ON REFINED METRIC AND HERMITIAN STRUCTURES IN ARITHMETIC, I: GALOIS-GAUSS SUMS AND WEAK RAMIFICATION

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Abstract. We use relative K-theory to develop a common refinement of the existing theories of metrized and hermitian Galois structures in arithmetic. As a first application, we derive new results concerning the arithmetic properties of Galois-Gauss sums for weakly ramified Galois extensions of number fields.

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1. Introduction

This article has two main purposes. Firstly, to develop a natural algebraic formalism that gives a common refinement of the theory of 'hermitian modules' and 'hermitian class groups' described by Fröhlich in [22] and of the theory of 'metrised modules' and complexes and 'arithmetic classgroups' introduced by Chinburg, Pappas and Taylor in [16] and, secondly, to show that this refined theory gives new insight on the arithmetic properties of wildly ramified Galois-Gauss sums.

To give a few more details we fix a finite group Γ and recall that an hermitian Γ-module is a pair comprising a finitely generated projective Γ-module together with a non-degenerate Γ-invariant pairing on this module. Fröhlich showed that such modules are naturally classified by a 'discriminant' invariant that lies in the Hermitian classgroup $\mathrm{HCl}(\Gamma)$ of Γ and is defined in terms of idelic-valued functions on the ring R_Γ of $\mathbb{Q}^c$-valued virtual characters of Γ.

This theory was developed with arithmetic applications in mind since for any tamely ramified Galois extension of number fields L/K with $\mathrm{Gal}(L/K) = \Gamma$ the ring of algebraic integers of L constitutes an hermitian Γ-module when endowed with its natural trace pairing. In this setting, Fröhlich conjectured, and Cassou-Noguès and Taylor subsequently proved ([13]), that the corresponding discriminant element uniquely characterises the Artin root numbers of irreducible complex symplectic characters of Γ. The latter result is commonly regarded as the highlight of classical 'Galois module theory', as had been developed in the 1970's and 1980's (for more details see [22])

To develop an analogous theory in the setting of arithmetic schemes admitting a tame action of Γ, Chinburg, Pappas and Taylor subsequently defined a metrised Γ-module (respectively, complex of Γ-modules) to be a pair comprising a finitely generated projective Γ-module and a collection of suitable metrics on the isotypic components of the complexified module (respectively, a perfect complex of Γ-modules together with metrics on the isotypic components of the complexified cohomomology modules). To classify such structures they defined the Arithmetic classgroup $A(\Gamma)$ of Γ in terms of idelic-valued functions on R_Γ and showed each metrised Γ-module (respectively complex) gives rise to an associated invariant in $A(\Gamma)$.

To describe a common refinement of the above algebraic theories we construct canonical homomorphisms $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ from the relative algebraic K_0-group $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ of the ring inclusion $\mathbb{Z}[\Gamma] \to \mathbb{Q}^c[\Gamma]$ to the group $A(\Gamma)$ and to a natural extension of the group $\mathrm{HCl}(\Gamma)$ respectively. We then show that $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ send each of the natural generating elements of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$, respectively of the subgroup $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$, to the difference of the natural invariants of two metrised modules in $A(\Gamma)$, respectively of the discriminants of two hermitian modules in $\mathrm{HCl}(\Gamma)$. To define the homomorphisms $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ we rely on a description of the group $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ in terms of idelic-valued functions on R_Γ that is proved by Agboola and the first author in [1].

The strategy to apply this theory in arithmetic settings is then twofold. Firstly, in any given setting, one hopes to identify a canonical element of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ that at least one of $\Pi_\Gamma^{\mathrm{met}}$ or $\Pi_\Gamma^{\mathrm{herm}}$ sends to arithmetic invariants that have been considered previously. Then one can hope to prove, or at least to formulate conjecturally, a precise relation in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ that projects (under either $\Pi_\Gamma^{\mathrm{met}}$ or $\Pi_\Gamma^{\mathrm{herm}}$ or both) to recover pre-existing results, or conjectures, in $\mathrm{A}(\Gamma)$ and $\mathrm{HCl}(\Gamma)$.

In any case in which this can be achieved one can expect three significant outcomes. Firstly, one will obtain strong refinements of earlier results since both $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ have large kernels. Secondly, one will obtain an explanation of any parallel aspects of the nature of earlier results in $\mathrm{A}(\Gamma)$ and $\mathrm{HCl}(\Gamma)$. Thirdly, and perhaps most importantly, since $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}[\Gamma])$ has a canonical direct sum decomposition as $\bigoplus_\ell K_0(\mathbb{Z}_\ell[\Gamma],\mathbb{Q}_\ell[\Gamma])$, where ℓ runs over all primes, theorems and conjectures in $\mathrm{A}(\Gamma)$ and $\mathrm{HCl}(\Gamma)$ that appeared to be intrinsically global in nature are replaced by problems that can admit natural local decompositions and hence become easier to study.

Whilst there is no guarantee that this strategy can work in all natural settings, in this article we show that it works well in the setting of hermitian and metrised modules that arise from fractional ideals of number fields and their links to classical Galois-Gauss sums.

In addition, in a separate article it will be shown that the same approach can also be used to refine the theory of Chinburg, Pappas and Taylor related to connections between the Zariski cohomology complexes of sheaves of differentials on arithmetic schemes with a tame action of a finite group and the associated epsilon constants and, in particular, to explain the strong similarity between the results obtained in [16] and [17].

A little more precisely, in the present article we first use the above approach in the setting of tamely ramified extensions of number fields to quickly both refine and extend previous results of Chinburg and the first author in [12] related to the links between Galois-Gauss sums and the hermitian modules comprising fractional powers of the different of L/K endowed with the natural trace pairing.

In the main body of the article we then consider wildly ramified Galois-Gauss sums. Whilst the arithmetic properties of such sums are still in general poorly understood, significant progress has been made by Erez and others (see, for example, [20]) in the case of Galois extensions L/K that are both of odd degree and 'weakly ramified' in the sense of [19]. We recall, in particular, that under these hypotheses there exists a unique fractional ideal $\mathcal{A}_{L/K}$ of L the square of which is equal to the inverse of the different of L/K and that the Hermitian-Galois structure of $\mathcal{A}_{L/K}$ has been shown in special cases to be closely linked to the properties of Galois-Gauss sums twisted by second Adams operators.

Following the general strategy described above, we shall now show that for any such extension L/K, with $\mathrm{Gal}(L/K)=\Gamma$, there exists a canonical element $\mathfrak{a}_{L/K}$ of $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma])$ that simultaneously controls both the hermitian and metrised structures that are naturally associated to $\mathcal{A}_{L/K}$.

We prove that $\mathfrak{a}_{L/K}$ belongs to, and has finite order in, $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}[\Gamma])$ and also that it behaves well functorially under change of extension. We also show that $\mathfrak{a}_{L/K}$ has a canonical decomposition as a sum of elements constructed from local fields and then use this decomposition both to compute $\mathfrak{a}_{L/K}$ explicitly in several important cases and, in the general case, to formulate (in Conjecture 7.2) a precise conjectural description of $\mathfrak{a}_{L/K}$ in terms of local 'Galois-Jacobi' sums and canonical classes. We show that this conjecture is equivalent to a special case of the 'local epsilon constant conjecture' formulated by Breuning [10] and hence provides a first concrete link between the theory of the square root of the inverse different and the general framework of Tamagawa number conjectures that originated with Bloch and Kato.

By using these results we derive several unconditional results concerning the hermitian and metrised structures associated to $\mathcal{A}_{L/K}$ and thereby, for example, extend the main results of Erez and Taylor in [20].

At the same time, we find that the approach gives a new, and effective, strategy for proving the epsilon constant conjecture formulated by the first and second authors in [2] for certain infinite families of wildly ramified Galois extensions of number fields.

To give a yet more concrete example of the insight that arises in this way we recall that in [37] Vinatier conjectures $\mathcal{A}_{L/K}$ is a free Γ-module when $K = \mathbb{Q}$ and is able, by using the connection to twisted Galois-Gauss sums, to prove this conjecture if the decomposition groups in $\mathrm{Gal}(L/\mathbb{Q})$ of each wildly ramified prime are abelian (see [35]). The conjecture is also known to hold if $L/\mathbb{Q}$ is tamely ramified by the work of Erez in [19]. However, aside from numerical verifications in a small (finite) number of cases (see [36]), there is still essentially nothing known about this conjecture in the non-abelian weakly and wildly ramified case.

By contrast, applying our approach in this setting now allows us to show easily that Vinatier's Conjecture naturally decomposes into a family of corresponding conjectures concerning extensions of local fields. This observation leads directly to a general 'finiteness result' for Vinatier's Conjecture and hence renders the conjecture accessible to effective computation. In particular, in this way we are able to prove the conjecture for several infinite families of non-abelian wildly ramified Galois extensions.

Although we do not pursue it here, we believe it likely that the same local approach would also shed light on several of the explicit questions that were recently raised by Caputo and Vinatier in the introduction to [15].

Finally, we would like to note that much of this work grew out of the King's College London PhD Thesis [23] of the third author.

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Part I: The general approach and first examples

In this part of the article we shall first review some basic facts concerning relative algebraic K-theory and the theories of both Arithmetic and Hermitian classgroups. We then establish a new link between these theories that will play a key role in subsequent arithmetic applications.

Throughout the section we illustrate abstract definitions and results by means of arithmetic examples that are motivated by our later applications.

For any Galois extension of fields F/E we set $G(F/E) := \mathrm{Gal}(F/E)$. We write $\mathbb{Q}^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and for any number field $E \subseteq \mathbb{Q}^c$ we also write Ω_E for the absolute Galois group $G(\mathbb{Q}^c/E)$.

For any finite group Γ we write $\widehat{\Gamma}$ for the set of irreducible $\mathbb{Q}^c$-valued characters of Γ. If ℓ denotes a rational prime, then we write $\widehat{\Gamma}_\ell$ for the set of irreducible $\mathbb{Q}_\ell^c$-valued characters.

2. Relative K-theory, metric structures and hermitian structures

2.1. **Relative algebraic K-theory.** We fix a finite group Γ and a Dedekind domain R of characteristic zero and write F for the field of fractions of R.

For any extension field E of F and any $R[\Gamma]$-module M we set $M_E := E \otimes_R M$ and for any homomorphism $\phi : M \to N$ of $R[\Gamma]$-modules we write $\phi_E : M_E \to N_E$ for the induced homomorphism of $E[\Gamma]$-modules.

2.1.1. We write $K_0(R[\Gamma],E[\Gamma])$ for the relative algebraic K_0-group that arises from the inclusion of rings $R[\Gamma]\subset E[\Gamma]$ and we use the description of this group in terms of explicit generators and relations that is given by Swan in [32, p. 215].

We recall in particular that in this description each element of $K_0(R[\Gamma],E[\Gamma])$ is represented by a triple $[P,\phi,Q]$ where P and Q are finitely generated projective left $R[\Gamma]$-modules and $\phi\colon P_E\longrightarrow Q_E$ is an isomorphism of (left) $E[\Gamma]$-modules.

We write $\mathrm{Cl}(R[\Gamma])$ for the reduced projective classgroup of $R[\Gamma]$ (as discussed in [18, §49]) and often use the fact that there exists a canonical exact commutative diagram

$$\begin{array}{ccccccc} K_1(R[\Gamma]) & \longrightarrow & K_1(E[\Gamma]) & \xrightarrow{\partial^1_{R,E,\Gamma}} & K_0(R[\Gamma],E[\Gamma]) & \xrightarrow{\partial^0_{R,E,\Gamma}} & \mathrm{Cl}(R[\Gamma]) \\ \| & & {\scriptstyle\iota}\uparrow & & {\scriptstyle\iota'}\uparrow & & \| \\ K_1(R[\Gamma]) & \longrightarrow & K_1(F[\Gamma]) & \xrightarrow{\partial^1_{R,F,\Gamma}} & K_0(R[\Gamma],F[\Gamma]) & \xrightarrow{\partial^0_{R,F,\Gamma}} & \mathrm{Cl}(R[\Gamma]). \end{array} \tag{1}$$

Here the map ι is induced by the inclusion $F[\Gamma]\subseteq E[\Gamma]$ and ι' sends each element $[P,\phi,Q]$ to $[P,E\otimes_F\phi,Q]$. These maps are injective and will usually be regarded as inclusions. The map $\partial^0_{R,E,\Gamma}$ sends each element $[P,\phi,Q]$ to $[P]-[Q]$. (For details of all the other homomorphisms that occur above see [32, Th. 15.5].)

We write $K_0T(R[\Gamma])$ for the Grothendieck group of finite $R[\Gamma]$-modules that are of finite projective dimension and recall that there are natural isomorphisms of abelian groups

$$K_0T(R[\Gamma])\cong K_0(R[\Gamma],F[\Gamma])\cong\bigoplus_v K_0(R_v[\Gamma],F_v[\Gamma]). \tag{2}$$

We choose the normalisation of the first isomorphism so that for any finite $R[\Gamma]$-module M of finite projective dimension, and any resolution of the form $0\to P\xrightarrow{\theta}P'\to M\to 0$, where the modules P and P' are finitely generated and projective, the class of M in $K_0T(R[\Gamma])$ is sent to $[P,\theta_F,P']$. In addition, the direct sum in (2) runs over all non-archimedean places v of F and the second isomorphism is the diagonal map induced by the homomorphisms

$$\pi_{\Gamma,v}:K_0(R[\Gamma],F[\Gamma])\to K_0(R_v[\Gamma],F_v[\Gamma])$$

that sends each element $[X,\xi,Y]$ to $[X_v,\xi_v,Y_v]$, where we set $X_v:=R_v\otimes_R X$ and $\xi_v:=F_v\otimes_F\xi$.

We write $\zeta(A)$ for the centre of a ring A. Then to compute in $K_1(E[\Gamma])$ one uses the 'reduced norm' homomorphism

$$\mathrm{Nrd}_{E[\Gamma]}:K_1(E[\Gamma])\to\zeta(E[\Gamma])^\times$$

which sends the class of each pair (V,ϕ), where V is a finitely generated free $E[\Gamma]$-module and ϕ is an automorphism of V (as $E[\Gamma]$-module), to the reduced norm of ϕ, considered as an element of the semisimple E-algebra $\mathrm{End}_{E[\Gamma]}(V)$. If $E\subseteq\mathbb{Q}^c$ is a number field and $|\Gamma|$ is odd, then $\mathrm{Nrd}_{E[\Gamma]}$ is bijective by the Hasse-Schilling-Maass Norm Theorem (cf. [18, Th. (45.3)]). The same is true for algebraically closed fields and p-adic fields. In particular we write

$$\delta_\Gamma:\zeta(\mathbb{Q}^c[\Gamma])^\times\to K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma]) \tag{3}$$

for the composite $\partial^1_{\mathbb{Z},\mathbb{Q}^c,\Gamma}\circ(\mathrm{Nrd}_{\mathbb{Q}^c[\Gamma]})^{-1}$. For a rational prime ℓ we write

$$\delta_{\Gamma,\ell}\colon\zeta(\mathbb{Q}^c_\ell[\Gamma])^\times\to K_0(\mathbb{Z}_\ell[\Gamma],\mathbb{Q}^c_\ell[\Gamma])$$

for the composite $\partial^1_{\mathbb{Z}_\ell,\mathbb{Q}^c_\ell,\Gamma} \circ (\mathrm{Nrd}_{\mathbb{Q}^c_\ell[\Gamma]})^{-1}$.

2.1.2. In the sequel we make much use of the fact that $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma])$ can be explicitly described in terms of idelic-valued functions on the characters of Γ.

To recall this description we write R_Γ for the free abelian group on $\widehat{\Gamma}$. Then the Galois group $\Omega_\mathbb{Q}$ acts on R_Γ via the rule $(\omega\circ\chi)(\gamma) = \omega(\chi(\gamma))$ for every $\omega \in \Omega_\mathbb{Q}$, $\chi \in \widehat{\Gamma}$ and $\gamma \in \Gamma$.

For each a in $\mathrm{GL}_n(\mathbb{Q}^c[\Gamma])$ we define an element $\mathrm{Det}(a)$ of $\mathrm{Hom}(R_\Gamma,\mathbb{Q}^{c\times})$ in the following way: if T is a representation over $\mathbb{Q}^c$ which has character ϕ, then $\mathrm{Det}(a)(\phi) := \det(T(a))$. This definition depends only on ϕ and not on the choice of representation T. Analogously, if w denotes a finite place of $\mathbb{Q}^c$, then each element a of $\mathrm{GL}_n(\mathbb{Q}^c_w[\Gamma])$ defines a homomorphism $\mathrm{Det}(a)\colon R_\Gamma \longrightarrow (\mathbb{Q}^c_w)^\times$.

We write $J_f(\mathbb{Q}^c[\Gamma])$ for the group of finite ideles of $\mathbb{Q}^c[\Gamma]$ and view $\mathbb{Q}[\Gamma]^\times$ as a subgroup of $J_f(\mathbb{Q}^c[\Gamma])$ via the natural diagonal embedding. In particular, if a is any element of $\mathrm{GL}_n(J_f(\mathbb{Q}^c[\Gamma]))$ the above approach allows one to define an element $\mathrm{Det}(a)$ of $\mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))$ which is easily seen to be $\Omega_\mathbb{Q}$-equivariant. We set

$$U_f(\mathbb{Z}[\Gamma]) := \prod_\ell \mathbb{Z}_\ell[\Gamma]^\times \subset J_f(\mathbb{Q}[\Gamma]),$$

with the product taken over all primes ℓ, and then define a homomorphism

$$\Delta^{\mathrm{rel}}_\Gamma\colon \mathrm{Det}(\mathbb{Q}[G]^\times) \to \frac{\mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_\mathbb{Q}}}{\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))} \times \mathrm{Det}(\mathbb{Q}^c[\Gamma]^\times); \qquad \theta \mapsto ([\theta], \theta^{-1}) \tag{4}$$

where $[\theta]$ denotes the class of θ modulo $\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))$. We recall that by the Hasse-Schilling-Maass norm theorem

$$\mathrm{Det}(\mathbb{Q}[G]^\times) = \mathrm{Hom}^+(R_\Gamma, \mathbb{Q}^{c\times})^{\Omega_\mathbb{Q}}$$

where the right hand expression denotes Galois equivariant homomorphisms whose values on R^s_Γ, the group of virtual symplectic characters, are totally positive. In particular, if Γ has odd order, then $\mathrm{Det}(\mathbb{Q}[G]^\times) = \mathrm{Hom}(R_\Gamma, \mathbb{Q}^{c\times})^{\Omega_\mathbb{Q}}$

It is shown in [1, Th. 3.5] that there is a natural isomorphism of abelian groups

$$h^{\mathrm{rel}}_\Gamma : K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma]) \xrightarrow{\sim} \mathrm{Cok}(\Delta^{\mathrm{rel}}_\Gamma).$$

We shall often use the explicit description of this map given in the following result (taken from [1, Rem. 3.8]).

In the sequel for any ordered set of d elements $\{e^j\}_{1\le j\le d}$ we write $\underline{e^j}$ for the $d\times 1$ column vector with j-th entry e^j.

In addition, for any Γ-modules X and Y we write $\mathrm{Is}_{\mathbb{Q}[\Gamma]}(X_\mathbb{Q}, Y_\mathbb{Q})$ for the set of isomorphisms of $\mathbb{Q}[\Gamma]$-modules $X_\mathbb{Q} \to Y_\mathbb{Q}$.

Lemma 2.1. *Let $c = [X,\xi,Y]$ be an element of $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma])$ with locally free $\mathbb{Z}[\Gamma]$-modules X and Y of rank d. Choose a $\mathbb{Q}[\Gamma]$-basis $\{y^j_0\}$ of $Y_\mathbb{Q}$ and, for each rational prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{y^j_p\}$ of Y_p and an $\mathbb{Z}_p[\Gamma]$-basis $\{x^j_p\}$ of X_p and define μ_p to be the element of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ which satisfies $\underline{y^j_p} = \mu_p\cdot\underline{y^j_0}$. Fix θ in $\mathrm{Is}_{\mathbb{Q}[\Gamma]}(X_\mathbb{Q}, Y_\mathbb{Q})$, note $\{\theta^{-1}(y^j_0)\}$ is a $\mathbb{Q}[\Gamma]$-basis of $X_\mathbb{Q}$ and write λ_p for the matrix in $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ with $\underline{x^j_p} = \lambda_p\cdot\underline{\theta^{-1}(y^j_0)}$. Finally, write μ for the matrix in $\mathrm{GL}_d(\mathbb{Q}^c[\Gamma])$ that represents $\xi\circ(\theta^{-1}\otimes_\mathbb{Q}\mathbb{Q}^c)$ with respect to the $\mathbb{Q}^c[\Gamma]$-basis $\{y^j\}$ of $Y_{\mathbb{Q}^c}$.*

Then the element $h^{\mathrm{rel}}_\Gamma(c)$ is represented by the homomorphism pair

$$\left(\prod_p \mathrm{Det}(\lambda_p\cdot\mu_p^{-1})\right)\times\mathrm{Det}(\mu)\in\mathrm{Hom}(R_\Gamma,J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}}\times\mathrm{Det}(\mathbb{Q}^c[\Gamma]^\times).$$

2.1.3. We give a first example of elements of relative algebraic K-groups that naturally arise in arithmetic contexts.

To do this we fix a finite Galois extension of number fields L/K and set $G := G(L/K)$. Since $\mathbb{Q}^c\subset\mathbb{C}$ we identity the set $\Sigma(L)$ of field embeddings $L\to\mathbb{Q}^c$ with the set of embeddings $L\to\mathbb{C}$ and we write $H_L := \prod_{\Sigma(L)}\mathbb{Z}$.

