mathrm{id}_\mathcal{H}} = \mathrm{id}_{\mathcal{F}_\mathcal{S}}$.*

In complete analogy to the operation from the left, we introduce next an operation from the right. Let ℓ' be another Hilbert space, and let $\mathrm{GL}_-(\ell',\ell)$ denote the set of all bounded invertible linear operators $R : \ell' \to \ell$ with the property $R^*R \in \mathrm{id}_{\ell'} + \mathrm{I}_1(\ell')$. Note that $\mathrm{GL}_-(\ell) := \mathrm{GL}_-(\ell,\ell)$ is a group with respect to composition.

Operation from the Right

Construction 7.21.

1. *The following operation from the right is well-defined:*

$$\mathtt{Seas}_\ell(\mathcal{H}) \times \mathrm{GL}_-(\ell',\ell) \to \mathtt{Seas}_{\ell'}(\mathcal{H}), \quad (\Phi,R) \mapsto \Phi R.$$

2. *This operation from the right is compatible with the equivalence relations $\sim$: For $\Phi,\Psi \in \mathtt{Seas}_\ell(\mathcal{H})$ and $R \in \mathrm{GL}_-(\ell',\ell)$, one has $\Phi \sim \Psi$ if and only if $\Phi R \sim \Psi R$ in $\mathtt{Seas}_{\ell'}(\mathcal{H})$. Thus, the operation of R from the right induces also an operation from the right on equivalence classes modulo $\sim$ as follows: For $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\!\sim$ and $R \in \mathrm{GL}_-(\ell',\ell)$,*

$$\mathcal{S}R = \{\Phi R \mid \Phi \in \mathcal{S}\} \in \mathtt{Seas}_{\ell'}(\mathcal{H})/\!\sim.$$

3. *For $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\!\sim$ and $R \in \mathrm{GL}_-(\ell',\ell)$ the induced operation from the right $\mathcal{R}_R : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(\mathcal{S}R)}$ given by*

$$\mathcal{R}_R\left(\sum_{\Phi\in\mathcal{S}} \alpha(\Phi)[\Phi]\right) = \sum_{\Phi\in\mathcal{S}} \alpha(\Phi)[\Phi R],$$

is an isometry up to scaling with respect to the hermitian forms $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{(\mathcal{S})}$ and on $\mathbb{C}^{(\mathcal{S}R)}$. More precisely, one has for all $\alpha,\beta \in \mathbb{C}^{(\mathcal{S})}$:

$$\langle \mathcal{R}_R\alpha, \mathcal{R}_R\beta\rangle = \det(R^*R)\,\langle\alpha,\beta\rangle\,.$$

In particular one has $\mathcal{R}_R[N_\mathcal{S}] \subseteq N_{\mathcal{S}R}$.

4. *For every $R \in \mathrm{GL}_-(\ell',\ell)$, the operation $\mathcal{R}_R : \mathbb{C}^{(\mathcal{S})} \to \mathbb{C}^{(\mathcal{S}R)}$ induces a bounded linear map, again called $\mathcal{R}_R : \mathcal{F}_\mathcal{S} \to \mathcal{F}_{\mathcal{S}R}$, characterized by $\mathcal{R}_R(\Lambda\Phi) = \Lambda(\Phi R)$ for $\Phi \in \mathcal{S}$. Up to scaling, this map is unitary. More precisely, for $\Phi,\Psi \in \mathcal{F}_\mathcal{S}$, one has*

$$\langle \mathcal{R}_R\Phi, \mathcal{R}_R\Psi\rangle = \det(R^*R)\,\langle\Phi,\Psi\rangle\,.$$

The operation $\mathcal{R}_R$ is contra-variantly functorial in the following sense. Let ℓ'' be another Hilbert space. For $Q \in \mathrm{GL}_-(\ell'',\ell')$, $R \in \mathrm{GL}_-(\ell',\ell)$ and $\mathcal{S} \in \mathtt{Seas}(\mathcal{H},\ell)/\!\sim$, one has $\mathcal{R}_Q\mathcal{R}_R = \mathcal{R}_{RQ} : \mathcal{F}_\mathcal{S} \to \mathcal{F}_{\mathcal{S}RQ}$ and $\mathcal{R}_{\mathrm{id}_\ell} = \mathrm{id}_{\mathcal{F}_\mathcal{S}}$.

The associativity of composition $(U\Phi)R = U(\Phi R)$ immediately yields:

Lemma 7.22. *The operations from the left and from the right commute: For $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$, $R \in \mathrm{GL}_-(\ell',\ell)$, and $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\sim$, one has $\mathcal{L}_U\mathcal{R}_R = \mathcal{R}_R\mathcal{L}_U : \mathcal{F}_\mathcal{S} \to \mathcal{F}_{U\mathcal{S}R}$.*

Left and right operations commute

We conclude this subsection with a last lemma that states an important property of the infinite wedge spaces. Essentially, it says that for any $R \in \mathrm{GL}_-(l)$ such that R has a determinant, we have $\mathcal{F}_\mathcal{S} = \mathcal{F}_{\mathcal{S}R}$. We introduce $\mathrm{SL}(\ell)$ to denote the set of all operators $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ with the property $\det R = 1$.

Lemma 7.23.

Uniqueness up to a phase

1. *For all $R \in \mathrm{GL}_-(\ell)$ and $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\sim$, one has $\mathcal{S} = \mathcal{S}R$ if and only if R has a determinant. In this case, $\mathcal{R}_R(\Psi) = (\det R)\Psi$ holds for all $\Psi \in \mathcal{F}_\mathcal{S}$. As a special case, if $R \in \mathrm{SL}(\ell)$, then $\mathcal{R}_R : \mathcal{F}_\mathcal{S} \to \mathcal{F}_\mathcal{S}$ is the identity map.*

2. *For all $Q, R \in \mathrm{GL}_-(\ell',\ell)$ and $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\sim$, we have $\mathcal{S}R = \mathcal{S}Q$ if and only if $Q^{-1}R \in \mathrm{GL}_-(\ell')$ has a determinant. In this case, one has for all $\Psi \in \mathcal{F}_\mathcal{S}$:*

$$\mathcal{R}_R\Psi = \det(Q^{-1}R)\mathcal{R}_Q\Psi$$

Proof. (a) Given $R \in \mathrm{GL}_-(\ell)$ and $\mathcal{S} \in \mathtt{Seas}_\ell(\mathcal{H})/\sim$, take any $\Phi \in \mathcal{S}$. Then, as $\Phi^*\Phi$ has a determinant, $\Phi^*\Phi R$ has a determinant if and only if R has a determinant. This is equivalent to $\Phi \sim \Phi R$ and to $\mathcal{S} = \mathcal{S}R$. In this case, Lemma 7.18p.124 (Null space) implies $\mathcal{R}_R\Psi = (\det R)\Psi$ for all $\Psi \in \mathcal{F}_\mathcal{S}$.

(b) Let $\Phi \in \mathcal{S} \cap \mathtt{Seas}^\perp_\ell(\mathcal{H})$. Then $\mathcal{S}R = \mathcal{S}Q$ holds if and only if $\Phi R \sim \Phi Q$, i.e. if and only if $Q^*R = (\Phi Q)^*\Phi R$ has a determinant. Since Q^*Q has a determinant and is invertible, this is equivalent to $Q^{-1}R \in \mathrm{id}_{\ell'} + \mathrm{I}_1(\ell')$. Using part (a), for any $\Psi \in \mathcal{F}_\mathcal{S}$, we have in this case: $\mathcal{R}_R\Psi = \mathcal{R}_{Q^{-1}R}\mathcal{R}_Q\Psi = \det(Q^{-1}R)\mathcal{R}_Q\Psi$.

□

REMARK 7.24. *The difference between the two operations is that for a Φ in $\mathtt{Seas}_\ell(\mathcal{H})$ the left operation in general changes the range of Φ while the right operation does not. The operation from the left will later be used to implement the lift of unitary one-particle operators like the Dirac time-evolution on $\mathcal{H}$. The right operation will be used to adjust the vacuum state; this will be discussed in detail when interpreting the pair creation rates in Subsection 7.4**p.151**.*

7.2.3 Lift Condition

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ and two polarization classes $C \in \mathtt{Pol}(\mathcal{H})/\approx_0$ and $C' \in \mathtt{Pol}(\mathcal{H})/\approx_0$, we now identify conditions under which a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ can be lifted to a unitary map between two wedge spaces.

By Lemma 7.2p.117 (Properties of ≈) it is clear how any unitary $U : \mathcal{H} \to \mathcal{H}'$ acts on polarization classes, and we shall not prove the following simple lemma:

Lemma 7.25. *The natural operation*

Action of U on polarization classes

$$\mathrm{U}(\mathcal{H},\mathcal{H}') \times \mathtt{Pol}(\mathcal{H}) \to \mathtt{Pol}(\mathcal{H}'), \quad (U, V) \mapsto UV = \{Uv \mid v \in V\}$$

is compatible with the equivalence relations ≈ in the following sense: For $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$ and $V, W \in \mathtt{Pol}(\mathcal{H})$, one has $V \approx W$ if and only if $UV \approx UW$. As a consequence, this operation from the left induces a natural operation on polarization classes $\mathrm{U}(\mathcal{H},\mathcal{H}') \times (\mathtt{Pol}(\mathcal{H})/\approx) \to \mathtt{Pol}(\mathcal{H}')/\approx$, $(U, [V]_\approx) \mapsto [UV]_\approx$.

We use the knowledge about the action of $U \in \mathrm{U}(\mathcal{H},\mathcal{H}')$ on the polarization classes to give $\mathrm{U}(\mathcal{H},\mathcal{H}')$ a finer structure, the restricted set of unitary operators:

Restricted set of unitary operators

Definition 7.26. *Given the polarization classes $C \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$ and $C' \in \mathtt{Pol}(\mathcal{H}')/{\approx_0}$ we define*

$$\begin{aligned}\mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}',C') :&= \{U \in \mathrm{U}(\mathcal{H},\mathcal{H}') \mid \textit{for all } V \in C \textit{ holds } UV \in C'\}\\ &= \{U \in \mathrm{U}(\mathcal{H},\mathcal{H}') \mid \textit{there exists } V \in C \textit{ such that } UV \in C'\}.\end{aligned}$$

As a special case, we yield a group $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C) := \mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H},C)$.

Note that for a third Hilbert space $\mathcal{H}''$ with a polarization class $C'' \in \mathtt{Pol}(\mathcal{H}'')/{\approx_0}$, one has $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H}',C';\mathcal{H}'',C'')\mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}',C') = \mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}'',C'')$. In fact, one could now define $\mathrm{U}^c_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}',C')$ for unitary operations that change the relative charge of two polarizations by $c \in \mathbb{Z}$.

Now we have all what is needed to prove the main result of this section: The following theorem is our version of the classical Shale-Stinespring theorem [SS65], and hence not completely new.

Lift condition

Theorem 7.27. *For given polarization classes $C \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$ and $C' \in \mathtt{Pol}(\mathcal{H}')/{\approx_0}$, let $\mathcal{S} \in \mathtt{Ocean}_\ell(C)/{\sim}$ and $\mathcal{S}' \in \mathtt{Ocean}_\ell(C')/{\sim}$. Then, for any unitary map $U : \mathcal{H} \to \mathcal{H}'$, the following are equivalent:*

1. *There is $R \in U(\ell)$ such that $U\mathcal{S}R = \mathcal{S}'$, and hence $\mathcal{R}_R\mathcal{L}_U$ maps $\mathcal{F}_\mathcal{S}$ to $\mathcal{F}_{\mathcal{S}'}$.*

(a') *There is $R \in \mathrm{GL}_-(\ell)$ such that $U\Phi R \sim \Phi'$.*

(b) *$U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}',C')$.*

Proof.

$(a) \Rightarrow (b)$: Take $R \in U(\ell)$ such that $U\mathcal{S}(\Phi)R = \mathcal{S}(\Phi')$. In particular, $U\Phi R \sim \Phi'$, and hence $\Phi'^*U\Phi R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$. This implies

$$(\Phi'^*U\Phi R)^*\Phi'^*U\Phi R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell). \tag{7.42}$$

Because $U\Phi R : \ell \to UV$ is unitary and $\Phi'\Phi'^* = P_{V'}$, we conclude that $P_{UV}P_{V'}P_{UV} = P_{UV}\Phi'\Phi'^*P_{UV}|_{UV\to UV}$ has a determinant. Similarly,

$$\Phi'^*P_{UV}\Phi^* = \Phi'^*U\Phi R(\Phi'^*U\Phi R)^* \in \mathrm{id}_\ell + \mathrm{I}_1(\ell) \tag{7.43}$$

implies that $P_{V'}P_{UV}P_{V'}|_{V'\to V'}$ has also a determinant. Together this yields $UV \approx V'$ by Lemma 7.2p.117 (Properties of $\approx$).

Furthermore, because of $U\Phi R \sim \Phi'$, we know that $\Phi'^*U\Phi R$ is a Fredholm operator with index 0. Since $\Phi R : \ell \to V$ and $\Phi' : \ell \to V'$ are unitary, $P_{V'}|_{UV\to V'}$ is also a Fredholm operator with index 0, i.e. $\mathrm{charge}(UV, V') = 0$. This shows $UV \approx_0 V'$, and the claim $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}',C')$ follows.

$(b) \Rightarrow (a')$: We abbreviate $A = P_{V'}|_{UV\to V'}$. The assumption $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H},C;\mathcal{H}',C')$ implies $A^*A \in \mathrm{id}_{UV} + \mathrm{I}_1(UV)$, and A is a Fredholm operator with index $\mathrm{ind}\, A = 0$. Using that $\Phi : \ell \to V$ and $\Phi' : \ell \to V'$ are unitary maps, we rewrite this in the form $(\Phi'^*U\Phi)^*\Phi'^*U\Phi \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$, and $\Phi'^*U\Phi$ is a Fredholm operator with $\mathrm{ind}(\Phi'^*U\Phi) = 0$. We now use a polar decomposition of $\Phi'^*U\Phi$ in the form $\Phi'^*U\Phi = BQ$, where $B : \ell \to \ell$ is positive semi-definite and $Q : \ell \to \ell$ is unitary. Note that we can take Q to be *u*nitary, not only a partial isometry, as $\Phi'^*U\Phi$ has the Fredholm index 0. Taking $R = Q^{-1}$, we get $\Phi'^*U\Phi R = B$. Now $B^2 = B^*B$ has a determinant because $Q^*B^*BQ = (\Phi'^*U\Phi)^*\Phi'^*U\Phi$ has a determinant. Since $B \geq 0$, this implies that B has also a determinant. We conclude $U\Phi R \sim \Phi'$.

$(a') \Rightarrow (a)$: We take $R \in \mathrm{GL}_{-}(\ell)$ with $U\Phi R \sim \Phi'$. By polar decomposition, we write R in the form $R = R'Q$, where $R' : \ell \to \ell$ is unitary and $Q : \ell \to \ell$ is invertible, positive definite, and has a determinant. As $\Phi'^* U\Phi R = \Phi'^* U\Phi R'Q$ and Q both have determinants, $\Phi'^* U\Phi R'$ has also a determinant. This shows $U\Phi R' \sim \Phi'$ and hence $\mathcal{S}(U\Phi R') = U\mathcal{S}(\Phi)R' = \mathcal{S}(\Phi')$. In particular, $\mathcal{R}_R\mathcal{L}_U$ maps $\mathcal{F}_{\mathcal{S}(\Phi)}$ to $\mathcal{F}_{U\mathcal{S}(\Phi)R'} = \mathcal{F}_{\mathcal{S}(\Phi')}$.

□

For $U = \mathrm{id}_{\mathcal{H}}$ we immediately get:

Corollary 7.28. *Let* $C \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$ *and* $\mathcal{S} \in \mathtt{Ocean}(C)/{\sim}$ *then we have* *Orbits in* Ocean

$$\mathtt{Ocean}(C)/{\sim} = \{\mathcal{S}R \mid R \in \mathrm{U}(\ell)\}.$$

This is the counterpart to Lemma 7.12$_{\text{p.121}}$ (Connection between $\sim$ and $\approx_0$) which stated that for every polarization class C there is a whole "ocean of seas" $\mathtt{Ocean}(C)/{\sim}$ which belongs to it. For every $S \in \mathtt{Ocean}(C)/{\sim}$ we constructed a wedge space $\mathcal{F}_{\mathcal{S}}$. Now the corollary above states that all these wedge spaces $\{\mathcal{F}_{\mathcal{S}} \mid \mathcal{S} \in \mathtt{Ocean}(C)/{\sim}\}$ are related to each other by unitary operations from the right; this is illustrated in Figure 7.1$_{\text{p.130}}$.

Furthermore, together with Lemma 7.23$_{\text{p.127}}$ (Uniqueness up to a phase) one gets:

Corollary 7.29. *For* $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}', C')$ *let* $R \in \mathrm{U}(\ell)$ *be as in Theorem 7.27*$_{\text{p.128}}$ *(Lift condition). Then the elements of the set* *Uniqueness of the lift up to a phase*

$$\{\mathcal{R}_Q\mathcal{R}_R\mathcal{L}_U \mid Q \in \mathrm{U}(\ell) \cap (\mathrm{id}_\ell + \mathrm{I}_1(\ell))\} = \{e^{i\varphi}\mathcal{R}_R\mathcal{L}_U \mid \varphi \in \mathbb{R}\}$$

are the only unitary maps from $\mathcal{F}_{\mathcal{S}}$ *to* $\mathcal{F}_{\mathcal{S}'}$ *in the set* $\{\mathcal{R}_T\mathcal{L}_U \mid T \in \mathrm{U}(\ell)\}$.

In this sense we refer to the lift $\mathcal{L}_U\mathcal{R}_R$ as being unique up to a phase. A typical situation is this: Consider, for example, the one-particle Dirac time-evolution $U : \mathcal{H} \to \mathcal{H}$ and assume that $U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C')$ for two given polarization classes $C, C' \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$. We choose $\Phi, \Phi' \in \mathtt{Seas}^{\perp}(\mathcal{H})$ such that $\operatorname{range}\Phi \in C$ and $\operatorname{range}\Phi' \in C'$. By Lemma 7.12$_{\text{p.121}}$ (Connection between $\sim$ and $\approx_0$) it follows that $\mathcal{S} = \mathcal{S}(\Phi)_{\sim} \in \mathtt{Ocean}(C)/{\sim}$ and $\mathcal{S}' = \mathcal{S}(\Phi) \in \mathtt{Ocean}(C')/{\sim}$ from which we built our wedge spaces $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{F}_{\mathcal{S}'}$ which elements represent the "in" and "out" states, respectively. Theorem 7.27$_{\text{p.128}}$ (Lift condition) and Corollary 7.29 (Uniqueness of the lift up to a phase) assure for the $\mathcal{S}, \mathcal{S}'$ that there is an $R \in \mathrm{U}(\ell)$ such that

$$\mathcal{F}_{\mathcal{S}} \xrightarrow{\mathcal{L}_U} \mathcal{F}_{U\mathcal{S}} \xrightarrow{\mathcal{R}_R} \mathcal{F}_{U\mathcal{S}R} = \mathcal{F}_{\mathcal{S}'} \circlearrowleft e^{i\varphi} \qquad \varphi \in \mathbb{R}.$$

We have illustrated this situation in Figure 7.1$_{\text{p.130}}$.

In Subsection 7.3.1$_{\text{p.130}}$ we show that the one-particle Dirac time-evolution $U^{\mathsf{A}}(t_1, t_0)$ for times t_0 and t_1 in any external, smooth and compactly supported field A is in $\mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C')$ for specific $C, C' \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$. We will show in Subsection 7.3.2$_{\text{p.144}}$ that C and C' are uniquely identified by the magnetic components of A at the times t_0 and t, respectively. In this sense, for $U = U^{\mathsf{A}}$ there exists a natural lift $\mathcal{R}_R\mathcal{L}_U$ which is unique up to a phase. This will be summarized in Subsection 7.3.3$_{\text{p.150}}$.

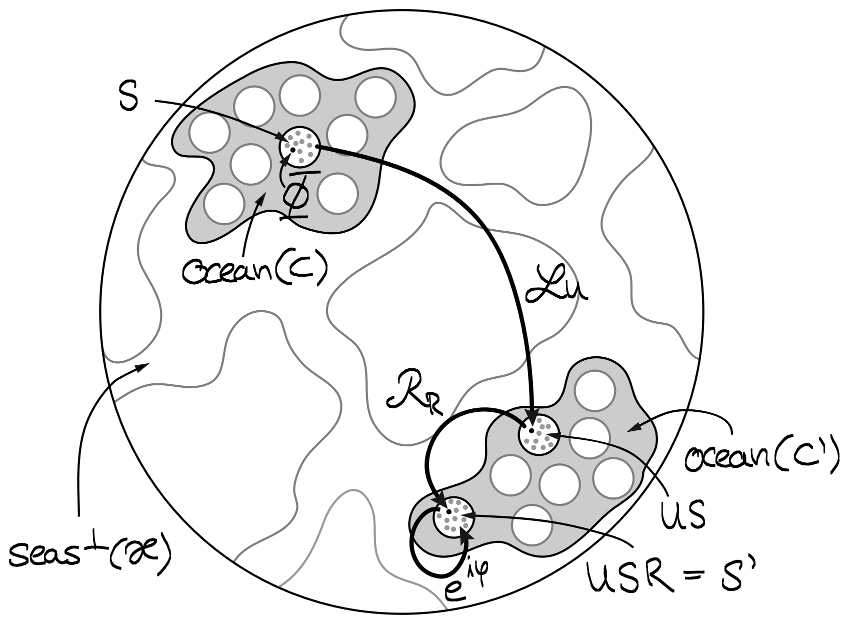


Figure 7.1: A sketch of the time-evolution.

7.3 The Time-Evolution of Dirac Seas

We now come to the one-particle Dirac time-evolution in an external four-vector field $\mathsf{A} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$, i.e. the set of infinitely often differentiable $\mathbb{R}^4$ valued functions on $\mathbb{R}^4$ with compact support. Recall the discussion at the end of Subsection 7.2.3p.127: In order to apply Theorem 7.27p.128 (Lift condition) to the one-particle Dirac time-evolution $U^{\mathsf{A}}(t_1, t_0)$ for fixed $t_0, t_1 \in \mathbb{R}$ and in this way to obtain a lift to unitary maps from one wedge space to another (the second quantized time-evolution) we need to show that $U^{\mathsf{A}}(t_1, t_0) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1))$ for appropriate $C(t_0), C(t_1) \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$. To ensure this condition holds is the main content of this last section.