Then the natural action of G on $\Sigma(L)$ endows H_L with the structure of a G-module (explicitly, if $\{w_\sigma : \sigma\in\Sigma(L)\}$ is the canonical $\mathbb{Z}$-basis of H_L, then $\gamma w_\sigma = w_{\sigma\circ\gamma^{-1}}$). This module is free of rank $[K:\mathbb{Q}]$ since, if one fixes an extension $\hat\sigma$ in $\Sigma(L)$ of each σ in $\Sigma(K)$, then the set $\{w_{\hat\sigma}\}_{\sigma\in\Sigma(K)}$ is a basis of H_L over $\mathbb{Z}[G]$.

In addition, the map

$$\kappa_L : \mathbb{Q}^c\otimes_{\mathbb{Q}} L\to\prod_{\Sigma(L)}\mathbb{Q}^c=\mathbb{Q}^c\otimes_{\mathbb{Z}} H_L$$

that sends each element $z\otimes\ell$ to $(\sigma(\ell)z)_{\sigma\in\Sigma(L)}$ is then an isomorphism of $\mathbb{Q}^c[G]$-modules.

As a result, any full projective $\mathbb{Z}[G]$-sublattice $\mathcal{L}$ of L gives rise to an associated element

$$[\mathcal{L},\kappa_L,H_L]$$

of $K_0(\mathbb{Z}[G],\mathbb{Q}^c[G])$.

In the case that $\mathcal{L}$ is an $\mathcal{O}_K[G]$-module the recipe in Lemma 2.1 gives rise to a useful description of this element that we record in the next result.

In this result (and the sequel) we use the following notation. For each element b of L with $L = K[G]\cdot b$ and each character χ in $\widehat{G}$ that is represented by a homomorphism of the form $T_\chi : G\to\mathrm{GL}_{n_\chi}(\mathbb{Q}^c)$, one defines a resolvent element

$$(b\mid\chi) := \det(\sum_{g\in G} g(b)T_\chi(g^{-1}))$$

and then an associated 'norm-resolvent' by setting

$$\mathcal{N}_{K/\mathbb{Q}}(b\mid\chi) := \prod_\omega (b\mid\chi^{\omega^{-1}})^\omega,$$

where ω runs through a transversal of $\Omega_{\mathbb{Q}}$ modulo Ω_K.

For each finite place v of K we write K_v for the completion of K at v and note that $L_v := L\otimes_K K_v\simeq\prod_{w|v}L_w$ is a free $K_v[G]$-module of rank one. Then, in the same way as above, for each element b_v in L_v such that $L_v = K_v[G]\cdot b_v$ we define an idelic-valued resolvent $(b_v\mid\chi)$ and an idelic-valued norm resolvent $\mathcal{N}_{K/\mathbb{Q}}(b_v\mid\chi)$ (for more details see [12, § 4.1]). For an $\mathcal{O}_K$-module $\mathcal{L}$ we also set $\mathcal{L}_v := \mathcal{L}\otimes_{\mathcal{O}_K}\mathcal{O}_{K_v}$.

Lemma 2.2. *Fix a $\mathbb{Z}$-basis $\{a_\sigma\}_{\sigma\in\Sigma(K)}$ of $\mathcal{O}_K$, an element b of L such that $L = K[G]\cdot b$ and, for each finite place v of K, an element b_v of L_v such that $\mathcal{L}_v = \mathcal{O}_{K_v}[G]\cdot b_v$.*

Then the element $h^{\mathrm{rel}}_G([\mathcal{L},\kappa_L,H_L])$ is represented by the homomorphism pair $(\theta_1\theta_2^{-1},\theta_2\theta_3)$ where for χ in $\widehat{G}$ one has

$$\theta_1(\chi) := \prod_v\mathcal{N}_{K/\mathbb{Q}}(b_v\mid\chi),\quad \theta_2(\chi) := \mathcal{N}_{K/\mathbb{Q}}(b\mid\chi),\quad \theta_3(\chi) := \delta_K^{\chi(1)}$$

with $\delta_K := \det(\tau(a_\sigma))_{\sigma,\tau\in\Sigma(K)}$.

Proof. Since H_L is a free G-module, in terms of the notation of Lemma 2.1 we can and will use the basis $\{y_0^j\} = \{y_p^j\} = \{w_{\hat\sigma}\}_{\sigma\in\Sigma(K)}$ so that μ_p is the identity matrix for every prime p.

We write $\theta_b : L \longrightarrow H_{L,\mathbb{Q}}$ for the $\mathbb{Q}[G]$-linear isomorphism that sends each element $a_\sigma \cdot b$ to $w_{\hat\sigma}$.

For each prime p we set $\mathcal{O}_{K,p} := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_K \simeq \prod_{v|p} \mathcal{O}_{K_v}$ and $\mathcal{L}_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{L} \simeq \prod_{v|p} \mathcal{L}_v$. We note that the element $b_p := (b_v)_{v|p}$ is a $\mathcal{O}_{K,p}[G]$-generator of $\mathcal{L}_p$ and that the homomorphism of $\mathbb{Z}_p[G]$-modules $\theta_{b_p} : \mathcal{L}_p \longrightarrow H_{L,p}$ that sends each element $a_\sigma \cdot b_p$ to $w_{\hat\sigma}$ is bijective.

For the basis $\{x_p^j\}$ that occurs in the statement of Lemma 2.1 we choose $\{a_\sigma \cdot b_p\}_{\sigma\in\Sigma(K)}$ and then write λ_p for the matrix in $\mathrm{GL}_d(\mathbb{Q}_p[G])$ which satisfies $\underline{a_\sigma \cdot b_p} = \lambda_p \cdot \underline{\theta_b^{-1}(w_{\hat\sigma})}$. We note, in particular, that λ_p is the coordinate matrix of the $\mathbb{Q}_p[G]$-linear map $(\mathbb{Q}_p^c \otimes_{\mathbb{Q}} \theta_b) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \theta_{b_p})^{-1}$ with respect to the basis $\{w_{\hat\sigma}\}$.

Then Lemma 2.1 implies that $h_G^{\mathrm{rel}}([\mathcal{L}, \kappa_L H_L])$ is represented by the homomorphism pair $\left(\prod_p \mathrm{Det}(\lambda_p)\right) \times \mathrm{Det}(\mu)$ where μ is the coordinate matrix in $\mathrm{GL}_d(\mathbb{Q}^c[G])$ of $\kappa_L \circ (\mathbb{Q}^c \otimes_{\mathbb{Q}} \theta_b)^{-1}$ with respect to the basis $\{w_{\hat\sigma}\}$.

In addition, from [2, (16) and (17)], one knows that $\mathrm{Det}(\mu)(\chi) = \delta_K^{\chi(1)} \cdot \mathcal{N}_{K/\mathbb{Q}}(b \mid \chi)$ for each character χ.

Finally to compute each homomorphism $\mathrm{Det}(\lambda_p)$ we note that

$$(\mathbb{Q}_p^c \otimes_{\mathbb{Q}} \theta_b) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \theta_{b_p})^{-1} = ((\mathbb{Q}_p^c \otimes_{\mathbb{Q}} \theta_b) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Q}^c} \kappa_L)^{-1}) \circ ((\mathbb{Q}_p^c \otimes_{\mathbb{Q}^c} \kappa_L) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \theta_{b_p})^{-1})$$

and write $\lambda_{p,2}$ for the coordinate matrix of $(\mathbb{Q}_p^c \otimes_{\mathbb{Q}^c} \kappa_L) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \theta_{b_p})^{-1}$.

Then using similar computations to those used to derive [2, (16) and (17)] one finds that for each character χ one has

$$\mathrm{Det}(\lambda_{p,2})(\chi) = \mathcal{N}_{K/\mathbb{Q}}(b_p \mid \chi) = \prod_{v|p} \mathcal{N}_{K/\mathbb{Q}}(b_v \mid \chi),$$

as required to complete the proof. □

2.2. **Hermitian Modules and Classgroups.** In this section we recall some of the basic theory of Hermitian Modules and Classgroups. For more details see Fröhlich [22, Chap. II]. Note however, that in contrast to the convention used in loc. cit. we consider all modules as left modules.

Definition 2.3. A *Hermitian form* on a Γ-module X is a nondegenerate bilinear map

$$h\colon X_{\mathbb{Q}} \times X_{\mathbb{Q}} \to \mathbb{Q}[\Gamma]$$

that is $\mathbb{Q}[\Gamma]$-linear in the first variable and satisfies $h(x, y) = h(y, x)^\sharp$ with $z \mapsto z^\sharp$ the $\mathbb{Q}$-linear anti-involution of $\mathbb{Q}[\Gamma]$ which inverts elements of Γ.

A *Hermitian* Γ-module is a pair (X, h) comprising a finitely generated projective Γ-module X and a hermitian form h on X.

Example 2.4. For any number field K and any finite group Γ we extend the field-theoretic trace $\mathrm{tr}_{K/\mathbb{Q}} : K \to \mathbb{Q}$ to a linear map $K[\Gamma] \to \mathbb{Q}[\Gamma]$ by applying it to the coefficients of each element of $K[\Gamma]$.

This action then gives rise to a hermitian form

$$t_{K[\Gamma]}\colon K[\Gamma]\times K[\Gamma]\to \mathbb{Q}[\Gamma]$$

by setting $t_{K[\Gamma]}(x,y) = \mathrm{tr}_{K/\mathbb{Q}}(xy)$. In particular, since $\mathcal{O}_K$ is a free $\mathbb{Z}$-module the pair $(\mathcal{O}_K[\Gamma], t_{K[\Gamma]})$ is a hermitian Γ-module.

Example 2.5. For any finite Galois extension L/K of number fields, with $G = G(L/K)$, one obtains a hermitian form

$$t_{L/K} : L\times L \to \mathbb{Q}[G]$$

by setting $t_{L/K}(x,y) = \sum_{g\in G}\mathrm{tr}_{L/\mathbb{Q}}(x\cdot g(y))g$. For each full projective G-sublattice $\mathcal{L}$ of L the pair $(\mathcal{L}, t_{L/K})$ is then a hermitian G-module.

Example 2.6. Let X_1 and X_2 be finitely generated projective Γ-modules and ξ an isomorphism of $\mathbb{Q}[\Gamma]$-modules $X_{2,\mathbb{Q}}\cong X_{1,\mathbb{Q}}$. For any hermitian form h on X_1 we define the 'pullback of h through ξ' to be the hermitian form $\xi^*(h)$ on X_2 that satisfies

$$\xi^*(h)(x_2,y_2) = h(\xi(x_2),\xi(y_2))$$

for all $x_2, y_2\in X_2$.

To classify general hermitian Γ-modules Fröhlich defined (see, for example, [22, Chap. II, (5.3)]) the 'hermitian classgroup' $\mathrm{HCl}(\Gamma)$ of Γ to be the cokernel of the homomorphism

$$\Delta_\Gamma^{\mathrm{herm}}\colon \mathrm{Det}(\mathbb{Q}[\Gamma]^\times)\to \frac{\mathrm{Hom}(\mathrm{R}_\Gamma, \mathrm{J_f}(\mathbb{Q}^c))^{\Omega_\mathbb{Q}}}{\mathrm{Det}(\mathrm{U_f}(\mathbb{Z}[\Gamma]))}\times \mathrm{Hom}(\mathrm{R}^s_\Gamma, \mathbb{Q}^{c\times})^{\Omega_\mathbb{Q}};\quad \theta\mapsto([\theta]^{-1},\theta^s) \tag{5}$$

where R^s_Γ denotes the subgroup of R_Γ generated by the set of irreducible symplectic characters of Γ and θ^s denotes the restriction of θ to R^s_Γ.

To each hermitian Γ-module (X,h) Fröhlich then associated a canonical 'discriminant' element $\mathrm{Disc}(X,h)$ in $\mathrm{HCl}(\Gamma)$ that is defined explicitly as follows.

Definition 2.7. Let (X,h) be a hermitian Γ-module and write d for the rank of the free $\mathbb{Q}[\Gamma]$-module $X_\mathbb{Q}$. Choose a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}$ of $X_\mathbb{Q}$ and, for each prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p. Then there exists an element λ_p of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ with $\underline{x_p^j} = \lambda_p\cdot\underline{x_0^j}$ and the 'discriminant class' $\mathrm{Disc}(X,h)$ is the element of $\mathrm{HCl}(\Gamma)$ represented by the pair

$$\left(\prod_p \mathrm{Det}(\lambda_p), \mathrm{Pf}(h(x_0^i,x_0^j))\right).$$

Here Pf is the 'Pfaffian' function in $\mathrm{Hom}(\mathrm{R}^s_\Gamma,\mathbb{Q}^{c\times})$ defined in [22, Chap. II, Prop. 4.3].

We end this section with a new definition that will be useful in the sequel.

Definition 2.8. The 'extended Hermitian classgroup' $\mathrm{eHCl}(\Gamma)$ of Γ is defined to the cokernel of the homomorphism that is defined just as $\Delta_\Gamma^{\mathrm{herm}}$ except that the term $\mathrm{Hom}(\mathrm{R}^s_\Gamma,\mathbb{Q}^{c\times})^{\Omega_\mathbb{Q}}$ on the right hand side of (5) is replaced by $\mathrm{Hom}(\mathrm{R}^s_\Gamma,\mathbb{Q}^{c\times})$. We regard $\mathrm{HCl}(\Gamma)$ as a subgroup of $\mathrm{eHCl}(\Gamma)$ in the obvious way.

2.3. **Metrised modules and Classgroups.** We quickly recall the definition of metrised modules and class groups. For further details we refer the reader to [16, §2 and §3.1].

For each ϕ in $\widehat{\Gamma}$ we write W_ϕ for the Wedderburn component of $\mathbb{Q}^c[\Gamma]$ which corresponds to the contragredient character $\overline{\phi}$ of ϕ. Thus W_ϕ has character $\phi(1)\overline{\phi}$.

For any $\mathbb{Q}^c[\Gamma]$-module X we then set

$$X_\phi := \bigwedge\nolimits_{\mathbb{Q}^c}^{\mathrm{top}} (X \otimes_{\mathbb{Q}^c} W_\phi)^\Gamma,$$

where '$\bigwedge_{\mathbb{Q}^c}^{\mathrm{top}}$' denotes the highest exterior power over $\mathbb{Q}^c$ which is non-zero, and Γ acts diagonally on the tensor product. We recall from [16, Lem. 2.3] that $X_\phi \simeq \overline{W}_\phi X$.

Recall that $\mathbb{Q}^c$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We write $\sigma_\infty : \mathbb{Q}^c \to \mathbb{C}$ for the inclusion and $\overline{z}$ for the conjugate of a complex number z.

Definition 2.9. A *metrised* Γ-module is a pair $\left(X, \{||\cdot||_\phi\}_{\phi\in\widehat{\Gamma}}\right)$ comprising a finitely generated projective Γ-module X and a set $\{||\cdot||_\phi\}_{\phi\in\widehat{\Gamma}}$ of metrics on the complex lines $\mathbb{C}\otimes_{\mathbb{Q}^c} X_\phi$ induced by positive definite hermitian forms μ_ϕ on the spaces $\mathbb{C}\otimes_{\mathbb{Q}^c} X_\phi$.

In this situation, we usually abbreviate $\left(X, \{||\cdot||_\phi\}_{\phi\in\widehat{\Gamma}}\right)$ to $(X, \mu_\bullet)$ and note that for each ϕ in $\widehat{\Gamma}$ and each element x of $\mathbb{C}\otimes_{\mathbb{Q}^c} X_\phi$ one has $||x||_\phi^2 = \mu_\phi(x,x)$.

Example 2.10. An important special case occurs when μ_ϕ arises as the 'highest exterior power' of a positive definite hermitian form $\tilde{\mu}_\phi$ on the space

$$(X\otimes_{\mathbb{Z}} W_\phi)^\Gamma \otimes_{\mathbb{Q}^c} \mathbb{C} = ((X\otimes_{\mathbb{Z}}\mathbb{C})\otimes_{\mathbb{C}} (W_\phi\otimes_{\mathbb{Q}^c}\mathbb{C}))^\Gamma.$$

In this case, for any $\mathbb{C}$-basis $v_1,\dots,v_d$ of this space one has

$$||v_1\wedge\ldots\wedge v_d||_\phi^2 = \det((\tilde{\mu}_\phi(v_i,v_j)_{1\le i,j\le d})).$$

Let Γ be a finite group. Then the standard Γ-equivariant positive definite hermitian form $\mu_{\mathbb{C}[\Gamma]}$ on $\mathbb{C}[\Gamma]$ is defined (for example, in [16, § 2.1]) by setting

$$\mu_{\mathbb{C}[\Gamma]}\left(\sum_{g\in\Gamma} x_g g, \sum_{h\in\Gamma} y_h h\right) = \sum_{g\in\Gamma} x_g\overline{y_g}.$$

The associated $\mathbb{C}[\Gamma]$-valued hermitian form is the so-called 'multiplication form'

$$\hat{\mu}_{\mathbb{C}[\Gamma]}\colon \mathbb{C}[\Gamma]\times\mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[\Gamma]$$

that sends each pair (x,y) to $x\cdot\overline{y}$, where we extend complex conjugation to an anti-involution on $\mathbb{C}[\Gamma]$ by setting

$$\overline{\sum_{\gamma\in\Gamma} a_\gamma\gamma} := \sum_{\gamma\in\Gamma}\overline{a_\gamma}\gamma^{-1}.$$

Example 2.11. In this example we use the hypotheses and notation of §2.1.3.

(i) We write μ_L for the (unique) Γ-equivariant positive definite hermitian form on $\mathbb{C}\otimes_{\mathbb{Z}} H_L$ that satisfies

$$\mu_L\left(\sum_{\sigma\in\Sigma(L)} x_\sigma w_\sigma, \sum_{\sigma\in\Sigma(L)} y_\sigma w_\sigma\right) = \sum_{\sigma\in\Sigma(L)} x_\sigma\overline{y_\sigma}.$$

For each $\phi \in \widehat{\Gamma}$ the form μ_L together with the restriction of $\mu_{\mathbb{C}[\Gamma]}$ on $\mathbb{C} \otimes_{\mathbb{Q}^c} W_\phi$ induces a positive definite hermitian form $\tilde{\mu}_{L,\phi}$ on the tensor product

$$((\mathbb{C} \otimes_{\mathbb{Z}} H_L) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{Q}^c} W_\phi))^\Gamma = \mathbb{C} \otimes_{\mathbb{Q}^c} (H_L \otimes_{\mathbb{Z}} W_\phi)^\Gamma .$$

We then write $\mu_{L,\phi}$ for the positive definite hermitian form on

$$\mathbb{C} \otimes_{\mathbb{Q}^c} \bigwedge\nolimits_{\mathbb{Q}^c}^{\mathrm{top}} (H_L \otimes_{\mathbb{Z}} W_\phi)^\Gamma = \bigwedge\nolimits_{\mathbb{C}}^{\mathrm{top}} \left(\mathbb{C} \otimes_{\mathbb{Q}^c} (H_L \otimes_{\mathbb{Z}} W_\phi)^\Gamma\right)$$

that is obtained as the highest exterior power of $\tilde{\mu}_{L,\phi}$ (as per the discussion in Example 2.10). The induced metric

$$\mu_{L,\bullet} := \{\mu_{L,\phi}\}_{\phi \in \widehat{\Gamma}}$$

on H_L plays an important role in the sequel.

(ii) There is a Γ-equivariant positive definite hermitian form h_L on $\mathbb{C} \otimes_{\mathbb{Q}} L$ defined by

$$h_L(z_1 \otimes m, z_2 \otimes n) = z_1 \overline{z_2} \sum_{\sigma \in \Sigma(L)} \sigma(m)\overline{\sigma(n)}.$$

(This form is a scalar multiple of the 'Hecke form' defined by Chinburg, Pappas and Taylor in [16, §5.2].) For each ϕ in $\widehat{\Gamma}$ we write $h_{L,\phi}$ for the positive definite hermitian form on $(\mathbb{C} \otimes_{\mathbb{Q}} L)_\phi$ that is obtained as the highest exterior power of the form on $(L \otimes_{\mathbb{Q}} W_\phi)^G$ which is induced by h_L on $\mathbb{C} \otimes_{\mathbb{Q}} L$ and by the restriction of $\mu_{\mathbb{C}[\Gamma]}$ on $\mathbb{C} \otimes_{\mathbb{Q}^c} W_\phi$.

We set

$$h_{L,\bullet} := \{h_{L,\phi}\}_{\phi \in \widehat{\Gamma}}$$

and note that if $\mathcal{L}$ is any full projective $\mathbb{Z}[\Gamma]$-sublattice of L, then the pair $(\mathcal{L}, h_{L,\bullet})$ is naturally a metrised Γ-module.

Example 2.12. Let $E \subseteq \mathbb{Q}^c$ be a subfield and let X_1 and X_2 be finitely generated locally-free $\mathbb{Z}[\Gamma]$-modules. Let ξ denote an isomorphism of $E[\Gamma]$-modules $X_{2,E} \cong X_{1,E}$. For each ϕ in $\widehat{\Gamma}$ we write

$$\xi_\phi : (X_2 \otimes_{\mathbb{Z}} \mathbb{Q}^c)_\phi \otimes_{\mathbb{Q}^c} \mathbb{C} \cong (X_1 \otimes_{\mathbb{Z}} \mathbb{Q}^c)_\phi \otimes_{\mathbb{Q}^c} \mathbb{C}$$

for the isomorphism of complex lines which is induced by ξ. If h is any metric on X_1, then we define the 'pullback' of h under ξ to be the (unique) metric $\xi^*(h)$ on X_2 which satisfies

$$\xi^*(h)_\phi(z) = h_\phi(\xi_\phi(z))$$

for all $\phi \in \widehat{\Gamma}$ and $z \in (X_2 \otimes_{\mathbb{Z}} \mathbb{Q}^c)_\phi \otimes_{\mathbb{Q}^c} \mathbb{C}$.