This section is structured as follows: In the first subsection we show that for any $t_0, t_1 \in \mathbb{R}$ there always exist $C(t_0), C(t_1) \in \mathtt{Pol}(\mathcal{H})/{\approx_0}$, depending only on $\mathsf{A}(t_0)$ and $\mathsf{A}(t_1)$, respectively, such that $U^{\mathsf{A}}(t_1, t_0) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1))$. In the second subsection we identify the polarization classes $C(t)$ uniquely by the magnetic components of $\mathsf{A}(t)$ for all $t \in \mathbb{R}$. The third subsection combines these results with Section 7.2p.116 and shows the existence of the second quantized Dirac time-evolution for the external field problem in quantum electrodyanmics. Finally, the fourth subsection then concludes with the analysis of second quantized gauge transformations as unitary maps between varying Fock spaces.

7.3.1 One-Particle Time-Evolution

Throughout this section we work with $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$, with $\mathbb{R}^3$ being interpreted as momentum space. The free Dirac equation in momentum representation is given by

$$i\frac{d}{dt}\psi^0(t) = H^0\psi^0(t) \tag{7.44}$$

for $\psi^0(t) \in \text{domain}(H^0) \subset \mathcal{H}$ where

$$H^0(p) = \alpha \cdot p + \beta m = \sum_{\mu=1}^{3} \alpha^\mu p_\mu + \beta m, \quad p \in \mathbb{R}^3 \tag{7.45}$$

and the $\mathbb{C}^{4\times4}$ Dirac matrices β and α^μ, $\mu = 1, 2, 3$, fulfill

$$\begin{aligned} \beta^2 &= 1 & \{\alpha^\mu, \beta\} &:= \alpha^\mu \beta + \beta \alpha^\mu = 0 \\ (\alpha^\mu)^2 &= 1 & \{\alpha^\mu, \alpha^\nu\} &:= \alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = 2\delta^{\mu\nu}. \end{aligned} \tag{7.46}$$

Which specific representation of this matrix algebra with hermitian matrices is used does not affect any of the following arguments. For convenience we introduce also

$$\alpha^0 = 1.$$

$H^0(p)$ is a self-adjoint multiplication operator which therefore gives rise to a one-parameter group of operators

$$U^0(t_1, t_0) := \exp(-iH^0(t_1 - t_0)) \tag{7.47}$$

on $\mathcal{H}$ for all $t_0, t_1 \in \mathbb{R}$. For every solution $\psi^0(t)$ of the free Dirac equation (7.44p.130) one has $\psi^0(t_1) = U^0(t_1, t_0)\psi^0(t_0)$. The matrix $H^0(p)$ has double eigenvalues $\pm E(p)$, where $E(p) = \sqrt{|p|^2 + m^2} > 0$, $p \in \mathbb{R}^3$. Therefore, the spectrum of the free Dirac operator is $\sigma(H^0) = (-\infty, -m] \cup [+m, +\infty)$ and the corresponding free spectral projectors $P_\pm$ are multiplication operators with the matrices

$$P_\pm(p) = \frac{1}{2}\left(1 \pm \frac{H^0(p)}{E(p)}\right). \tag{7.48}$$

We define $\mathcal{H}_\pm := P_\pm \mathcal{H}$ for which $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$. For any linear operator L on $\mathcal{H}$ and signs $\sigma, \tau \in \{+, -\}$ we write $L_{\sigma\tau} = P_\sigma L P_\tau$. Furthermore, $L_{\text{ev}} = L_{++} + L_{--}$ denotes the even (diagonal) part, and $L_{\text{odd}} = L_{+-} + L_{-+}$ for the odd (non-diagonal) part of L. If L has an integral kernel $(q, p) \mapsto L(p, q)$ it follows by equation (7.48) that the kernel of $L_{\sigma\tau}$ is given by $(p, q) \mapsto L_{\sigma\tau}(p, q) = P_\sigma(p) L(p, q) P_\tau(q)$.

Now let $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ be a smooth, compactly supported, external four-vector field. We denote its time slice at time $t \in \mathbb{R}$ by $\mathsf{A}(t) = (\mathbb{R}^3 \ni x \mapsto (\mathsf{A}_\mu(t, x))_{\mu=0,1,2,3})$. The Dirac equation with the external field A in momentum representation is then given by

$$i\frac{d}{dt}\psi(t) = H^{\mathsf{A}(t)}\psi(t) = \left(H^0 + iZ^{\mathsf{A}(t)}\right)\psi(t) \tag{7.49}$$

where for $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A}) \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, the operator Z^A on $\mathcal{H}$ is defined as follows:

$$iZ^A = \sum_{\mu=0}^{3} \alpha^\mu \widehat{A_\mu}. \tag{7.50}$$

Here we understand $\widehat{A_\mu}$, $\mu = 0, 1, 2, 3$, as convolution operators

$$(\widehat{A_\mu}\psi)(p) = \int_{\mathbb{R}^3} \widehat{A_\mu}(p - q)\psi(q)\, dq, \quad p \in \mathbb{R}^3. \tag{7.51}$$

for $\psi \in \mathcal{H}$ and $\widehat{A_\mu}$ being the Fourier transform of A_μ given by

$$\widehat{A_\mu}(p) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ipx} A_\mu(x)\, dx. \tag{7.52}$$

Therefore, in momentum representation, Z^A is an integral operator with integral kernel

$$(p,q) \mapsto Z^A(p-q) = -i\sum_{\mu=0}^{3} \alpha^\mu \widehat{A_\mu}(p-q), \quad p,q \in \mathbb{R}^3. \tag{7.53}$$

The Dirac equation with external field A gives also rise to a family of unitary operators $(U^{\mathsf{A}}(t_1,t_0))_{t_0,t_1\in\mathbb{R}}$ on $\mathcal{H}$ which fulfill

$$\frac{\partial}{\partial t_1} U^{\mathsf{A}}(t_1,t_0) = -iH^{\mathsf{A}(t_1)}U^{\mathsf{A}}(t_1,t_0), \tag{7.54}$$

$$\frac{\partial}{\partial t_0} U^{\mathsf{A}}(t_1,t_0) = iU^{\mathsf{A}}(t_1,t_0)H^{\mathsf{A}(t_0)} \tag{7.55}$$

on the appropriate domains, such that for every solution $\psi(t)$ of equation (7.49p.131) one has $\psi(t_1) = U^{\mathsf{A}}(t_1,t_0)\psi(t_0)$; see [Tha93].

In order to present the main result of this section in a short form we introduce:

Induced polarization classes

Definition 7.30. *For $A \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^4)$, we define the integral operator $Q^A : \mathcal{H} \to \mathcal{H}$ by its integral kernel, also denoted by Q^A:*

$$\mathbb{R}^3\times\mathbb{R}^3 \ni (p,q) \mapsto Q^A(p,q) := \frac{Z^A_{+-}(p,q) - Z^A_{-+}(p,q)}{i(E(p)+E(q))} \tag{7.56}$$
$$\text{with } Z^A_{\pm\mp}(p,q) := P_\pm(p)Z^A(p-q)P_\mp(q).$$

Furthermore, we define the polarization class $C(0) := [\mathcal{H}_-]_{\approx_0}$ belonging to the negative spectral space $\mathcal{H}_-$ of the free Dirac operator H^0, and therewith the polarization classes

$$C(A) := e^{Q^A}C(0) = \{e^{Q^A}V \mid V \in C(0)\}. \tag{7.57}$$

The operators Q^A are bounded and skew-adjoint. They will appear naturally in the iterative scheme that we use to control the time-evolution, and their origin will become clear as we go along (Lemma 7.35p.137).

We now state the main result of this section, using the notation of Section 7.2p.116.

Dirac time-evolution with external field

Theorem 7.31. *For all $\mathsf{A} \in C_c^\infty(\mathbb{R}^4,\mathbb{R}^4)$ and for all $t_1,t_0 \in \mathbb{R}$ it is true that*

$$U^{\mathsf{A}}(t_1,t_0) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(\mathsf{A}(t_0));\ \mathcal{H}, C(\mathsf{A}(t_1))).$$

We do *n*ot focus on finding the weakest regularity conditions on the external four-vector potential A under which this theorem holds, although much weaker conditions will suffice. Actually, the theorem and also its proof remain valid for four-vector potentials A in the following class $\mathcal{A} \supset C_c^\infty(\mathbb{R}^4,\mathbb{R}^4)$:

Class of external four-vector potentials

Definition 7.32. *Let $\mathcal{A}$ be the class of four-vector potentials $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} : \mathbb{R}^4 \to \mathbb{R}^4$ such that for all $\mu = 0,1,2,3$, $m = 0,1,2$ and $p = 1,2$ the integral*

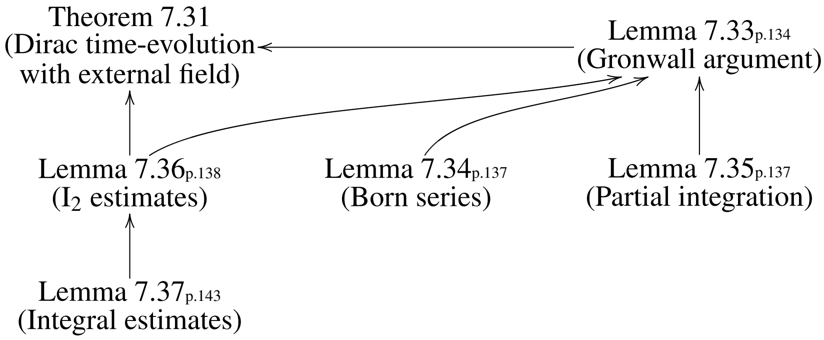
$$\int_{\mathbb{R}} \left\| \frac{d^m}{dt^m}\widehat{\mathsf{A}}_\mu(t)\right\|_p dt \tag{7.58}$$

exists and is finite. Here $\widehat{\mathsf{A}}_\mu(t)$ denotes the Fourier transform of a time slice $\mathsf{A}_\mu(t)$ with respect to the spatial coordinates.

This class of four-vector potentials has also been considered by Scharf in his analysis of the second-quantized scattering operator in an external potential (Theorem 5.1 in [Sch95]). We remark that the class $\mathcal{A}$ does *not* contain the Coulomb potential, not even when one truncates it at large times.

Skeleton of the proof of Theorem 7.31

Since quite some computation is involved in the proof of the above theorem, we split it up into a series of small lemmas, to separate technicalities from ideas. Here is the skeleton of the proof:



The key ideas are worked out in Lemma 7.33$_{\text{p.134}}$ (Gronwall argument). The other lemmas have a more technical character.

In the following, when dealing with a given external vector potential $\mathsf{A} \in \mathcal{A}$, we abbreviate $U(t_1, t_0) = U^{\mathsf{A}}(t_1, t_0)$, $H(t) = H^{\mathsf{A}(t)}$, $Z(t) = Z^{\mathsf{A}(t)}$, and $Q(t) = Q^{\mathsf{A}(t)}$. We start with putting things together:

Dirac time-evolution with external field

Proof of Theorem 7.31$_{p.132}$. By Lemma 7.2$_{\text{p.117}}$(b), we need only to show that for some $V \in C(\mathsf{A}(t_0))$ and some $W \in C(\mathsf{A}(t_1))$ it is true that

$$P_{V^\perp} U(t_1, t_0) P_W, P_V U(t_1, t_0) P_{W^\perp} \in \mathrm{I}_2(\mathcal{H}). \tag{7.59}$$

Let us choose $V = e^{Q(t_0)}\mathcal{H}_- \in C(\mathsf{A}(t_0))$ and $W = e^{Q(t_1)}\mathcal{H}_- \in C(\mathsf{A}(t_1))$. Then, claim (7.59) is equivalent to

$$e^{Q(t_1)} P_\pm e^{-Q(t_1)} U(t_1, t_0) e^{Q(t_0)} P_\mp e^{-Q(t_0)} \in \mathrm{I}_2(\mathcal{H}). \tag{7.60}$$

Since $e^{Q(t_1)}$ and $e^{-Q(t_0)}$ are both unitary operators, this claim is equivalent to

$$P_\pm e^{-Q(t_1)} U(t_1, t_0) e^{Q(t_0)} P_\mp \in \mathrm{I}_2(\mathcal{H}). \tag{7.61}$$

Now $Q(t)$ is a bounded operator, and Lemma 7.36$_{\text{p.138}}$ (I_2 estimates) states that $Q^2(t) \in \mathrm{I}_2(\mathcal{H})$ for any time $t \in \mathbb{R}$. Therefore, by expanding $e^{\pm Q(t)}$ in its series, we find that

$$e^{\pm Q(t)} - (\mathrm{id}_{\mathcal{H}} \pm Q(t)) \in \mathrm{I}_2(\mathcal{H}). \tag{7.62}$$

Hence it suffices to prove

$$P_\pm(\mathrm{id}_{\mathcal{H}} - Q(t_1)) U(t_1, t_0)(\mathrm{id}_{\mathcal{H}} + Q(t_0)) P_\mp \in \mathrm{I}_2(\mathcal{H}). \tag{7.63}$$

This is just the claim (7.66$_{\text{p.134}}$) of Lemma 7.33$_{\text{p.134}}$ (Gronwall Argument) and concludes the proof. □

The following stenographic notation will be very convenient: For families of operators $A = (A(t_1, t_0))_{t_1 \geq t_0}$ and $B = (B(t_1, t_0))_{t_1 \geq t_0}$, indexed by time intervals $[t_0, t_1] \subset \mathbb{R}$, we set

$$AB = \left(\int_{t_0}^{t_1} A(t_1, t) B(t, t_0)\, dt \right)_{t_1 \geq t_0},$$

whenever this is well-defined. Furthermore, if $C = (C(t))_{t\in\mathbb{R}}$ and $D = (D(t))_{t\in R}$ denote families of operators indexed by time points, we use the abbreviations $AC = (A(t_1,t_0)C(t_0))_{t_1\geq t_0}$, $CA = (C(t_1)A(t_1,t_0))_{t_1\geq t_0}$, and $CD = (C(t)D(t))_{t\in\mathbb{R}}$. Recall that $\|\cdot\|$ denotes the operator norm on bounded operators on $\mathcal{H}$. We set

$$\begin{aligned} \|A\|_\infty &:= \sup_{s,t\in\mathbb{R}:\, s\geq t} \|A(s,t)\|, & \|A\|_{\mathrm{I}_2,\infty} &:= \sup_{s,t\in\mathbb{R}:\, s\geq t} \|A(s,t)\|_{\mathrm{I}_2}, \\ \|C\|_1 &:= \int_{\mathbb{R}} \|C(t)\|\, dt, & \|C\|_{\mathrm{I}_2,\infty} &:= \sup_{t\in\mathbb{R}} \|C(t)\|_{\mathrm{I}_2}, \end{aligned} \tag{7.64}$$

whenever these quantities exist. Recall Definition 7.30p.132 (Induced polarization classes) of the operators $Q(t) = Q^{\mathsf{A}(t)}$, and let $(Q'(t) : \mathcal{H} \to \mathcal{H})_{t\in\mathbb{R}}$ denote their time derivative, defined by using the time derivative of the corresponding kernels

$$(p,q) \mapsto Q'(t,p,q) := \frac{\partial}{\partial t} Q^{\mathsf{A}(t)}(p,q), \quad p,q\in\mathbb{R}^3, t\in\mathbb{R}. \tag{7.65}$$

We now state and prove the lemmas in the above diagram.

Gronwall argument

Lemma 7.33. *For all $t_0, t_1 \in \mathbb{R}$, the following holds:*

$$P_\pm(\mathrm{id}_{\mathcal{H}} - Q(t_1))U(t_1,t_0)(\mathrm{id}_{\mathcal{H}} + Q(t_0))P_\mp \in \mathrm{I}_2(\mathcal{H}) \tag{7.66}$$

Proof. Without loss of generality and to simplify the notation, we treat only the case $t_1 \geq t_0$. Let

$$R := (\mathrm{id}_{\mathcal{H}} - Q)U(\mathrm{id}_{\mathcal{H}} + Q). \tag{7.67}$$

The strategy is to expand R in a series and to check the Hilbert-Schmidt properties of the non-diagonal part term by term. Lemma 7.34p.137 (Fixed point form of the Dirac equation) states that the Dirac time-evolution U fulfills the fixed-point equation $U = U^0 + U^0ZU$ (equation (7.91p.137) below); recall that U^0 is the free Dirac time-evolution introduced in (7.47p.131). Iterating this fixed point equation once yields

$$U = U^0 + U^0ZU^0 + U^0ZU^0ZU.$$

Before going into the details, let us explain informally some ideas behind the subsequent proof. The first-order term U^0ZU^0 appears over and over again. Therefore, one may expect that its properties will be inherited by all other orders within the perturbation series. We therefore take a closer look at this term in Lemma 7.35p.137 (Partial integration). Equation (7.94p.137) in this lemma states

$$U^0ZU^0 = QU^0 - U^0Q - U^0Q'U^0 + U^0Z_{\mathrm{ev}}U^0.$$

One finds that the non-diagonal part $(U^0ZU^0)_{\mathrm{odd}}$ does in general not consist of Hilbert-Schmidt operators because of the first two terms $QU^0 - U^0Q$ on the right hand side, which are the boundary terms of the partial integration. However, we show now that the transformation induced by Q remedies these terms such that the non-diagonal part of R consists of Hilbert-Schmidt operators.

Substituting the formula (7.94p.137), cited above, into the fixed point equation $U = U^0 + U^0ZU$, we get

$$\begin{aligned} U &= U^0 + QU^0 - U^0Q - U^0Q'U^0 + U^0Z_{\mathrm{ev}}U^0 \\ &\quad + QU^0ZU - U^0QZU - U^0Q'U^0ZU + U^0Z_{\mathrm{ev}}U^0ZU \\ &= U^0 + QU - U^0Q - U^0Q'U + U^0Z_{\mathrm{ev}}U - U^0QZU. \end{aligned} \tag{7.68}$$

We rewrite this as

$$(\mathrm{id}_{\mathcal{H}} - Q)U = U^0(\mathrm{id}_{\mathcal{H}} - Q) + U^0(-Q' + Z_{\mathrm{ev}} - QZ)U. \tag{7.69}$$

Multiplying (7.69) with $\mathrm{id}_{\mathcal{H}} + Q$ from the right and using the equation

$$U(\mathrm{id}_{\mathcal{H}} + Q) = (\mathrm{id}_{\mathcal{H}} + Q)R + Q^2 U(\mathrm{id}_{\mathcal{H}} + Q) = U(\mathrm{id}_{\mathcal{H}} + Q),$$

which follows from the Definition (7.67) of R, we get

$$\begin{aligned} R &= U^0(\mathrm{id}_{\mathcal{H}} - Q^2) + U^0(-Q' + Z_{\mathrm{ev}} - QZ)U(\mathrm{id}_{\mathcal{H}} + Q) \\ &= U^0(-Q' + Z_{\mathrm{ev}} - QZ)(\mathrm{id}_{\mathcal{H}} + Q)R \\ &\quad + U^0(\mathrm{id}_{\mathcal{H}} - Q^2) + U^0(-Q' + Z_{\mathrm{ev}} - QZ)Q^2 U(\mathrm{id}_{\mathcal{H}} + Q). \end{aligned} \tag{7.70}$$

We view (7.70) also as a fixed point equation for R. In order to control the Hilbert-Schmidt norm of the non-diagonals of R, we solve this fixed point equation for R by iteration. Using the abbreviation

$$F := (-Q' + Z_{\mathrm{ev}} - QZ)(\mathrm{id}_{\mathcal{H}} + Q), \tag{7.71}$$

$$G := -U^0 Q^2 + U^0(-Q' + Z_{\mathrm{ev}} - QZ)Q^2 U(\mathrm{id}_{\mathcal{H}} + Q), \tag{7.72}$$

we rewrite (7.70) as $R = U^0 FR + U^0 + G$ and define recursively for $n \in \mathbb{N}_0$:

$$R^{(0)} := 0, \qquad R^{(n+1)} := U^0 F R^{(n)} + U^0 + G. \tag{7.73}$$

Although our main interest is to control the Hilbert-Schmidt norm $\|R(t_1, t_0)_{\mathrm{odd}}\|_{\mathrm{I}_2}$, we need also some control of the $R^{(n)}$ in the operator norm. We show first that $\|R^{(n)} - R\|_\infty \to 0$ as $n \to \infty$. We have for all $n \in \mathbb{N}_0$

$$R^{(n+1)} - R = U^0 F(R^{(n)} - R), \tag{7.74}$$

which implies

$$R^{(n)} - R = (U^0 F)^n (R^{(0)} - R) = -(U^0 F)^n R. \tag{7.75}$$

Now for $s \geq t$, we know $\|U^0(s,t)F(t)\| = \|F(t)\|$, because $U^0(s,t)$ is unitary. Let $t_1 \geq t_0$. Using the abbreviation

$$I(t_1, t_0) := \{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid t_1 > s_n > \ldots > s_1 > t_0\}, \tag{7.76}$$

we get

$$\begin{aligned} \|[R^{(n)} - R](t_1, t_0)\| &= \|[(U^0 F)^n R](t_1, t_0)\| \\ &\leq \int_{I(t_0, t_1)} \|F(s_n)\|\|F(s_{n-1})\| \ldots \|F(s_1)\|\|R(s_1, t_0)\|\, ds_1 \ldots ds_n \\ &\leq \frac{\|F\|_1^n}{n!}\|R\|_\infty \overset{n\to\infty}{\longrightarrow} 0; \end{aligned} \tag{7.77}$$

we use here the bounds $\|F\|_1 < \infty$ and $\|R\|_\infty < \infty$ from (7.100p.138) in Lemma 7.36p.138, below. Note that the convergence in (7.77) is uniform in the time variables t_0 and t_1. This proves the claim

$$\|R^{(n)} - R\|_\infty \overset{n\to\infty}{\longrightarrow} 0. \tag{7.78}$$

As a consequence, we find

$$\sup_{n\in\mathbb{N}_0} \|R^{(n)}\|_\infty \leq \sup_{n\in\mathbb{N}_0} \|R^{(n)} - R\|_\infty + \|R\|_\infty < \infty. \tag{7.79}$$

Now we split F into its diagonal and non-diagonal parts: $F = F_{\mathrm{ev}} + F_{\mathrm{odd}}$, where

$$F_{\mathrm{ev}} = Z_{\mathrm{ev}} - QZ_{\mathrm{odd}} - QQ' - QZ_{\mathrm{ev}}Q, \tag{7.80}$$

$$F_{\mathrm{odd}} = Z_{\mathrm{ev}}Q - QZ_{\mathrm{ev}} - Q' - QZ_{\mathrm{odd}}Q; \tag{7.81}$$

recall that Q is odd: $Q = Q_{\text{odd}}$. We calculate for $n \geq 1$:

$$
\begin{aligned}
R^{(n+1)} &= U^0 F R^{(n)} + U^0 + G \\
&= U^0 F_{\text{ev}} R^{(n)} + U^0 F_{\text{odd}} R^{(n)} + U^0 + G \\
&= U^0 F_{\text{ev}} R^{(n)} + U^0 F_{\text{odd}} U^0 F R^{(n-1)} + U^0 F_{\text{odd}} G + U^0 F_{\text{odd}} U^0 + U^0 + G.
\end{aligned} \tag{7.82}
$$

Estimating the Hilbert-Schmidt norm for the non-diagonals in each summand on the right hand side in (7.82p.136) now gives:

$$
\begin{aligned}
\|R^{(n+1)}(t_1, t_0)_{\text{odd}}\|_{\mathrm{I}_2} \leq &\int_{t_0}^{t_1} \|[U^0 F_{\text{ev}}](t_1, t)\| \|R^{(n)}(t, t_0)_{\text{odd}}\|_{\mathrm{I}_2}\, dt \\
&+ \int_{t_0}^{t_1} \|[U^0 F_{\text{odd}} U^0](t_1, t)\|_{\mathrm{I}_2} \|[F R^{(n-1)}](t, t_0)\|\, dt \\
&+ \int_{t_0}^{t_1} \|[U^0 F_{\text{odd}}](t_1, t)\| \|G(t, t_0)\|_{\mathrm{I}_2}\, dt \\
&+ \|[U^0 F_{\text{odd}} U^0](t_1, t_0)\|_{\mathrm{I}_2} + \|G(t_1, t_0)\|_{\mathrm{I}_2} \\
\leq &\int_{t_0}^{t_1} \|F_{\text{ev}}(t)\| \|R^{(n)}(t, t_0)_{\text{odd}}\|_{\mathrm{I}_2}\, dt + C_{40},
\end{aligned} \tag{7.83}
$$

where we have abbreviated

$$
\begin{aligned}
C_{40} :=& \|U^0 F_{\text{odd}} U^0\|_{\mathrm{I}_2,\infty} \|F\|_1 \sup_{n \in \mathbb{N}} \|R^{(n-1)}\|_\infty + \|F_{\text{odd}}\|_1 \|G\|_{\mathrm{I}_2,\infty} \\
&+ \|U^0 F_{\text{odd}} U^0\|_{\mathrm{I}_2,\infty} + \|G\|_{\mathrm{I}_2,\infty}.
\end{aligned} \tag{7.84}
$$

Lemma 7.36p.138 (I_2 estimates) states that $U^0 F_{\text{odd}} U^0$ and G consist of Hilbert-Schmidt operators, with $\|U^0 F_{\text{odd}} U^0\|_{\mathrm{I}_2,\infty} < \infty$ and $\|G\|_{\mathrm{I}_2,\infty} < \infty$. Furthermore, it also states that $\|F\|_1 < \infty$, which implies also $\|F_{\text{ev}}\|_1 < \infty$ and $\|F_{\text{odd}}\|_1 < \infty$. Combining these facts with the bound (7.79p.135), it follows that

$$
C_{40} < \infty. \tag{7.85}
$$

We claim that the following bound holds for all $n \geq 1$:

$$
\|R^{(n)}(t_1, t_0)_{\text{odd}}\|_{\mathrm{I}_2} \leq C_{40} \exp\left(\int_{t_0}^{t_1} \|F_{\text{ev}}(t)\|\, dt \right). \tag{7.86}
$$

We prove it by induction. For $n = 1$, we have $R^{(1)} = U^0 + G$. Using $U^0_{\text{odd}} = 0$ and $t_1 \geq t_0$, we conclude

$$
\|R^{(1)}(t_1, t_0)_{\text{odd}}\|_{\mathrm{I}_2} \leq \|G\|_{\mathrm{I}_2,\infty} \leq C_{40} \leq C_{40} \exp\left(\int_{t_0}^{t_1} \|F_{\text{ev}}(t)\|\, dt \right). \tag{7.87}
$$

For the induction step $n \rightsquigarrow n + 1$, we calculate, using the estimate (7.83) in the first step and the induction hypothesis in the second step:

$$
\begin{aligned}
\|R^{(n+1)}(t_1, t_0)_{\text{odd}}\|_{\mathrm{I}_2} &\leq \int_{t_0}^{t_1} \|F_{\text{ev}}(t)\| \|R^{(n)}(t, t_0)_{\text{odd}}\|_{\mathrm{I}_2}\, dt + C_{40} \\
&\leq C_{40} \int_{t_0}^{t_1} \|F_{\text{ev}}(t)\| \exp\left(\int_{t_0}^{t} \|F_{\text{ev}}(s)\|\, ds \right) dt + C_{40} \\
&= C_{40} \exp\left(\int_{t_0}^{t_1} \|F_{\text{ev}}(t)\|\, dt \right).
\end{aligned} \tag{7.88}
$$

Finally, we get $\|R^{(n)}_{\text{odd}}\|_{\mathrm{I}_2,\infty} \leq C_{40} e^{\|F_{\text{ev}}\|_1} < \infty$, which is a uniform bound in n. We now use following general fact, which follows from Fatou's lemma: If $(L_n)_{n \in \mathbb{N}}$ is a sequence of Hilbert-Schmidt operators converging to a bounded operator L with respect to the operator norm, then the following bound holds:

$$
\|L\|_{\mathrm{I}_2} \leq \liminf_{n \to \infty} \|L_n\|_{\mathrm{I}_2}. \tag{7.89}
$$

An application of this fact to the sequence $(R^{(n)}_{\mathrm{odd}}(t_1, t_0))_{n\in\mathbb{N}}$, using the uniform convergence stated in (7.78p.135), yields the result:

$$\sup_{t_1\geq t_0} \|[(\mathrm{id}_{\mathcal{H}} - Q(t_1))U(t_1, t_0)(\mathrm{id}_{\mathcal{H}} + Q(t_0))]_{\mathrm{odd}}\|_{\mathrm{I}_2} = \|R_{\mathrm{odd}}\|_{\mathrm{I}_2,\infty} \leq C_{40}e^{\|F_{\mathrm{ev}}\|_1} < \infty. \tag{7.90}$$

This proves the claim (7.66p.134). □

For our purposes, the following fixed point form (7.91) of the Dirac equation is technically more convenient to handle than the Dirac equation in its differential form (7.49p.131), as the fixed point equation gives rise to iterative approximation methods and deals only with bounded operators. We could have used it as our starting point.

Fixed point form of the Dirac equation

Lemma 7.34. *The one-particle Dirac time-evolution U fulfills the fixed point equation*

$$U = U^0 + U^0 ZU. \tag{7.91}$$

As the fixed-point form (7.91) of the Dirac equation is well-known, we only sketch its proof:

Proof. Using the Dirac equation in the form (7.54p.132-7.55p.132), we get for $t_0, t_1 \in \mathbb{R}$ on an appropriate domain:

$$\frac{\partial}{\partial t}[U^0(t_1, t)U(t, t_0)] = -iU^0(t_1, t)[H^{\mathsf{A}(t)} - H^0]U(t, t_0) = U^0(t_1, t)Z(t)U(t, t_0). \tag{7.92}$$

Note that although $H^{\mathsf{A}(t)}$ and H^0 are unbounded operators, their difference $iZ(t)$ is a bounded operator. Integrating (7.92), and using $U(t, t) = \mathrm{id}_{\mathcal{H}} = U^0(t, t)$, we get

$$U(t_1, t_0) = U^0(t_1, t_0) + \int_{t_0}^{t_1} U^0(t_1, t)Z(t)U(t, t_0)\, dt. \tag{7.93}$$

This equation recast in our stenographic notation is the fixed point equation (7.91) for U. □

Partial integration

Lemma 7.35. *The following integration-by-parts formula holds true:*

$$U^0 ZU^0 = QU^0 - U^0 Q - U^0 Q' U^0 + U^0 Z_{\mathrm{ev}} U^0. \tag{7.94}$$

Proof. We split $Z = Z_{\mathrm{ev}} + Z_{\mathrm{odd}}$ into even and odd pieces:

$$U^0 ZU^0 = U^0 Z_{\mathrm{odd}} U^0 + U^0 Z_{\mathrm{ev}} U^0 \tag{7.95}$$

Now $U^0 Z_{\mathrm{odd}} U^0 = U^0 Z_{+-} U^0 + U^0 Z_{-+} U^0$ consists of integral operators with the following integral kernels: The component $U^0 Z_{+-} U^0$ has the integral kernel

$$\begin{aligned} (p, q) \mapsto & \int_{t_0}^{t_1} e^{-i(t_1-t)H^0(p)} P_+(p) Z^{\mathsf{A}(t)}(p - q) P_-(q) e^{-i(t-t_0)H^0(q)}\, dt \\ = & \int_{t_0}^{t_1} e^{-i(t_1-t)E(p)} P_+(p) Z^{\mathsf{A}(t)}(p - q) P_-(q) e^{+i(t-t_0)E(q)}\, dt \\ = & e^{-it_1 E(p)} P_+(p) \int_{t_0}^{t_1} e^{it(E(p)+E(q))} Z^{\mathsf{A}(t)}(p - q)\, dt\, P_-(q) e^{-it_0 E(q)}. \end{aligned} \tag{7.96}$$

Recall that the function $E : \mathbb{R}^3 \to \mathbb{R}$ is defined by $E(p) = +\sqrt{m^2 + p^2}$. The crucial point is that the frequencies $E(p)$ and $E(q)$ have equal signs; they do *n*ot partially cancel each other, giving rise to a highly oscillatory integral at high momenta. Note that this works only for the odd part of Z. Integrating by parts, the right hand side in (7.96) equals

$$\begin{aligned}\dots = & \frac{P_+(p)Z^{\mathsf{A}(t)}(p-q)P_-(q)}{i(E(p)+E(q))}e^{i(t_1-t_0)E(q)} - e^{-i(t_1-t_0)E(p)}\frac{P_+(p)Z^{\mathsf{A}(t)}(p-q)P_-(q)}{i(E(p)+E(q))} \\ & - e^{-it_1E(p)}P_+(p)\int_{t_0}^{t_1}\frac{e^{it(E(p)+E(q))}}{i(E(p)+E(q))}\frac{\partial}{\partial t}Z^{\mathsf{A}(t)}(p-q)\,dt\,P_-(q)e^{-it_0E(q)}. \end{aligned} \tag{7.97}$$

Similarly, the integral kernel of the $-+$ component $U^0Z_{-+}U^0$ can be rewritten by an integration by parts as

$$\begin{aligned} & \frac{P_-(p)Z^{\mathsf{A}(t)}(p-q)P_+(q)}{-i(E(p)+E(q))}e^{-i(t_1-t_0)E(q)} - e^{i(t_1-t_0)E(p)}\frac{P_-(p)Z^{\mathsf{A}(t)}(p-q)P_+(q)}{-i(E(p)+E(q))} \\ & -e^{it_1E(p)}P_-(p)\int_{t_0}^{t_1}\frac{e^{-it(E(p)+E(q))}}{-i(E(p)+E(q))}\frac{\partial}{\partial t}Z^{\mathsf{A}(t)}(p-q)\,dt\,P_+(q)e^{it_0E(q)}. \end{aligned} \tag{7.98}$$

The sum of (7.97) and (7.98) is just the integral kernel of $QU^0 + U^0Q - U^0Q'U^0$. Substituting this into (7.95p.137) proves the claim (7.94p.137). □

For measurable functions $f : \mathbb{R}^3 \to \mathbb{C}$ and $g : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{C}$, we use the notation $\|f(p)\|_{2,p} := \|f\|_2$ and $\|g(p,q)\|_{2,(p,q)} := \|g\|_2$. The same notation is used for matrix-valued functions f and g. Recall that the class $\mathcal{A} \supset C_c^\infty(\mathbb{R}^4,\mathbb{R}^4)$ of vector potentials was introduced in Definition 7.32p.132 (Class of external four-vector potentials).

I_2 *estimates* **Lemma 7.36.** *Assume that the external vector potential* A *belongs to the class* $\mathcal{A}$*. Then the operators* $U^0Z_{\text{ev}}QU^0$*,* $U^0QZ_{\text{ev}}U^0$*,* $U^0Q'U^0$*,* Q^2*,* $Q'Q$ *and* QZQ*, constructed with this potential* A*, are Hilbert-Schmidt operators. Furthermore, their Hilbert-Schmidt norm is uniformly bounded in the time variables. Finally, the family of operators* $F = (-Q' + Z_{\text{ev}} - QZ)(\text{id}_{\mathcal{H}} + Q)$*,* $G = -U^0Q^2 + U^0(-Q' + Z_{\text{ev}} - QZ)Q^2U(\text{id}_{\mathcal{H}} + Q)$ *and* $R = (\text{id}_{\mathcal{H}} - Q)U(\text{id}_{\mathcal{H}} + Q)$*, introduced in (7.71p.135), (7.72p.135), and (7.67p.134), respectively, fulfill the following bounds in the Hilbert-Schmidt norm:*

$$\|U^0F_{\text{odd}}U^0\|_{I_2,\infty} < \infty \quad \textit{and} \quad \|G\|_{I_2,\infty} < \infty, \tag{7.99}$$

and the following bounds in the operator norm:

$$\|F\|_1 < \infty \quad \textit{and} \quad \|R\|_\infty < \infty. \tag{7.100}$$

Proof. Preliminarily, we estimate for any $\mathsf{A} \in \mathcal{A}$, $\mu = 0,1,2,3$, $m = 0,1$, and $n = 1,2$, using the fundamental theorem of calculus and averaging the starting point s uniformly over the unit interval:

$$\begin{aligned}\sup_{t\in\mathbb{R}}\left\|\frac{d^m}{dt^m}\widehat{\mathsf{A}}_\mu(t)\right\|_n &= \sup_{t\in\mathbb{R}}\left\|\int_0^1\left[\frac{d^m}{ds^m}\widehat{\mathsf{A}}_\mu(s) + \int_s^t\frac{d^{m+1}}{du^{m+1}}\widehat{\mathsf{A}}_\mu(u)\,du\right]ds\right\|_n \\ &\le \int_{\mathbb{R}}\left\|\frac{d^m}{dt^m}\widehat{\mathsf{A}}_\mu(t)\right\|_n dt + \int_{\mathbb{R}}\left\|\frac{d^{m+1}}{dt^{m+1}}\widehat{\mathsf{A}}_\mu(t)\right\|_n dt < \infty.\end{aligned} \tag{7.101}$$

At first let us examine the operators $U^0Z_{\text{ev}}QU^0$, $U^0QZ_{\text{ev}}U^0$, $U^0Q'U^0$. All of these operators have in common that the operator Q or its derivative are sandwiched between two free time-evolution operators U^0. The kernel of Q, equation (7.56p.132), appeared the first time after a partial integration in the time variable, Lemma 7.94p.137 (Partial Integration), which gave rise to the factor $[i(E(p)+E(q))]^{-1}$. The idea is that with another partial integration in the time variable, we will gain another such factor, giving enough decay to see the Hilbert-Schmidt property of the kernel.

In order to treat a part of the cases simultaneously, let V denote $Z_{\text{ev}}Q$, QZ_{ev}, or Q'. Note that in each of these cases, for $t \in \mathbb{R}$, $V(t) : \mathcal{H} \to \mathcal{H}$ is an odd integral operator. We denote its integral kernel by $(p, q) \mapsto V(t, p, q)$. For any $t_0, t_1 \in \mathbb{R}$, we have

$$\|(U^0 V U^0)(t_1, t_0)\|_2 \leq \|(U^0 V_{+-} U^0)(t_1, t_0)\|_2 + \|(U^0 V_{-+} U^0)(t_1, t_0)\|_2, \tag{7.102}$$

$$\|(U^0 V_{\pm\mp} U^0)(t_1, t_0)\|_2 = \left\| \int_{t_0}^{t_1} dt\; e^{\mp iE(p)(t_1 - t)} V_{\pm\mp}(t, p, q) e^{\pm iE(q)(t - t_0)} \right\|_{2,(p,q)}. \tag{7.103}$$

Using a partial integration, the last expression (7.103) is estimated as follows.

$$\begin{aligned} \ldots &= \left\| \int_{t_0}^{t_1} dt \left[\frac{d}{dt} \frac{e^{\mp i[E(p)+E(q)]t}}{\mp i[E(p) + E(q)]} \right] V_{\pm\mp}(t, p, q) e^{\mp iE(p)t_1} e^{\pm iE(q)t_0} \right\|_{2,(p,q)} \\ &\leq 2 \sup_{t\in\mathbb{R}} \left\| \frac{V_{\pm\mp}(t, p, q)}{E(p) + E(q)} \right\|_{2,(p,q)} + \int_{\mathbb{R}} dt \left\| \frac{V'_{\pm\mp}(t, p, q)}{E(p) + E(q)} \right\|_{2,(p,q)} \\ &=: f[V_{\pm\mp}] + g[V'_{\pm\mp}]. \end{aligned} \tag{7.104}$$

The first summand comes from the two boundary terms for $t = t_0$ and $t = t_1$. In the following, we show that $f[V_{\pm\mp}]$ and $g[V'_{\pm\mp}]$ are finite. Then, $U^0 Z_{\text{ev}} Q U^0$, $U^0 Q Z_{\text{ev}} U^0$, $U^0 Q' U^0$ are in I_2 with a Hilbert-Schmidt norm uniformly bounded in the time variable.

Case $V = Z_{\text{ev}}Q$: The 2-norm of the kernel of $V_{\pm\mp}(t)$ is estimated as follows:

$$\begin{aligned} |V_{\pm\mp}(t, p, q)| &= \left| \int_{\mathbb{R}^3} dk\; \frac{Z_{\pm\pm}(t, p, k) Z_{\pm\mp}(t, k, q)}{E(k) + E(q)} \right| \\ &= \left| \int_{\mathbb{R}^3} dk \sum_{\mu,\nu=0}^{3} \frac{P_\pm(p)\alpha^\mu P_\pm(k)\alpha^\nu P_\mp(q) \widehat{\mathsf{A}}_\mu(t, p - k) \widehat{\mathsf{A}}_\nu(t, k - q)}{E(k) + E(q)} \right| \\ &\leq \sum_{\mu,\nu=0}^{3} \int_{\mathbb{R}^3} dk\; |P_\pm(p)\alpha^\mu P_\pm(k)\alpha^\nu P_\mp(q)| \frac{|\widehat{\mathsf{A}}_\mu(t, p - k) \widehat{\mathsf{A}}_\nu(t, k - q)|}{E(k) + E(q)} \\ &\leq C_{41} \sum_{\mu,\nu=0}^{3} \int_{\mathbb{R}^3} dk\; \frac{|\widehat{\mathsf{A}}_\mu(t, p - k) \widehat{\mathsf{A}}_\nu(t, k - q)|}{E(k) + E(q)} \end{aligned} \tag{7.105}$$

with the constant

$$C_{41} := \sum_{\mu,\nu=0}^{3} \sup_{p,k,q\in\mathbb{R}^3} |P_\pm(p)\alpha^\mu P_\pm(k)\alpha^\nu P_\mp(q)| < \infty; \tag{7.106}$$

note that $\sup_{p\in\mathbb{R}^3} |P_\pm(p)| < \infty$ holds, because $P_\pm$ are orthogonal projections. An analogous argument for V' yields

$$\begin{aligned} |V'_{\pm\mp}(t, p, q)| &= \left| \int_{\mathbb{R}^3} dk\; \frac{Z'_{\pm\pm}(t, p, k) Z_{\pm\mp}(t, k, q) + Z_{\pm\pm}(t, p, k) Z'_{\pm\mp}(t, k, q)}{E(k) + E(q)} \right| \\ &\leq C_{41} \sum_{\mu,\nu=0}^{3} \int_{\mathbb{R}^3} dk\; \frac{|\widehat{\mathsf{A}}'_\mu(t, p - k) \widehat{\mathsf{A}}_\nu(t, k - q)| + |\widehat{\mathsf{A}}_\mu(t, p - k) \widehat{\mathsf{A}}'_\nu(t, k - q)|}{E(k) + E(q)}. \end{aligned} \tag{7.107}$$

With the bound (7.105), we compute

$$\begin{aligned} f[V_{\pm\mp}] &= 2 \sup_{t\in\mathbb{R}} \left\| \frac{V_{\pm\mp}(t, p, q)}{E(p) + E(q)} \right\|_{2,(p,q)} \\ &\leq 2C_{41} \sum_{\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \left\| \int_{\mathbb{R}^3} dk\; \frac{|\widehat{\mathsf{A}}_\mu(t, p - k) \widehat{\mathsf{A}}_\nu(t, k - q)|}{[E(p) + E(q)][E(k) + E(q)]} \right\|_{2,(p,q)}. \end{aligned} \tag{7.108}$$

Lemma 7.37p.143(iip.143) (Integral estimates), applied to the present situation, states that the norm in the last expression is bounded by $C_{42}\|\widehat{\mathsf{A}}_\mu(t,\cdot)\|_1\|\widehat{\mathsf{A}}_\nu(t,\cdot)\|_2$ with a finite constant C_{42}. Applying this yields

$$f[V_{\pm\mp}] \le 2C_{41}C_{42}\sum_{\mu,\nu=0}^{3}\sup_{t\in\mathbb{R}}\|\widehat{\mathsf{A}}_\mu(t,\cdot)\|_1\|\widehat{\mathsf{A}}_\nu(t,\cdot)\|_2. \tag{7.109}$$

The fact $\mathsf{A}\in\mathcal{A}$ and inequality (7.101p.138) ensure that this expression is finite.