In order to classify metrised Γ-modules Chinburg, Pappas and Taylor ([16, §3.1 and §3.2]) defined the *arithmetic classgroup* $A(\Gamma)$ of Γ to be the cokernel of the homomorphism

$$\Delta_\Gamma^{\mathrm{met}} \colon \mathrm{Det}(\mathbb{Q}[\Gamma]^\times) \to \frac{\mathrm{Hom}(\mathrm{R}_\Gamma, \mathrm{J}_\mathrm{f}(\mathbb{Q}^c))^{\Omega_\mathbb{Q}}}{\mathrm{Det}(\mathrm{U}_\mathrm{f}(\mathbb{Z}[\Gamma]))} \times \mathrm{Hom}(\mathrm{R}_\Gamma, \mathbb{R}^\times_{>0}); \quad \theta \mapsto ([\theta], |\theta|)$$

where we write $|\theta|$ for the homomorphism which sends each character ϕ in $\hat{\Gamma}$ to $|\theta(\phi)|^{-1}$. Note that we adopt here the convention of [1, §4.2 and Rem. 4.4], i.e. our $|\theta|$ is the inverse of the map $|\theta|$ used in [16].

To each metrised Γ-module (X, h) one can then associate a canonical 'arithmetic class' $[X, h]$ in $A(\Gamma)$.

We next recall the explicit definition of this element from [16, §3.2] (see also [1, Rem. 4.6]) and to do this we use the notation of Lemma 2.1.

Definition 2.13. Let $(X,\mu_\bullet)$ be a metrised Γ-module, with X locally-free over $\mathbb{Z}[\Gamma]$ of rank d. Choose a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}$ of $X_\mathbb{Q}$ and, for each prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p. Then there exists an element λ_p of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ such that $\underline{x_p^j} = \lambda_p\cdot\underline{x_0^j}$.

For each x in $X_\mathbb{Q}$ we set $r(x) := \sum_{\gamma\in\Gamma}\gamma(x)\otimes\gamma \in X\otimes_\mathbb{Z}\mathbb{Q}^c[\Gamma]$ and note that for each w in W_ϕ one has $r(x)(1\otimes w)\in (X\otimes_\mathbb{Z} W_\phi)^\Gamma$ where for each w in W_ϕ the action of $r(x)$ on $1\otimes w$ is defined by $r(x)(1\otimes w) := \sum_{\gamma\in\Gamma}\gamma(x)\otimes\gamma(w)$.

Let $\{w_{\phi,k}\}_{1\le k\le\phi(1)^2}$ be a $\mathbb{Q}^c$-basis of W_ϕ that is orthonormal with respect to the restriction of $\mu_{\mathbb{C}[\Gamma]}$ to W_ϕ. Then the set $\{r(x_0^j)(1\otimes w_{\phi,k})\}_{j,k}$ is an $\mathbb{Q}^c$-basis of $(X\otimes_\mathbb{Z} W_\phi)^\Gamma$ and so $\bigwedge_j\bigwedge_k r(x_0^j)(1\otimes w_{\phi,k})$ is an $\mathbb{Q}^c$-basis of $(X\otimes_\mathbb{Z}\mathbb{Q}^c)_\phi$.

We then define $[X,\mu_\bullet]$ to be the element of $\mathrm{A}(\Gamma)$ that is represented by the homomorphism on R_Γ which sends each character $\phi\in\widehat{\Gamma}$ to

$$\prod_p \mathrm{Det}(\lambda_p)(\phi)\times\left\|\left(\bigwedge_j\bigwedge_k r(x_0^j)(1\otimes w_{\phi,k})\right)\otimes 1\right\|_\phi^{1/\phi(1)} \in J_f(\mathbb{Q}^c)\times\mathbb{R}^\times_{>0}. \tag{6}$$

We note that it is straightforward to show that $[X,\mu]$ is independent of the precise choices of bases $\{x_0^j\}$, $\{x_p^j\}$ and $\{w_{\phi,k}\}$.

As a simple example, we apply the above recipe in the setting of Example 2.11(i). The following result will play an important role in a later argument.

Lemma 2.14. *In terms of the notation of Example 2.11(i) the element $[H_L,\mu_{L,\bullet}]$ of $A(G)$ is represented by the pair $(1,\theta)$ where θ sends each character ϕ of $\widehat{G}$ to $|G|^{[K:\mathbb{Q}]\frac{\phi(1)}{2}}$.*

Proof. If $X=H_L$ and $\mu_\bullet=\mu_{L,\bullet}$, then in the notation of Definition 2.13 we can take both $\{x_0^j\}$ and $\{x_p^j\}$ to be the basis $\{w_{\hat\sigma}\}_{\sigma\in\Sigma(K)}$ described in §2.1.3 and so $\lambda_p=1$.

In addition, for a character ϕ in $\widehat{G}$, embeddings σ and τ in $\Sigma(K)$ and integers k and ℓ with $1\le k,\ell\le\phi(1)^2$ one has

$$\begin{aligned}
&(\mu_L\otimes\mu_{\mathbb{C}[G]})(r(w_{\hat\sigma})(1\otimes w_{\phi,k}), r(w_{\hat\tau})(1\otimes w_{\phi,\ell}))\\
&=(\mu_L\otimes\mu_{\mathbb{C}[G]})(\sum_{g\in G} g(w_{\hat\sigma})\otimes g(w_{\phi,k}), \sum_{h\in G} h(w_{\hat\tau})\otimes h(w_{\phi,\ell}))\\
&=\sum_{g,h}\mu_L(g(w_{\hat\sigma}),h(w_{\hat\tau}))\cdot\mu_{\mathbb{C}[G]}(g(w_{\phi,k}),h(w_{\phi,\ell}))\\
&=\sum_{g,h}\delta_{g,h}\delta_{\hat\sigma,\hat\tau}\cdot\mu_{\mathbb{C}[G]}(g(w_{\phi,k}),h(w_{\phi,\ell}))\\
&=\delta_{\hat\sigma,\hat\tau}\delta_{k,\ell}\cdot|G|.
\end{aligned}$$

From the explicit description given in Example 2.10 it thus follows that the second component of the representative (6) is equal to the $\phi(1)$-st root of

$$\det\big((\delta_{\hat\sigma,\hat\tau}\delta_{k,\ell}\cdot|G|)_{(\sigma,k),(\tau,\ell)}\big)^{1/2} = |G|^{[K:\mathbb{Q}]\cdot\phi(1)^2/2},$$

as suffices to give the claimed result. ∎

3. Canonical homomorphisms and the universal diagram

In this section we establish a direct link between relative algebraic K-theory and the theories of metrised and hermitian modules. The existence of such a link plays a key role in subsequent arithmetic results.

For any finite group Γ we abbreviate $\mathrm{Cl}(\mathbb{Z}[\Gamma])$ to $\mathrm{Cl}(\Gamma)$ and we recall that there is a natural isomorphism of abelian groups

$$h_\Gamma^{\mathrm{red}} : \mathrm{Cl}(\Gamma) \cong \mathrm{Cok}(\Delta_\Gamma^{\mathrm{red}})$$

where $\Delta_\Gamma^{\mathrm{red}}$ denotes the homomorphism

$$\Delta_\Gamma^{\mathrm{red}} \colon \mathrm{Hom}(R_\Gamma, \mathbb{Q}^{c\times})^{\Omega_{\mathbb{Q}}} \to \frac{\mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}}}{\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))}; \qquad \theta \mapsto [\theta].$$

Remark 3.1. We normalise the isomorphism h_Γ^{red} as in [21, Rem. 1, p. 21]. To be specific, if X is a finitely generated projective $\mathbb{Z}[\Gamma]$-module, then one can give an explicit representative of the class $h_\Gamma^{\mathrm{red}}([X])$ as follows. We choose a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}$ of $X_{\mathbb{Q}}$ and, for each rational prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p. Let λ_p be the matrix in $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ which satisfies $\underline{x_p^j} = \lambda_p \cdot \underline{x_0^j}$. Then $h_\Gamma^{\mathrm{red}}([X])$ is represented by the function $(\prod_p \mathrm{Det}(\lambda_p))$.

In the next result we shall use the canonical homomorphisms (of abelian groups)

$$\begin{array}{lcll}
\partial_\Gamma^{1,1} & : & \mathrm{Cok}(\Delta_\Gamma^{\mathrm{rel}}) \longrightarrow \mathrm{A}(\Gamma), & ([\theta_1],\theta_2) \mapsto ([\theta_1], |\theta_2|) \\
\partial_\Gamma^{2,1} & : & \mathrm{Cok}(\Delta_\Gamma^{\mathrm{rel}}) \longrightarrow \mathrm{eHCl}(\Gamma), & ([\theta_1],\theta_2) \mapsto ([\theta_1], \theta_2^s) \\
\partial_\Gamma^{1,2} & : & \mathrm{A}(\Gamma) \longrightarrow \mathrm{Cok}(\Delta_\Gamma^{\mathrm{red}}), & ([\theta_1],\theta_2) \mapsto [\theta_1] \\
\partial_\Gamma^{2,2} & : & \mathrm{eHCl}(\Gamma) \longrightarrow \mathrm{Cok}(\Delta_\Gamma^{\mathrm{red}}), & ([\theta_1],\theta_2) \mapsto [\theta_1].
\end{array}$$

We shall also use the following composite homomorphisms (defined using the isomorphisms h_Γ^{rel} and h_Γ^{red})

$$\begin{array}{lcl}
\Pi_\Gamma^{\mathrm{met}} := \partial_\Gamma^{1,1} \circ h_\Gamma^{\mathrm{rel}} & : & K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \longrightarrow \mathrm{A}(\Gamma), \\
\Pi_\Gamma^{\mathrm{herm}} := \partial_\Gamma^{2,1} \circ h_\Gamma^{\mathrm{rel}} & : & K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \longrightarrow \mathrm{eHCl}(\Gamma), \\
\partial_\Gamma^{\mathrm{met}} := \left(h_\Gamma^{\mathrm{red}}\right)^{-1} \circ \partial_\Gamma^{1,2} & : & \mathrm{A}(\Gamma) \longrightarrow \mathrm{Cl}(\Gamma), \\
\partial_\Gamma^{\mathrm{herm}} := \left(h_\Gamma^{\mathrm{red}}\right)^{-1} \circ \partial_\Gamma^{2,2} & : & \mathrm{eHCl}(\Gamma) \longrightarrow \mathrm{Cl}(\Gamma),
\end{array}$$

For convenience we shall use the same notation $\partial_\Gamma^{\mathrm{herm}}$ to denote the restriction of $\partial_\Gamma^{\mathrm{herm}}$ to the subgroup $\mathrm{HCl}(\Gamma)$.

Theorem 3.2.

(i) *The homomorphism* $\Pi_\Gamma^{\mathrm{met}}$ *sends each class* $[X,\xi,Y]$ *to* $[X,\xi^*(\mu)]-[Y,\mu]$ *for any choice of metric* μ *on* Y.

(ii) *The homomorphism* $\Pi_\Gamma^{\mathrm{herm}}$ *sends each element* $[X,\xi,Y]$ *of the subgroup* $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}[\Gamma])$ *to* $\mathrm{Disc}(X,\xi^*(h)) - \mathrm{Disc}(Y,h)$ *for any choice of hermitian form* h *on* Y.

(iii) *The homomorphism* $\partial_\Gamma^{\mathrm{met}}$ *sends the class* $[X,h]$ *of a metrized module* (X,h) *to the class* $[X]$.

(iv) *The homomorphism* $\partial_\Gamma^{\mathrm{herm}}$ *sends the discriminant* $\mathrm{Disc}(X,h)$ *of a hermitian module* (X,h) *to the class* $[X]$.

(v) *The following diagram commutes.*

$$\begin{tikzcd}
 & & \mathrm{A}(\Gamma) \arrow[dr, "\partial_\Gamma^{\mathrm{met}}"] \arrow[drr, "\partial_\Gamma^{1,2}"] & & \\
\mathrm{Cok}(\Delta_\Gamma^{\mathrm{rel}}) \arrow[urr, "\partial_\Gamma^{1,1}"] \arrow[dddrr, "\partial_\Gamma^{2,1}"'] & K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma]) \arrow[l, "\sim", "h_\Gamma^{\mathrm{rel}}"'] \arrow[ur, "\Pi_\Gamma^{\mathrm{met}}"] \arrow[rr, "\partial^0_{\mathbb{Z},\mathbb{Q}^c,\Gamma}"] \arrow[dddr, "\Pi_\Gamma^{\mathrm{herm}}"'] & & \mathrm{Cl}(\Gamma) \arrow[r, "\sim", "h_\Gamma^{\mathrm{red}}"'] & \mathrm{Cok}(\Delta_\Gamma^{\mathrm{red}}) \\
 & & \mathrm{K}_0\mathrm{T}(\Gamma) \arrow[ul, "\iota_\Gamma"'] \arrow[ur, "\partial_\Gamma"] \arrow[d, "\Upsilon_\Gamma"'] & & \\
 & & \mathrm{HCl}(\Gamma) \arrow[uur, bend left, "\delta''_\Gamma"'] \arrow[d] & & \\
 & & \mathrm{eHCl}(\Gamma) \arrow[uuur, bend left, "\partial_\Gamma^{\mathrm{herm}}"'] \arrow[uuurr, "\partial_\Gamma^{2,2}"'] & &
\end{tikzcd}$$

Here the unlabeled arrow is the natural inclusion $\mathrm{HCl}(\Gamma) \to \mathrm{eHCl}(\Gamma)$ *and the remaining homomorphisms that are not defined above are as follows.*

- ι_Γ *is the composition of the first isomorphism in (2) and the natural inclusion* $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}[\Gamma]) \to K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma])$*,*
- Υ_Γ *is the homomorphism defined by Fröhlich in* [22, Chap. 2, §6]*,*
- ∂_Γ *is the canonical map (as described in* [22, Chap. 1, (1.3)]*) ,*
- δ''_Γ *the homomorphism described in* [22, Chap. 2, (6.16)].

(For further details of these maps see the argument below.)

Proof. Claim (i) is proved by Agboola and the first author in [1, Th. 4.11].

To prove claim (ii) we write d for the $\mathbb{Q}[\Gamma]$-rank of $X_\mathbb{Q} \cong Y_\mathbb{Q}$ and, just as in Definition 2.13, we fix a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}_{1\le j\le d}$ of $X_\mathbb{Q}$ and also, for each prime p, $\mathbb{Z}_p[\Gamma]$-bases $\{x_p^j\}_{1\le j\le d}$ of X_p and $\{y_p^j\}_{1\le j\le d}$ of Y_p.

We write λ_p and μ_p for the (unique) elements of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ with $\underline{x_p^j} = \lambda_p \cdot \underline{x_0^j}$ and $\underline{y_p^j} = \mu_p \cdot \underline{\xi(x_0^j)}$, where in the last equality we use the fact that $\{\xi(x_0^j)\}_{1\le j\le d}$ is a $\mathbb{Q}[\Gamma]$-basis of $Y_\mathbb{Q}$.

Then the explicit definition of h_Γ^{rel} as described in Lemma 2.1 ensures that $h_\Gamma^{\mathrm{rel}}([X,\xi,Y])$ is represented by the pair

$$\left(\prod_p \mathrm{Det}(\lambda_p)\cdot\mathrm{Det}(\mu_p)^{-1}\right) \times 1 \in \mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_F} \times \mathrm{Hom}(R_\Gamma,(\mathbb{Q}^c)^\times).$$

The assertion of claim (ii) thus follows because Definition 2.7 implies that for any hermitian form h on X the element $\mathrm{Disc}(X,\xi^*(h)) - \mathrm{Disc}(Y,h)$ is also represented by

$$\begin{aligned}
&(\prod_p \mathrm{Det}(\lambda_p), \mathrm{Pf}(\xi^*(h)(x_0^i, x_0^j))) \times (\prod_p \mathrm{Det}(\mu_p)^{-1}, \mathrm{Pf}(h(\xi(x_0^i), \xi(x_0^j)))^{-1})\\
=&(\prod_p \mathrm{Det}(\lambda_p)\cdot \mathrm{Det}(\mu_p)^{-1}, \mathrm{Pf}(h(\xi(x_0^i), \xi(x_0^j)))\mathrm{Pf}(h(\xi(x_0^i), \xi(x_0^j)))^{-1})\\
=&(\prod_p \mathrm{Det}(\lambda_p)\cdot \mathrm{Det}(\mu_p)^{-1}, 1)
\end{aligned}$$

where the first equality follows immediately from the definition of the pullback $\xi^*(h)$.

Claim (iii) and (iv) are immediate consequences of the respective Hom-descriptions of the groups $\mathrm{A}(\Gamma), \mathrm{eHCl}(\Gamma)$ and $\mathrm{Cl}(\Gamma)$.

Turning to claim (v) we note at the outset that the upper and lower left and right hand most triangles commute by definition of the maps involved and that the outer quadrilateral commutes since both of the composites $\partial_\Gamma^{1,2}\circ\partial_\Gamma^{1,1}$ and $\partial_\Gamma^{2,2}\circ\partial_\Gamma^{2,1}$ send each pair $([\theta_1], \theta_2)$ to the class of $[\theta_1]$.

We next note that the commutativity of the upper central triangle, namely the equality $\partial^0_{\mathbb{Z},\mathbb{Q}^c,\Gamma} = \partial_\Gamma^{\mathrm{met}}\circ\pi_\Gamma^{\mathrm{met}}$, will follow if we show that the composites $\partial_\Gamma^{1,2}\circ\partial_\Gamma^{1,1}\circ h_\Gamma^{\mathrm{rel}}$ and $h_\Gamma^{\mathrm{red}}\circ\partial^0_{\mathbb{Z},\mathbb{Q}^c,\Gamma}$ coincide.

This is true because the explicit description of h_Γ^{rel} implies that $\partial_\Gamma^{1,2}\circ\partial_\Gamma^{1,1}\circ h_\Gamma^{\mathrm{rel}}$ sends each element $[X,\xi,Y]$ of $K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma])$ to the class represented by the homomorphism

$$\prod_p(\mathrm{Det}(\lambda_p)\cdot\mathrm{Det}(\mu_p)^{-1}) = (\prod_p \mathrm{Det}(\lambda_p))\cdot(\prod_p\mathrm{Det}(\mu_p))^{-1}$$

whilst

$$(h_\Gamma^{\mathrm{red}}\circ\partial^0_{\mathbb{Z},\mathbb{Q}^c,\Gamma})([X,\xi,Y]) = h_\Gamma^{\mathrm{red}}([X]-[Y]) = h_\Gamma^{\mathrm{red}}([X])h_\Gamma^{\mathrm{red}}([Y])^{-1}$$

and Remark 3.1 implies that the classes $h_\Gamma^{\mathrm{red}}([X])$ and $h_\Gamma^{\mathrm{red}}([Y])$ are respectively represented by the products $\prod_p \mathrm{Det}(\lambda_p)$ and $\prod_p\mathrm{Det}(\mu_p)$.

The above facts combine to directly imply commutativity of the lower central triangle, namely the equality $\partial^0_{\mathbb{Z},\mathbb{Q}^c,\Gamma} = \partial_\Gamma^{\mathrm{herm}}\circ\pi_\Gamma^{\mathrm{herm}}$, and so it only suffices to prove commutativity of the four triangles inside this triangle.

We shall now discuss these triangles clockwise, starting from the uppermost.

The commutativity of the first triangle follows directly from the fact that for any finite Γ-module M of finite projective dimension, and any resolution of the form

$$0\to P\xrightarrow{\theta} P'\to M\to 0,$$

where P is finitely generated and locally-free and P' is finitely generated and free, the class of M in $K_0T(\mathbb{Z}[\Gamma])$ is sent by ι_Γ to $[P,\theta_\mathbb{Q},P']$ and by ∂_Γ to $[P]-[P']$ ($=[P]$ as P' is free).

If for the above sequence we fix a $\mathbb{Z}[\Gamma]$-basis $\{x^i\}$ of P' and then for each prime p choose a matrix λ_p in $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ so that the components of the vector $\lambda_p\cdot\underline{\theta_\mathbb{Q}^{-1}(x^i)}$ are a $\mathbb{Z}_p[\Gamma]$-basis of P_p, then the image of the class of M in $K_0T(\mathbb{Z}[\Gamma])$ under Υ_Γ is represented by $(\prod_p\mathrm{Det}(\lambda_p),1)$. This implies the commutativity of the second triangle since Remark 3.1 implies the class of $\partial_\Gamma(M) = [P]$ is represented by $(\prod_p\mathrm{Det}(\lambda_p))$ whilst the definition of δ''_Γ implies that it is induced by sending each pair $([\theta_1],\theta_2)$ to $(h_\Gamma^{\mathrm{red}})^{-1}([\theta_1])$.

The latter fact also directly implies commutativity of the third triangle and the fourth triangle commutes since, in terms of the above notation, the composite $h_\Gamma^{\rm rel}\circ\iota_\Gamma$ sends the class of M to the element represented by the pair $((\prod_p \mathrm{Det}(\lambda_p)),1)$. □

In the next result we describe an explicit link between the elements in relative algebraic K-theory constructed in §2.1.3, the hermitian modules described in Example 2.5 and the metrised modules defined in Example 2.11. This link explains the relevance of Theorem 3.2 to our later results.