The second summand on the right hand side of (7.104p.139) is estimated with the help of the bound (7.107p.139) as follows:

$$\begin{aligned} g[V'_{\pm\mp}] &= \int_{\mathbb{R}} dt\,\left\|\frac{V'_{\pm\mp}(t,p,q)}{E(p)+E(q)}\right\|_{2,(p,q)} \\ &\le C_{41}\sum_{\mu,\nu=0}^{3}\int_{\mathbb{R}} dt\,\left\|\int_{\mathbb{R}^3} dk\,\frac{|\widehat{\mathsf{A}}'_\mu(t,p-k)\widehat{\mathsf{A}}_\nu(t,k-q)|}{[E(p)+E(q)][E(k)+E(q)]}\right\|_{2,(p,q)} \\ &\quad + C_{41}\sum_{\mu,\nu=0}^{3}\int_{\mathbb{R}} dt\,\left\|\int_{\mathbb{R}^3} dk\,\frac{|\widehat{\mathsf{A}}_\mu(t,p-k)\widehat{\mathsf{A}}'_\nu(t,k-q)|}{[E(p)+E(q)][E(k)+E(q)]}\right\|_{2,(p,q)}. \end{aligned} \tag{7.110}$$

Again by Lemma 7.37p.143(ii) (Integral estimate) we then find

$$g[V'_{\pm\mp}(t)] \le C_{41}C_{42}\sum_{\mu,\nu=0}^{3}\int_{\mathbb{R}} dt\,\left(\|\widehat{\mathsf{A}}'_\mu(t)\|_1\|\widehat{\mathsf{A}}_\nu(t)\|_2 + \|\widehat{\mathsf{A}}_\mu(t)\|_1\|\widehat{\mathsf{A}}'_\nu(t)\|_2\right), \tag{7.111}$$

while the fact $\mathsf{A}\in\mathcal{A}$ together with its consequence (7.101p.138) ensure the finiteness of this expression. Summarizing, we have shown that $\|U^0 Z_{\text{ev}} Q U^0\|_{\mathrm{I}_2,\infty} < \infty$.

Case $V = QZ_{\text{ev}}$: We reduce this case to the case $V = Z_{\text{ev}}Q$, which we treated already. For any linear operator A on $\mathcal{H}$, $\|A\|_{\mathrm{I}_2} = \|A^*\|_{\mathrm{I}_2}$ holds. Using this and recalling that Z is self-adjoint and Q is skew-adjoint, we compute

$$\|U^0 Q Z_{\text{ev}} U^0\|_{\mathrm{I}_2,\infty} = \|-(U^0 Z_{\text{ev}} Q U^0)^*\|_{\mathrm{I}_2,\infty} = \|U^0 Z_{\text{ev}} Q U^0\|_{\mathrm{I}_2,\infty}, \tag{7.112}$$

which we have already shown to be finite.

Case $V = Q'$: In this case we get

$$\begin{aligned} |V_{\pm\mp}(t,p,q)| = \frac{|Z'_{\pm\mp}(t,p,q)|}{E(p)+E(q)} &\le \sum_{\mu=0}^{3}|P_\pm(p)\alpha^\mu P_\mp(q)|\frac{|\widehat{\mathsf{A}}'_\mu(t,p-q)|}{E(p)+E(q)} \\ &\le C_{43}\sum_{\mu=0}^{3}\frac{|\widehat{\mathsf{A}}'_\mu(t,p-q)|}{E(p)+E(q)} \end{aligned} \tag{7.113}$$

with the finite constant

$$C_{43} := \sum_{\mu=0}^{3}\sup_{p,q\in\mathbb{R}^3}|P_\pm(p)\alpha^\mu P_\mp(q)|, \tag{7.114}$$

A similar bound holds for the derivative

$$|V'_{\pm\mp}(t,p,q)| \le C_{43}\sum_{\mu=0}^{3}\frac{|\widehat{\mathsf{A}}''_\mu(t,p-q)|}{E(p)+E(q)}. \tag{7.115}$$

Lemma 7.37p.143(i) (Integral estimates), applied to the present situation, states the following bound:

$$\left\| \frac{\widehat{\mathsf{A}}'_\mu(t, p-q)}{[E(p)+E(q)]^2} \right\|_{2,(p,q)} \le C_{42} \|\widehat{\mathsf{A}}'_\mu(t)\|_2. \tag{7.116}$$

Using this yields the following estimate:

$$\begin{aligned} f[V_{\pm\mp}(t)] &= 2 \sup_{t\in\mathbb{R}} \left\| \frac{V_{\pm\mp}(t,p,q)}{E(p)+E(q)} \right\|_{2,(p,q)} \le 2C_{43} \sum_{\mu=0}^{3} \sup_{t\in\mathbb{R}} \left\| \frac{\widehat{\mathsf{A}}'_\mu(t,p-q)}{[E(p)+E(q)]^2} \right\|_{2,(p,q)} \\ &\le 2C_{43}C_{42} \sum_{\mu=0}^{3} \sup_{t\in\mathbb{R}} \|\widehat{\mathsf{A}}'_\mu(t)\|_2. \end{aligned} \tag{7.117}$$

The fact that $\mathsf{A} \in \mathcal{A}$ and inequality (7.101p.138) ensures the finiteness of this expression. Furthermore, we estimate

$$g[V'_{\pm\mp}(t)] = \int_{\mathbb{R}} dt \left\| \frac{V'_{\pm\mp}(t,p,q)}{E(p)+E(q)} \right\|_{2,(p,q)} \le C_{43} \sum_{\mu=0}^{3} \int_{\mathbb{R}} dt \left\| \frac{\widehat{\mathsf{A}}''_\mu(t,p-q)}{[E(p)+E(q)]^2} \right\|_{2,(p,q)} \tag{7.118}$$

Again Lemma 7.37p.143(i) (Integral estimates) gives that the last expression is bounded as follows:

$$\ldots \le C_{43}C_{42} \sum_{\mu=0}^{3} \int_{\mathbb{R}} dt \, \|\widehat{\mathsf{A}}''_\mu(t)\|_2 \tag{7.119}$$

which is also finite since $\mathsf{A} \in \mathcal{A}$. Summarizing, we have shown $\|U^0 Q' U^0\|_{\mathrm{I}_2,\infty} < \infty$.

Next we examine the operators Q^2, $Q'Q$ and QZQ. All of them have in common that Q or its derivatives appear twice, and therefore we have two of such factors $[E(p)+E(q)]^{-1}$ in the kernel of these operators. We shall see that these factors give enough decay to ensure the finiteness of the Hilbert-Schmidt norms of these operators.

Cases Q^2 and $Q'Q$: We denote the nth derivative with respect to time t by a superscript (n). For $n = 0, 1$ we estimate

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \|Q^{(n)}(t)Q(t)\|_2 \\ &\le \sup_{t\in\mathbb{R}} \sum_{\mu,\nu=0}^{3} \sum_{\pm} \left\| \int_{\mathbb{R}^3} dk \, |P_\pm(p)\alpha^\mu P_\mp(k)\alpha^\nu P_\pm(q)| \frac{|\widehat{\mathsf{A}}^{(n)}_\mu(t,p-k)\widehat{\mathsf{A}}_\nu(t,k-q)|}{[E(p)+E(k)][E(k)+E(q)]} \right\|_{2,(p,q)} \\ &\le C_{44} \sum_{\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \left\| \int_{\mathbb{R}^3} dk \, \frac{|\widehat{\mathsf{A}}^{(n)}_\mu(t,p-k)\widehat{\mathsf{A}}_\nu(t,k-q)|}{[E(p)+E(k)][E(k)+E(q)]} \right\|_{2,(p,q)} \end{aligned} \tag{7.120}$$

with the finite constant

$$C_{44} := \sum_{\mu,\nu=0}^{3} \sum_{\pm} \sup_{p,k,q\in\mathbb{R}^3} |P_\pm(p)\alpha^\mu P_\mp(k)\alpha^\nu P_\pm(q)|. \tag{7.121}$$

Lemma 7.37p.143(iii) (Integral estimates) provides the upper bound $C_{42}\|\widehat{\mathsf{A}}^{(n)}_\mu(t)\|_1 \|\widehat{\mathsf{A}}_\nu(t)\|_2$ for the norm of the integral on the right hand side of (7.120). Thus, the right hand side of (7.120) is bounded by:

$$\ldots \le C_{42}C_{44} \sum_{\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \|\widehat{\mathsf{A}}^{(n)}_\mu(t)\|_1 \|\widehat{\mathsf{A}}_\nu(t)\|_2, \tag{7.122}$$

which is finite because of $\mathsf{A} \in \mathcal{A}$ and inequality (7.101p.138). Hence, we have shown $\|Q^2\|_{\mathrm{I}_2,\infty} < \infty$ and $\|Q'Q\|_{\mathrm{I}_2,\infty} < \infty$.

Case QZQ: In this case we find

$$\begin{aligned}
&\|Q(t)ZQ(t)\|_2 \\
&= \left\| \sum_{\sigma,\tau\in\{-,+\}} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj\; Q_{-\sigma,\sigma}(t,p,k) Z_{\sigma,\tau}(t,k,j) Q_{\tau,-\tau}(t,j,q) \right\|_{2,(p,q)} \\
&\leq 4 \sup_{\sigma,\tau\in\{-,+\}} \sum_{\lambda,\mu,\nu=0}^{3} \left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj\, |P_{-\sigma}(p)\alpha^\lambda P_\sigma(k)\alpha^\mu P_\tau(j)\alpha^\nu P_{-\tau}(q)| \times \right. \\
&\qquad\qquad \left. \times \frac{|\widehat{\mathsf{A}}_\lambda(t,p-k)\widehat{\mathsf{A}}_\mu(t,k-j)\widehat{\mathsf{A}}_\nu(t,j-q)|}{[E(p)+E(k)][E(j)+E(q)]} \right\|_{2,(p,q)} \\
&\leq C_{45} \left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj\, \frac{|\widehat{\mathsf{A}}_\lambda(t,p-k)\widehat{\mathsf{A}}_\mu(t,k-j)\widehat{\mathsf{A}}_\nu(t,j-q)|}{[E(p)+E(k)][E(j)+E(q)]} \right\|_{2,(p,q)}
\end{aligned} \tag{7.123}$$

with the finite constant

$$C_{45} := 4 \sup_{\sigma,\tau\in\{-,+\}} \sum_{\lambda,\mu,\nu=0}^{3} \sup_{p,k,j,q\in\mathbb{R}^3} |P_{-\sigma}(p)\alpha^\lambda P_\sigma(k)\alpha^\mu P_\tau(j)\alpha^\nu P_{-\tau}(q)|. \tag{7.124}$$

By Lemma 7.37p.143(iv) (Integral estimates) we find the following bound for the right hand side in (7.123):

$$\ldots \leq C_{45} C_{42} \sum_{\lambda,\mu,\nu=0}^{3} \sup_{t\in\mathbb{R}} \|\mathsf{A}_\lambda(t)\|_1 \|\mathsf{A}_\mu(t)\|_2 \|\mathsf{A}_\nu(t)\|_2 \tag{7.125}$$

which is finite because $\mathsf{A} \in \mathcal{A}$ and inequality (7.101p.138). This proves the claim $\|QZQ\|_{\mathrm{I}_2,\infty} < \infty$.

Finally, we prove the claims (7.99p.138) and (7.100p.138). As a consequence of $\mathsf{A} \in \mathcal{A}$ and the bound (7.101p.138), using the definition of the operators $Z(t)$, $Q(t)$, and $Q'(t)$ by their integral kernels given in the equations (7.53p.132), (7.56p.132), and (7.65p.134), we observe the following operator norm bounds:

$$\|L\|_1 < \infty \quad \text{and} \quad \|L\|_\infty < \infty \quad \text{for} \quad L \in \{Z, Z_{\mathrm{ev}}, Q, Q'\}; \tag{7.126}$$

recall the definition (7.64p.134) of the norms used here. Furthermore, we know $\|U\|_\infty = 1$, since the one-particle Dirac time-evolution U consists of unitary operators. Combining these facts proves the claim (7.100p.138). To prove the first claim in (7.99p.138), we calculate:

$$U^0 F_{\mathrm{odd}} U^0 = U^0 Z_{\mathrm{ev}} Q U^0 - U^0 Q Z_{\mathrm{ev}} U^0 - U^0 Q' U^0 - U^0 Q Z_{\mathrm{odd}} Q U^0; \tag{7.127}$$

see also equation (7.81p.135) Using the bounds in the Hilbert-Schmidt norm proven before, this implies the claim $\|U^0 F_{\mathrm{odd}} U^0\|_{\mathrm{I}_2,\infty} < \infty$. Finally, using $\|U^0\|_\infty = 1$, $\|Q^2\|_{\mathrm{I}_2,\infty} < \infty$, $\|U\|_\infty = 1$, and the bounds (7.126), the second claim $\|G\|_{\mathrm{I}_2,\infty} < \infty$ in (7.99p.138) follows also. This finishes the proof of the lemma. □

We now state and prove the integral estimates that were used in the previous proof. Recall that the function $E : \mathbb{R}^3 \to \mathbb{R}$ is defined by $E(p) = \sqrt{|p|^2 + m^2}$.

Integral estimates

Lemma 7.37. *For $C_{42} := \|E^{-2}\|_2 < \infty$, the following bounds hold for all $A_1, A_3 \in L_1(\mathbb{R}^3, \mathbb{C})$ and $A_2 \in L^2(\mathbb{R}^3, \mathbb{C})$:*

$$\left\| \frac{A_2(p-q)}{[E(p)+E(q)]^2} \right\|_{2,(p,q)} \le C_{42} \|A_2\|_2 \tag{i}$$

$$\left\| \int_{\mathbb{R}^3} dk \, \frac{A_1(p-k)A_2(k-q)}{[E(p)+E(q)][E(k)+E(q)]} \right\|_{2,(p,q)} \le C_{42} \|A_1\|_1 \|A_2\|_2 \tag{ii}$$

$$\left\| \int_{\mathbb{R}^3} dk \, \frac{A_1(p-k)A_2(k-q)}{[E(p)+E(k)][E(k)+E(q)]} \right\|_{2,(p,q)} \le C_{42} \|A_1\|_1 \|A_2\|_2 \tag{iii}$$

$$\left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \, \frac{A_1(p-j)A_2(j-k)A_3(k-q)}{[E(p)+E(j)][E(k)+E(q)]} \right\|_{2,(p,q)} \le C_{42} \|A_1\|_1 \|A_2\|_2 \|A_3\|_1 \tag{iv}$$

*Proof. I*nequality (i): Substituting $r := p - q$ and using $E(p) + E(q) \ge E(p)$, one finds

$$\left\| \frac{A_2(p-q)}{[E(p)+E(q)]^2} \right\|_{2,(p,q)} \le \left\| \frac{A_2(r)}{E(p)^2} \right\|_{2,(p,r)} = \left\| E^{-2} \right\|_2 \|A_2\|_2. \tag{7.128}$$

*I*nequality (ii): Let $B = \{\chi \in L^2(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C}) \mid \|\chi\|_2 \le 1\}$ denote the unit ball in $L^2(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C})$. Using a dual representation of the norm $\|\cdot\|_2$, we get

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} dk \, \frac{A_1(p-k)A_2(k-q)}{[E(p)+E(q)][E(k)+E(q)]} \right\|_{2,(p,q)} \le \left\| \int_{\mathbb{R}^3} dk \, \frac{|A_1(p-k)A_2(k-q)|}{E(q)^2} \right\|_{2,(p,q)} \\ &\le \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \left| \frac{A_1(p-k)A_2(k-q)}{E(q)^2} \chi(p,q) \right|. \end{aligned} \tag{7.129}$$

Substituting $j := p - k$, we bound the right hand side in (7.129) as follows:

$$\begin{aligned} \ldots &= \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dj \left| \frac{A_1(j)A_2(p-j-q)}{E(q)^2} \chi(p,q) \right| \\ &\le \|A_1\|_1 \sup_{\chi \in B} \sup_{j \in \mathbb{R}^3} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \left| \frac{A_2(p-j-q)}{E(q)^2} \chi(p,q) \right|. \end{aligned} \tag{7.130}$$

Substituting $r := p - j - q$ and changing the order of integration turns this into

$$\ldots = \|A_1\|_1 \sup_{\chi \in B} \sup_{j \in \mathbb{R}^3} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dr \left| \frac{A_2(r)}{E(q)^2} \chi(r+q+j,q) \right|. \tag{7.131}$$

By the Cauchy-Schwarz inequality, we bound the last expression as follows:

$$\begin{aligned} \ldots &\le \|A_1\|_1 \sup_{\chi \in B} \sup_{j \in \mathbb{R}^3} \left\| \frac{A_2(r)}{E(q)^2]} \right\|_{2,(q,r)} \|\chi(r+q+j,q)\|_{2,(q,r)} \\ &= \left\| E^{-2} \right\|_2 \|A_1\|_1 \|A_2\|_2. \end{aligned} \tag{7.132}$$

*I*nequality (iii): Similarly, we estimate

$$\begin{aligned} &\left\| \int_{\mathbb{R}^3} dk \, \frac{A_1(p-k)A_2(k-q)}{[E(p)+E(k)][E(k)+E(q)]} \right\|_{2,(p,q)} \le \left\| \int_{\mathbb{R}^3} dk \, \frac{|A_1(p-k)A_2(k-q)|}{E(k)^2} \right\|_{2,(p,q)} \\ &\le \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \left| \frac{A_1(p-k)A_2(k-q)}{E(k)^2} \chi(p,q) \right|. \end{aligned} \tag{7.133}$$

Although these terms looks similar to (7.129), there seems to be no substitution which enables us to use the result (7.129) directly.

Interchanging the order of integration and substituting first $j := p - k$ and then $r = k - q$, the right hand side in (7.133) equals

$$\begin{aligned}
\ldots &= \sup_{\chi\in B} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \left| \frac{A_1(j)A_2(k-q)}{E(k)^2} \chi(j+k,q) \right| \\
&\le \|A_1\|_1 \sup_{\chi\in B} \sup_{j\in\mathbb{R}} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \left| \frac{A_2(k-q)}{E(k)^2} \chi(j+k,q) \right| \\
&= \|A_1\|_1 \sup_{\chi\in B} \sup_{j\in\mathbb{R}} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dr \left| \frac{A_2(r)}{E(k)^2} \chi(j+k,k-r) \right| \\
&\le \|A_1\|_1 \sup_{\chi\in B} \sup_{j\in\mathbb{R}^3} \left\| \frac{A_2(r)}{E(k)^2]} \right\|_{2,(r,k)} \|\chi(j+k,k-r)\|_{2,(r,k)} \\
&= \left\|E^{-2}\right\|_2 \|A_1\|_1 \|A_2\|_2. \qquad (7.134)
\end{aligned}$$

*I*nequality (iv): Again, we get

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \, \frac{A_1(p-j)A_2(j-k)A_3(k-q)}{[E(p)+E(j)][E(k)+E(q)]} \right\|_{2,(p,q)} \\
&\le \left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \, \frac{|A_1(p-j)A_2(j-k)A_3(k-q)|}{E(j)E(k)} \right\|_{2,(p,q)} \\
&= \sup_{\chi\in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \left| \frac{A_1(p-j)A_2(j-k)A_3(k-q)}{E(j)E(k)} \chi(p,q) \right|. \qquad (7.135)
\end{aligned}$$

Interchanging the integration and substituting $r := p - j$ and $s := k - q$, this equals

$$\ldots = \sup_{\chi\in B} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} ds \left| \frac{A_1(r)A_2(j-k)A_3(s)}{E(j)E(k)} \chi(r+j,k-s) \right|. \qquad (7.136)$$

We apply Hölder's inequality twice to bound (7.136) as follows:

$$\ldots \le \|A_1\|_1 \|A_3\|_1 \sup_{\chi\in B} \sup_{r,s\in\mathbb{R}^3} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \left| \frac{A_2(j-k)}{E(j)E(k)} \chi(r+j,k-s) \right| \qquad (7.137)$$

Using the Cauchy-Schwarz inequality and then the substitution $u := j - k$, this term is bounded from above by

$$\begin{aligned}
\ldots &\le \|A_1\|_1 \|A_3\|_1 \left\| \frac{A_2(j-k)}{E(j)E(k)} \right\|_{2,(j,k)} \\
&\le \|A_1\|_1 \|A_3\|_1 \left\| \frac{A_2(u)}{E(u+k)E(k)} \right\|_{2,(u,k)} \\
&\le \|A_1\|_1 \|A_2\|_2 \|A_3\|_1 \sup_{u\in\mathbb{R}^3} \left\| \frac{1}{E(u+k)E(k)} \right\|_{2,k} \\
&\le \|A_1\|_1 \|A_2\|_2 \|A_3\|_1 \left\|E^{-2}\right\|_2. \qquad (7.138)
\end{aligned}$$

In the last step, we have once more used the Cauchy-Schwarz inequality. □

7.3.2 Identification of Polarization Classes

In this subsection we show that there is a one-to-one correspondence of the magnetic components $\vec{A}$ of the four-vector field $A = (A_\mu)_{\mu=0,1,2,3} = (A^0, -\vec{A})$ onto the set of physically relevant polarization classes $C(A)$, introduced in Definition 7.30p.132 (Induced polarization classes).

Identification of the polarization classes

Theorem 7.38. *For $A, A' \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, the following are equivalent:*

(a) $C(A) = C(A')$

(b) $\vec{A} = \vec{A'}$

On this ground the following notation makes sense:

Physical polarization classes

Definition 7.39. *For $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A})$ in $C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, we define*

$$C(\vec{A}) := C(A).$$

For this subsection it is convenient to use the four-vector notation of special relativity. To avoid confusion, in this section, three-vectors are labeled with an arrow. Define the Lorentz metric $(g_{\mu\nu})_{\mu,\nu=0,1,2,3} = \mathrm{diag}(1,-1,-1,-1)$. Raising and lowering of Lorentz indices is performed with respect to this metric. The inner product of two four-vectors $a = (a^\mu)_{\mu=0,1,2,3} = (a_0, \vec{a})$ and $b = (b^\nu)_{\nu=0,1,2,3} = (b_0, \vec{b})$ is given by

$$a \cdot b := a_\mu b^\mu = \sum_{\mu,\nu=0}^{3} a^\mu g_{\mu\nu} b^\nu = a_0 b_0 - \vec{a} \cdot \vec{b} \tag{7.139}$$

where the $\cdot$ on the right hand side above is the Euclidian inner product on $\mathbb{R}^3$. Raising and lowering of Lorentz indices is performed with respect to this metric. Within this four-vector notation it is more convenient to write the Dirac $\mathbb{C}^{4\times4}$ matrices (7.46p.131) as

$$(\gamma^\mu)_{\mu=0,1,2,3} = \beta\alpha^\mu \tag{7.140}$$

which then fulfill

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{7.141}$$

Recall that the Fourier transform $\widehat{A}$ of a vector potential $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A})$ in $C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$ was introduced in equation (7.52p.131). Using Feynman's dagger $\not{A} = \gamma^\mu A_\mu$, the integral kernel $Z = Z^A$, introduced in equation (7.53p.132), reads

$$Z(\vec{p}, \vec{q}) = -ie\gamma^0 \widehat{\not{A}}(\vec{p} - \vec{q}), \quad \vec{p}, \vec{q} \in \mathbb{R}^3. \tag{7.142}$$

Abbreviating again $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$, we define two momentum four-vectors p_+, p_- for $\vec{p} \in \mathbb{R}^3$ by

$$p_+ = (p_{+\mu})_{\mu=0,1,2,3} = (E(\vec{p}), -\vec{p}), \tag{7.143}$$
$$p_- = (p_{-\mu})_{\mu=0,1,2,3} = (-E(\vec{p}), -\vec{p}) \tag{7.144}$$

such that the corresponding projection operators introduced in (7.48p.131) then read

$$P_\pm(\vec{p}) = \frac{1}{2p_{\pm 0}} (\not{p}_\pm + m)\gamma^0. \tag{7.145}$$

Proof of Theorem 7.38

Proof of Theorem 7.38. Note that e^{Q^A} and $e^{Q^{A'}}$ are unitary maps on $\mathcal{H}$, because Q^A and $Q^{A'}$ are skew-adjoint.