Proposition 3.3. *Let L/K be a finite Galois extension of number fields with group G. Then for any full projective $\mathcal{O}_K[G]$-submodule $\mathcal{L}$ of L the following claims are valid.*

(i) *The image of $[\mathcal{L},\kappa_L,H_L]$ under $\Pi_G^{\rm met}$ is equal to $[\mathcal{L},h_{L,\bullet}]-[H_L,\mu_{L,\bullet}]$.*
(ii) *The image of $[\mathcal{L},\kappa_L,H_L]$ under $\Pi_G^{\rm herm}$ is equal to $\mathrm{Disc}(\mathcal{L},t_{L/K})$.*

Proof. The pullback with respect to κ_L of the metric $\mu_{L,\bullet}$ defined in Example 2.11 is equal to $h_{L,\bullet}$ (cf. [1, Exam. 4.10(i)]). This fact combines with Theorem 3.2(i) to directly imply the equality in claim (i).

To prove claim (ii) we use the representative $(\theta_1\theta_2^{-1},\theta_2\theta_3)$ of $h_G^{\rm rel}([\mathcal{L},\kappa_L,H_L])$ described in Lemma 2.2. We also recall that, with this notation, the general result of Fröhlich in [22, Cor. to Th. 27] implies the element $\mathrm{Disc}(\mathcal{L},t_{L/K})-\mathrm{Disc}(\mathcal{O}_K[G],t_{K[G]})$ of $\mathrm{HCl}(G)$ is represented by $(\theta_1\cdot\theta_2^{-1},\theta_2^s)$, where the form $t_{K[G]}$ is as defined in Example 2.4.

Comparing these results one deduces that the element

$$\Pi_G^{\rm herm}([\mathcal{L},\kappa_L,H_L])-\mathrm{Disc}(\mathcal{L},t_{L/K})+\mathrm{Disc}(\mathcal{O}_K[G],t_{K[G]})$$

of $\mathrm{HCl}(G)$ is represented by the pair $(1,\theta_3^s)$.

To deduce claim (ii) from this it is thus enough to show that the pair $(1,\theta_3^s)$ also represents the element $\mathrm{Disc}(\mathcal{O}_K[G],t_{K[G]})$.

To check this we need only note that, in the terminology of [22, Chap. II, §5], the Pfaffian of the matrix $(t_{K[G]}(a_\sigma,a_\tau))_{\sigma,\tau\in\Sigma(K)}$ sends each character χ in R_G^s to $\delta_K^{\chi(1)}=\theta_3(\chi)$.

Then, by applying the recipe of Definition 2.7 with $\{x_0^j\}=\{x_p^j\}=\{a_\sigma\}_{\sigma\in\Sigma(K)}$ one finds that $\mathrm{Disc}(\mathcal{O}_K[G],t_{K[G]})$ is indeed represented by the pair $(1,\theta_3^s)$, as required. □

Part II: Weak ramification and Galois-Gauss sums

In this part of the article we describe a first arithmetic application of the approach described in earlier sections by using Theorem 3.2 (and Proposition 3.3) to refine existing results concerning links between Galois-Gauss sums and certain metric and hermitian structures that arise naturally in arithmetic.

In this way, in §4 we refine the main results of Chinburg and the second author in [12] concerning relations between hermitian-metric structures involving fractional powers of the inverse different of a tamely ramified Galois extension of number fields and the associated Galois-Gauss sums (twisted by appropriate Adams operations).

In the remainder of the article we then focus on weakly ramified Galois extensions (of odd degree) and use Theorem 3.2 to refine key aspects of the extensive existing theory of the square root of the inverse different for such extensions.

4. Tamely ramified Galois-Gauss sums

4.1. **Galois-Gauss sums, Adams operators and Galois-Jacobi sums.** For the reader's convenience in this section we fix notation regarding various variants of Galois-Gauss sums that will play a role in the sequel.

To do this we fix an arbitrary finite Galois extension L/K of number fields in $\mathbb{Q}^c$ and set $G := G(L/K)$.

We use the fact that each element of $\zeta(\mathbb{Q}^c[G])$ can be written uniquely in the form

$$x = \sum_{\chi \in \widehat{G}} e_\chi \cdot x_\chi \tag{7}$$

with each x_χ in $\mathbb{Q}^c$. For convenience we extend the assignment $x \mapsto x_\chi$ to arbitrary elements χ of R_G by multiplicativity.

4.1.1. We define the 'equivariant global Galois-Gauss sum' for L/K by setting

$$\tau_{L/K} := \sum_{\chi \in \widehat{G}} e_\chi \cdot \tau(K, \chi) \in \zeta(\mathbb{Q}^c[G])$$

where each (global) Galois-Gauss sum $\tau(K, \chi)$ belongs to $\mathbb{Q}^c$ and is as defined, for example, by Fröhlich in [21, Ch. I, (5.22)].

We also define an 'equivariant unramified characteristic' in $\zeta(\mathbb{Q}[G])$ by setting

$$y_{L/K} := \sum_{\chi \in \widehat{G}} e_\chi \cdot \prod_{v | d_L} y(K_v, \chi_v).$$

Here χ_v is the restriction of χ to the decomposition subgroup of some fixed place w of L above v and (following [21, Ch. IV, §1]) for any finite Galois extension of local fields F/E of group D and each ϕ in $\widehat{D}$ we set

$$y(E, \phi) := \begin{cases} 1, & \text{if } \phi|_I \neq 1, \\ -\phi(\sigma), & \text{if } \phi|_I = 1, \end{cases} \tag{8}$$

where I is the inertia subgroup of D and σ is a lift to D of the Frobenius element in D/I.

We then define the 'modified equivariant (global) Galois-Gauss sum' for L/K by setting

$$\tau'_{L/K} := \tau_{L/K} \cdot y_{L/K}^{-1}.$$

Since we rely on certain results from [2] we will also use the 'absolute (global) Galois-Gauss sum for L/K' that is obtained by setting

$$\tau^\dagger_{L/K} := \sum_{\chi \in \hat{G}} e_\chi \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \chi) \in \zeta(\mathbb{Q}^c[G])^\times.$$

In particular, it is useful to note that the inductivity property of Galois-Gauss sums combines with the fact $\tau(K, 1_K) = 1$ to imply

$$\tau^\dagger_{L/K} = \tau_K^G \cdot \tau_{L/K} \tag{9}$$

where τ_K^G is the invertible element of $\zeta(\mathbb{Q}^c[G])$ obtained by setting

$$\tau_K^G := \mathrm{Nrd}_{\mathbb{Q}[G]}(\tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \mathbf{1}_K))$$

so that $(\tau_K^G)_\chi = \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}}\mathbf{1}_K)^{\chi(1)}$ for all χ in $\widehat{G}$.

4.1.2. For each integer k that is coprime to $|G|$ we write ψ_k for the k-th Adams operator on R_G (for the relevant properties of which we refer to [12, Lem. 3.1]).

We use this operator to construct endomorphisms of $\zeta(\mathbb{Q}^c[G])$ in the following way. For each pair of integers m and n we then write $(m+n\cdot\psi_{k,*})(x)$ for the unique element of $\zeta(\mathbb{Q}^c[G])$ with $(m+n\cdot\psi_{k,*})(x)_\chi := (x_\chi)^m\cdot(x_{\psi_k(\chi)})^n$ for every χ in $\widehat{G}$.

We then define the 'k-th Galois-Jacobi sum' for the extension L/K by setting

$$J_{k,L/K} := (\psi_{k,*}-k)(\tau_{L/K}).$$

In the sequel we shall often use the following key property of these sums.

Lemma 4.1. *For each integer k prime to $|G|$ one has $J_{k,L/K}\in\zeta(\mathbb{Q}[G])^\times$.*

Proof. An element x of $\zeta(\mathbb{Q}^c[G])$ belongs to $\zeta(\mathbb{Q}[G])$ if and only if one has $(x_\chi)^\omega = x_{\chi^\omega}$ for all $\chi\in\hat{G}$ and all $\omega\in\Omega_{\mathbb{Q}}$.

To verify that the elements $J_{k,L/K}$ satisfy this criterion we recall how the absolute Galois group acts on Gauss sums. We let $\mathrm{Ver}_{K/\mathbb{Q}}\colon \Omega_{\mathbb{Q}}^{ab}\longrightarrow\Omega_K^{ab}$ denote the transfer map and write $v_{K/\mathbb{Q}}$ for the cotransfer map from abelian characters of Ω_K to abelian characters of $\Omega_{\mathbb{Q}}$. Thus, for each $\chi\in\widehat{G}$ the map $v_{K/\mathbb{Q}}\det_\chi$ is an abelian character of $\Omega_{\mathbb{Q}}$. Then, by [21, Th. 20B(ii)], one has $\tau(K,\chi^{\omega^{-1}})^\omega = \tau(K,\chi)\cdot\big(v_{K/\mathbb{Q}}\det_\chi\big)(\omega)$ for all $\chi\in\widehat{G}$ and all $\omega\in\Omega_{\mathbb{Q}}$.

Hence it suffices to show that $\big(\big(v_{K/\mathbb{Q}}\det_\chi\big)(\omega)\big)^k = \big(v_{K/\mathbb{Q}}\det_{\psi_k(\chi)}\big)(\omega)$ and this is true because $\det_{\psi_k(\chi)} = (\det_\chi)^k$ (see [12, Lem. 3.1]). $\square$

With the results of [2] in mind we finally note that if G has odd order, then an explicit comparison of the respective definitions shows that

$$\tau_K^G\cdot(\psi_{2,*}-1)(\tau'_{L/K})\cdot(\tau^\dagger_{L/K})^{-1} = J_{2,L/K}\cdot(\psi_{2,*}-1)(y_{L/K}^{-1}). \tag{10}$$

Remark 4.2. If F/E is a finite Galois extensions of p-adic fields (for some p) with group D, then one can use the canonical local Gauss sum $\tau(E,\phi)$ (as discussed, for example, in [21, Ch. III, §2, Th. 18 and Rem. 1]) for each ϕ in $\widehat{D}$ to define natural analogues $\tau_{F/E}, y_{F/E}, \tau'_{F/E}, \tau^\dagger_{F/E}, \tau_E^D$ and $J_{k,F/E}$ in $\zeta(\mathbb{Q}^c[D])$ of the elements defined above. Then in the same way as above one can show that for each integer k that is coprime to $|D|$ the element $J_{k,F/E}$ belongs to $\zeta(\mathbb{Q}[D])$ and can also prove the following local analogues of the equalities (9) and (10):

$$\tau^\dagger_{F/E} = \tau_E^D\cdot\tau_{F/E} \tag{11}$$

and

$$\tau_E^D\cdot(\psi_{2,*}-1)(\tau'_{F/E})\cdot(\tau^\dagger_{F/E})^{-1} = J_{2,F/E}\cdot(\psi_{2,*}-1)(y_{F/E}^{-1}). \tag{12}$$

4.2. **Tame Galois-Gauss sums and fractional powers of the different.** We now assume the Galois extension L/K is tamely ramified and fix a natural number k that is both coprime to $|G|$ and so that the order of each inertia subgroup of G is congruent to 1 modulo k.

In any such case it follows immediately from Hilbert's formula for the different in terms of ramification invariants (cf. [30, Ch. IV, Prop. 4]) that there exists a unique fractional ideal

$\mathfrak{D}_{L/K}^{-1/k}$ of $\mathcal{O}_L$ whose k-th power is equal to the inverse of the different $\mathfrak{D}_{L/K}$ of L/K and for any integer i we set $\mathfrak{D}_{L/K}^{-i/k} = (\mathfrak{D}_{L/K}^{-1/k})^i$.

Each ideal $\mathfrak{D}_{L/K}^{-i/k}$ is stable under the natural action of $\mathcal{O}_K[G]$ and, since L/K is assumed to be tamely ramified, the $\mathcal{O}_K[G]$-module $\mathfrak{D}_{L/K}^{-i/k}$ is known to be locally-free (by Ullom [34]).

In particular, since $\mathfrak{D}_{L/K}^{-i/k}$ is a full sublattice of L, the construction of §2.1.3 gives rise to a well-defined element $[\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L]$ of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$.

We can now state the main result of this section. This result uses the invertible elements τ_K^G and $\tau'_{L/K}$ of $\zeta(\mathbb{Q}^c[G])$ that are defined in §4.1.

Theorem 4.3. *Let L/K be a tamely ramified Galois extension of number fields with group G and k any natural number as specified above. Then in $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ one has*

$$\sum_{i=0}^{k-1} [\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L] = \delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K})). \tag{13}$$

Before proving this result we use it to derive certain explicit consequences concerning the metric and hermitian structures that arise in this setting. In particular, the following result extends to all integers k as above the results on the hermitian modules $(\mathcal{O}_L, t_{L/K})$, corresponding to $k = 0$, and $(\mathfrak{D}_{L/K}^{-1/2}, t_{L/K})$, corresponding to $k = 2$ and G of odd order, that are obtained by Erez and Taylor in [20].

We recall the definition of δ_K from Lemma 2.2 and write d_K for the discriminant of $\mathcal{O}_K$. In the sequel we will often use the fact that

$$\tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \mathbf{1}_K)^2 = d_K = \delta_K^2, \tag{14}$$

as follows by combining [27, Th. (11.7)(iii)] together with [21, (5.23)].

Corollary 4.4. *Assume the notation and hypotheses of Theorem 4.3. Then both of the following claims are valid.*

(i) *In* $\mathrm{A}(G)$ *one has*

$$\sum_{i=0}^{k-1} [\mathfrak{D}_{L/K}^{-i/k}, h_{L,\bullet}] = \varepsilon_{L/K,k}^{\mathrm{met}}$$

where $h_{L,\bullet}$ is the metric defined in Example 2.11 and $\varepsilon_{L/K,k}^{\mathrm{met}}$ is represented by the pair $(1, |\theta_k|)$ with $\theta_k(\phi) = (|G|^{[K:\mathbb{Q}]}|d_K|)^{k\frac{\phi(1)}{2}} \cdot \tau(K, \psi_k(\phi))$ for all ϕ in R_G.

(ii) *In* $\mathrm{HCl}(G)$ *one has*

$$\sum_{i=0}^{k-1} \mathrm{Disc}(\mathfrak{D}_{L/K}^{-i/k}, t_{L/K}) = \varepsilon_{L/K,k}^{\mathrm{herm}}$$

where the hermitian form $t_{L/K}$ is as defined in Example 2.5 and $\varepsilon_{L/K,k}^{\mathrm{herm}}$ is represented by the pair $(1, \tilde{\theta}_k)$ with $\tilde{\theta}_k(\phi) = d_K^{k\frac{\phi(1)}{2}} \cdot \tau(K, \psi_k(\phi))$ for all ϕ in R_G^s.

Proof. To prove claim (i) we note first Proposition 3.3(i) implies that for each i one has

$$\Pi_G^{\mathrm{met}}([\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L]) = [\mathfrak{D}_{L/K}^{-i/k}, h_{L,\bullet}] - [H_L, \mu_{L,\bullet}].$$

We next recall that for $\alpha = (\alpha_\chi)_{\chi\in\widehat{G}} \in \zeta(\mathbb{Q}^c[G])^\times$ the element $h_G^{\mathrm{rel}}(\delta_G(\alpha))$ is represented by the function $\chi \mapsto (1, \alpha_\chi)$.

This implies, in particular, that $h_G^{\mathrm{rel}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K})))$ is represented by the pair $(1, \theta'_k)$ with $\theta'_k(\phi) := \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}}\mathbf{1}_K)^{k\phi(1)} \cdot \tau'(K, \psi_k(\phi))$ for each ϕ in $\widehat{G}$.

Finally we recall that the element $[H_L, \mu_{L,\bullet}]$ has been explicitly computed in Lemma 2.14.

Putting these facts together with Theorem 4.3 one finds that the element

$$\begin{aligned}\sum_{i=0}^{k-1}[\mathfrak{D}_{L/K}^{-i/k}, h_{L,\bullet}] &= k \cdot [H_L, \mu_{L,\bullet}] + \sum_{i=0}^{k-1}\Pi_G^{\mathrm{met}}([\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L]) \\ &= k \cdot [H_L, \mu_{L,\bullet}] + \Pi_G^{\mathrm{met}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K})))\end{aligned}$$

of $A(G)$ is represented by the homomorphism pair $(1, |\theta_k|)$ where for each ϕ in $\widehat{G}$ one has

$$\theta_k(\phi) := |G|^{[K:\mathbb{Q}]\frac{k\phi(1)}{2}} \cdot \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}}\mathbf{1}_K)^{k\phi(1)} \cdot \tau'(K, \psi_k(\phi)).$$

But, taking account of both (14) and the fact that $y(K_v, \phi_v)$ is a root of unity for all ϕ in $\widehat{G}$, one finds that

$$|\theta_k|(\phi) = (|G|^{[K:\mathbb{Q}]}|d_K|)^{k\frac{\phi(1)}{2}} \cdot |\tau(K, \psi_k(\phi))|$$

and this proves claim (i).

It is enough to prove the equality of claim (ii) in $\mathrm{eHCl}(G)$ and to do this we note that the description in Proposition 3.3(ii) combines with Theorem 4.3 to imply that

$$\sum_{i=0}^{k-1}\mathrm{Disc}(\mathfrak{D}_{L/K}^{-i/k}, t_{L/K}) = \Pi_G^{\mathrm{herm}}\left(\sum_{i=0}^{k-1}[\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L]\right) = \Pi_G^{\mathrm{herm}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K}))).$$

By the definition of Π_G^{herm} one has the following equality in $\mathrm{eHCl}(G)$

$$\Pi_G^{\mathrm{herm}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K}))) = (\partial_G^{2,1} \circ h_G^{\mathrm{rel}})(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K}))).$$

Hence one deduces that $\sum_{i=0}^{k-1}\mathrm{Disc}(\mathfrak{D}_{L/K}^{-i/k}, t_{L/K})$ is represented by the pair $(1, (\theta'_k)^s)$, where θ'_k is as defined in the proof of claim (i).

To deduce claim (ii) from this it is now enough to note that for ϕ in R_G^s one has

$$\theta'_k(\phi) = \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}}\mathbf{1}_K)^{k\phi(1)} \cdot \tau'(K, \psi_k(\phi)) = d_K^{k\phi(1)/2} \cdot \tau(K, \psi_k(\phi)), \tag{15}$$

where, to derive the second equality, we have used (14) and the fact that for every ϕ in R_G^s the integer $\phi(1)$ is even and $y(K_v, \phi_v) = 1$ by [21, Th. 29 (i)]. □

In the remainder of this section we shall prove Theorem 4.3 by combining results of the first and second author from [2] and of Chinburg and the second author from [12].

To do this we fix a $K[G]$-generator b of L and a $\mathbb{Z}$-basis $\{a_\sigma\}_{\sigma\in\Sigma(K)}$ of $\mathcal{O}_K$. For each integer i with $0 \le i < k$ and each non-archimedean place v of K we also fix an $\mathcal{O}_{K,v}[G]$-generator $b_{i,v}$ of $(\mathfrak{D}_{L/K}^{-i/k})_v$. Then by Lemma 2.2 the element $h_G^{\mathrm{rel}}([\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L])$ is represented by the pair of homomorphisms $(\theta_{i,1} \cdot \theta_2^{-1}, \theta_2 \cdot \theta_3)$ where for each χ in R_G one has

$$\theta_{i,1}(\chi) := \prod_v \mathcal{N}_{K/\mathbb{Q}}(b_{i,v} \mid \chi), \;\; \theta_2(\chi) := \mathcal{N}_{K/\mathbb{Q}}(b \mid \chi), \;\; \theta_3(\chi) := \delta_K^{\chi(1)}. \tag{16}$$

We next write $\psi_k^\vee$ for the map which sends a function h on R_G to $h\circ\psi_k$ and recall that, since k is prime to $|G|$, Cassou-Noguès and Taylor in [14] have shown that the assignment $(\theta,\theta')\mapsto(\psi_k^\vee(\theta),\psi_k^\vee(\theta'))$ induces a well-defined endomorphism Ψ_k of $K_0(\mathbb{Z}[G],\mathbb{Q}^c[G])$.

With this notation, it is straightforward to check that

$$(\theta_2^{-k},\theta_2^k)\equiv(\psi_k^\vee(\theta_2)^{-1},\psi_k^\vee(\theta_2)) \pmod{\mathrm{im}(\Delta_G^{\mathrm{rel}})},$$

(see, for example, the end of the proof of [12, Prop. 3.3]) and it is also clear $\psi_k^\vee(\theta_3)=\theta_3$.

In particular, if we denote the sum on the left hand side of (13) by Σ_k, then these observations combine with the above description of each element $h_G^{\mathrm{rel}}([\mathfrak{D}_{L/K}^{-i/k},\kappa_L,H_L])$ and the congruence proved in Lemma 4.5 below to imply that $h_G^{\mathrm{rel}}(\Sigma_k)$ is represented by the pair

$$(\psi_k^\vee(\theta_{0,1}\cdot\theta_2^{-1}),\psi_k^\vee(\theta_2\cdot\theta_3^k))=(\psi_k^\vee(\theta_{0,1}\cdot\theta_2^{-1}),\psi_k^\vee(\theta_2\cdot\theta_3))\cdot(1,\theta_3^{k-1}).$$

It follows that, writing x_k for the element of $K_0(\mathbb{Z}[G],\mathbb{Q}^c[G])$ for which $h_G^{\mathrm{rel}}(x_k)$ is represented by the pair $(1,\theta_3^{k-1})$, one has $\Sigma_k=\Psi_k([\mathcal{O}_L,\kappa_L,H_L])+x_k$.