Let $V = e^{Q^A}\mathcal{H}_-$ and $W = e^{Q^{A'}}\mathcal{H}_-$. By definition, $V \in C(A)$ and $W \in C(A')$ hold. We need to show that $V \approx_0 W$ holds if and only if $\vec{A} = \vec{A}'$.

Now $P_V = e^{Q^A} P_- e^{-Q^A}$ holds. Just as in (7.62p.133), we know that $e^{\pm Q^A} - (\mathrm{id}_{\mathcal{H}} \pm Q^A)$ are Hilbert-Schmidt operators. As a consequence, P_V differs from $(\mathrm{id}_{\mathcal{H}}+Q^A)P_-(\mathrm{id}_{\mathcal{H}}-Q^A)$ only by a Hilbert-Schmidt operator. Using that Q^A is odd, we know $Q^A P_- Q^A = [(Q^A)^2]_{++}$. Because $(Q^A)^2$ is a Hilbert-Schmidt operator by Lemma 7.36p.138 (I_2 estimates), it follows that $Q^A P_- Q^A \in \mathrm{I}_2(\mathcal{H})$. We conclude that $P_V - \mathrm{id}_{\mathcal{H}} - Q^A P_- + P_- Q^A \in \mathrm{I}_2(\mathcal{H})$. The same argument, applied to A', shows that $P_W - \mathrm{id}_{\mathcal{H}} - Q^{A'} P_- + P_- Q^{A'} \in \mathrm{I}_2(\mathcal{H})$. Taking the difference, this implies

$$\begin{aligned} P_V - P_W &\in (Q^A - Q^{A'})P_- - P_-(Q^A - Q^{A'}) + \mathrm{I}_2(\mathcal{H}) \\ &= Q^{A-A'} P_- - P_- Q^{A-A'} + \mathrm{I}_2(\mathcal{H}) = Q^{A-A'}_{+-} - Q^{A-A'}_{-+} + \mathrm{I}_2(\mathcal{H}); \end{aligned} \tag{7.146}$$

recall that Q^A is linear in the argument A. Using once more that $Q^{A-A'}$ is odd, this yields the following equivalences:

$$V \approx W \Leftrightarrow P_V - P_W \in \mathrm{I}_2(\mathcal{H}) \Leftrightarrow Q^{A-A'}_{+-} - Q^{A-A'}_{-+} \in \mathrm{I}_2(\mathcal{H}) \Leftrightarrow Q^{A-A'} \in \mathrm{I}_2(\mathcal{H}) \tag{7.147}$$

Now Lemma 7.40 (Hilbert-Schmidt Condition for Q) below, applied to $A-A'$, states that $Q^{A-A'} \in \mathrm{I}_2(\mathcal{H})$ is equivalent to $\vec{A} = \vec{A}'$. Summarizing, we have shown that $V \approx W$ holds if and only if $\vec{A} = \vec{A}'$.

In order to show that in this case $V \approx_0 W$ holds also, it remains to show $\mathrm{charge}(V, W) = 0$. Now because $e^{Q^A}|_{\mathcal{H}_- \to V}$ and $e^{-Q^{A'}}|_{W \to \mathcal{H}_-}$ are unitary maps, we get

$$\begin{aligned} \mathrm{charge}(V, W) &= \mathrm{ind}(P_W|_{V\to W}) = \mathrm{ind}\left(e^{-Q^{A'}} P_W e^{Q^A}\Big|_{\mathcal{H}_-\to\mathcal{H}_-}\right) \\ &= \mathrm{ind}\left(P_- e^{-Q^{A'}} e^{Q^A}\Big|_{\mathcal{H}_-\to\mathcal{H}_-}\right) = \mathrm{ind}\left((e^{-Q^{A'}} e^{Q^A})_{--}\Big|_{\mathcal{H}_-\to\mathcal{H}_-}\right). \end{aligned} \tag{7.148}$$

Because Q^A is skew-adjoint and its square $(Q^A)^2$ is a Hilbert-Schmidt operator, e^{Q^A} is a compact perturbation of the identity $\mathrm{id}_{\mathcal{H}}$. The same argument shows that $e^{-Q^{A'}}$ is also a compact perturbation of the identity. Hence, $(e^{-Q^{A'}} e^{Q^A})_{--}|_{\mathcal{H}_-\to\mathcal{H}_-}$ is a compact perturbation of $\mathrm{id}_{\mathcal{H}_-}$ and thus has Fredholm index 0. This shows that $\mathrm{charge}(V, W) = 0$ and finishes the proof. □

The lemma used in the proof of Theorem 7.38p.145 (Identification of the polarization classes) is:

Hilbert-Schmidt condition for Q

Lemma 7.40. *For $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A})$ in $C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, the following are equivalent:*

1. $Q^A \in \mathrm{I}_2(\mathcal{H})$,
2. $\vec{A} = 0$.

Proof. We calculate the squared Hilbert-Schmidt norm $\|Q^A\|^2_{\mathrm{I}_2}$ of Q^A. Using the abbreviations $Q^A_{+-}(\vec{p}, \vec{q}) = P_+(\vec{p}) Q^A(\vec{p}, \vec{q}) P_-(\vec{q})$ and $Q^A_{-+}(\vec{p}, \vec{q}) = P_-(\vec{p}) Q^A(\vec{p}, \vec{q}) P_+(\vec{q})$, we get

$$\begin{aligned} \|Q^A\|^2_{\mathrm{I}_2} &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{tr}[Q^A(\vec{p}, \vec{q}) Q^A(\vec{p}, \vec{q})^*]\, dp\, dq \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(\mathrm{tr}[Q^A_{+-}(\vec{p}, \vec{q}) Q^A_{+-}(\vec{p}, \vec{q})^*] + \mathrm{tr}[Q^A_{-+}(\vec{p}, \vec{q}) Q^A_{-+}(\vec{p}, \vec{q})^*]\right) dp\, dq. \end{aligned} \tag{7.149}$$

Inserting the Definition (7.56p.132) of Q^A, using that $[\gamma^0 \not{A}(\vec{p} - \vec{q})]^* = \gamma^0 \not{A}(\vec{q} - \vec{p})$ and that $P_+(p)$ and $P_-(q)$ are orthogonal projections having the representation (7.145p.145), we express the first summand as follows:

$$\begin{aligned}
&\operatorname{tr}[Q^A_{+-}(\vec{p},\vec{q})Q^A_{+-}(\vec{p},\vec{q})^*] \\
&= \frac{e^2}{4p_{+0}q_{-0}(p_{+0}-q_{-0})^2}\operatorname{tr}\Big([(\not{p}_+ + m)\gamma^0][\gamma^0\widehat{\not{A}}(\vec{p}-\vec{q})][(\not{q}_- + m)\gamma^0] \\
&\qquad \cdot [(\not{q}_- + m)\gamma^0]^*[\gamma^0\widehat{\not{A}}(\vec{p}-\vec{q})]^*[(\not{p}_+ + m)\gamma^0]^*\Big) \\
&= \frac{e^2}{4p_{+0}q_{-0}(p_{+0}-q_{-0})^2}\operatorname{tr}\Big((\not{p}_+ + m)\widehat{\not{A}}(\vec{p}-\vec{q})(\not{q}_- + m)\widehat{\not{A}}(\vec{q}-\vec{p})\Big) \qquad (7.150)
\end{aligned}$$

Now we use the following formulas for traces of products of γ-matrices:

$$\operatorname{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \qquad (7.151)$$

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\kappa) = 0, \qquad (7.152)$$

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda) = 4(g^{\mu\nu}g^{\kappa\lambda} + g^{\mu\lambda}g^{\kappa\nu} - g^{\mu\kappa}g^{\nu\lambda}). \qquad (7.153)$$

We obtain

$$\begin{aligned}
0 &\le \operatorname{tr}[Q^A_{+-}(\vec{p},\vec{q})Q^A_{+-}(\vec{p},\vec{q})^*] \\
&= \frac{e^2}{4p_{+0}q_{-0}(p_{+0}-q_{-0})^2}\operatorname{tr}\Big((\not{p}_+ + m)\widehat{\not{A}}(\vec{p}-\vec{q})(\not{q}_- + m)\widehat{\not{A}}(\vec{q}-\vec{p})\Big) \\
&= \frac{e^2}{p_{+0}q_{-0}(p_{+0}-q_{-0})^2}\Big((m^2 - p_+\cdot q_-)\widehat{A}(\vec{p}-\vec{q})\cdot\widehat{A}(\vec{q}-\vec{p}) \\
&\quad + (p_+\cdot\widehat{A}(\vec{p}-\vec{q}))(q_-\cdot\widehat{A}(\vec{q}-\vec{p})) + (p_+\cdot\widehat{A}(\vec{q}-\vec{p}))(q_-\cdot\widehat{A}(\vec{p}-\vec{q}))\Big). \qquad (7.154)
\end{aligned}$$

The second summand on the right hand side in (7.149p.146) can be calculated in a similar way by exchanging the indices "+" and "−":

$$\begin{aligned}
0 &\le \operatorname{tr}[Q^A_{-+}(\vec{p},\vec{q})Q^A_{-+}(\vec{p},\vec{q})^*] \\
&= \frac{e^2}{p_{-0}q_{+0}(p_{-0}-q_{+0})^2}\Big((m^2 - p_-\cdot q_+)\widehat{A}(\vec{p}-\vec{q})\cdot\widehat{A}(\vec{q}-\vec{p}) \\
&\quad + (p_-\cdot\widehat{A}(\vec{p}-\vec{q}))(q_+\cdot\widehat{A}(\vec{q}-\vec{p})) + (p_-\cdot\widehat{A}(\vec{q}-\vec{p}))(q_+\cdot\widehat{A}(\vec{p}-\vec{q}))\Big) \\
&= \operatorname{tr}[Q^A_{+-}(\vec{q},\vec{p})Q^A_{+-}(\vec{q},\vec{p})^*]. \qquad (7.155)
\end{aligned}$$

Thus, the two summands in (7.149p.146) are the same up to exchanging $\vec{p}$ and $\vec{q}$. In particular, this yields

$$\begin{aligned}
\|Q^A\|^2_{\mathrm{I}_2} &= 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\operatorname{tr}[Q^A_{+-}(\vec{p},\vec{q})Q^A_{+-}(\vec{p},\vec{q})^*]\,dp\,dq \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{2e^2}{p_{+0}q_{-0}(p_{+0}-q_{-0})^2}\Big((m^2 - p_+\cdot q_-)\widehat{A}(\vec{p}-\vec{q})\cdot\widehat{A}(\vec{q}-\vec{p}) \\
&\quad + (p_+\cdot\widehat{A}(\vec{p}-\vec{q}))(q_-\cdot\widehat{A}(\vec{q}-\vec{p})) + (p_+\cdot\widehat{A}(\vec{q}-\vec{p}))(q_-\cdot\widehat{A}(\vec{p}-\vec{q}))\Big)\,dp\,dq. \qquad (7.156)
\end{aligned}$$

Let us now use this to prove that $\vec{A} = 0$ implies $Q^A \in \mathrm{I}_2(H)$. In the case $\vec{A} = 0$, formula (7.156) boils down to

$$\begin{aligned}\|Q^A\|_{I_2}^2 &= 2e^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{E(\vec p)E(\vec q) - \vec p\cdot\vec q - m^2}{E(\vec p)E(\vec q)(E(\vec p)+E(\vec q))^2} |\widehat{A_0}(\vec p-\vec q)|^2\,dp\,dq\\
&= 2e^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{E(\vec p)E(\vec p-\vec k) - \vec p\cdot(\vec p-\vec k) - m^2}{E(\vec p)E(\vec p-\vec k)(E(\vec p)+E(\vec p-\vec k))^2} |\widehat{A_0}(\vec k)|^2\,dp\,dk\\
&= 2e^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{E(\vec p)(E(\vec p-\vec k) - E(\vec p)) + \vec p\cdot\vec k}{E(\vec p)E(\vec p-\vec k)(E(\vec p)+E(\vec p-\vec k))^2} |\widehat{A_0}(\vec k)|^2\,dp\,dk\\
&\le 2e^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{E(\vec p-\vec k) - E(\vec p) + \vec p\cdot\vec k/E(\vec p)}{E(\vec p-\vec k)E(\vec p)^2} |\widehat{A_0}(\vec k)|^2\,dp\,dk,\end{aligned} \tag{7.157}$$

where we have used $E(\vec p)^2 - |\vec p|^2 = m^2$. We expand $E(\vec p-\vec k)$ around $\vec k = 0$: For $t \in \mathbb{R}$, one has

$$\frac{\partial}{\partial t}E(\vec p - t\vec k) = -\frac{\vec k\cdot(\vec p - t\vec k)}{E(\vec p - t\vec k)}, \tag{7.158}$$

$$\frac{\partial^2}{\partial t^2}E(\vec p - t\vec k) = \frac{|\vec k|^2}{E(\vec p - t\vec k)} - \frac{[\vec k\cdot(\vec p - t\vec k)]^2}{E(\vec p - t\vec k)^3}. \tag{7.159}$$

Using

$$0 \le [\vec k\cdot(\vec p - t\vec k)]^2 \le |\vec k|^2|\vec p - t\vec k|^2 \le |\vec k|^2 E(\vec p - t\vec k)^2 \tag{7.160}$$

we conclude

$$0 \le \frac{\partial^2}{\partial t^2}E(\vec p - t\vec k) \le \frac{|\vec k|^2}{E(\vec p - t\vec k)}. \tag{7.161}$$

By Taylor's formula, we get for some $t_{\vec p,\vec k} \in [0,1]$:

$$0 \le E(\vec p - \vec k) - E(\vec p) + \frac{\vec p\cdot\vec k}{E(\vec p)} \le \frac{|\vec k|^2}{2E(\vec p - t_{\vec p,\vec k}\vec k)}. \tag{7.162}$$

Now using the variable $\vec q_t := \vec p - t\vec k$ with $0 \le t \le 1$, we estimate

$$\begin{aligned}E(\vec p)^2 &= E(\vec q_t + t\vec k)^2 = |\vec q_t + t\vec k|^2 + m^2 \le 2|\vec q_t|^2 + 2t^2|\vec k|^2 + m^2\\
&\le \frac{2}{m^2}(|\vec q_t|^2 + m^2)(t^2|\vec k|^2 + m^2) \le \frac{2}{m^2}E(\vec q_t)^2E(\vec k)^2.\end{aligned} \tag{7.163}$$

This yields for $0 \le t \le 1$:

$$\frac{1}{E(\vec p - t\vec k)} \le \frac{\sqrt{2}}{m}\frac{E(\vec k)}{E(\vec p)}. \tag{7.164}$$

Substituting the bounds (7.162) and (7.164) for $t = 1$ and for $t = t_{\vec p,\vec k}$ in (7.157), we conclude

$$\|Q^A\|_{I_2}^2 \le \frac{2e^2}{m^2}\int_{\mathbb{R}^3}\frac{dp}{E(\vec p)^4}\int_{\mathbb{R}^3}|\vec k|^2E(\vec k)^2|\widehat{A_0}(\vec k)|^2\,dk < \infty. \tag{7.165}$$

Thus $\vec A = 0$ implies $\|Q^A\|_{I_2} < \infty$.

We now prove that $\vec A \neq 0$ implies $\|Q^A\|_{I_2} = \infty$. We split $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec A)$ into $A = (A_0, \vec 0) + (0, -\vec A)$. Abbreviating $Q^{A_0} = Q^{(A_0,\vec 0)}$ and $Q^{\vec A} := Q^{(0,-\vec A)}$, we conclude

$$Q^A = Q^{A_0} + Q^{\vec A}. \tag{7.166}$$

As we have just shown, the first summand Q^{A_0} is a Hilbert-Schmidt operator. Hence, Q^A is a Hilbert-Schmidt operator if and only if $Q^{\vec{A}}$ is a Hilbert-Schmidt operator. Thus it remains to show that $\vec{A} \neq 0$ implies $\|Q^{\vec{A}}\|_{\mathrm{I}_2} = \infty$.

Equation (7.156p.147) in the special case of a vanishing 0-component of the vector potential can be rewritten as follows:

$$\|Q^{\vec{A}}\|_{\mathrm{I}_2}^2 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{2e^2}{E(\vec{p})E(\vec{q})(E(\vec{p})+E(\vec{q}))^2}\Big((m^2 + E(\vec{p})E(\vec{q}) + \vec{p}\cdot\vec{q})|\widehat{\vec{A}}(\vec{p}-\vec{q})|^2 - (\vec{p}\cdot\widehat{\vec{A}}(\vec{p}-\vec{q}))(\vec{q}\cdot\widehat{\vec{A}}(\vec{q}-\vec{p})) - (\vec{p}\cdot\widehat{\vec{A}}(\vec{q}-\vec{p}))(\vec{q}\cdot\widehat{\vec{A}}(\vec{p}-\vec{q}))\Big)\,dp\,dq \tag{7.167}$$

Using (7.155p.147), we see that the integrand in this integral is positive. We substitute $\vec{k} := \vec{p} - \vec{q}$. For any measurable set $S \subseteq \mathbb{R}^3 \times \mathbb{R}^3$, we get a lower bound by restricting the integration to S:

$$\|Q^{\vec{A}}\|_{\mathrm{I}_2}^2 \geq \int_S \frac{2e^2}{E(\vec{p})E(\vec{p}-\vec{k})(E(\vec{p})+E(\vec{p}-\vec{k}))^2}\Big((m^2 + E(\vec{p})E(\vec{p}-\vec{k}) + \vec{p}\cdot(\vec{p}-\vec{k}))|\widehat{\vec{A}}(\vec{k})|^2 - (\vec{p}\cdot\widehat{\vec{A}}(\vec{k}))((\vec{p}-\vec{k})\cdot\widehat{\vec{A}}(-\vec{k})) - (\vec{p}\cdot\widehat{\vec{A}}(-\vec{k}))((\vec{p}-\vec{k})\cdot\widehat{\vec{A}}(\vec{k}))\Big)\,dp\,dk. \tag{7.168}$$

The following considerations serve to find an appropriate choice of the set S. By the assumption $\vec{A} \neq 0$ we can take $\vec{l} \in \mathbb{R}^3$ such that $\widehat{\vec{A}}(\vec{l}) \neq 0$. For every $\vec{a} \in \mathbb{C}^3 \setminus \{0\}$, there exists a unit vector $\vec{b} \in \mathbb{R}^3$, $|\vec{b}| = 1$, such that $|\vec{b}\cdot\vec{a}| \leq |\vec{a}|/\sqrt{2}$. One can see this as follows. We define $\vec{c} = \vec{a}$ if $|\operatorname{Re}\vec{a}| \geq |\operatorname{Im}\vec{a}|$, and $\vec{c} = i\vec{a}$ otherwise. In particular, $|\vec{c}| = |\vec{a}|$ and

$$2|\operatorname{Im}\vec{c}|^2 \leq |\operatorname{Re}\vec{a}|^2 + |\operatorname{Im}\vec{a}|^2 = |\vec{a}|^2. \tag{7.169}$$

Take any unit vector $\vec{b} \in \mathbb{R}^3$ orthogonal to $\operatorname{Re}\vec{c}$. Using (7.169), we get

$$|\vec{b}\cdot\vec{a}| = |\vec{b}\cdot\vec{c}| = |\vec{b}\cdot\operatorname{Im}\vec{c}| \leq |\vec{b}||\operatorname{Im}\vec{c}| = |\operatorname{Im}\vec{c}| \leq \frac{|\vec{a}|}{\sqrt{2}}. \tag{7.170}$$

We apply this to $\vec{a} = \widehat{\vec{A}}(\vec{l})$, taking a unit vector $\vec{b} \in \mathbb{R}^3$ with $|\vec{b}\cdot\widehat{\vec{A}}(\vec{l})| \leq |\widehat{\vec{A}}(\vec{l})|/\sqrt{2}$. Take any fixed number C_{46} such that $1/\sqrt{2} < C_{46} < 1$; then $|\vec{b}\cdot\widehat{\vec{A}}(\vec{l})| < C_{46}|\vec{b}||\widehat{\vec{A}}(\vec{l})|$ holds because of $|\vec{b}| = 1$ and $|\widehat{\vec{A}}(\vec{l})| > 0$. Now $\widehat{A}$ is a continuous function. Therefore, there is a compact ball $\bar{B}_r(\vec{l}) = \{\vec{k} \in \mathbb{R}^3 \mid |\vec{k}-\vec{l}| \leq r\}$, centered at $\vec{l}$ with some radius $r > 0$, such that

$$C_{47} := \inf_{\vec{k}\in\bar{B}_r(\vec{l})} |\widehat{\vec{A}}(\vec{k})| > 0 \tag{7.171}$$

is true and $|\vec{b}\cdot\widehat{\vec{A}}(\vec{k})| < C_{46}|\vec{b}||\widehat{\vec{A}}(\vec{k})|$ holds for all $\vec{k} \in \bar{B}_r(\vec{l})$. By compactness of the ball, using continuity of the function $\mathbb{R}^3 \times \mathbb{R}^3 \ni (\vec{p},\vec{k}) \mapsto |\vec{p}\cdot\widehat{\vec{A}}(\vec{k})| - C_{46}|\vec{p}||\widehat{\vec{A}}(\vec{k})|$, the set

$$S_1 := \{\vec{p} \in \mathbb{R}^3 \mid \text{for all } \vec{k} \in \bar{B}_r(\vec{l}) \text{ holds } |\vec{p}\cdot\widehat{\vec{A}}(\vec{k})| < C_{46}|\vec{p}||\widehat{\vec{A}}(\vec{k})|\} \tag{7.172}$$

is an open subset of $\mathbb{R}^3$. The set S_1 is nonempty because of $\vec{b} \in S_1$. Furthermore, S_1 is a homogeneous set in the following sense: For all $\vec{p} \in \mathbb{R}^3$ and all $\lambda \in \mathbb{R} \setminus \{0\}$, $\vec{p} \in S_1$ is equivalent to $\lambda\vec{p} \in S_1$. Note that $|\vec{p}\cdot\widehat{\vec{A}}(\vec{k})| = |\vec{p}\cdot\widehat{\vec{A}}(-\vec{k})|$ holds, as $\widehat{\vec{A}}(-\vec{k})$ and $\widehat{\vec{A}}(\vec{k})$ are complex conjugate to each other.