The key fact now is that, from [2, Cor. 7.7], one knows $[\mathcal{O}_L,\kappa_L,H_L]=\delta_G(\tau_K^G\cdot\psi_{k,*}(\tau'_{L/K}))$ (where we use (9) to convert the result of loc. cit. to the notation used here) and hence that $h_G^{\mathrm{rel}}(\Psi_k([\mathcal{O}_L,\kappa_L,H_L]))$ is represented by the homomorphism pair $(1,\theta'_3\cdot\psi_k^\vee(\theta_4))$ where for each χ in R_G one has $\theta'_3(\chi)=\tau(\mathbb{Q},\mathrm{ind}_K^{\mathbb{Q}}\mathbf{1}_K)^{\chi(1)}$ and $\theta_4(\chi)=\tau'(K,\chi)$.

Substituting this fact into the last displayed equality one finds that $h_G^{\mathrm{rel}}(\Sigma_k)$ is represented by the pair $(1,\theta_3^{k-1}\cdot\theta'_3\cdot\psi_k^\vee(\theta_4))$.

Now from (14) we deduce $\tau(\mathbb{Q},\mathrm{ind}_K^{\mathbb{Q}}\mathbf{1}_K)=\pm\delta_K$ and hence that $(1,\theta_3)\equiv(1,\theta'_3)\pmod{\mathrm{im}(\Delta_G^{\mathrm{rel}})}$.

This implies $h_G^{\mathrm{rel}}(\delta_G((\tau_K^G)^k\cdot\psi_{k,*}(\tau_{L/K})))$ is also represented by the pair $(1,\theta_3^{k-1}\cdot\theta'_3\cdot\psi_k^\vee(\theta_4))$ and therefore that $\Sigma_k=\delta_G((\tau_K^G)^k\cdot\tau_{L/K,k})$, as claimed.

This completes the proof of Theorem 4.3.

Lemma 4.5. *For the homomorphisms $\theta_{i,1}$ for $0\le i<k$ that are defined in (16) one has*

$$\prod_{i=0}^{k-1}\theta_{i,1}\equiv\psi_k^\vee(\theta_{0,1}) \pmod{\mathrm{Det}(U_f(\mathbb{Z}[G]))}.$$

Proof. This is proved by a slight adaptation of the arguments in [12] (and is implicitly used in the proof of Corollary 2.2 in loc. cit.). To be precise, we shall use the notation of [12, §4.3.1] with our integer k corresponding the integer ℓ used in loc. cit.

Then the present hypotheses (on k) allow us to choose the integer ℓ' to be $(1-e)/\ell$. In particular, if we set $N:=0$, then $N_\ell=0$ and, for each i with $0\le i<\ell$, also $N_i=-i\ell'(e-1)=-i\ell'e+N'_i$ with $N'_i:=-i(e-1)/\ell$. Each element a_{N_i} can therefore be written as $c_i\cdot a_{N'_i}$ with c_i an element of B with $v_{\mathfrak{p}}(c_i)=-i\ell'$.

With this choice of ℓ' an explicit computation shows that the integer $M_{\mathfrak{p},\ell,\ell'}$ defined in (2.4) of loc. cit. is equal to $\sum_{i=0}^{\ell-1}i\ell'$ and so one can take the element c chosen in [loc. cit., Cor. 4.5] to be the product $\prod_{i=0}^{\ell-1}c_i$. For this element there is for every χ in $\mathrm{Hom}(\Lambda,B^{c\times})$ an equality

$$(ca_0\mid\psi_\ell\chi)\prod_{i=0}^{\ell-1}(a_{N_i}\mid\chi)^{-1}=(a_0\mid\psi_\ell\chi)\prod_{i=0}^{\ell-1}(a_{N'_i}\mid\chi)^{-1}.$$

and so [loc. cit., Cor. 4.5] asserts that the $\mathfrak{p}$-adic valuation of this element is zero.

It is now straightforward to derive the claimed congruence by combining this fact with the argument of [loc. cit., §5]. □

5. Weakly ramified Galois-Gauss sums and the relative element $\mathfrak{a}_{L/K}$

In the remainder of the article we study links between Galois-Gauss sums and hermitian and metric structures that arise in weakly ramified Galois extensions of odd degree. In this first section we define a canonical element in relative algebraic K-theory that is key to the theory we develop and then state some of the main results about this element that we establish in later sections.

At the outset we fix a finite odd degree Galois extension of number fields L/K that is 'weakly ramified' in the sense of Erez [19] (that is, the second lower ramification subgroups in G of each place of L are trivial) and set $G := G(L/K)$.

Since L/K is of odd degree there exists a unique fractional $\mathcal{O}_L$-ideal $\mathcal{A}_{L/K}$ whose square is the inverse of the different $\mathfrak{D}_{L/K}$ (see the discussion at the beginning of §4.2).

In addition, since L/K is weakly ramified, Erez has shown that $\mathcal{A}_{L/K}$ is a locally-free module with respect to the restriction of the natural action of $\mathcal{O}_K[G]$ on L (see [19]).

We may therefore use the general construction of §2.1.3 to define a canonical element of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ by setting

$$\mathfrak{a}_{L/K} := [\mathcal{A}_{L/K}, \kappa_L, H_L] - \delta_G(\tau_K^G \cdot (\psi_{2,*} - 1)(\tau'_{L/K})) \tag{17}$$

where the Galois-Gauss sums τ_K^G and $\tau'_{L/K}$ are as defined in §4.1.

Proposition 3.3 implies the projection of $[\mathcal{A}_{L/K}, \kappa_L, H_L]$ to each of the groups $A(G), \mathrm{HCl}(G)$ and $\mathrm{Cl}(G)$ recovers arithmetical invariants related to $\mathcal{A}_{L/K}$ that have been studied in previous articles. By using this fact explicit information about the element $\mathfrak{a}_{L/K}$ can often constitute a strong refinement of pre-existing results or conjectures concerning the metric and hermitian structures that are associated to $\mathcal{A}_{L/K}$ and this observation motivates the systematic study of $\mathfrak{a}_{L/K}$ that we undertake in later sections.

In the next result (which will be proved in §8.2) we collect some of the main results that we prove concerning $\mathfrak{a}_{L/K}$.

In the sequel we write $\mathcal{W}_{L/K}$ for the set of finite places v of K that ramify wildly in an extension L/K and $\mathcal{W}_{L/K}^{\mathbb{Q}}$ for the set of rational primes that lie below any place in $\mathcal{W}_{L/K}$.

We also let A_{tor} denote the torsion subgroup of an abelian group A.

Theorem 5.1. *Let L/K be a finite odd degree weakly ramified Galois extension of number fields of group G. Then the following assertions are valid.*

(i) *The element $\mathfrak{a}_{L/K}$ belongs to the subgroup*

$$\bigoplus_{\ell \in \mathcal{W}_{L/K}^{\mathbb{Q}}} K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])_{\mathrm{tor}}$$

of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$. In particular, if L/K is tamely ramified, then $\mathfrak{a}_{L/K} = 0$.

(ii) *In $A(G)$ one has*

$$[\mathcal{A}_{L/K}, h_{L,\bullet}] = \Pi_G^{\mathrm{met}}(\mathfrak{a}_{L/K}) + \varepsilon_{L/K}^{\mathrm{met}}$$

where the metric $h_{L,\bullet}$ is as defined in Example 2.11 and $\varepsilon_{L/K}^{\mathrm{met}}$ is represented by the pair $(1, \theta)$ with $\theta(\phi) = (|G|^{[K:\mathbb{Q}]}|\mathrm{d}_K|)^{\phi(1)/2} \cdot |\tau(K, \psi_2(\phi) - \phi)|$ for all ϕ in R_G.

(iii) *In* $\mathrm{HCl}(G)$ *one has*

$$\mathrm{Disc}(\mathcal{A}_{L/K}, t_{L/K}) = \Pi_G^{\mathrm{herm}}(\mathfrak{a}_{L/K}) + \varepsilon_{L/K}^{\mathrm{herm}}$$

where the hermitian form $t_{L/K}$ *is as defined in Example 2.5 and* $\varepsilon_{L/K}^{\mathrm{herm}}$ *is represented by the pair* $(1, \tilde{\theta})$ *with* $\tilde{\theta}(\phi) = \mathrm{d}_K^{\phi(1)/2} \cdot \tau(K, \psi_2(\phi) - \phi)$ *for all* ϕ *in* R_G^s.

(iv) *In* $\mathrm{Cl}(G)$ *one has* $\partial^0_{\mathbb{Z},\mathbb{Q}^c,G}(\mathfrak{a}_{L/K}) = [\mathcal{A}_{L/K}]$.

Remark 5.2. In addition to the result of Theorem 5.1(i) it is also possible to explicitly compute $\mathfrak{a}_{L/K}$ for certain (weakly) wildly ramified extensions L/K (see, for example, Theorem 8.3 below). These results show, in particular, that $\mathfrak{a}_{L/K}$ does not in general vanish.

In Conjecture 7.2 below we shall offer a precise conjectural description of $\mathfrak{a}_{L/K}$ in terms of local (second) Galois-Jacobi sums and invariants related to canonical classes arising in local class field theory. This description is related to certain 'epsilon constant conjectures' that are already in the literature and hence to the general philosophy of Tamagawa number conjectures that originated with Bloch and Kato.

This connection gives a new perspective to the theory of the square root of the inverse different but does not itself help to compute $\mathfrak{a}_{L/K}$ explicitly in any degree of generality.

Nevertheless, our methods combine with numerical experiments to suggest that, rather surprisingly, it might be possible in general to describe $\mathfrak{a}_{L/K}$ explicitly (see, in particular, Remark 10.4). This possibility seems worthy of further investigation, not least because it could be used to obtain significant new evidence in the context of certain wildly ramified Galois extensions in support of the formalism of Tamagawa number conjectures.

In a different direction, Theorem 5.1 leads to effective 'finiteness results' on the natural arithmetic invariants related to $\mathfrak{a}_{L/K}$ that arise as the extension L/K varies.

To give a simple example of such a result, for each number field K and finite abstract group Γ of odd order we write $\mathrm{WR}_K(\Gamma)$ for the set of fields L that are weakly ramified odd degree Galois extensions of K and for which there exists an isomorphism of groups $\iota\colon G(L/K) \cong \Gamma$.

For each field $L \in \mathrm{WR}_K(\Gamma)$ we then write $\mathrm{Is}_L(\Gamma)$ for the set of group isomorphisms $\iota : G(L/K) \cong \Gamma$, and for each $\iota \in \mathrm{Is}_L(\Gamma)$ we consider the induced isomorphism of relative algebraic K-groups

$$\iota_* : K_0(\mathbb{Z}[G(L/K)], \mathbb{Q}^c[G(L/K)]) \cong K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]).$$

We then define a subset of 'realisable classes' in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ by setting

$$R_K^{\mathrm{wr}}(\Gamma) := \{\iota_*(\mathfrak{a}_{L/K})\colon L \in \mathrm{WR}_K(\Gamma), \iota \in \mathrm{Is}_L(\Gamma)\}.$$

Recalling that the group $K_0(\mathbb{Z}[G], \mathbb{Q}[G])_{\mathrm{tor}}$ is finite (see, for example, [7, Cor. 2.5]) the result of Theorem 5.1(i) leads directly to the following result.

Corollary 5.3. *The set* $R_K^{\mathrm{wr}}(\Gamma)$ *is finite.*

In §9 we explain how the set $R_K^{\mathrm{wr}}(\Gamma)$ can be computed effectively and then apply the general theory in the setting of an explicit conjecture of Vinatier (from [37, §1, Conj.]) concerning the Galois structure of $\mathcal{A}_{L/K}$.

To end this section we prove an important preliminary result.

Proposition 5.4. *Let* L/K *be a finite odd degree weakly ramified Galois extension of number fields of group* G. *Then* $\mathfrak{a}_{L/K}$ *belongs to the subgroup* $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ *of* $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$.

Proof. For x and y in $K_0(\mathbb{Z}[G],\mathbb{Q}^c[G])$ we write $x \equiv y$ if $x-y$ belongs to $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$.

Then $\mathfrak{a}_{L/K}$ is equal to

$$\begin{aligned}[\mathcal{A}_{L/K},\kappa_L,H_L] - \delta_G(\tau_K^G\cdot(\psi_{2,*}-1)(\tau'_{L/K})) &\equiv \Big([\mathcal{A}_{L/K},\kappa_L,H_L] - \delta_G(\tau^\dagger_{L/K})\Big) - \delta_G(J_{2,L/K}) \\ &\equiv [\mathcal{A}_{L/K},\kappa_L,H_L] - \delta_G(\tau^\dagger_{L/K})\end{aligned}$$

where the first equivalence follows from (10) and the obvious containment $(\psi_{2,*}-1)(y_{L/K}) \in \zeta(\mathbb{Q}[G])$ and the second from Lemma 4.1 (with $k=2$).

It thus suffices to note that the computations in [2, p. 555-556] (which rely heavily on a result of Fröhlich in [21, §9 (i),(ii)]) show that $[\mathcal{A}_{L/K},\kappa_L,H_L] \equiv \delta_G(\tau^\dagger_{L/K})$. □

6. Functoriality properties of $\mathfrak{a}_{L/K}$

Following Proposition 5.4 we know that each element $\mathfrak{a}_{L/K}$ belongs to $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$. In this section we prove the following result which establishes the basic functorial properties of these elements as the extension L/K varies.

Theorem 6.1. *Let L/K be a weakly ramified odd degree Galois extension of number fields of group G, fix an intermediate field F of L/K and set $J := G(L/F)$.*

- (i) *The restriction map $\rho^G_J : K_0(\mathbb{Z}[G],\mathbb{Q}[G]) \to K_0(\mathbb{Z}[J],\mathbb{Q}[J])$ sends $\mathfrak{a}_{L/K}$ to $\mathfrak{a}_{L/F}$.*
- (ii) *Assume J is normal in G and write Γ for the quotient $G/J \cong G(F/K)$. Then the natural coinflation map $\pi^G_\Gamma : K_0(\mathbb{Z}[G],\mathbb{Q}[G]) \to K_0(\mathbb{Z}[\Gamma],\mathbb{Q}[\Gamma])$ sends $\mathfrak{a}_{L/K}$ to $\mathfrak{a}_{F/K}$.*

Proof. It is convenient to first prove claim (ii) in the statement of Theorem 6.1. To do this we use the commutative diagram

$$\begin{array}{ccc} \zeta(\mathbb{Q}^c[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G],\mathbb{Q}^c[G]) \\ {\scriptstyle\tilde{\pi}^G_\Gamma}\downarrow & & {\scriptstyle\pi^G_\Gamma}\downarrow \\ \zeta(\mathbb{Q}^c[\Gamma])^\times & \xrightarrow{\delta_\Gamma} & K_0(\mathbb{Z}[\Gamma],\mathbb{Q}^c[\Gamma]). \end{array} \tag{18}$$

in which $\tilde{\pi}^G_\Gamma(z)_\phi = z_{\mathrm{inf}^G_\Gamma(\phi)}$ for all z in $\zeta(\mathbb{Q}^c[G])^\times$ and ϕ in $\widehat{\Gamma}$ (see, for example, [2, p. 577]).

Then both of the equalities $\tilde{\pi}^G_\Gamma(\tau^G_K) = \tau^\Gamma_K$ and $\tilde{\pi}^G_\Gamma((\psi_{2,*}-1)(\tau'_{L/K})) = (\psi_{2,*}-1)(\tau'_{F/K})$ follow easily from the (well-known) facts that Gauss sums and unramified characteristics are invariant under inflation and Adams operations commute with inflation.

Hence the key point in proving claim (ii) is to prove $\pi^G_\Gamma([\mathcal{A}_{L/K},\kappa_L,H_L]) = [\mathcal{A}_{F/K},\kappa_F,H_F]$. To show this we write $\mathrm{tr}_{L/F}$ for the field theoretic trace map $L\to F$. Since $\mathcal{A}_{L/K}$ is $\mathbb{Z}[G]$-projective it is also cohomologically trivial and so $\mathcal{A}^J_{L/K} = \mathrm{tr}_{L/F}(\mathcal{A}_{L/K}) = \mathcal{A}_{F/K}$, where the last equality follows, for example, from the explicit computations of Erez in [19, p. 246].

In addition, the natural identification of H^J_L with H_F induces a commutative diagram of $\mathbb{Q}^c[\Gamma]$-modules

$$\begin{array}{ccc}(\mathbb{Q}^c\otimes_{\mathbb{Q}} L)^J & \xrightarrow{\kappa_L^J} & (\mathbb{Q}^c\otimes_{\mathbb{Z}} H_L)^J\\ \| & & \| \\ \mathbb{Q}^c\otimes_{\mathbb{Q}} F & \xrightarrow{\kappa_F} & \mathbb{Q}^c\otimes_{\mathbb{Z}} H_F\end{array}$$

and, taken together, these facts imply that

$$\pi_\Gamma^G([\mathcal{A}_{L/K},\kappa_L,H_L]) = [\mathcal{A}_{L/K}^J,\kappa_L^J,H_L^J] = [\mathcal{A}_{F/K},\kappa_F,H_F],$$

as required to complete the proof of claim (ii) of Theorem 6.1.

To prove Theorem 6.1(i) we use the commutative diagram (see, for example, [2, p. 575])

$$\begin{array}{ccc}\zeta(\mathbb{Q}^c[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G],\mathbb{Q}^c[G])\\ {\scriptstyle\tilde\rho_J^G}\big\downarrow & & {\scriptstyle\rho_J^G}\big\downarrow\\ \zeta(\mathbb{Q}^c[J])^\times & \xrightarrow{\delta_J} & K_0(\mathbb{Z}[J],\mathbb{Q}^c[J]).\end{array} \tag{19}$$

Here, for each z in $\zeta(\mathbb{Q}^c[G])^\times$ and ϕ in $\widehat{J}$, one has $\tilde\rho_J^G(z)_\phi = \prod_{\chi\in\widehat{G}} z_\chi^{\langle\chi,\mathrm{ind}_J^G(\phi)\rangle_G}$ where we write $\langle\cdot,\cdot\rangle_G$ for the natural pairing on R_G.

For each number field E we now set $\tau_E := \tau(\mathbb{Q},\mathrm{ind}_E^{\mathbb{Q}}\mathbf{1}_E)$. We claim that

$$\tilde\rho_J^G(\tau_K^G) = \mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_K^{[G:J]}). \tag{20}$$

In fact, for all $\phi\in\widehat{J}$ one has

$$\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_K^{[G:J]})_\phi = \tau_K^{\phi(1)[G:J]} \text{ and } \tilde\rho_J^G(\tau_K^G)_\phi = \prod_{\chi\in\widehat{G}}\tau_K^{\chi(1)\langle\chi,\mathrm{ind}_J^G\phi\rangle_G}$$

and so the claimed equality is valid since $\sum_{\chi\in\widehat{G}}\chi(1)\langle\chi,\mathrm{ind}_J^G\phi\rangle_G = \phi(1)[G:J]$.

We next note that, since $|G|$ is odd, one has

$$\mathrm{ind}_J^G(\psi_2(\phi)) = \psi_2(\mathrm{ind}_J^G(\phi))$$

for all ϕ in $\widehat{G}$ (see, for example, [19, Prop.-Def. 3.5]). Thus, given the commutativity of (19) and the (well-known) inductivity in degree zero of both Galois-Gauss sums and non-ramified characteristics one deduces that

$$\rho_J^G(\delta_G((\psi_{2,*}-1)(\tau'_{L/K}))) = \delta_J(\tilde\rho_J^G((\psi_{2,*}-1)(\tau'_{L/K}))) = \delta_J((\psi_{2,*}-1)(\tau'_{L/F})). \tag{21}$$

By combining (20) and (21) we obtain an equality

$$\begin{aligned}\rho_J^G(\delta_G(\tau_K^G\cdot(\psi_{2,*}-1)(\tau'_{L/K}))) &= \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_K^{[G:J]})) + \delta_J((\psi_{2,*}-1)(\tau'_{L/F}))\\ &= \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_F^{-1}\cdot\tau_K^{[G:J]}) + \delta_J(\tau_F^J\cdot(\psi_{2,*}-1)(\tau'_{L/F})).\end{aligned}$$

To consider the corresponding behaviour of the term $[\mathcal{A}_{L/K},\kappa_L,H_L]$ under restriction the key point is that in the subgroup $K_0(\mathbb{Z}[J],\mathbb{Q}[J])$ of $K_0(\mathbb{Z}[J],\mathbb{Q}^c[J])$ there are equalities

$$\begin{aligned}\rho_J^G([\mathcal{A}_{L/K},\kappa_L,H_L]) - [\mathcal{A}_{L/F},\kappa_L,H_L] &= [\mathcal{A}_{L/K},\kappa_L,H_L] - [\mathcal{A}_{L/F},\kappa_L,H_L] = [\mathcal{A}_{L/K},\mathrm{id},\mathcal{A}_{L/F}]\\ &= [\mathcal{A}_{L/F}\mathcal{A}_{F/K},\mathrm{id},\mathcal{A}_{L/F}] = \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_F^{-1}\cdot\tau_K^{[G:J]})).\end{aligned}$$

Here the first equality is obvious, the second is by the defining relations of $K_0(\mathbb{Z}[J], \mathbb{Q}^c[J])$, the third follows from the (well-known) multiplicativity property $\mathcal{A}_{L/K} = \mathcal{A}_{L/F}\mathcal{A}_{F/K}$ and the fourth from the result of Lemma 6.2 below.