We set $S = S_1 \times \bar{B}_r(\vec{l})$. For the following considerations, note that $|E(\vec{p}-\vec{k}) - E(\vec{p})| \leq |\vec{k}|$, $|\vec{p}| \leq E(\vec{p})$, and $(\vec{p}\cdot\widehat{\vec{A}}(\vec{k}))(\vec{p}\cdot\widehat{\vec{A}}(-\vec{k})) = |\vec{p}\cdot\widehat{\vec{A}}(\vec{k})|^2$ hold for all $\vec{p},\vec{k} \in \mathbb{R}^3$, and that $\widehat{\vec{A}}$ is bounded on the ball $\bar{B}_r(\vec{l})$. Using this, one sees that there is a constant $C_{48} > 0$, depending only on the potential $\widehat{\vec{A}}$ and on the choice of the compact ball $\bar{B}_r(\vec{l})$, such that for all $\vec{p} \in \mathbb{R}^3$ and all $\vec{k} \in \bar{B}_r(\vec{l})$, one has

$$\Big\| \Big[(m^2 + E(\vec{p})E(\vec{p}-\vec{k}) + \vec{p}\cdot(\vec{p}-\vec{k}))|\widehat{\vec{A}}(\vec{k})|^2 - (\vec{p}\cdot\widehat{\vec{A}}(\vec{k}))((\vec{p}-\vec{k})\cdot\widehat{\vec{A}}(-\vec{k})) - (\vec{p}\cdot\widehat{\vec{A}}(-\vec{k}))((\vec{p}-\vec{k})\cdot\widehat{\vec{A}}(\vec{k})) \Big] - \Big[2E(\vec{p})^2|\widehat{\vec{A}}(\vec{k})|^2 - 2|\vec{p}\cdot\widehat{\vec{A}}(\vec{k})|^2 \Big] \Big\| \le C_{48}E(\vec{p}). \tag{7.173}$$

Furthermore, there is another constant $C_{49} > 0$, depending only on the choice of the compact ball $\bar{B}_r(\vec{l})$, such that for all $\vec{p} \in \mathbb{R}^3$ and all $\vec{k} \in \bar{B}_r(\vec{l})$, one has

$$E(\vec{p}-\vec{k})(E(\vec{p}) + E(\vec{p}-\vec{k}))^2 \le C_{49}E(\vec{p})^3. \tag{7.174}$$

Substituting the bounds (7.173), (7.174), the choice ($7.172_{\text{p.149}}$) of S_1, and the lower bound ($7.171_{\text{p.149}}$) of $|\widehat{\vec{A}}|$ on $\bar{B}_r(\vec{l})$ in the lower bound ($7.168_{\text{p.149}}$) of $\|Q^{\vec{A}}\|^2_{\mathrm{I}_2}$, we obtain

$$\begin{aligned} \|Q^{\vec{A}}\|^2_{\mathrm{I}_2} &\ge \int_S \frac{2e^2}{C_{49}E(\vec{p})^4}\Big(2E(\vec{p})^2|\widehat{\vec{A}}(\vec{k})|^2 - 2|\vec{p}\cdot\widehat{\vec{A}}(\vec{k})|^2 - C_{48}E(\vec{p})\Big)\,dp\,dk \\ &\ge \int_S \frac{2e^2}{C_{49}E(\vec{p})^4}\Big(2(1-C_{46})E(\vec{p})^2|\widehat{\vec{A}}(\vec{k})|^2 - C_{48}E(\vec{p})\Big)\,dp\,dk \\ &\ge |\bar{B}_r(\vec{l})| \int_{S_1} \frac{2e^2}{C_{49}E(\vec{p})^4}\Big(2(1-C_{46}){C_{47}}^2E(\vec{p})^2 - C_{48}E(\vec{p})\Big)\,dp = \infty. \end{aligned} \tag{7.175}$$

We have used that $1 - C_{46} > 0$, and that S_1 is a nonempty, open homogeneous subset of $\mathbb{R}^3$. Thus the lemma is proven. □

7.3.3 The Second Quantized Time-Evolution

Let us summarize. For a given vector potential $\mathsf{A} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and any time $t \in \mathbb{R}$, let $C(\vec{\mathsf{A}}(t)) \in \mathtt{Pol}(\mathcal{H})/\approx_0$ be the polarization class belonging to time t as introduced in Definitions $7.30_{\text{p.132}}$ (Induced polarization classes) and $7.39_{\text{p.145}}$ (Physical polarization classes). Combining Theorem $7.38_{\text{p.145}}$ (Identification of the polarization classes), Theorem $7.31_{\text{p.132}}$ (Dirac time-evolution with external field), Theorem $7.27_{\text{p.128}}$ (Lift condition), and Corollary $7.29_{\text{p.129}}$ (Uniqueness of the lift up to a phase), we have proven the following theorem.

Second quantized Dirac time-evolution

Theorem 7.41. *Let* $\mathsf{A} = (\mathsf{A}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_0, -\vec{\mathsf{A}}) \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ *be an external vector potential,* $t_1, t_0 \in \mathbb{R}$ *be two time points,* $U = U^{\mathsf{A}}(t_1, t_0)$ *be the one-particle Dirac time-evolution,* ℓ *be a (separable, infinite dimensional) Hilbert space, and* $\Phi \in \mathtt{Ocean}_\ell(C(0))$. *Set*

$$\mathcal{S}(t) := [e^{Q^{\mathsf{A}(t)}}\Phi]_\sim \in \mathtt{Ocean}_\ell(C(\vec{\mathsf{A}}(t)))/\sim \tag{7.176}$$

for $t \in \mathbb{R}$. *Then, one has*

$$U \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(\vec{\mathsf{A}}(t_0)); \mathcal{H}, C(\vec{\mathsf{A}}(t_1))). \tag{7.177}$$

As a consequence, there is $R \in \mathrm{U}(\ell)$ *such that*

$$\mathcal{R}_R\mathcal{L}_U : \mathcal{F}_{\mathcal{S}(t_0)} \to \mathcal{F}_{\mathcal{S}(t_1)} \tag{7.178}$$

is a unitary map between the wedge spaces $\mathcal{F}_{\mathcal{S}(t_0)}$ *and* $\mathcal{F}_{\mathcal{S}(t_1)}$. *This second-quantized Dirac time-evolution is unique up to a phase in the following sense. For any two such choices* $R_1, R_2 \in \mathrm{U}(\ell)$ *with* $\mathcal{R}_{R_1}\mathcal{L}_U, \mathcal{R}_{R_2}\mathcal{L}_U : \mathcal{F}_{\mathcal{S}(t_0)} \to \mathcal{F}_{\mathcal{S}(t_1)}$, *the operator* $R_1^{-1}R_2$ *has a determinant* $\det(R_1^{-1}R_2) = e^{i\varphi}$ *for some* $\varphi \in \mathbb{R}$, *and*

$$\mathcal{R}_{R_2}\mathcal{L}_U = e^{i\varphi}\mathcal{R}_{R_1}\mathcal{L}_U \tag{7.179}$$

holds.

An application of this theorem is the computation of transition amplitudes. Consider given $\Lambda\Psi^{in} \in \mathcal{F}_{\mathcal{S}(t_0)}$ and $\Lambda\Psi^{out} \in \mathcal{F}_{\mathcal{S}(t_1)}$, which represent "in" and "out" states at times t_0 and t_1, respectively. The transition amplitude is according to the above theorem given by

$$\mathbb{P}\left(X \overset{A}{\rightsquigarrow} \begin{matrix} \nearrow \Lambda\Psi^{OUT} \\ \searrow \Lambda\Psi^{IN} \end{matrix} \right) := \left| \left\langle \Lambda\Psi^{out}, \mathcal{R}_{R_1} \mathcal{L}_U \Lambda\Psi^{in} \right\rangle \right|^2 = |e^{i\varphi}|^2 \left| \left\langle \Lambda\Psi^{out}, \mathcal{R}_{R_2} \mathcal{L}_U \Lambda\Psi^{in} \right\rangle \right|^2$$
$$= \left| \left\langle \Lambda\Psi^{out}, \mathcal{R}_{R_2} \mathcal{L}_U \Lambda\Psi^{in} \right\rangle \right|^2$$

which is therefore independent on our specific choice of the matrix R.

7.4 Pair Creation Probabilities

So far we have constructed Dirac seas and implemented their time-evolution between time-varying Fock spaces in a way that allows to compute well-defined and unique transition amplitudes. However, describing the Dirac sea of infinitely many electrons under the negligence of all electron-electron interactions is in general a crude approximation, and the only way to make sense of such an external field model is by the equilibrium assumption discussed in 6p.105. For this we need to introduce Dirac seas which shall represent the equilibrium (i.e. vacuum) states. Since we do not model quantum interaction between the electrons, we do not have means to distinguish a vacuum state, using the condition that all electron-electron interaction vanishes. Therefore, in a model of pair creation we rather have to introduce this state a posteriori. The line of reasoning is the following: Coming from a hypothetically fully-interacting quantum electrodynamic theory, one looks for states for which the net-electron-electron interaction for each electron evens out to zero. These states are the vacuum states. Negligence of the electron-electron interaction is then justified if we only describe vacuum states and small deviations from those vacuum states – e.g. states with a small number of pairs with respect to N (in the limit $N \to \infty$ which we consider here one can read "small" to mean finite).

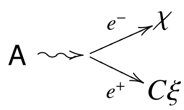
We define the following model for the computation of pair creation rates for an external field that is only non-zero only within the time interval (t_0, t_1):

1. In the absence of an external field the vacuum state, i.e. a state for which the net-electron-electron interactions vanish, can be modeled by an infinite wedge product of one-particle wave functions in $V \in C(0) = [\mathcal{H}_-]_{\approx 0}$. We choose any basis $(\varphi_n)_{n\in\mathbb{N}}$ of V and define $\Phi : \ell^2 \to \mathcal{H}$, $\Phi e_n := \varphi_n$ for the canonical basis $(e_n)_{n\in\mathbb{N}}$ of ℓ^2. The state
$$\Lambda\Phi = \varphi_1 \wedge \varphi_2 \wedge \dots \tag{7.180}$$
then represents a vacuum state. Built this way, Φ is some sea in `Ocean`$(C(0))$.

2. Again in the absence of an external field all states in the wedge space $\mathcal{F}_{\mathcal{S}}$ for $\mathcal{S} = [\Phi]_\sim$ are considered small deviations from the equilibrium state $\Lambda\Phi$. We interpret a basis of $\mathcal{F}_{\mathcal{S}}$ as follows: If we take the vacuum state $\Lambda\Phi$ and replace $n \in \mathbb{N}$ of the one-particle wave functions in the wedge product by one-particle wave functions in $V^\perp$ (observing the Pauli exclusion principle), the corresponding state represents a disturbance of the equilibrium by the presence of n pairs. For example, for $\chi, \xi \in V^\perp$ the state
$$\varphi_1 \wedge \chi \wedge \varphi_3 \wedge \varphi_4 \wedge \xi \wedge \varphi_6 \wedge \varphi_7 \wedge \dots$$
would describe the presence of two pairs: One having electron wave function χ and positron wave function $C\varphi_2$ where C denotes the charge conjugation. And another one with electron wave function ξ and positron wave function $C\varphi_5$.

Note, however, that the choice of V along with the assumption that the vacuum state is given by a state that has product structure has to be compatible with the physical situation one wants to describe. For the situation above, where A was zero before time t_0 and after time t_1, energetic considerations suggest that for these times the equilibrium state occupies $V = \mathcal{H}_-$. In the presence of a non-zero but static external field A for these times (e.g. for pair creation processes near an ion) one of course would have to choose V to lie in the correct polarization class $C(\vec{\mathsf{A}}(t_0))$ or $C(\vec{\mathsf{A}}(t_1))$, respectively; cf. the Furry picture [FS79]. However, in more general situations only a fully interacting theory can give an a priori inside of how to model the vacuum (since the vacuum was introduced as an equilibrium state one still expects non-uniqueness as there are many micro-states fulfilling the equilibrium property that all net-electron-electron vanishes).

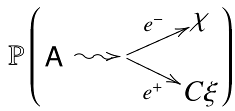
With these model assumptions let us come back to the example given at the end of the introductory Chapter 6p.105 and compute the probability for the creation of one pair with electron wave function $\chi \in V^\perp$ and positron wave function $C\xi$ for $\xi \in V$ out of a vacuum induced by the external potential A:



At times smaller t_0 and times larger t_1 the external field is zero, and we assume that at these times the electrons are relaxed to an equilibrium state, or at most to a small perturbation of this equilibrium. According to model assumption 1, we can model the equilibrium state by $\Lambda\Phi$ for an appropriate choice of polarization $V \in C(0)$ as well as a basis $(\varphi_n)_{n\in\mathbb{N}}$ of V such that $\Phi e_n = \varphi_n$ for all $n \in \mathbb{N}$. Let $\mathcal{S} := \mathcal{S}(\Phi)$ so that $\Lambda\Phi \in \mathcal{F}_\mathcal{S}$. Since we want to model the electron-positron pair state in $\mathcal{F}_\mathcal{S}$ we have to make sure that we chose V and its basis in such a way that $\chi \in V^\perp$ and $\xi \in V$ while one basis element is equal ξ, say for example $\varphi_1 = \xi$. According to model assumption 2, we may model the outgoing state by $\Lambda\Psi^{\text{out}} \in \mathcal{F}_\mathcal{S}$ for a $\Psi^{\text{out}} \in \texttt{Ocean}(C(0))$ with $\Psi^{\text{out}} : \ell^2 \to \mathcal{H}$, $\Psi^{\text{out}} e_1 := \chi$ and $\Psi^{\text{out}} e_n := \varphi_n$ for $n > 1$, i.e. written symbolically

$$\Lambda\Psi^{\text{out}} = \chi \wedge \varphi_2 \wedge \varphi_3 \wedge \dots .$$

In the notation of Subsection 7.3.3p.150 we then have: $C(\vec{\mathsf{A}}(t_0)) = C(0) = C(\vec{\mathsf{A}}(t_1))$ since $A(t_0) = 0 = A(t_1)$. Therefore, the one-particle time-evolution $U^{\mathsf{A}}(t_1, t_0)$ is in $\mathrm{U}^0_{\text{res}}(\mathcal{H}, C(0); \mathcal{H}, C(0))$. By Theorem 7.41p.150 () we know that there is an $R \in \mathrm{U}(\ell^2)$ being unique up to a phase such that the transition amplitude for the considered process is given by the square modulus of the inner product in $\mathcal{F}_\mathcal{S}$:



$$\mathbb{P}\left(\cdots \right) := |\langle \Lambda\Psi^{\text{out}}, \mathcal{R}_R \mathcal{L}_{U^A(t_1,t_0)} \Lambda\Phi \rangle|^2.$$

The matrix R of Theorem 7.41p.150 is explicitly given by the polar decomposition of $\Phi^* U \Phi = BR$ for $B = |\Phi^* U^{\mathsf{A}}(t_0, t_1)\Phi|$; cf. Proof of Theorem 7.27p.128 (Lift condition). Note that R defined in this way is the time-evolution of the sea $P_V U^{\mathsf{A}}(t_0, t_1) P_V$, since $\Phi : \ell^2 \to V$ is a unitary transformation and $P_V = \Phi\Phi^*$. So one should read the transition probability above as follows: We want to compute the transition amplitude of one vacuum state at time t_0 evolving into the definite state $\Lambda\Psi^{\text{out}}$ at time t_1, i.e. one electron has wave function χ, another has wave function φ_2, another has the wave function φ_3 and so on. Since this state is only a small deviation from the vacuum, we can guess what the vacuum state is at time t_1, namely $\Lambda\Phi$. If we evolve this state backwards to t_0 by the unitary part of $P_V U^{\mathsf{A}}(t_0, t_1) P_V$, we find the vacuum state at time t_0 and call this state

$$\Lambda\Psi^{\text{in}} = \mathcal{R}_R \Lambda\Phi$$

The transition amplitude is then given by

$$|\langle \Lambda\Psi^{\text{out}}, \mathcal{L}_{U^A(t_1,t_0)}\Lambda\Psi^{\text{out}}\rangle|^2 = |\langle \Lambda\Psi^{\text{out}}, \mathcal{R}_R\mathcal{L}_{U^A(t_1,t_0)}\Lambda\Phi\rangle|^2$$

where the equality holds because the above guess is unique up to a phase. The guess together with the backward time-evolution of the vacuum state is encoded in the matrix R. Mathematically the role of R can be understand best when writing the inner product in matrix notation:

$$\langle \Lambda\Psi^{\text{out}}, \mathcal{R}_R\mathcal{L}_{U^A(t_1,t_0)}\Lambda\Phi\rangle = \det\begin{pmatrix} \langle\chi, U_{+-}\xi\rangle & \langle\chi, U_{+-}\varphi_2\rangle & \langle\chi, U_{+-}\varphi_3\rangle & \dots \\ \langle\varphi_2, U_{--}\xi\rangle & \langle\varphi_2, U_{--}\varphi_2\rangle & \langle\varphi_2, U_{--}\varphi_3\rangle & \dots \\ \langle\varphi_3, U_{--}\xi\rangle & \langle\varphi_3, U_{--}\varphi_2\rangle & \langle\varphi_3, U_{--}\varphi_3\rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot R \qquad (7.181)$$

where we use the notation $U = U^A(t_1, t_0)$, $U_{+-} = P_{V^\perp}UP_V$ and $U_{--} = P_VUP_V$. The Fredholm determinant is only well-defined if the matrix is a trace class perturbation of the identity matrix which for general U_{--} and without the matrix R will not be the case. This is due to the fact that the one-particle wave functions of the vacuum, i.e. φ_n for $n \in \mathbb{N}$, are also time-evolved. The matrix R undoes this time-evolution of the one-particle wave functions in V.

We emphasize again that there is no absolute number of pairs as the phenomena of pair creation is only an artefact arising from the adaption of the effective description of a many particle system. The best we can do is to define the absolute value of the difference of the number of pairs between two Dirac seas:

Definition 7.42. *Let $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$ then we define the relative number of pairs*

Relative number of pairs

$$\# : \mathcal{S}\times\mathcal{S} \to \mathbb{N}_0,\ (\Phi,\Psi) \mapsto \#(\Phi,\Psi) := \dim\ker\Phi^*\Psi.$$

Physically one should read $\#(\Phi,\Psi)$ in the following way: If $\Lambda\Phi$ is a vacuum state, then with respect to it $\Lambda\Psi$ has $\#(\Phi,\Psi)$ pairs. In the example above the relative number of pairs of Ψ^{out} with respect to a vacuum Ψ^{IN} is $\#(\Psi^{IN}, \Psi^{\text{out}}) = 1$. In particular, the relative number of pairs is symmetric and invariant under the operations from the right:

Lemma 7.43. *Let $\mathcal{S} \in \mathtt{Seas}(\mathcal{H})/\sim$ and $\Phi, \Psi \in \mathcal{S}$:*

(i) $\#(\Phi,\Psi)$ *is well-defined and finite.*

(ii) $\#(\Phi,\Psi) = \#(\Psi,\Phi)$.

(iii) For $R \in \mathrm{U}(\ell)$ *we have* $\#(\Phi,\Psi) = \#(\Psi R, \Phi R)$.

Proof. (i) Since $\Phi \sim \Psi$ we know $\Phi^*\Psi \in \mathrm{id}(\ell) + \mathrm{I}_1(\ell)$ which means that it is a Fredholm operator with Fredholm index $\mathrm{ind}(\Phi^*\Psi) = 0$. Hence, the dimension of its kernel is finite.

(ii) $\#(\Phi,\Psi) = \dim\ker\Phi^*\Psi = \dim\ker\Phi^*\Psi + \mathrm{ind}(\Phi^*\Psi) = \dim\ker\Psi^*\Phi = \#(\Psi,\Phi)$.

(iii) holds because the dimension is invariant under unitary transformations. □

In particular, property (iii) is important as it implies that if we had chosen a different equivalence classes $\mathcal{S}'$ instead of $\mathcal{S} = \mathcal{S}(\Psi^{\text{in}})$ this would then have had no effect on the relative number of pairs since by Lemma 7.28p.129 (Orbits in `Ocean`) we know that there is an $R \in \mathrm{U}(\ell)$ such that $\mathcal{S}' = \mathcal{S}R$. In order to yield expectation values of the relative number of pairs, one could lift the map # to wedge spaces in the usual manner.

7.5 Gauge Transformations

As an addendum we briefly discuss gauge transformations. Let $\vec{A} \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ be a vector potential and $\vec{A}^{\sim} = \vec{A} + \nabla Y$ be a gauge transform of it with $Y \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. Let $e^{iY} : \mathcal{H} \to \mathcal{H}$ the multiplication operator with e^{iY}. We prove:

Gauge transformations

Theorem 7.44. *The gauge transformation e^{iY} fulfills:*

$$e^{iY} \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(\vec{A}); \mathcal{H}, C(\vec{A}^{\sim})) \tag{7.182}$$

Although the statement of this theorem does not involve time, we prove it using the time-evolution from Subsection 7.3.1p.130. A “direct” proof, avoiding time-evolution and using similar techniques as in Subsection 7.3.1p.130, is possible. However, the approach presented here avoids additional analytical considerations.

Proof. We switch the gauge transformation on between the times 0 and 1, using a smooth function $f : \mathbb{R} \to [0, 1]$ with $f(t) = 0$ and $f(t) = 1$ for t in a neighborhood of 0 and 1, respectively. We define $\mathsf{Y} : \mathbb{R}^4 \ni (t, \vec{x}) \mapsto f(t)Y(\vec{x}) \in \mathbb{R}$. Take the static vector potential $\mathsf{A} : \mathbb{R}^4 \ni (t, \vec{x}) \mapsto (0, -\vec{A}(\vec{x})) \in \mathbb{R}^4$ and its gauge-transformed version $\mathsf{A}^{\sim} = (\mathsf{A}^{\sim}_\mu)_{\mu=0,1,2,3} = (\mathsf{A}_\mu - \partial_\mu \mathsf{Y})_{\mu=0,1,2,3} = (A_0 - \partial_0 \mathsf{Y}, -\vec{A} - \nabla \mathsf{Y})$. In other words,

$$\mathsf{A}^{\sim}(t, \vec{x}) = (-f'(t)Y(\vec{x}), -\vec{A}(\vec{x}) - f(t)\nabla Y(\vec{x})). \tag{7.183}$$

(It is no problem that the vector potentials used here do in general not have compact support in time, because we use only times $t \in [0, 1]$.) Note that at time $t = 0$, the gauge transformation is turned off: $\mathsf{A}(0) = \mathsf{A}^{\sim}(0) = (0, -\vec{A})$, and at time $t = 1$ it is completely turned on: $\mathsf{A}(1) = (0, -\vec{A})$ and $\mathsf{A}^{\sim}(1) = (0, -\vec{A}^{\sim})$. The one-particle Dirac time-evolutions U^{A} and $U^{\mathsf{A}'}$ are also related by a gauge transformation as follows:

$$e^{i\mathsf{Y}(t_1)} U^{\mathsf{A}}(t_1, t_0) = U^{\mathsf{A}^{\sim}}(t_1, t_0) e^{i\mathsf{Y}(t_0)}, \quad t_1, t_0 \in [0, 1]. \tag{7.184}$$

In particular, this includes $e^{iY} U^{\mathsf{A}}(1, 0) = U^{\mathsf{A}^{\sim}}(1, 0)$. By Theorem 7.31p.132 (Dirac time-evolution with external field), we have the following:

$$U^{\mathsf{A}}(0, 1) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(\vec{A}); \mathcal{H}, C(\vec{A})) \tag{7.185}$$

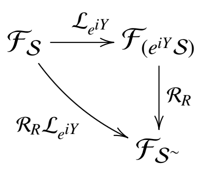
$$U^{\mathsf{A}^{\sim}}(1, 0) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(\vec{A}); \mathcal{H}, C(\vec{A}^{\sim})) \tag{7.186}$$

This implies the following:

$$e^{iY} = U^{\mathsf{A}^{\sim}}(1, 0) U^{\mathsf{A}}(0, 1) \in \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C(\vec{A}); \mathcal{H}, C(\vec{A}^{\sim})) \tag{7.187}$$

Thus the claim is proven. □

We infer that in general the gauge transformation e^{iY} changes the polarization class. Using varying wedge spaces, it can be second quantized as follows. Let $\mathcal{S} \in \mathtt{Ocean}(C(\vec{A}))$ and $\mathcal{S}^{\sim} \in \mathtt{Ocean}(C(\vec{A}^{\sim}))$. By Theorem 7.27p.128 (Lift condition), there exists $R \in \mathrm{U}(\ell)$ such that we have the following second-quantized gauge transformation from $\mathcal{F}_{\mathcal{S}}$ to $\mathcal{F}_{\mathcal{S}^{\sim}}$:



7.6 Conclusion and Outlook

There are two apparent near-term goals:

First, as we have shown the construction of the second quantized Dirac time-evolution is unique only up to a phase which depends on external field A. This phase carries physical information which can be seen in the following argument: Let $U^{\mathsf{A}}(t_1, t_0)$ be the one-particle Dirac time-evolution on $\mathcal{H}$ for an external potential which is non-zero only in (t_1, t_0). Then one can express the usual charge current by

$$j^\mu(x) = -iU(t_0, t_1)\frac{\delta}{\delta A_\mu(x)}U(t_1, t_0)$$

which is a distribution-valued operator on $\mathcal{H}$. Furthermore, let $\widetilde{U}^{\mathsf{A}}(t_1, t_0) : \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{\mathcal{S}'}$ be one possible lift of the one-particle time-evolution between the wedge spaces $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{F}_{\mathcal{S}'}$. One would then expect to be able to define the charge current in a similar way:

$$J^\mu(x) = -i\widetilde{U}(t_0, t_1)\frac{\delta}{\delta A_\mu(x)}\widetilde{U}(t_1, t_0).$$

However, the lift is only unique up to a phase. Therefore, a different lift $\bar{U}^{\mathsf{A}}(t_1, t_0) = \widetilde{U}^{\mathsf{A}}(t_1, t_0)e^{i\varphi_{\mathsf{A}}(t_1, t_0)}$ would yield a different current

$$\bar{J}^\mu(x) = \frac{\delta}{\delta A_\mu(x)}\varphi_{\mathsf{A}}(t_1, t_0) + J^\mu(x).$$

The question is how we can get our hands on the functional derivative of the phase. Due to Scharf [Sch95] one idea is the following: In analogy to the one-particle case, we would expect that a disturbance of the external field at space-time point y outside the backward light-cone of x would not change the current at x and

$$\frac{\delta}{\delta A_\nu(y)}J^\mu(x) = 0, \text{ for such } y.$$

For such x and y we can determine $\frac{\delta^2}{\delta A_\nu(y)\delta A_\mu(x)}\varphi_{\mathsf{A}}(t_1, t_0)$. Using the symmetry in x and y we can thus determine the second functional derivative of $\varphi_{\mathsf{A}}(t_1, t_0)$ everywhere except for $x = y$. If we continue this distribution in a sensible way to be defined for all x and y, an integration of the space of fields would yield the desired term $\frac{\delta}{\delta A_\mu(x)}\varphi_{\mathsf{A}}(t_1, t_0)$. The question is how much freedom of choice is left in the continuation when physical input like gauge and Lorentz invariance, the continuity equation and the condition that the vacuum expectation value of the charge current subject to a zero external potential is zero. It is conjectured that this freedom is the manifestation of the charge renormalization.

Second, it seems natural to generalize the second quantized Dirac time-evolution between equal time hyper-surfaces to a time-evolution between smooth space-like hypersurfaces. For this, one would need to construct wedge spaces corresponding to these space-like hyper surfaces. The idea for the construction is the following. Let $\psi : \mathbb{R}^4 \to \mathbb{C}^4$ be a spinor field on Minkowski space $\mathbb{M}$ that solves the Dirac equation subject to an external potential A, i.e.

$$(i\partial\!\!\!/ - m)\psi(x) = e\mathsf{A}\!\!\!/(x)\psi(x).$$

Any solution gives rise to the divergence free one-particle charge current $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ so that Stoke's Theorem yields for two space-like hypersurfaces Σ_1, Σ_2

$$\int_{\Sigma_1} d\sigma(x)\bar{\psi}(x)n^\mu(x)\gamma_\mu\psi(x) = \int_{\Sigma_2} d\sigma(x)\bar{\psi}(x)n^\mu(x)\gamma_\mu\psi(x)$$

where we denoted the surface measure on the space-like hypersurfaces by $d\sigma(x)$ and the surface normal four-vectors by $n^\mu(x)$. This gives rise to a candidate of a Hilbert space of wave functions on the space-like hypersurface Σ, namely the space of $\psi : \Sigma \to \mathbb{C}^4$ with the inner product

$$\langle \psi, \phi \rangle := \int_{\Sigma_1} d\sigma(x) \overline{\psi}(x) n^\mu(x) \gamma_\mu \phi(x).$$

We now have a similar construction as in Subsection 7.2.1p.116 in mind to yield an infinite wedge space for Σ. The only work involved in doing this is the definition of the polarization classes which so far depend on the equal time hyper-surfaces and, hence, need to be generalized. Note that via this construction one could also replace the metric tensor by a metric tensor field on $\mathbb{M}$ to account for general relativistic effects.

The long-term goal is of course the introduction of electron-electron interaction.

Part III

Outlook

Chapter 8

Electrodynamic Absorber Theory

After presenting the mathematical results we want to conclude this work by given an informal outlook on our perspective of an electrodynamic theory for point-like charges which should be divergence free and capable of describing the phenomena of radiation reaction as well as of pair creation. Let us briefly review the Chapters $2_{\text{p.7}}$ and $7_{\text{p.111}}$:

We introduced the classical electrodynamic absorber theory as a theory about N charges and N fields which obey the ML-SI equations ($2.1_{\text{p.8}}$) and ($2.8_{\text{p.9}}$) as described in Chapter $2_{\text{p.7}}$. By an argument due to Wheeler and Feynman we retrieved the mechanism of radiation reaction under the condition that the absorber assumption ($2.9_{\text{p.9}}$) holds, and we found that for special initial values ($2.15_{\text{p.12}}$) this theory is equivalent to Wheeler-Feynman electrodynamics. For rigid charges we have shown that the initial value problem of the ML-SI dynamics is well-posed in Chapter $3_{\text{p.15}}$. As there seems to be no obstruction in taking the point particle limit apart from the crossing of trajectories the dynamical theory is expected to be divergence free for almost all initial values. In Chapter $7_{\text{p.111}}$ we further discussed how subsystems of the absorber medium for large N can be treated. For a system of N electrons which is initially in an equilibrium state so that the net interaction between the electrons vanishes ($6.3_{\text{p.106}}$) the effective description of a subsystem of the absorber medium gives rise to creation and annihilation processes of fields as in ($6.4_{\text{p.107}}$) whenever this equilibrium state is disturbed. We further developed a quantum theoretic description of such a situation for N electrons where, because of the equilibrium assumption, we neglected the interactions between the electrons completely. Following the idea of Dirac the effective description of this equilibrium state of the N electrons gives rise to pair creation and annihilation. In this sense, both phenomena, radiation reaction and pair creation, emerge from the same physical assumption, namely that an absorber medium or Dirac sea with many particles, i.e. with large N, is present.

The next thing missing is a quantum mechanical analogue of the ML-SI equations. In the following last section we address this issue and give an outlook on possible implications of this electrodynamic absorber theory.

8.1 Short Review of Steps Towards an Absorber Quantum Electrodynamics

The most interesting case are ML-SI equations for the special initial values ($2.15_{\text{p.12}}$) whence they are equivalent to Wheeler-Feynman electrodynamics. First, because Wheeler-Feynman electrodynamics is a theory only about world lines without the need of fields, and second, because of the state-dependent advanced and delayed terms in the equation of motion. Since we have no

Hamiltonian (as it is generically the case for the ML-SI equations), we cannot rely on some correspondence principle in order to find a "quantization" and therefore we rather let us guide by classical field theory.

Action integral for a Dirac field

The Dirac equation subject to a given external potential A can be deduced from a variational principle: Consider the action

$$S[\psi,\overline{\psi}] := \int d^4x \left[\overline{\psi}(x)\,(i\partial\!\!\!/ - m)\,\psi(x) - e\overline{\psi}(x)A\!\!\!/(x)\psi(x)\right]$$

for $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ and $A\!\!\!/(x) = \gamma_\mu A^\mu(x)$. Any extremum needs to fulfill

$$\frac{d}{d\epsilon}S[\psi,\overline{\psi}+\epsilon\overline{f}]|_{\epsilon=0} = 0 \qquad \text{and} \qquad \frac{d}{d\epsilon}S[\psi+\epsilon f,\overline{\psi}]|_{\epsilon=0} = 0 \tag{8.1}$$

for all test functions $f \in C_c^\infty(\mathbb{R}^4,\mathbb{R}^4)$. One finds that at the extremum the fields ψ and $\overline{\psi}$ obey the equations

$$(i\partial\!\!\!/ - m)\psi(x) = eA\!\!\!/(x)\psi(x) \qquad \text{and} \qquad -i\partial_\mu\overline{\psi}(x)\gamma^\mu - \overline{\psi}(x)m = e\overline{\psi}(x)A\!\!\!/(x)$$

which for $\overline{\psi}(x) = \psi^*(x)\gamma^0$ are the Dirac equations for an external potential A. The goal of this section is to generalize this action principle step by step to yield a classical field theory of Dirac fields which interact with each other by an action-at-a-distance principle like in Wheeler-Feynman electrodynamics.

Generalization of the action to N Dirac fields

The first generalization is to allow for N fields which yields the action

$$S[\psi_k,\overline{\psi}_k;1\le k\le N] := \sum_{k=1}^N \int d^4x \left[\overline{\psi}_k(x)\,(i\partial\!\!\!/ - m)\,\psi_k(x) - e\overline{\psi}_k(x)A\!\!\!/(x)\psi_k(x)\right]$$

and the field equations

$$(i\partial\!\!\!/ - m)\psi_k(x) = eA\!\!\!/(x)\psi_k(x) \qquad \text{and} \qquad -i\partial_\mu\overline{\psi}_k(x)\gamma^\mu - \overline{\psi}_k(x)m = e\overline{\psi}_k(x)A\!\!\!/(x). \tag{8.2}$$

The N Dirac fields can be interpreted as a Hartree-Fock approximation of the N particle wave function which we describe later. Next we need to introduce an interaction between the fields. Note that every solution to the equations (8.2) gives rise to a four-vector current

$$j_l^\mu(x) := e\overline{\psi}_l(x)\gamma^\mu\psi_l(x) \tag{8.3}$$

which fulfills the continuity equation $\partial_\mu j_l^\mu(x) = 0$. For every such current we introduce a potential A_l^μ defined by

$$A_l^\mu(x) = \int d^4y\ \triangle(x-y)j_l^\mu(y)$$

where we choose $\triangle(x) = \delta(x_\mu x^\mu)$, i.e. the time-symmetric Green's function of the d'Alembert operator. Hence, these potentials fulfill the electrodynamic wave equation $\square A_l^\mu(x) = -4\pi j_l^\mu(x)$. Using this input we define the action integral

Generalization of N Dirac fields with Wheeler-Feynman type interaction

$$\begin{aligned} S[\psi_k,\overline{\psi}_k;1\le k\le N] &:= \sum_{k=1}^N \int d^4x \left[\overline{\psi}_k(x)\,(i\partial\!\!\!/ - m)\,\psi_k(x) - \frac{e}{2}\sum_{l\neq k}\overline{\psi}_k(x)A\!\!\!/_l(x)\psi_k(x)\right] \\ &= \sum_{k=1}^N \int d^4x \left[\overline{\psi}_k(x)\,(i\partial\!\!\!/ - m)\,\psi_k(x) - \frac{e^2}{2}\sum_{l\neq k}\int d^4y\ \triangle(x-y)\,\overline{\psi}_l(x)\gamma^\mu\psi_l(x)\,\overline{\psi}_k(x)\gamma_\mu\psi_k(x)\right]. \end{aligned} \tag{8.4}$$

As already observed by Fokker when formulating classical Wheeler-Feynman electrodynamics in terms of an action integral [Fok29], it is important to note that only the choice of the time-symmetric Green's function $\triangle(x) = \delta(x_\mu x^\mu)$ allows to derive an extremum via the computation of (8.1) which yields the equations

$$(i\partial\!\!\!/ - m)\psi_k(x) = e^2 \sum_{l\neq k} \int d^4y \; \triangle(x-y)\overline{\psi}_l(x)\gamma^\mu\psi_l(x)\gamma_\mu\psi_k(x) \tag{8.5}$$

together with their conjugated form for $\overline{\psi}_k(x)$ for $1 \leq k \leq N$.

Before we continue let us appreciate what we have got so far. Equations (8.5) give rise to an relativistically interacting theory of N Dirac fields. The interaction is, as in the case of Wheeler-Feynman electrodynamics, of the action-at-a-distance type, i.e. the only interactions that occur are between points of the supports of two fields ψ_l and ψ_k for $l \neq k$ which have Minkowski distance zero. As in the case of ML-SI or Wheeler-Feynman electrodynamics we can again formulate the analogue of the absorber assumption: At some space-time point x in some distance to the supports of the N Dirac fields we demand that the net interaction with some test charge vanishes which would be fulfilled for

Corresponding absorber assumption and its consequences

$$\sum_{k=1}^{N} A_k^\mu(x) = 0. \tag{8.6}$$

Borrowing the argument of Wheeler and Feynman as in Chapter 2p.7 we can deduce from this that for all x in Minkowski space

$$\sum_{k=1}^{N} (A_{k,-}^\mu(x) - A_{k,+}^\mu(x)) = 0 \tag{8.7}$$

for the advanced and retarded fields $A_{k,-}^\mu$ which are given by

$$A_{k,\pm}^\mu(x) = \int d^4y \triangle^\pm(x-y) j_k^\mu(y) \qquad \text{and} \qquad A_k^\mu = A_{k,+}^\mu + A_{k,-}^\mu$$

where $\triangle^\pm(x) = \frac{\delta(\|\mathbf{x}\|\pm t)}{4\pi\|\mathbf{x}\|}$ represent the advanced and retarded Green's functions of the d'Alembert operator, cf. (4.10p.61). Using (8.7) we find that the effective equation on the kth Dirac field is given by

Dirac-Barut equation

$$(i\partial\!\!\!/ - m)\psi_k(x) = \frac{e}{2}[A\!\!\!/_{k,-}(x) - A\!\!\!/_{k,+}(x)]\psi_k(x) + e\sum_{l\neq k} A\!\!\!/_{l,-}(x)\psi_k(x). \tag{8.8}$$

In analogy to the classical ML-SI equations we may interpret the first term on the right-hand side as radiation reaction felt by the kth Dirac field while the second term constitutes the retarded interaction with all $l \neq k$ Dirac fields. As in Chapter 7p.111 let us further assume that the system of N Dirac fields is in equilibrium in the sense that the sum $\sum_{l\neq k} A_{l,-}^\mu(x)$ vanishes. The equations for the kth field then read

$$(i\partial\!\!\!/ - m)\psi_k(x) = \frac{e^2}{2}\int d^4y \;\; (\triangle^-(x-y) - \triangle^+(x-y))\,\overline{\psi}_k(y)\gamma^\mu\psi_k(y)\;\gamma_\mu\psi_k(x). \tag{8.9}$$

This equation exhibits the following nice features: First, the right-hand side gives rise to a radiation reaction term analogous to the one we have found in the ML-SI and, respectively, Wheeler-Feynman, equations. Second, note that in contrast to $\triangle^\pm(x)$ the distribution $\triangle^-(x-y) - \triangle^+(x-y)$ is a smooth function so that no extra regularity of ψ is needed to evaluate the integral on the right-hand side. In other words, no ultraviolet divergence is expected. Third, this equation was studied intensely by Asim O. Barut et. al. in a series of 19 (or even more) papers. In particular, he computed the energy corrections in a H-atom (an similar systems like muonium and positronium) due to spontaneous emission, vacuum fluctuations, anomalous magnetic moment and Lamb shift by means of equations (8.9) only which are in great agreement with experimental results; cf. the overview article [Bar88]. And fourth, as we have shown, this equation can be deduced from the action-at-a-distance integral (8.4p.160) and Wheeler and Feynman's absorber assumption (8.7) – for some strange reason this connection was never mentioned in Barut's works although he was very well familiar with Wheeler-Feynman electrodynamics [Bar80] and most probably aware it.

Generalization to N particle wave functions

Nevertheless, the action integral (8.4p.160) gives only rise to a Hatree-Fock description of the time-evolution in terms of product wave functions. To find an action integral for N particle Dirac fields and thus allowing for non-local, quantum mechanical interactions, there are several ways to proceed from here. One of them shall be explained in the following. Recall that the charge currents are divergence free which gives rise to the following property: Let Σ_1, Σ_2 be two space-like hypersurfaces and let M denote the volume of Minkowski space between these two hypersurface. By Stoke's theorem we get

$$0 = \int_M d^4x\, \partial_\mu j_k^\mu(x) = \int_{\partial M} d\sigma(x) n_\mu(x) j^\mu(x) = \int_{\Sigma_2} d\sigma(x) n_\mu(x) j^\mu(x) - \int_{\Sigma_1} d\sigma(x) n_\mu(x) j^\mu(x)$$

where we denoted the surface measure on the space-like hypersurfaces by $d\sigma(x)$ and the surface normal four-vectors by $n^\mu(x)$. This means in particular that

$$\|\psi_k\|_\Sigma^2 := \int_\Sigma d\sigma(x) \overline{\psi}_k(x) \not{n}(x) \psi_k(x) \tag{8.10}$$

is invariant under the choice of the space-like hypersurface Σ. For a fixed foliation of Minkowski space-time by space-like hypersurfaces let us further denote the unique space-like hypersurface which includes the space-time event x by $\Sigma(x)$. Using the notation $X = (x_1, \ldots, x_N)$,

$$\int_\Sigma d\sigma_k(X) = \int_\Sigma d\sigma(x_1) \int_\Sigma \sigma(x_2) \ldots \int_\Sigma \sigma(x_{k-1}) \int_\Sigma \sigma(x_{k+1}) \ldots \int_\Sigma \sigma(x_N) \tag{8.11}$$

and

$$(i\not{\partial} - m)_k(X) := \not{n}(x_1)\not{n}(x_2) \ldots \not{n}(x_{k-1})(i\not{\partial}_k - m)\not{n}(x_{k+1}) \ldots \not{n}(x_N)$$

as well as

$$\gamma_k^\mu(X) := \not{n}(x_1)\not{n}(x_2) \ldots \not{n}(x_{k-1})\gamma^\mu \not{n}(x_{k+1}) \ldots \not{n}(x_N)$$

we can define an action integral for N particle Dirac fields $\Psi(X)$ and $\overline{\Psi}(X)$ by

$$\begin{aligned} S[\Psi, \overline{\Psi}] := \sum_{k=1}^N \int d^4x_k \int_{\Sigma(x_k)} d\sigma_k(X) \Big[\overline{\Psi}(X)(i\not{\partial} - m)_k \Psi(X) + \\ - \frac{e^2}{2} \sum_{l \neq k} \int d^4y_l \int_{\Sigma(y_l)} d\sigma_l(Y)\, \triangle(x_k - y_l)\, \overline{\Psi}(Y)\gamma_k^\mu(Y)\Psi(Y)\, \overline{\Psi}(X)\gamma_{k,\mu}(X)\Psi(X) \Big]. \end{aligned} \tag{8.12}$$