Comparing the last two displayed equalities it follows directly that $\rho_J^G(\mathfrak{a}_{L/K}) = \mathfrak{a}_{L/F}$, as claimed. □

Lemma 6.2. *With the subgroup J and field F as above one has $[\mathcal{A}_{L/F}\mathcal{A}_{F/K}, \mathrm{id}, \mathcal{A}_{L/F}] = \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_F^{-1} \cdot \tau_K^{[G:J]}))$ in $K_0(\mathbb{Z}[J], \mathbb{Q}[J])$.*

Proof. By Lemma 6.3 below it suffices to show that $N_{F/\mathbb{Q}}(\mathcal{A}_{F/K}) = \pm\tau_F^{-1} \cdot \tau_K^{[G:J]}$.

This equality is, in turn, a direct consequence of the fact that

$$\begin{aligned} N_{F/\mathbb{Q}}(\mathcal{A}_{F/K})^2 = N_{F/\mathbb{Q}}(\mathfrak{D}_{F/K})^{-1} = N_{F/\mathbb{Q}}(\mathfrak{D}_{F/\mathbb{Q}}^{-1}\mathfrak{D}_{K/\mathbb{Q}}) &= d_{F/\mathbb{Q}}^{-1} \cdot N_{K/\mathbb{Q}}(\mathfrak{D}_{K/\mathbb{Q}})^{[F:K]} \\ &= d_{F/\mathbb{Q}}^{-1} \cdot d_{K/\mathbb{Q}}^{[F:K]} = \tau_F^{-2} \cdot \tau_K^{2[G:J]}, \end{aligned}$$

where the last equality follows from (14). □

Lemma 6.3. *Let E be a number field and G a finite group. Let N be a locally free $\mathcal{O}_E[G]$-module of rank one. Let $\mathfrak{a}$ denote a fractional $\mathcal{O}_E$-ideal. Then in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has*

$$[\mathfrak{a}N, \mathrm{id}, N] = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(N_{E/\mathbb{Q}}(\mathfrak{a}))).$$

Proof. Recall that for each prime p and each $\mathbb{Z}$-module X we write X_p for the $\mathbb{Z}_p$-module $\mathbb{Z}_p \otimes_{\mathbb{Z}} X$.

In particular, there is an isomorphism $N_p \simeq (\mathcal{O}_{E,p}[G])^d$ of $\mathcal{O}_{E,p}[G]$-modules and hence in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$ an equality

$$[(\mathfrak{a}N)_p, \mathrm{id}, N_p] = [\mathfrak{a}_p[G]^d, \mathrm{id}, \mathcal{O}_{E,p}[G]^d] = d[\mathfrak{a}_p[G], \mathrm{id}, \mathcal{O}_{E,p}[G]].$$

It follows that $[\mathfrak{a}N, \mathrm{id}, N] = d[\mathfrak{a}[G], \mathrm{id}, \mathcal{O}_E[G]]$ in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.

Now set $n := [E : \mathbb{Q}]$ and choose $\mathbb{Z}$-bases $\omega_1, \dots, \omega_n$ for $\mathcal{O}_E$ and $\alpha_1, \dots, \alpha_n$ for $\mathfrak{a}$. Then

$$\mathfrak{a}[G] = \bigoplus_{i=1}^n \mathbb{Z}[G]\alpha_i, \quad \mathcal{O}_E[G] = \bigoplus_{i=1}^n \mathbb{Z}[G]\omega_i.$$

With respect to these bases the identity is represented by the matrix $B \in \mathrm{GL}_n(\mathbb{Q}) \subseteq \mathrm{GL}_n(\mathbb{Q}[G])$ defined by $B = (b_{ji})$ where $\alpha_i = \sum_{j=1}^n b_{ji}\omega_j$. Note that $|\det(B)| = N_{E/\mathbb{Q}}(\mathfrak{a})$.

By the defining relations in relative K-groups and the definitions of $\partial_{\mathbb{Z},\mathbb{Q},G}$ and δ_G we obtain

$$[\mathfrak{a}[G], \mathrm{id}, \mathcal{O}_E[G]] = [\mathbb{Z}[G]^n, B, \mathbb{Z}[G]^n] = \partial_{\mathbb{Z},\mathbb{Q},G}([\mathbb{Q}[G]^n, B]) = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(B)).$$

Now $\mathrm{Nrd}_{\mathbb{Q}[G]}(B) = \sum_{\chi \in \widehat{G}} x_\chi e_\chi$ with $x_\chi = \det(T_\chi(B)) = \det(B)^{\chi(1)}$, where T_χ is a representation with character χ. Hence $\mathrm{Nrd}_{\mathbb{Q}[G]}(B) = \mathrm{Nrd}_{\mathbb{Q}[G]}(\det(B))$ and

$$[\mathfrak{a}[G], \mathrm{id}, \mathcal{O}_E[G]] = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(\det(B))) = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(|\det(B)|)) = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(N_{E/\mathbb{Q}}(\mathfrak{a}))).$$

where the second equality follows from $\delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(-1)) = 0$. □

7. A canonical local decomposition of $\mathfrak{a}_{L/K}$

In this section we follow the approach of Breuning in [10] to give a canonical decomposition of the term $\mathfrak{a}_{L/K}$ as a sum of terms which depend only upon the local extensions L_w/K_v for places v of K which ramify wildly (and weakly) in L/K.

7.1. **The canonical local elements.** We first define the canonical local terms that will occur in the decomposition of $\mathfrak{a}_{L/K}$.

To do this we fix a rational prime ℓ and an odd degree weakly ramified Galois extension F/E of fields which are contained in $\mathbb{Q}_\ell^c$ and of finite degree over $\mathbb{Q}_\ell$ and we set $\Gamma := G(F/E)$.

We also fix an embedding of fields $j_\ell : \mathbb{Q}^c \longrightarrow \mathbb{Q}_\ell^c$ and by abuse of notation also write $j_\ell : \zeta(\mathbb{Q}^c[\Gamma]) \longrightarrow \zeta(\mathbb{Q}_\ell^c[\Gamma])$ for the induced ring embedding. We then write

$$j_{\ell,*} : K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \longrightarrow K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$$

for the homomorphism of abelian groups that sends each element $[P, \iota, Q]$ to $[P_\ell, \mathbb{Q}_\ell^c \otimes_{\mathbb{Q}^c, j_\ell} \iota, Q_\ell]$ and we note that $j_{\ell,*} \circ \delta_\Gamma = \delta_{\Gamma,\ell} \circ j_\ell$.

We write $\Sigma(F)$ for the set of embeddings $F \hookrightarrow \mathbb{Q}_\ell^c$ and

$$\kappa_F : \mathbb{Q}_\ell^c \otimes_{\mathbb{Q}_\ell} F \to \prod_{\Sigma(F)} \mathbb{Q}_\ell^c$$

for the isomorphism of $\mathbb{Q}_\ell^c[\Gamma]$-modules sending $x \otimes f$ to $(\sigma(f)x)_{\sigma \in \Sigma(F)}$ for $f \in F$ and $x \in \mathbb{Q}_\ell^c$.

We also write H_F for the submodule $\prod_{\Sigma(F)} \mathbb{Z}_\ell$ of $\prod_{\Sigma(F)} \mathbb{Q}_\ell^c$ and note that $[\mathcal{A}_{F/E}, \kappa_F, H_F]$ is then a well-defined element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$.

We next write $U_{F/E}$ for the canonical 'unramified' element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$ defined (for any Galois extension of local fields) by Breuning in [10] and then define an element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$ by setting

$$\mathfrak{a}_{F/E} := [\mathcal{A}_{F/E}, \kappa_F, H_F] - \delta_{\Gamma,\ell}(j_\ell(\tau_E^\Gamma \cdot (\psi_{2,*} - 1)(\tau'_{F/E}))) - U_{F/E},$$

where the elements τ_E^Γ and $\tau'_{F/E}$ of $\zeta(\mathbb{Q}^c[\Gamma])^\times$ are constructed from local Galois-Gauss sums as in Remark 4.2.

Proposition 7.1. *$\mathfrak{a}_{F/E}$ is independent of the choice of j_ℓ and belongs to $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$.*

Proof. The first assertion follows immediately from [10, Lem. 2.2] and the containment

$$\tau_E^\Gamma \cdot (\psi_{2,*} - 1)(\tau'_{F/E}) \cdot (\tau^\dagger_{F/E})^{-1} \in \zeta(\mathbb{Q}[\Gamma])^\times,$$

which itself follows directly from (12) and the local analugue of Lemma 4.1 (see Remark 4.2).

The second claim follows by combining the same containment with the containment

$$[\mathcal{A}_{F/E}, \kappa_F, H_F] - \delta_{\Gamma,\ell}(j_\ell(\tau^\dagger_{F/E})) - U_{F/E} \in K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma]).$$

proved by Breuning's argument in [10, Prop. 4.26]. □

7.2. **An explicit local conjecture.** In this section we reformulate the local epsilon constant conjecture formulated by Breuning in [10, Conj. 3.2] in terms of the explicit element $\mathfrak{a}_{F/E}$.

To this end we recall that for any finite Galois extension of ℓ-adic fields F/E, of group Γ, Breuning's conjecture is an equality in $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ of the form

$$T_{F/E} + C_{F/E} + U_{F/E} - M_{F/E} = 0. \tag{22}$$

Here, in addition, to the element $U_{F/E}$ used in the previous section, the following elements also occur: where

- $T_{F/E} := \delta_{\Gamma,\ell}(j_\ell(\tau^\dagger_{F/E}))$ is the equivariant local epsilon constant.

- $C_{F/E} = \mathcal{E}(\exp_\ell(\mathcal{L}))_\ell - [\mathcal{L}, \kappa_F, H_F]$, with $\mathcal{L}$ is any full projective $\mathbb{Z}_\ell[\Gamma]$-sublattice of $\mathcal{O}_F$ that is contained in a sufficiently large power of the maximal ideal $\mathfrak{p}_F$ of $\mathcal{O}_F$ to ensure the ℓ-adic exponential map $\exp_\ell$ converges on $\mathcal{L}$. For the precise definition of $\mathcal{E}(\exp_\ell(\mathcal{L}))_\ell$ we refer the reader to [10, § 2.4] and [2, § 3.2]. For the moment, we point out only that this element relies on local fundamental classes and is very difficult to compute explicitly in any degree of generality.
- $M_{F/E}$ is a simple and explicitly defined correction term (see [10, § 2.6]).

To reinterpret (22) we assume F/E is weakly ramified. In this case the lattice $\mathcal{L}$ that occurs above can be taken to be $p^N \cdot \mathcal{A}_{F/E}$ for any sufficiently large integer N and the element

$$\mathcal{E}_{F/E} := \mathcal{E}(\exp_\ell(p^N \cdot \mathcal{A}_{F/E})) - \delta_{\Gamma,\ell}(\mathrm{Nrd}_{\mathbb{Q}_\ell[\Gamma]}(p^{N[E:\mathbb{Q}_\ell]}))$$

of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ is easily seen to be independent of the choice of N.

We next define an element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ by setting

$$\mathfrak{c}_{F/E} := \delta_{\Gamma,\ell}((\psi_{2,*} - 1)(y_{F/E})). \tag{23}$$

Then by combining Lemma 6.3 with (12) one finds that Breuning's conjectural equality (22) is equivalent to the following conjecture.

Conjecture 7.2. *Let F/E be a weakly ramified Galois extension of ℓ-adic fields with group Γ. Then in $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ one has*

$$\mathfrak{a}_{F/E} = \mathcal{E}_{F/E} - \delta_{\Gamma,\ell}(J_{2,F/E}) + \mathfrak{c}_{F/E} - M_{F/E},$$

where the second Galois-Jacobi sum $J_{2,F/E}$ of F/E is as discussed in Remark 4.2.

Remark 7.3. We will later present evidence that the elements $\mathfrak{a}_{F/E}$ and $\mathfrak{c}_{F/E}$ are themselves very closely related (see, in particular Remark 10.4) and hence that Conjecture 7.2 suggests, rather strikingly, that the class-field theoretic term $\mathcal{E}_{F/E}$ should have a simple description in terms of the Galois-Jacobi sum $J_{2,F/E}$.

Remark 7.4. For later purposes we note that (8) implies $(\psi_{2,*} - 1)(y_{F/E}) = (1 - e_{\Gamma_0}) + \sigma e_{\Gamma_0}$, with Γ_0 the inertia subgroup of Γ and σ an element of Γ that projects to the Frobenius in Γ/Γ_0, and hence that $\mathfrak{c}_{F/E} = \delta_{\Gamma,l}((1 - e_{\Gamma_0}) + \sigma e_{\Gamma_0})$. In particular, $\mathfrak{c}_{F/E}$ vanishes if F/E is tame (since then $(1 - e_{\Gamma_0}) + \sigma e_{\Gamma_0} \in \mathbb{Z}_\ell[\Gamma]^\times$) and, in all cases, Γ/Γ_0 is abelian and so $\mathrm{Nrd}_{\mathbb{Q}_\ell[\Gamma]}((1 - e_{\Gamma_0}) + \sigma e_{\Gamma_0})) = (1 - e_{\Gamma_0}) + \sigma e_{\Gamma_0}$.

7.3. **The decomposition result.** We can now state and prove the main result of this section. In this result we use for each prime ℓ, each extension E of $\mathbb{Q}_\ell$ and each subgroup H of G, the natural induction map $\mathrm{i}_{H,E}^G : K_0(\mathbb{Z}_\ell[H], E[H]) \to K_0(\mathbb{Z}_\ell[G], E[G])$ on relative K-groups.

Theorem 7.5. *Let L/K be a weakly ramified odd degree Galois extension of number fields of group G. Then in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has an equality*

$$\mathfrak{a}_{L/K} = \sum_\ell \sum_{v|\ell} \mathrm{i}_{G_w,\mathbb{Q}_\ell}^G(\mathfrak{a}_{L_w/K_v})$$

where the sum is over all primes ℓ and for each place v of K we fix a place w of L lying above v and identify the Galois group of L_w/K_v with the decomposition subgroup G_w of w in G.

Proof. Proposition 5.4 implies $\mathfrak{a}_{L/K}$ decomposes naturally as a sum $\sum_\ell \mathfrak{a}_{L/K,\ell}$ of ℓ-primary components and so it suffices to prove that for each ℓ there is in $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell[G])$ an equality

$$\mathfrak{a}_{L/K,\ell} = \sum_{v|\ell} \mathrm{i}^G_{G_w,\mathbb{Q}_\ell}(\mathfrak{a}_{L_w/K_v}). \tag{24}$$

To do this we fix a prime ℓ and an embedding $j_\ell : \mathbb{Q}^c \to \mathbb{Q}^c_\ell$ and write $\mathcal{O}^t_\ell$ for the valuation ring of the maximal tamely ramified extension of $\mathbb{Q}_\ell$ in $\mathbb{Q}^c_\ell$.

We recall that Taylor's Fixed Point Theorem for group determinants (see [33, Chap. 8, Th. 1.1]) implies that the following composite homomorphism is injective

$$K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell[G]) \to K_0(\mathbb{Z}_\ell[G],\mathbb{Q}^c_\ell[G]) \xrightarrow{j^t_{\ell,*}} K_0(\mathcal{O}^t_\ell[G],\mathbb{Q}^c_\ell[G]) \tag{25}$$

where the first arrow is the natural inclusion and $j^t_{\ell,*}$ sends $[X,\xi,Y]$ to $[\mathcal{O}^t_\ell\otimes_{\mathbb{Z}_\ell} X,\xi,\mathcal{O}^t_\ell\otimes_{\mathbb{Z}_\ell} Y]$. It is therefore enough to show that the equality (24) holds after applying $j^t_{\ell,*}$.

The key ingredients required to prove this fact are due to Breuning and are stated in Lemma 7.6 below.

In the sequel we abbreviate $\mathrm{i}^G_{G_w,\mathbb{Q}_\ell}$ and $\mathrm{i}^G_{G_w,\mathbb{Q}^c_\ell}$ to $\mathrm{i}_{w,\ell}$ and $\mathrm{i}^c_{w,\ell}$ respectively.

In particular, if for any finite Galois extension F/E of either local fields or number fields we set $\tau_{F/E,2} := \tau_E^{G(F/E)}\cdot(\psi_{2,*}-1)(\tau'_{F/E})$, then Breuning's results as stated below combine with the explicit definitions of the terms $\mathfrak{a}_{L/K,\ell}$ and $\mathfrak{a}_{L_w/K_v}$ to imply that

$$\begin{aligned}
&j^t_{\ell,*}(\mathfrak{a}_{L/K,\ell} - \sum_{v|\ell}\mathrm{i}_{w,\ell}(\mathfrak{a}_{L_w/K_v}))\\
=&j^t_{\ell,*}\left(j_{\ell,*}([\mathcal{A}_{L/K,\ell},\kappa_{L,\ell},H_{L,\ell}]) - j_{\ell,*}(\delta_G(\tau_{L/K,2}))\right)\\
&\quad -\sum_{v|\ell} j^t_{\ell,*}\left(\mathrm{i}^c_{w,\ell}([\mathcal{A}_{L_w/K_v},\kappa_{L_w},H_{L_w}]) - \mathrm{i}^c_{w,\ell}(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v,2}))) - \mathrm{i}^c_{w,\ell}(U_{L_w/K_v})\right)\\
=&-j^t_{\ell,*}(j_{\ell,*}(\delta_G(\tau_{L/K,2}))) + \sum_{v|\ell} j^t_{\ell,*}(\mathrm{i}^c_{w,\ell}(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v,2}))))\\
=&-j^t_{\ell,*}(j_{\ell,*}(\delta_G(\tau_{L/K,2}(\tau^\dagger_{L/K})^{-1}))) + \sum_{v|\ell} j^t_{\ell,*}(\mathrm{i}^c_{w,\ell}(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v,2}(\tau^\dagger_{L_w/K_v})^{-1}))))\\
&\qquad\qquad -j^t_{\ell,*}(\delta_{G,\ell}(\prod_{\substack{v|d_L\\ v\nmid\ell}}\tilde{i}_w(y_{L_w/K_v}))).
\end{aligned}\tag{26}$$

Here the first equality follows directly from the definitions and the second uses Lemma 7.6(i) and (ii). In addition, the third equality follows from Lemma 7.6(iii) below and uses the map $\tilde{i}_w : \zeta(\mathbb{Q}^c_\ell[G_w])^\times \to \zeta(\mathbb{Q}^c_\ell[G])^\times$ which satisfies $\tilde{i}_w(x)_\chi = \prod_{\varphi\in\widehat{G}_w} x_\varphi^{\langle \mathrm{res}^G_{G_w}\chi,\varphi\rangle_{G_w}}$ for all x in $\zeta(\mathbb{Q}^c_\ell[G_w])^\times$ and χ in $\widehat{G}$.

Now, by (10), the first term in the expression (26) is equal to

$$-(j^t_{\ell,*}\circ\delta_{G,\ell}\circ j_\ell)(J_{2,L/K}\cdot(\psi_{2,*}-1)(y^{-1}_{L/K})).$$

In the same way, equality (12) implies that the second term in (26) is

$$(j^t_{\ell,*}\circ i^c_{w,l}\circ\delta_{G_w,\ell}\circ j_\ell)(\prod_{v|\ell} J_{2,L_w/K_v}\cdot(\psi_{2,*}-1)(y^{-1}_{L_w/K_v})).$$

These two expressions combine with the commutative diagram

$$\begin{array}{ccc} \zeta(\mathbb{Q}^c_\ell[G_w])^\times & \xrightarrow{\delta_{G_w,\ell}} & K_0(\mathbb{Z}_\ell[G_w],\mathbb{Q}^c_\ell[G_w]) \\ {\scriptstyle \tilde{i}_w}\downarrow & & \downarrow{\scriptstyle i^c_{w,\ell}} \\ \zeta(\mathbb{Q}^c_\ell[G])^\times & \xrightarrow{\delta_{G,\ell}} & K_0(\mathbb{Z}_\ell[G],\mathbb{Q}^c_\ell[G]), \end{array} \tag{27}$$

the fact that $j_\ell(J_{2,L/K})=\prod_{v|d_L}\tilde{i}_w(j_\ell(J_{2,L_w/K_v}))$ by the decomposition of global Galois-Gauss sums as a product of local Galois-Gauss sums and the explicit definition of $y_{L/K}$ to show that the sum in (26) is equal to the image under $j^t_{\ell,*}\circ\delta_{G,\ell}$ of

$$\begin{aligned} & j_\ell(J_{2,L/K})^{-1}\cdot\prod_{v|\ell}\tilde{i}_w(j_\ell(J_{2,L_w/K_v}))\cdot\prod_{\substack{v|d_L\\ v\nmid\ell}}\tilde{i}_w(j_\ell((\psi_{2,*}-2)(y_{L_w/K_v}))) \\ = & \prod_{\substack{v|d_L\\ v\nmid\ell}}\tilde{i}_w(j_\ell(J^{-1}_{2,L_w/K_v}\cdot(\psi_{2,*}-2)(y_{L_w/K_v}))). \end{aligned}$$

It is thus enough to note the image under $j^t_{\ell,*}\circ\delta_{G,\ell}$ of the latter element vanishes as a consequence of [9, (9) and Lem. 5.3] and the second displayed equation on page 68 of loc. cit. This completes the proof of Theorem 7.5. □

Lemma 7.6.

(i) *For each prime ℓ one has*

$$j_{\ell,*}([\mathcal{A}_{L/K,\ell},\kappa_{L,\ell},H_{L,\ell}])=\sum_{v|\ell}\mathrm{i}^G_{G_w,\mathbb{Q}^c_\ell}([\mathcal{A}_{L_w/K_v},\kappa_{L_w},H_{L_w}]).$$

(ii) *For each $v\mid\ell$ the element $\mathrm{i}^G_{G_w,\mathbb{Q}^c_\ell}(U_{L_w/K_v})$ belongs to* $\ker(j^t_{\ell,*})$.

(iii) *One has*

$$j_{\ell,*}(\delta_G(\tau^\dagger_{L/K}))-\sum_{v|\ell}\mathrm{i}^G_{G_w,\mathbb{Q}_\ell}(\delta_{G_w,\ell}(j_\ell(\tau^\dagger_{L_w/K_v})))\equiv\delta_{G,\ell}(\prod_{\substack{v|d_L\\ v\nmid\ell}}\tilde{i}_w(y_{L_w/K_v}))\pmod{\ker(j^t_{\ell,*})}.$$

Proof. To prove claim (i) one can just follow the proof of [9, Lem. 5.4] verbatim, merely substituting $\mathcal{A}_{L/K}$ for the projective $\mathbb{Z}[G]$-sublattice $\mathcal{L}$ of $\mathcal{O}_L$ that is used in loc. cit.

The property stated in claim (ii) is part of the axiomatic characterisation used by Breuning to define the elements U_{L_w/K_v} in [9, Prop. 4.4].

To prove claim (iii) we note that elements $\delta_G(\tau^\dagger_{L/K})$ and $\delta_{G_w,\ell}(\tau^\dagger_{L_w/K_v})$ are respectively denoted by $\tau_{L/K}$ and T_{L_w/K_v} in [10] and that the claimed congruence is thus equivalent to the equality of [9, (36)]. □

8. Results in special cases

In this section we compute $\mathfrak{a}_{L/K}$ explicitly in some important special cases and also give a proof of Theorem 5.1.

8.1. **Local results.** The following result uses the element $\mathfrak{c}_{F/E}$ defined in (23).

Theorem 8.1. *Let $E/\mathbb{Q}_\ell$ be a finite extension and F/E a weakly ramified Galois extension of odd degree with Galois group $\Gamma = G(F/E)$. Then $\mathfrak{a}_{F/E} = \mathfrak{c}_{F/E}$ if either F/E is tamely ramified or if $E/\mathbb{Q}_\ell$ is unramified and F/E is both abelian and has cyclic ramification subgroup.*

Proof. We fix an embedding $j_\ell \colon \mathbb{Q}^c \longrightarrow \mathbb{Q}_\ell^c$ and use it to identify $\widehat{\Gamma}$ with the set of irreducible $\mathbb{Q}_\ell^c$-valued characters of Γ.

By Proposition 7.1 and Taylor's fixed point theorem it suffices to show that

$$j_{\ell,*}^t([\mathcal{A}_{F/E}, \kappa_F, H_F] - \delta_{\Gamma,\ell}(j_\ell(\tau_E^\Gamma \cdot (\psi_{2,*} - 1)(\tau'_{F/E}))) - U_{F/E} - \mathfrak{c}_{F/E}) = 0 \tag{28}$$

with $j_{\ell,*}^t$ as in (25).

At the outset we note that $j_{\ell,*}^t(U_{F/E}) = 0$ (by [9, Prop. 4.4]) and that if θ is any element of F with $\mathcal{A}_{F/E} = \mathcal{O}_E[\Gamma] \cdot \theta$, then [9, Lem. 4.16] implies

$$[\mathcal{A}_{F/E}, \kappa_F, H_F] = \delta_{\Gamma,\ell}\left(\sum_{\chi \in \widehat{\Gamma}} e_\chi \delta_E^{\chi(1)} \cdot \mathcal{N}_{E/\mathbb{Q}_\ell}(\theta \mid \chi)\right).$$

We now assume F/E is tamely ramified. In this case Remark 7.4 implies both that $\mathfrak{c}_{F/E}$ vanishes and $\delta_{\Gamma,\ell}((\psi_{2,*} - 1)\tau'_{F/E}) = \delta_{\Gamma,\ell}((\psi_{2,*} - 1)\tau_{F/E})$ and so the element on the left hand side of (28) is equal to the image under $j_{\ell,*}^t \circ \delta_{\Gamma,\ell}$ of $x_1 \cdot x_2$ where for each χ in $\widehat{\Gamma}$ one has (in terms of the notation in (7))

$$x_{1,\chi} := \frac{\delta_E^{\chi(1)}}{j_\ell(\tau(\mathbb{Q}_\ell, \mathrm{ind}_E^{\mathbb{Q}_\ell} 1_E)^{\chi(1)})} \quad \text{and} \quad x_{2,\chi} := \frac{\mathcal{N}_{E/\mathbb{Q}_\ell}(\theta \mid \chi)}{j_\ell(\tau(E, \psi_2(\chi) - \chi))}.$$

The equality (28) is therefore true in this case since both $(j_{\ell,*}^t \circ \delta_{\Gamma,\ell})(x_1) = 0$ (as a consequence of the obvious local analogue of (14)) and $(j_{\ell,*}^t \circ \delta_{\Gamma,\ell})(x_2) = 0$, as indicated by Erez in the proof of [19, Prop. 8.2].

In the remainder of the argument we assume that $E/\mathbb{Q}_\ell$ is unramified and F/E is both abelian and has cyclic ramification subgroup. The proof in this case will heavily rely on the computations of [5] (which in turn rely on the work of Pickett and Vinatier in [29]) and so, for convenience, we switch to the notation introduced in §3.1 of loc. cit. (so that F, E and $\Gamma = G(F/E)$ are now replaced by N, K and G respectively).

In particular, we define α_M as in [5, just before Lem. 5.1.4], let $\theta_2 \in K'$ be such that $\mathcal{O}_K[G] \cdot \theta_2 = \mathcal{O}_{K'}$ and $T_{K'/K}(\theta_2) = 1$ and recall that the product $\theta = \alpha_M \cdot \theta_2$ satisfies $\mathcal{A}_{N/K} = \mathcal{O}_K[G] \cdot \theta$. (In this regard we observe that the assumption made in [5] that $[K : \mathbb{Q}_p]$ and $[K' : K]$ are coprime is not needed for the results obtained in §5 of loc. cit.)

Each character $\psi \in \widehat{G}$ is of the form $\chi\phi$ with an unramified character ϕ of $G_{N/M}$ and χ a character of $G_{N/K'}$ and from [5, Prop. 5.1.5] one has

$$\frac{\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta \mid \chi\phi)}{j_\ell(\tau(K, \chi\phi))} = \begin{cases} \mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi), & \text{if } \chi = \chi_0, \\ p^{-m} \cdot \chi(4) \cdot \mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) \cdot \phi(p^2), & \text{if } \chi \neq \chi_0, \end{cases}$$

where here and in the following we omit each occurrence of j_ℓ in our notation.

Now the proof of [5, Prop. 5.2.1] shows that $\tau(K,\chi\phi)=\tau(\mathbb{Q}_\ell, i_K^{\mathbb{Q}_\ell}(\chi\phi))$ and so (12) implies

$$\tau_K^G\cdot(\psi_{2,*}-1)(\tau'_{N/K})=(\sum_{\phi,\psi}e_{\chi\psi}\tau(K,\chi\phi))\cdot J_{2,N/K}\cdot(\psi_{2,*}-1)(y_{N/K}^{-1}).$$

It follows that for each χ and ϕ one has

$$\begin{aligned}\frac{\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta\mid\chi\phi)}{(\tau_K^G\cdot(\psi_{2,*}-1)(\tau'_{N/K}))_{\chi\phi}} &= \frac{\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta\mid\chi\phi)}{\tau(K,\chi\phi)\cdot\tau(K,\psi_2(\chi\phi)-2\chi\phi)\cdot y(K,\chi\phi-\psi_2(\chi\phi))}\\ &= \begin{cases}\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2\mid\phi)\frac{\tau(K,2\chi\phi-\psi_2(\chi\phi))}{y(K,\chi\phi-\psi_2(\chi\phi))}, & \text{if } \chi=\chi_0,\\ p^{-m}\cdot\chi(4)\cdot\phi(p^2)\cdot\mathcal{N}_{K/\mathbb{Q}_\ell)}(\theta_2\mid\phi)\frac{\tau(K,2\chi\phi-\psi_2(\chi\phi))}{y(K,\chi\phi-\psi_2(\chi\phi))}, & \text{if } \chi\neq\chi_0.\end{cases}\end{aligned}$$

These facts combine with the definition of $\mathfrak{c}_{N/K}$ to imply that (28) is valid if $\ker(j^t_{\ell,*}\circ\delta_{G,\ell})$ contains the element x_3 of $\zeta(\mathbb{Q}_\ell^c[G])^\times$ for which for all χ and ϕ one has

$$x_{3,\chi\phi}=\begin{cases}\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2\mid\phi)\cdot\tau(K,2\chi\phi-\psi_2(\chi\phi)), & \text{if } \chi=\chi_0,\\ p^{-m}\cdot\chi(4)\cdot\phi(p^2)\cdot\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2\mid\phi)\cdot\tau(K,2\chi\phi-\psi_2(\chi\phi)), & \text{if } \chi\neq\chi_0.\end{cases}$$

Now, as in the proof of [5, Th. 6.1, p. 1243], one can show that $\ker(j^t_{\ell,*}\circ\delta_{G,\ell})$ contains the element x'_3 of $\zeta(\mathbb{Q}_\ell^c[G])^\times$ for which at all χ and ϕ one has

$$x'_{3,\chi\phi}=\begin{cases}\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2\mid\phi), & \text{if } \chi=\chi_0,\\ \chi(4)\cdot\phi(p^2)\cdot\mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2\mid\phi), & \text{if } \chi\neq\chi_0.\end{cases}$$

In addition, [5, Lem. 5.1.2] implies that for all χ and ϕ one has $\tau(K,2\chi\phi-\psi_2(\chi\phi))=\tau(K,2\chi-\chi^2)$.

The required equality $(j^t_{\ell,*}\circ\delta_{G,\ell})(x_3)=0$ is thus true if and only if $(j^t_{\ell,*}\circ\delta_{G,\ell})(x_4)=0$ with x_4 the element of $\zeta(\mathbb{Q}_\ell^c[G])^\times$ for which at each χ and ϕ one has

$$x_{4,\chi\phi}=\begin{cases}1, & \text{if } \chi=\chi_0,\\ p^{-m}\tau(K,2\chi-\chi^2), & \text{if } \chi\neq\chi_0.\end{cases}$$

But, by the last displayed formula in the proof of [29, Prop. 3.9], for each non-trivial character χ one has

$$\tau(K,\chi)=p^m\cdot\chi(c_\chi^{-1})\cdot\psi_K(c_\chi^{-1})\ \text{ and }\ \tau(K,\chi^2)=p^m\cdot\chi^2((c_\chi/2)^{-1})\cdot\psi_K((c_\chi/2)^{-1}),$$

with ψ_K the standard additive character and c_χ as described in [29, Prop. 3.9].

It follows that $\tau(K,2\chi-\chi^2)=p^m\cdot\chi(4)^{-1}$ for non-trivial characters χ and hence that $x_{4,\chi\phi}=\chi(4)^{-1}$ for all χ and ϕ. Given this description, it is clear that $x_4\in\ker(j^t_{\ell,*}\circ\delta_{G,\ell})$, as required to complete the proof of (28) in this case. □

Remark 8.2. The numerical results discussed in §10.1.3 (see, in particular, (35)) will show that the equality $\mathfrak{a}_{F/E}=\mathfrak{c}_{F/E}$ is not always valid.

8.2. **Global results.** In this section we derive several consequences of Theorem 8.1, including a proof of Theorem 5.1.

8.2.1. We shall first give a proof of Theorem 5.1.

Following Proposition 5.4, for each prime ℓ we write $\mathfrak{a}_{L/K,\ell}$ for the image of $\mathfrak{a}_{L/K}$ in $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell[G])$.

Then Theorem 7.5 combines with the vanishing of $\mathfrak{a}_{F/E}$ for each tamely ramified extension F/E of local fields (as proved in Theorem 8.1) to reduce the proof of Theorem 5.1(i) to showing that for each ℓ for which there is an ℓ-adic place v in $\mathcal{W}_{L/K}$ the element $\mathfrak{a}_{L/K,\ell}$ belongs to $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell[G])_{\rm tor}$.

In view of the explicit description of $K_0(\mathbb{Z}_\ell[G],\mathbb{Q}_\ell[G])_{\rm tor}$ given by the first author in [11, Th. 4.1], it is thus enough to prove that for each such prime ℓ one has $\pi^H_{H/J}(\rho^G_H(\mathfrak{a}_{L/K,\ell}))=0$ for every cyclic subgroup H of G and every subgroup J of H with $|H/J|$ prime to ℓ.

Invoking the result of Theorem 6.1 it is thus enough to show that $\mathfrak{a}_{F/E,\ell}$ vanishes for all towers of number fields $K\subseteq E\subseteq F\subseteq L$ with L/E cyclic and the degree $[F:E]$ prime to ℓ. However, in any such case, all ℓ-adic places of E are tamely ramified in F/E and so Theorem 8.1 in conjunction with Theorem 7.5 (or (24)) implies $\mathfrak{a}_{F/E,\ell}$ vanishes, as required.

Claims (ii) and (iii) of Theorem 5.1 will follow from the same argument used to prove Corollary 4.4.

Finally we note that claim (iv) follows directly from the definition of $\mathfrak{a}_{L/K}$ and the facts that H_L is a free G-module and $\partial_{\mathbb{Z},\mathbb{Q},G}\circ\delta_G$ is the zero homomorphism.

This completes the proof of Theorem 5.1.

8.2.2. In order to describe a global consequence of Theorem 8.1 we define an 'idelic twisted unramified characteristic' by setting

$$\mathfrak{c}_{L/K}:=\sum_\ell\sum_{v|\ell} i^G_{G_w,\mathbb{Q}_\ell}(\mathfrak{c}_{L_w/K_v}). \tag{29}$$

If v is at most tamely ramified in L/K, then $\mathfrak{c}_{L_w/K_v}$ vanishes. This shows $\mathfrak{c}_{L/K}$ is a well-defined element in $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$ and that

$$\mathfrak{c}_{L/K,\ell}=\begin{cases}0, & \text{if } \ell\notin\mathcal{W}^{\mathbb{Q}}_{L/K},\\ \sum_{v|\ell} i^G_{G_w,\mathbb{Q}_\ell}(\mathfrak{c}_{L_w/K_v}), & \text{if } \ell\in\mathcal{W}^{\mathbb{Q}}_{L/K}.\end{cases}$$

In particular, by combining Theorems 7.5 and 8.1 one obtains the following result.

Corollary 8.3. *Let L/K be a weakly ramified odd degree Galois extensions of number fields. Then $\mathfrak{a}_{L/K}=\mathfrak{c}_{L/K}$ whenever all of the following conditions are satisfied at each v in $\mathcal{W}_{L/K}$:*

(i) *The decomposition subgroup of v is abelian;*
(ii) *The inertia subgroup of v is cyclic;*
(iii) *The extension $K_v/\mathbb{Q}_\ell$ is unramified, where $\ell=\ell(v)$ denotes the residue characteristic.*

This result immediately combines with Theorem 5.1(ii) and (iii) to give the following explicit consequence concerning the structures discussed in Examples 2.5 and 2.11.

Corollary 8.4. *Under the hypotheses of Corollary 8.3 one has*

$$[\mathcal{A}_{L/K},h_{L,\bullet}]=\Pi^{\rm met}_G(\mathfrak{c}_{L/K})+\varepsilon^{\rm met}_{L/K}\ \textit{and}\ {\rm Disc}(\mathcal{A}_{L/K},t_{L/K})=\Pi^{\rm herm}_G(\mathfrak{c}_{L/K})+\varepsilon^{\rm herm}_{L/K}.$$

It is therefore of interest to know when the classes $\Pi^{\rm met}_G(\mathfrak{c}_{L/K})$ and $\Pi^{\rm herm}_G(\mathfrak{c}_{L/K})$ vanish and the next result shows that this is often the case.

Lemma 8.5. *The images of $\mathfrak{c}_{L/K}$ in each of the groups* $\mathrm{Cl}(G), A(G)$ *and* $\mathrm{HCl}(G)$ *all vanish if for each $v \in \mathcal{W}_{L/K}$ one has either $I_w = G_w$ or I_w is of prime power order.*

Proof. We show that each of the individual terms in the definition of $\mathfrak{c}_{L/K}$ projects to zero. We fix v in $\mathcal{W}_{L/K}$ and set $\ell := \ell(v)$ and $\lambda_w := (1 - e_{I_w}) + \sigma_w e_{I_w}$. If $G_w = I_w$, then $\lambda_w = 1$. In the other case I_w is necessarily of ℓ-power order. Hence for any prime $p \neq \ell$ we have $\delta_{G_w}(\lambda_w) = 0$ in $K_0(\mathbb{Z}_p[G_w], \mathbb{Q}_p[G_w])$ since $\lambda_w \in \mathrm{Nrd}_{\mathbb{Q}_p[G_w]}(\mathbb{Z}_p[G_w]^\times)$. Therefore $\pi_{G,\ell}(i^G_{G_w}(\delta_{G_w}(\lambda_w)) = i^G_{G_w}(\delta_{G_w}(\lambda_w))$ in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.

We next show that $\mathfrak{c}_v := i^G_{G_w}(\delta_{G_w}(\lambda_w))$ belongs to both $\ker(\partial^{1,1}_G \circ h^{\mathrm{rel}}_G)$ and $\ker(\partial^{2,1}_G \circ h^{\mathrm{rel}}_G)$.

To do this we recall first that for $\alpha = (\alpha_\chi)_{\chi \in \widehat{G}}$ in $\zeta(\mathbb{Q}^c[G])^\times$ the element $h^{\mathrm{rel}}_G(\delta_G(\alpha))$ is represented by the function $\chi \mapsto (1, \alpha_\chi)$. Thus the global analogue of the commutative diagram (27) implies that $h^{\mathrm{rel}}_G(\mathfrak{c}_v)$ is represented by the pair $(1, \theta)$ with

$$\theta(\chi) = \prod_{\phi \in \widehat{G_w/I_w}} \phi(\sigma_w)^{\langle \mathrm{res}^G_{G_w}(\chi), \phi \rangle_{G_w}}.$$

The elements $\partial^{1,1}_G(h^{\mathrm{rel}}_G(\mathfrak{c}_v))$ and $\partial^{2,1}_G(h^{\mathrm{rel}}_G(\mathfrak{c}_v))$ are therefore represented by the pairs $(1, |\theta|)$ and $(1, \theta^s)$, respectively, and so it is enough to show that the maps $|\theta|$ and θ^s are both trivial.

Since $\theta(\chi)$ is a root of unity one has $|\theta|(\chi) = |\theta(\chi)| = 1$, and so $|\theta|$ is trivial.

In addition, the triviality of θ^s follows from the fact that if χ is a symplectic character of G, then both $\langle \mathrm{res}^G_{G_w}(\chi), \phi \rangle_{G_w} = \langle \mathrm{res}^G_{G_w}(\chi), \overline{\phi} \rangle_{G_w}$ and $\phi(\sigma_w)\overline{\phi}(\sigma_w) = 1$. □

Remark 8.6. In connection with Lemma 8.5 we note that if L_w/K_v is weakly ramified and abelian, then class field theory implies I_w is of prime-power order (as a consequence of [30, Cor. 2, p. 70]). In fact, at this stage we know of no example in which the projection of $\mathfrak{c}_{L/K}$ to any of the groups $\mathrm{Cl}(G), A(G)$ and $\mathrm{HCl}(G)$ does not vanish. It is, however, not difficult to show that the element $\mathfrak{c}_{L/K}$ itself does not always vanish. For example, if G is abelian, then $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ identifies with the group of invertible $\mathbb{Z}[G]$-sublattices of $\mathbb{Q}[G]$. In particular, if $L/\mathbb{Q}$ is an abelian p-extension in which, for any p-adic place w of L, one has $I_w \subsetneq G_w = G$, then $(1 - e_{I_w}) + \sigma_w e_{I_w}$ does not belong to $\mathbb{Z}[G]$ and so $\mathfrak{c}_{L/\mathbb{Q}} \neq 0$.

Remark 8.7. The element $\mathfrak{c}_{L/K}$ is in general different from, and better behaved than, the simpler variant $\delta_G((\psi_{2,*} - 1)(y_{L/K}))$. In particular, whilst it is straightforward to show that $\mathfrak{c}_{L/K}$ enjoys the same functorial properties under change of extension as those described in Theorem 6.1, the same is not true of $\delta_G((\psi_{2,*} - 1)(y_{L/K}))$.

9. Effective computations and Vinatier's Conjecture

In this section we first refine Corollary 5.3 by explaining how to make an effective computation of the set of realisable classes $R^{\mathrm{wr}}_K(\Gamma)$.

We then apply this observation to consider a conjecture of Vinatier in the setting of two natural infinite families of extensions which will then be investigated numerically in §10.

In §9.2.2 we consider the family of extensions of smallest degree for which Vinatier's Conjecture is not currently known to be valid and, whilst studying this case, we obtain evidence (described in Proposition 10.2) that $\mathfrak{a}_{L/K}$ may be controlled by the idelic twisted unramified characteristic $\mathfrak{c}_{L/K}$ in cases beyond those considered in Corollary 8.3.

Motivated by this last rather surprising observation, we consider in §9.2.3 a family of extensions of smallest possible degree for which the projection of $\mathfrak{c}_{L/K}$ to $\mathrm{Cl}(G(L/K))$ might not vanish, and hence that a close link between $\mathfrak{a}_{L/K}$ and $\mathfrak{c}_{L/K}$ need not be consistent with the validity of Vinatier's Conjecture.

In all of the cases that we compute, however, we find both that Vinatier's Conjecture is valid and the projection of $\mathfrak{c}_{L/K}$ to $\mathrm{Cl}(G(L/K))$ vanishes.