If we replace $\Psi(X)$ by the product of functions $\prod_{k=1}^N \psi_k(x_k)$ for normalization $\|\psi_k\|_\Sigma = 1$ for an arbitrary space-like hypersurface Σ, we retrieve the action integral (8.4p.160). Hence, it seems that (8.12) is a natural generalization of the previous action integral (8.4p.160). The computation of its extremum via

$$\frac{d}{d\epsilon} S[\Psi, \overline{\Psi} + \epsilon \overline{f}]|_{\epsilon=0} = 0 \qquad \text{and} \qquad \frac{d}{d\epsilon} S[\Psi + \epsilon f, \overline{\Psi}]|_{\epsilon=0} = 0$$

yields for any $1 \leq k \leq N$ and Minkowski point x_k the equation

$$[(i\partial\!\!\!/ - m)_k(X)\Psi(X)]_{\Sigma(x_k)} = e^2 \sum_{l\neq k} \int d^4y_l \int_{\Sigma(y_l)} d\sigma_l(Y)\, \triangle(x_k - y_l)\overline{\Psi}(Y)\gamma_l^\mu(Y)\Psi(Y) \left[\gamma_{k,\mu}(X)\Psi(X)\right]_{\Sigma(x_k)} \tag{8.13}$$

together with its conjugate form for $\overline{\Psi}$ where the subscript $\Sigma(x_k)$ denotes that the equations need to hold only for X on the space-like hypersurface $\Sigma(x_k)$. Defining the kth charge current by

$$j_k^\mu(x_k)|_{\Sigma(x_k)} := e \int_{\Sigma(x_k)} d\sigma_k(X)\, \overline{\Psi}(X)\gamma_k^\mu(X)\Psi(X) \tag{8.14}$$

and therewith the kth retarded and advanced potential by

$$A_{k,\pm}^\mu(x) = \int d^4y\, \triangle^\pm(x-y) j_k^\mu(y) \tag{8.15}$$

Corresponding absorber assumption and its consequences

we yield the following generalization of the absorber assumption ($8.6_{\text{p.161}}$): For all x in some distance to the supports of the tensor components of Ψ we demand that the net interaction with a test charge at x vanishes, i.e. that

$$\sum_{k=1}^N \left(A_{k,+}^\mu(x) + A_{k,-}^\mu(x)\right) = 0.$$

Thus, we again get an effective equation which is the generalization of ($8.8_{\text{p.161}}$) and reads

$$[(i\partial\!\!\!/ - m)_k\Psi(X)]_{\Sigma(x_k)} = e\left[\frac{1}{2}\left(A_{k,-}^\mu(x_k) - A_{k,+}^\mu(x_k)\right) + \sum_{l\neq k} A_{l,-}^\mu(x_k)\right]\left[\gamma_{k,\mu}(X)\Psi(X)\right]_{\Sigma(x_k)}.$$

For the special case of equilibrium in the sense that the net interaction $\sum_{l\neq k} A_{l,-}^\mu(x_k)$ vanishes we find

$$\begin{aligned}[(i\partial\!\!\!/ - m)_k\Psi(X)]_{\Sigma(x_k)} = \frac{e^2}{2}\int d^4y_k \int_{\Sigma(y_k)} & d\sigma_k(Y)\left(\triangle^-(x_k-y_k) - \triangle^+(x_k-y_k)\right) \\ & \times \Psi(Y)\gamma_k^\mu(Y)\Psi(Y)\left[\gamma_{k,\mu}(X)\Psi(X)\right]_{\Sigma(x_k)}.\end{aligned} \tag{8.16}$$

Whenever the N particle Dirac fields Ψ can be written as $\prod_{k=1}^N \psi_k(x_k)$ we retrieve again the Dirac-Barut equation ($8.9_{\text{p.161}}$) by integrating the left- and right-hand side with respect to $\int_{\Sigma(x_k)} d\sigma_k(X)$ using ($8.11_{\text{p.162}}$) and the normalization $\|\psi_k\|_\Sigma = 1$ for any space-like hypersurface Σ. In this way we can regard (8.16) as a natural generalization of ($8.9_{\text{p.161}}$).

Second quantization

The last issue that needs to be addressed in this section is that the interaction in (8.13) is nonlinear in the Dirac field Ψ. This is due to the fact that we are still on the level of a classical field theory. The "quantization" needed here is to linearize the charge current. With ($8.10_{\text{p.162}}$) we can define a one-particle Hilbert space $\mathcal{H}_\Sigma := L^2(\Sigma, \mathbb{C}^4, d\sigma)$ consisting of functions $\varphi : \Sigma \to \mathbb{C}^4$ which have a finite norm $\|\varphi\|_\Sigma$ for any space-like hypersurface Σ. By forming the N-fold antisymmetric tensor product we yield the N particle Hilbert space $\mathcal{H}_\Sigma^N$ with an inner product

$$\langle\Psi, \Phi\rangle_\Sigma := \int_\Sigma d\sigma(x_1)\ldots\int_\Sigma d\sigma(x_N)\overline{\Psi}(X)n\!\!\!/(x_1)\ldots n\!\!\!/(x_2)\Phi(X)$$

for any $\Psi, \Phi \in \mathcal{H}_\Sigma^N$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a orthonormal basis in $\mathcal{H}_\Sigma$. Denoting the antisymmetric N-fold tensor product by $\varphi^\alpha = \varphi_{\alpha_1}\ldots\wedge\varphi_{\alpha_N}$ for multi-indices $\alpha \in \mathcal{I} := \{(\alpha_1,\ldots,\alpha_N) \in \mathbb{N}^N \mid \alpha_i <$

$\alpha_j, i < j\}$ the family $(\varphi^\alpha)_{\alpha\in\mathcal{I}}$ forms an orthonormal basis of $\mathcal{H}^N_\Sigma$. In Dirac's bra-ket notation we can represent any $|\Psi\rangle \in \mathcal{H}^N_\Sigma$ by

$$|\Psi\rangle_\Sigma = \sum_{\alpha\in\mathcal{I}} \langle\varphi^\alpha|\Psi\rangle_\Sigma \left|\varphi^\alpha\right\rangle_\Sigma \quad \text{and} \quad \Psi(X)|_{\Sigma(x_k)} = \langle X|\Psi\rangle_{\Sigma(x_k)} := \sum_{\alpha\in\mathcal{I}} \langle\varphi^\alpha|\Psi\rangle_{\Sigma(x_k)}\, \varphi^\alpha(X)_{\Sigma(x_k)}.$$

With this notation we can define a charge current operator for all Minkowski points x_k in analogy to (8.14) by

$$J^\mu_k(x_k)|_{\Sigma(x_k)} := e \sum_{\alpha,\beta\in\mathcal{I}} \left|\varphi^\alpha\right\rangle_{\Sigma(x_k)} \int_{\Sigma(x_k)} d\sigma_k(X)\, \overline{\varphi}^\alpha(X)\gamma^\mu_k(X)\varphi^\beta(X)\; \left\langle\varphi^\alpha\right|_{\Sigma(x_k)}.$$

By replacing j^μ with J^μ in the formulas for the potentials (8.15p.163) we yield a linear version of equations (8.13p.163). The connection to the nonlinear equation is that (8.14p.163) is the expectation value of $J^\mu_k(x_k)|_{\Sigma(x_k)}$, i.e. it holds that

$$j^\mu_k(x_k)|_{\Sigma(x_k)} = \left\langle\Psi\middle|J^\mu_k(x_k)|_{\Sigma(x_k)}\middle|\Psi\right\rangle_{\Sigma(x_k)}.$$

Furthermore, for a product state φ^σ with $\sigma \in \mathcal{I}$ we get

$$\left\langle\varphi^\sigma\middle|J^\mu_k(x_k)|_{\Sigma(x_k)}\middle|\varphi^\sigma\right\rangle_{\Sigma(x_k)} = \int_{\Sigma(x_k)} d\sigma_k(X)\, \overline{\varphi}^\sigma(X)\gamma^\mu_k(X)\varphi^\sigma(X)$$

which boils down to the form (8.3p.160) if we neglect the antisymmetry.

If it is possible to give the linearized version of equation (8.13p.163) a mathematical meaning, we would consider this equation to be a candidate for a wave equation for Wheeler-Feynman electrodynamics. Together with a Bohmian velocity law [DMZ99] on the same foliation $x \mapsto \Sigma(x)$ one could hope to yield a fundamental theory of electrodynamics for point-like charge which is divergence free and whose effective description (maybe in a thermodynamic limit $N \to \infty$) explains pair creation and annihilation. The role of the preferred foliation and the dependence of the theory thereon, however, is completely unclear. For example it is also conceivable to not assume a fixed foliation of space-time but to choose $x \mapsto \Sigma(x)$ as the light-cone of x.

Further Implications. As concluding remarks to these admittedly wild speculations we want to point out some features of such a theory with regards to physics: First, a not perfectly fulfilled absorber assumption (as well as the free fields for the ML-SI case) could account for the cosmic microwave background. Second, the assumption of an absorber medium in equilibrium such that net interactions vanish would predict the presence of dark matter – dark because the net interaction vanishes so that a spectator charge would not "see" any light, i.e. electrodynamic interaction. The reaction to disturbances of this equilibrium state manifests itself as pair creation or annihilation and gives rise to what is commonly called vacuum fluctuations. The condition that in equilibrium net interactions vanish can be formulated in a Lorentz invariant way. However, for a uniformly accelerated spectator charge this equilibrium condition changes which leads to effective pair creation or annihilation. This mechanism could account for the Unruh effect. The nicest feature is that this theory explains pair creation/annihilation as an artifact of our effective description. Nothing is ever created or annihilated. At all times there are $N \in \mathbb{N}$ charges. In this spirit we conclude with the words of Parmenides:

> There is a solitary word still left to say of a way: 'exists'; very many signs are on this road: that *Being* is ungenerated and imperishable, whole, unique, immovable, and complete. It was not once nor will it be, since it is now altogether, one, continuous. For, what origin could you search out for it? How and whence did it grow? Not from non-Being shall I allow You to say or to think, for it is not possible to say or to think that it is not. What need would have made it grow, beginning from non-Being, later or sooner?
>
> On Nature, Parmenides of Elea (ca. 5th century BCE). Translation taken from Leonardo Tarán: Parmenides, Princeton 1965.

Notation

$\mathbb{M}$	Minkowski space $\mathbb{R}\times\mathbb{R}^3$ with metric tensor $g = \mathrm{diag}(1,-1,-1,-1)$, page 7
$\mathrm{O}_{x\to y}(g(x))$	Standard Big-Oh notation meaning $f(x) = \mathrm{O}_{x\to y}(g(x))$ iff $\lim_{x\to y}\frac{f(x)}{g(x)}$ is a constant , page 169
$\mathtt{Bounds}$	Set of continuous and non-decreasing functions which are used as bounds for estimates, page 169
$\mathbb{C}^{(\mathcal{S})}$	The space of formal $\mathbb{C}$-linear combinations of elements of $\mathcal{S}$, page 122
$C^n(V,W), C^\infty(V,W)$	n times continuous differentiable, respectively infinitely differentiable, functions $V\to W$, page 169
$C^n_c(V,W)$	n times continuous differentiable, respectively infinitely differentiable, functions $V\to W$ with compact support, page 169
$\mathcal{D}$	Set of charge-current densities, page 60
$\mathcal{F}^1$	Space of initial condition for the strong Maxwell solutions, page 60
$\mathcal{F}_w$	Space of Maxwell fields, page 19
$\mathcal{H}_w$	Phase space for the ML±SI equations, page 19
$\mathcal{P}$	Newtonian phase space $\mathbb{R}^{6N}$, page 19
$\mathcal{T}^1_\triangledown, \mathcal{T}_\triangledown$	Set of strictly time-like once continuously differentiable charge trajectories, respectively the N-fold cartesian product thereof, page 60
$\mathcal{T}^1_\vee, \mathcal{T}_\vee$	Set of time-like once continuously differentiable charge trajectories, respectively the N-fold cartesian product thereof, page 60
$\mathcal{W}, \mathcal{W}^k, \mathcal{W}^\infty$	Spaces of weight functions, page 18
$\mathcal{H}$	One-particle Hilbert space, page 116
$\mathcal{S}(\Phi)$	The equivalence class w.r.t. $\sim$, page 120
$\mathrm{U}(\mathcal{H},\mathcal{H}')$	Set of unitary maps $U:\mathcal{H}\to\mathcal{H}'$, page 118
$\mathcal{V}(\Phi)$	For all $\Phi\in\mathtt{Seas}(\mathcal{H})$ we have $\mathcal{S}(\Phi) = \Phi + \mathcal{V}(\Phi)$, page 121
$\det T$	The Fredholm determinant of T, page 116
ℓ	Index Hilbert space, page 116
$\ell_2(\mathbb{N})$	The space of square summable sequences in $\mathbb{C}$, page 116

$\mathrm{GL}_-(\ell', \ell)$	The set of all bounded invertible linear operators $R : \ell' \to \ell$ with the property $R^*R \in \mathrm{id}_{\ell'} + \mathrm{I}_1(\ell')$, page 126
$\mathrm{I}_2 = \mathrm{I}_2(\ell, \mathcal{H})$	The space of Hilbert-Schmidt operators $T : \ell \to \mathcal{H}$, page 116
id_ℓ	Identity operator on some space ℓ, page 116
$\mathrm{ind}\, T$	Fredholm index of T, page 118
$\mathtt{Seas}^\perp(\mathcal{H}) = \mathtt{Seas}^\perp_\ell(\mathcal{H})$	Only Dirac seas out of $\mathtt{Seas}(\mathcal{H})$ which also are isometries, page 119
$\mathcal{L}_U$	Left operation $\mathcal{F}_S \to \mathcal{F}_{US}$ induced by U, page 126
∇	Gradient, page 169
$\nabla\cdot$	Divergence, page 15
$\nabla\wedge$	Curl, page 15
$\|\cdot\|_{\mathrm{I}_2}$	Hilbert-Schmidt norm, page 116
$\|\cdot\|_\Phi$	The norm $\|\cdot\|_\Phi := \|\cdot\|_{\mathrm{I}_1} + \|\cdot\|_{\mathrm{I}_2}$, page 121
$\|\cdot\|_{\mathrm{I}_1}$	Trace class norm, page 116
$\vert_{X\to Y}$	Restriction to the map $X \to Y$, page 117
$\mathtt{Ocean}(C) = \mathtt{Ocean}_\ell(C)$	The set of all $\Phi \in \mathtt{Seas}^\perp_\ell(\mathcal{H})$ such that range $\Phi \in C$, page 119
$\bar{z}$	Complex conjugate of z, page 123
∂_t	Partial derivative with respect to t, page 15
∂_{x_i}	Derivative with respect to x_i, page 169
$\Phi \sim \Psi$	$\Leftrightarrow \Psi^*\Phi \in 1 + \mathrm{I}_1(\ell)$, page 120
$\mathtt{Pol}(\mathcal{H})$	The set of all polarizations $V \subset \mathcal{H}$, page 116
$\mathcal{R}_R$	The operation from the right $\mathcal{F}_S \to \mathcal{F}_{SR}$ induced by R, page 126
$\mathtt{Seas}(\mathcal{H}) = \mathtt{Seas}_\ell(\mathcal{H})$	All possible Dirac seas, e.g. bounded operators $\Phi : \ell \to \mathcal{H}$ such that range $\Phi \in \mathtt{Pol}(\mathcal{H})$ and $\det \Phi^*\Phi$ exists., page 119
$\langle\cdot,\cdot\rangle$	Scalar products in respective Hilbert spaces, page 116
$\mathrm{SL}(\ell)$	The set of all operators $R \in \mathrm{id}_\ell + \mathrm{I}_1(\ell)$ with the property $\det R = 1$, page 127
$\Box$	D'Alembert operator, page 169
$\mathrm{I}_1 = \mathrm{I}_1(\ell)$	Space of trace class operators on ℓ, page 116
$\triangle$	Laplace operator, page 169
$\mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C)$	$:= \mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}, C)$, page 128
$\mathrm{U}^0_{\mathrm{res}}(\mathcal{H}, C; \mathcal{H}', C')$	The set of operators $U \in \mathrm{U}(\mathcal{H}, \mathcal{H}')$ such that for all $V \in C$ one has $UV \in C'$, page 128
D^α	Multi-index differentiation operators, page 169

H^0	Free Dirac hamiltonian, page 108
$H^{\#}_w$	Weighted Sobolev spaces w.r.t. operator #, page 25
$H^{\mathsf{A}(t)}$	Dirac hamiltonian with external field A, page 112
L^2_w	Hilbert space of weighted square integrable functions, page 18
$L^\infty(V, W)$	Functions $V \to W$ with a finite essential supremum, page 169
$L^p(V, W)$	Functions $V \to W$ whose absolute value to the power of p is integrable, page 169
$L^p_{loc}(V, W)$	Functions $V \to W$ whose absolute value to the power of p is integrable on any compact set, page 169
$M_L[\varphi^0](t, t_0)$	Maxwell-Lorentz time-evolution of initial value φ^0 from time t_0 to time t, page 41
$P_V : \mathcal{H} \to V$	Orthogonal projector of $\mathcal{H}$ on the subspace V, page 116
$t \mapsto M_{\varrho,m}[F^0, (\boldsymbol{q}, \boldsymbol{p})](t, t_0)$	Maxwell time-evolution of with initial F^0 from time t_0 to time t, page 65
T^*	The Hilbert space adjoint of the operator T, page 116
$V \approx W$	$\Leftrightarrow P_V - P_W \in \mathrm{I}_2(\mathcal{H})$ for $V, W \in \mathtt{Pol}(\mathcal{H})$, page 116
$V \approx_0 W$	$\Leftrightarrow V \approx W$ and $\mathrm{ind}(V, W) = 0$ for $V, W \in \mathtt{Pol}(\mathcal{H})$, page 119
$x = (x^0, \boldsymbol{x}) \in \mathbb{M}$	space-time point x with time coordinate $x^0 \in \mathbb{R}$ and space coordinate $\boldsymbol{x} \in \mathbb{R}^3$, page 8

General Notation. We stick to the standard notation of spaces of functions $V \to W$ for normed spaces V, W: We call functions integrable if their Riemann or Lebesgue integral is well-defined and finite. For $n \in \mathbb{N}$ let $L^p(V, W)$ denote the space of functions whose pth power is integrable, $L^\infty(V, W)$ the space of functions with a finite essential supremum. For $1 \leq p \leq \infty$ we write $L^p_{loc}(V, W)$ when the integrability condition, respectively the finiteness of the essential supremum, shall only be fulfilled on compact subsets of V. For any $n \in \mathbb{N} \cup \{0\}$ let $C^n(V, W), C^\infty(V, W)$ be the space of n times continuously differentiable functions, $C^\infty(V, W) := \{f \in C^n(V, W) \ \forall\, n \in \mathbb{N}\}$. Furthermore, for $0 \leq n \leq \infty$ let $C^n_c(V, W)$ denote $C^n(V, W)$ functions with compact support. If there is no ambiguity we sometimes omit the V and/or W in the notation of the function spaces. Limits and derivatives in V are always taken with respect to the norm in V unless we specify the sense otherwise, e.g. the point-wise sense. Function spaces

In addition to the introduced differential operators we denote the partial derivative with respect to x_i by ∂_{x_i}, the d'Alembert operator by $\square = \partial_t^2 - \triangle$, the Laplace by $\triangle = \nabla \cdot \nabla$ and the gradient by ∇. Multi-index differentiation operators are denoted by $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ for a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where ∂^0 is understood as no differentiation. Derivatives

Throughout we use the standard Big-Oh notation, i.e. $f(x) = \mathrm{O}_{x \to y}(g(x))$ iff $\lim_{x \to y} \frac{f(x)}{g(x)}$ is a constant. For $f \in L^2$ we mean by $D^\alpha f \in L^2$ that there exists a $g \in L^2$ which fulfills $g = D^\alpha f$ in the distribution sense. Big-Oh notation

We denote the set of functions $\mathbb{R}^n \to \mathbb{R}$ for any $n \in \mathbb{R}$ which are continuous and non-decreasing by $\mathtt{Bounds}$. This set is used to provide upper bounds for estimates which depend on certain parameters. Estimates

Continuation of equations or inequalities over several lines

In various places we use "..." in the first line of equations or inequalities which means: placeholder for the right-hand side of the last equations written.

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Wenn etwas IST, ist es ewig,
denn aus NICHTS kann nur NICHTS entstehen.

Wenn es ewig ist, ist es auch unendlich,
da es weder Anfang hat noch Ende.

Wenn es ewig ist und unendlich, ist es auch eines,
denn wäre es zwei, dann müsste das eine das andere begrenzen.

Wenn es ewig ist und unendlich und eines, ist es auch gleichartig,
denn wenn es nicht gleichartig wäre, unterschieden sich seine Teile
voneinander und es wäre also vielfältig.

Wenn es ewig ist, unendlich, eines und gleichartig, ist es auch bewegungslos,
da es außerhalb seiner selbst keinen Ort gibt wohin es sich bewegen könnte.

Wenn es ewig ist, unendlich, eines, gleichartig und bewegungslos kann es weder Leiden noch Schmerz empfinden, da es immer sich selbst gleich bleiben muss.

Luciano De Crescenzos Zusammenfassung des Textfragments
“Über die Natur oder über das Seiende” von Melissons von Samos
(ca. Mitte des 5. Jahrhunderts v.u.Z.)

Electrodynamic Absorber Theory
A Mathematical Study

This work deals with questions that arise in classical and quantum electrodynamics when describing the phenomena of *radiation reaction* and *pair creation*. The two guiding ideas are the absorber idea of Wheeler and Feynman (i.e. all emitted radiation will be again be absorbed by matter) and the electron sea idea of Dirac.

In the first part classical dynamics are studied which allow for a description of radiation reaction without the need of renormalization. The starting point are the coupled Maxwell and Lorentz equations without self-interaction. Based on the notion of absorber medium, it is shown how the so-called Lorentz-Dirac equation for radiation reaction emerges and the intimate connection to the famous Wheeler-Feynman action at a distance electrodynamics is explained. Based on this, the mathematical problem of the existence of solutions to the Wheeler-Feynman theory, which is given by a functional differential equation, is rigorously analyzed.

In the second part the phenomenon of pair creation is discussed from a thermodynamic perspective in which the Dirac sea satisfies the absorber condition. Taking Dirac's original idea seriously, the vacuum is to be regarded as an equilibrium state in which all net-electron-electron interactions vanish. Small departures of this equilibrium can be effectively described by introducing pair creation. For the mathematical discussion these seas are considered to consist of infinitely many electrons (in the thermodynamical limit). The mathematical implementation of the quantum mechanical time-evolution for such infinitely many electron seas subject to prescribed external four-vector fields is then carried out in detail. The main result is that the probability amplitudes induced by this time-evolution are well-defined and unique.

In a last part we give a perspective on the quantization of Wheeler-Feynman-like interaction. Based on the proposed equations, a derivation of the Dirac-Barut equation is given, which seems to predict QED corrections in accordance with the experiment.