At the same time, our methods also give a new proof of the epsilon constant conjecture formulated in [2] for an infinite family of wildly ramified Galois extensions of number fields.

9.1. **The general result.** Recall that for each number field K and finite abstract group Γ of odd order we write $\mathrm{WR}_K(\Gamma)$ for the set of fields L that are weakly ramified odd degree Galois extensions of K and for which $G(L/K)$ is isomorphic to Γ.

Theorem 9.1. *Let K be a number field and Γ a finite abstract group whose order is both odd and coprime to the number of roots of unity in K.*

Then there exists a finite set $\mathrm{WR}^*_K(\Gamma)$ of Galois extensions E of K which have all of the following properties:

(i) *there exists an injective homomorphism of groups $i_E : G(E/K) \to \Gamma$;*
(ii) *there exists a unique place v of K that ramifies both wildly and weakly in E and for which there exists a unique place w of E above v;*
(iii) *all places of K other than v that divide $|\Gamma|$ are completely split in E/K;*
(iv) *for each L in $\mathrm{WR}_K(\Gamma)$ and every E in $\mathrm{WR}^*_K(\Gamma)$ there exists an integer $n_{L,E} \in \{0,1\}$ so that in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ one has*

$$i_{L,*}(\mathfrak{a}_{L/K}) = \sum_{E \in \mathrm{WR}^*_K(\Gamma)} n_{L,E} \cdot \mathrm{i}^{\Gamma}_{\mathrm{im}(i_E)}(i_{E,*}(\mathfrak{a}_{E/K})).$$

Proof. We recall first that for each place v of K the set $R_v(K,\Gamma)$ of isomorphism classes of Galois extensions E/K_v for which $G(E/K_v)$ is isomorphic to a subgroup of Γ is finite.

We next fix a weakly ramified Galois extension L/K for which the group $G := G(L/K)$ is isomorphic to the given group Γ. We recall that Theorems 7.5 and 8.1 combine to imply that there is a finite sum decomposition

$$\mathfrak{a}_{L/K} = \sum_{\ell \in \mathcal{W}^{\mathbb{Q}}_{L/K}} \sum_{v|\ell} \mathrm{i}^{G}_{G_w,\mathbb{Q}_\ell}(\mathfrak{a}_{L_w/K_v}) \tag{30}$$

For each place v in this sum the (weakly ramified) Galois extension L_w/K_v is isomorphic to one of the Galois extensions E/K_v in the finite set $R_v(K,\Gamma)$.

Further, since we are assuming $|\Gamma|$ is coprime to the number of roots of unity in K a result of Neukirch [26, Cor. 2, p. 156] implies that there exists a finite Galois extension $\tilde{E}/K$ with both of the following properties:

(P1) $\tilde{E}$ has a unique place $\tilde{w}$ above v and the completion $\tilde{E}_{\tilde{w}}/K_v$ is isomorphic to E/K_v (and hence to L_w/K_v);
(P2) if v' is any place of K which divides $|\Gamma|$, and $v' \neq v$, then v' is totally split in $\tilde{E}/K$.

These conditions imply that the global extension $\tilde{E}/K$ is weakly ramified and that the isomorphism of $\tilde{E}_w/K_v$ with L_w/K_v induces a natural identification

$$G(\tilde{E}/K) \cong G(L_w/K_v) \cong G_w. \tag{31}$$

In addition, since v is the only place of K that is not tamely ramified in $\tilde{E}/K$ the results of Theorem 7.5 and 8.1(i) combine to imply

$$\mathfrak{a}_{\tilde{E}/K} = \mathfrak{a}_{L_w/K_v}. \tag{32}$$

We now define $\mathrm{WR}_K^*(\Gamma)$ to be the finite set of extensions $\tilde{E}/K$ that are obtained from the above construction as v runs over the places of K that divide $|\Gamma|$. We note that this set satisfies the claimed property (i) as a consequence of the isomorphisms (31), it satisfies property (ii) and (iii) as a consequence of the properties (P1) and (P2) above and it satisfies property (iv) as a consequence of the equalities (30) and (32). □

Remark 9.2. The above argument also shows that $|\mathrm{WR}_K^*(\Gamma)| \leq \sum_{v||\Gamma|} \tilde{\nu}(K_v, \Gamma)$ where $\tilde{\nu}(K_v, \Gamma)$ denotes the number of non-isomorphic Galois extensions of K_v whose Galois group is isomorphic to a subgroup of Γ. In this context we recall that if Γ is a p-group then $\tilde{\nu}(K_v, \Gamma)$ is explicitly computed by work of Shafarevich [31] and Yamagishi [38]. We also recall that Pauli and Roblot [28] have developed an algorithm for the computation of all extensions of a p-adic field of a given degree. One can therefore use the results of [31] and [38] to design an algorithm to compute all p-extensions with a given p-group (see [28, § 10]).

9.2. **Applications to Vinatier's Conjecture.** Vinatier has conjectured that for any weakly ramified odd degree Galois extension L of $\mathbb{Q}$ the $G(L/\mathbb{Q})$-module $\mathcal{A}_{L/\mathbb{Q}}$ is free (see [37, §1, Conj.]) and we now apply our techniques to study this conjecture.

9.2.1. We first reformulate the conjecture in terms of the elements $\mathfrak{a}_{L/K}$ (global) and $\mathfrak{a}_{F/E}$ (local).

If F/E is a Galois extension of ℓ-adic fields, then we use the decomposition (2) to view $\mathfrak{a}_{F/E}$ as an element of $K_0(\mathbb{Z}[G(F/E)], \mathbb{Q}[G(F/E)])$.

Proposition 9.3. *The following are equivalent:*

(i) *For all odd degree weakly ramified Galois extensions L/K of number fields the $G(L/K)$-module $\mathcal{A}_{L/K}$ is free.*
(ii) *For all odd degree weakly ramified Galois extensions L/K of number fields the element $\mathfrak{a}_{L/K}$ projects to zero in $\mathrm{Cl}(G(L/K))$.*
(iii) *For all odd degree weakly ramified Galois extensions F/E of local fields the element $\mathfrak{a}_{F/E}$ projects to zero in $\mathrm{Cl}(G(F/E))$.*

Proof. The equivalence of (i) and (ii) is Lemma 9.4 below and (ii) follows directly from (iii) and Theorem 7.5.

We finally assume (ii) and for a local extension F/E we choose a number field K and a place v of K such that K_v is isomorphic to E and $|G(F/E)|$ is coprime to the number of roots of unity in K. (Since $G(F/E)$ is of odd order the existence of such a field K is easily implied by the main result of Henniart [25].)

Then by the construction in the proof of Theorem 9.1 we find a global extension $\tilde{E}/K$ with the properties (P1) and (P2).

It follows that $\mathfrak{a}_{\tilde{E}/K} = \mathfrak{a}_{F/E}$, and hence that $\mathfrak{a}_{F/E}$ projects to zero in $\mathrm{Cl}(G(F/E))$, as required to prove (iii). □

Lemma 9.4. *Let L/K be an odd degree weakly ramified Galois extension of number fields of group G. Then the G-module $\mathcal{A}_{L/K}$ is free if and only if the image of $\mathfrak{a}_{L/K}$ in $\mathrm{Cl}(G)$ vanishes.*

Proof. By Theorem 5.1(iv) one has $\partial_{\mathbb{Z},\mathbb{Q},G}(\mathfrak{a}_{L/K}) = [\mathcal{A}_{L/K}]$ in $\mathrm{Cl}(G)$. Given this, the equivalence of the stated conditions follows immediately from the fact that, as G has odd order, a finitely generated projective G-module is free if and only if its class in $\mathrm{Cl}(G)$ vanishes. □

9.2.2. By [35] Vinatier's conjecture is known to be true for extensions $L/\mathbb{Q}$ with the property that the decomposition group of each wildly ramified prime is abelian. The family of non-abelian Galois extensions of degree p^3, for some odd prime p, is thus the family of smallest possible degree for which Vinatier's conjecture is not known to be valid. Such extensions were considered (in special cases) by Vinatier in [36].

In the following result we study the number of corresponding such extensions of the base field $\mathbb{Q}_p$. This result (which will be proved at the end of this section) shows that the bounds on the number of such extensions that are discussed in Remark 9.2 can be improved if one imposes ramification conditions.

Proposition 9.5. *Let p be an odd prime. Then there exist exactly p (non-isomorphic) weakly ramified non-abelian Galois extensions of $\mathbb{Q}_p$ of degree p^3. Exactly one of these extensions has exponent p and the remaining $p-1$ extensions have exponent p^2.*

As in the proof of Theorem 9.1 we find for each odd prime p and each weakly ramified non-abelian Galois extension F of $\mathbb{Q}_p$ of degree p^3 a global weakly ramified Galois extension $N/\mathbb{Q}$ of degree p^3 such that $N_w/\mathbb{Q}_p \simeq F/\mathbb{Q}_p$.

Definition 9.6. For each odd prime p we fix a set $\mathcal{F}(p)$ of global weakly ramified Galois extensions $N/\mathbb{Q}$ of degree p^3 such that the localizations give the set of local extensions as in Proposition 9.5 (so $|\mathcal{F}(p)| = p$). Let K be a number field. For a finite set $P = \{p_1, \dots, p_n\}$ of odd primes p_i we define $\mathcal{L}(P)$ to be the set of extensions L/K such that $\mathcal{W}^{\mathbb{Q}}_{L/K} \subseteq P$ and such that for each $v \in \mathcal{W}_{L/K}$ the following conditions hold:

(i) Set $p := \ell(v)$. Then the decomposition group G_w is a p-group of order at most p^3
(ii) $K_v = \mathbb{Q}_p$.

Theorem 9.7. *Let K be a number field and let $P = \{p_1, \dots, p_n\}$ be a finite set of odd primes. Then the following are equivalent:*

(i) *For all extensions L/K in $\mathcal{L}(P)$ the G-module $\mathcal{A}_{L/K}$ is free.*
(ii) *For all extensions $N/\mathbb{Q}$ in in the finite set $\cup_{i=1}^n \mathcal{F}(p_i)$ the G-module $\mathcal{A}_{N/K}$ is free.*

Proof. Obviously (i) implies (ii). For the other implication fix an extensions L/K in $\mathcal{L}(P)$. By Lemma 9.4 we have to show that the element $\mathfrak{a}_{L/K}$ projects to zero in $\mathrm{Cl}(G)$. By Theorem 7.5 together with Proposition 8.1(i) we have

$$\mathfrak{a}_{L/K} = \sum_{v \in \mathcal{W}_{L/K}} i^G_{G_w, \mathbb{Q}^c_{\ell(v)}}(\mathfrak{a}_{L_w/K_v}).$$

It is therefore enough to show that each of the terms $\mathfrak{a}_{L_w/K_v}$ projects to zero in $\mathrm{Cl}(\mathbb{Z}[G_w])$. By our assumptions $K_v = \mathbb{Q}_p$ for a prime $p \in P$ and $G(L_w/\mathbb{Q}_p)$ is a p-group of order at most p^3. If $|G(L_w/\mathbb{Q}_p)| \le p^2$ then $L_w/\mathbb{Q}_p$ is abelian and $\mathfrak{a}_{L_w/\mathbb{Q}_p} = 0$ by the relevant case of Theorem 8.1. If $|G_{L_w/\mathbb{Q}_p}| = p^3$ then by the definition of $\mathcal{L}(P)$ the local extension $L_w/\mathbb{Q}_p$ is the localisation of one of the extensions $N/\mathbb{Q}$ in $\mathcal{F}(p)$, so that we have $\mathfrak{a}_{N/\mathbb{Q}} = \mathfrak{a}_{L_w/\mathbb{Q}_p}$. The claim now follows from Lemma 9.4. □

In the rest of this section we give the postponed proof of Proposition 9.5.

As a first step we recall that there are two isomorphism classes of non-abelian groups of order p^3, with respective presentations

$$\left\{ \begin{array}{l} \langle a, b \mid a^{p^2} = 1 = b^p, b^{-1}ab = a^{1+p}\rangle, \\ \langle a, b, c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb\rangle, \end{array} \right. \tag{33}$$

the first having exponent p^2 and the second exponent p (see, for example, [24, §4.4]). In both cases the centre $Z(G)$ of the group G has order p (being generated by a^p and c respectively) and the quotient group $G/Z(G)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Any weakly ramified non-abelian Galois extension L of $\mathbb{Q}_p$ of degree p^3 must thus contain a subfield E that is Galois over $\mathbb{Q}_p$ and such that both $G(L/E)$ is central in $G(L/\mathbb{Q}_p)$ and $G(E/\mathbb{Q}_p)$ is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Since $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p$ has order p^2 local class field theory implies E is the compositum of the unique subextension E_1 of $\mathbb{Q}_p(\zeta_{p^2})$ of degree p over $\mathbb{Q}_p$ and of the unique unramified extension E_2 of $\mathbb{Q}_p$ of degree p (and hence is weakly ramified, as required). In the sequel we set $G := G(L/\mathbb{Q}_p), H := G(L/E), \Gamma := G(E/\mathbb{Q}_p)$ and $\Delta := G(E/E_1)$.

If L/E is a weakly ramified degree p extension such that $L/\mathbb{Q}_p$ is Galois, then $L/\mathbb{Q}_p$ is weakly ramified. Indeed, $G_2 \cap H = H_2 = 1$ and hence $G_2 \simeq G_2H/H$. By Herbrand's theorem we obtain $G_2H/H = (G/H)_2$, which is trivial since $E/\mathbb{Q}_p$ is weakly ramified. The required fields therefore correspond to weakly ramified degree p extensions L of E which are Galois over $\mathbb{Q}_p$.

For each subfield F of E we write $\mathfrak{p}_F$ for the maximal ideal of the valuation ring $\mathcal{O}_F$ of F, $U_F^{(i)}$ for each natural number i for the group $1 + \mathfrak{p}_F^i$ of i-th principal units of F and μ'_F for the maximal finite subgroup of $F^\times$ of order prime to p. If L/F is abelian we also write $\mathrm{rec}_{L/F}$ for the reciprocity map $F^\times \to G_{L/F}$.

If L/E is unramified, then the ramification degree of $L/\mathbb{Q}_p$ is p so that L contains both E_1 and the unramified extension of $\mathbb{Q}_p$ of degree p^2 and so L is abelian over $\mathbb{Q}_p$.

On the other hand, if L/E is ramified, then the inertia subgroup G_0 has order p^2. In addition, since $L/\mathbb{Q}_p$ is assumed to be weakly ramified, the group $G_0 = G_1$ identifies with G_1/G_2 and so is isomorphic to a subgroup of $U^1_{\mathbb{Q}_p}/U^2_{\mathbb{Q}_p}$ and therefore has exponent dividing p. It follows that G_0 is not cyclic and hence that $L/\mathbb{Q}_p$ is not abelian. We have therefore shown that $L/\mathbb{Q}_p$ is abelian if and only if L/E is unramified.

In summary, there is thus a field diagram of the following sort.

$$\begin{tikzcd}
& L \arrow[d, dash, "p", "\text{totally weakly ramified}"'] & \\
& E \arrow[dl, dash, "\text{unramified}"', "p"] \arrow[dr, dash, "p"', "\text{totally weakly ramified}"] & \\
E_1 \arrow[dr, dash, "p", "\text{totally weakly ramified}"'] & & E_2 \arrow[dl, dash, "p"', "\text{unramified}"] \\
& \mathbb{Q}_p &
\end{tikzcd}$$

By an easy exercise one checks that L/E is weakly ramified if and only if the upper ramification subgroup H^2 vanishes. By local class field theory, the desired extensions L are therefore in bijective correspondence with subgroups N of $E^\times$ that are Γ-stable (as $L/\mathbb{Q}_p$ is Galois), contain $U_E^{(2)}$ (by [30, Cor. 3 p. 228]), contain $E^{\times p}$ (as $E^\times/N$ has exponent p) and contain $I_\Gamma(E^\times)$ (as Γ acts trivial on $E^\times/N \simeq Z(G)$), where I_Γ denotes the augmentation ideal of $\mathbb{Z}[\Gamma]$.

We note next that there are isomorphisms of abelian groups

$$(U_E^{(1)}/U_E^{(2)})_\Gamma \cong \left(\left(\mathfrak{p}_E/\mathfrak{p}_E^2\right)_\Delta\right)_{\Gamma/\Delta} \cong \left(\mathfrak{p}_{E_1}/\mathfrak{p}_{E_1}^2\right)_{\Gamma/\Delta} \cong (\mathbb{Z}/p\mathbb{Z})_{\Gamma/\Delta} = \mathbb{Z}/p\mathbb{Z} \tag{34}$$

where the first map is induced by the natural isomorphism $U_E^{(1)}/U_E^{(2)} \cong \mathfrak{p}_E/\mathfrak{p}_E^2$. The second isomorphism is induced by the field-theoretic trace Tr_{E/E_1}. Indeed, since E/E_1 is unramified, the induced map $\left(\mathfrak{p}_E/\mathfrak{p}_E^2\right)_\Delta \cong \mathfrak{p}_{E_1}/\mathfrak{p}_{E_1}^2$ is surjective with kernel $\hat{H}^{-1}(\Delta, \mathfrak{p}_E/\mathfrak{p}_E^2)$, which is trivial since $\mathfrak{p}_E^i$ is $\mathbb{Z}_p[\Delta]$-free for each non-negative integer i. The third is induced by the fact that $\mathfrak{p}_{E_1}/\mathfrak{p}_{E_1}^2 \cong \mathcal{O}_{E_1}/\mathfrak{p}_{E_1}$ has order p (since $E_1/\mathbb{Q}_p$ is totally ramified).

To be explicit we fix a uniformizing parameter π of E_1 and recall that $E^\times = \langle\pi\rangle \times \mu_E' \times U_E^{(1)}$. Any $\gamma \in \Gamma$ can be written in the form $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in G(E/E_1), \gamma_2 \in G(E/E_2)$. The wild inertia group Γ_1 is equal to $G(E/E_2)$ and hence we obtain $\pi^{\gamma-1} = \pi^{\gamma_2-1} \in U_{E_1}^{(1)} \subseteq U_E^{(1)}$. In addition, by (34) and the fact that Tr_{E/E_1} acts as multiplication by p on $\mathfrak{p}_{E_1}$, we see that $\pi^{\gamma-1}$ has trivial image in $(U_E^{(1)}/U_E^{(2)})_\Gamma$.

We set

$$T := \langle \left(E^\times\right)^p, U_E^{(2)}, I_\Gamma(E^\times)\rangle = \langle \left(E^\times\right)^p, U_E^{(2)}, I_\Gamma(U_E^{(1)})\rangle$$

and note that the map

$$E^\times \to \langle\pi\rangle/\langle\pi^p\rangle \times (U_E^{(1)}/U_E^{(2)})_\Gamma, \quad \pi^a\epsilon y \mapsto (\pi^a \mod \langle\pi^p\rangle, yU_E^{(2)} \mod I_\Gamma(U_E^{(1)}/U_E^{(2)})),$$

where $a \in \mathbb{Z}$, $\epsilon \in \mu_E'$ and $y \in U_E^{(1)}$, induces an isomorphism of the quotient group $Q := E^\times/T$ with the direct product $\langle\pi\rangle/\langle\pi^p\rangle \times (U_E^{(1)}/U_E^{(2)})_\Gamma \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

In particular, if we fix an element u of $U_E^{(1)}$ that generates $(U_E^{(1)}/U_E^{(2)})_\Gamma$, then the order p subgroups of Q correspond to the subgroups generated by the classes of the elements u and $\pi \cdot u^i$ for $i \in \{0, 1, 2, \ldots, p-1\}$.

In addition, since L/E is ramified if and only if N does not contain u, the quotients that we require correspond to the subgroups $Q_i := \langle \pi \cdot u^i \mod T\rangle$ for $i \in \{0, 1, 2, \ldots, p-1\}$. The corresponding subgroups N_i of $E^\times$ are given by $N_i := \langle \pi u^i, T\rangle$ and we write L_i for the fields that correspond to N_i via local class field theory.

If $i \neq 0$, then Q_i does not contain the class of π so $G(L_i/E)$ is generated by $\mathrm{rec}_{L_i/E}(\pi) = \mathrm{rec}_{L_i/E_1}(\mathrm{N}_{E/E_1}(\pi)) = \mathrm{rec}_{L_i/E_1}(\pi)^p$ and hence $G(L_i/E_1)$ is cyclic of order p^2 (and so $G(L_i/\mathbb{Q}_p)$ has exponent p^2).

Finally we claim that $G(L_0/\mathbb{Q}_p)$ has exponent p. To prove this it is enough, in view of the possible presentations (33), to show $G(L_0/\mathbb{Q}_p)$ contains two non-cyclic subgroups of order p^2. Hence, since its inertia subgroup $G(L_0/E_2)$ is one such subgroup (as $L_0/\mathbb{Q}_p$ is weakly ramified), it is enough to prove $G(L_0/E_1)$ also has exponent p.

To do this we note $G(L_0/E)$ is generated by $\mathrm{rec}_{L_0/E}(u) = \mathrm{rec}_{L_0/E_1}(\mathrm{N}_{E/E_1}(u))$ and so it suffices to show $\mathrm{N}_{E/E_1}(u)$ is not contained in the subgroup $N_{L_0/E_1}(L_0^\times)$ of $E_1^\times$. But the above argument shows that the class of $\mathrm{N}_{E/E_1}(u)$ generates $U_{E_1}^{(1)}/U_{E_1}^{(2)}$ whilst one has

$$\begin{aligned} N_{L_0/E_1}(L_0^\times) &= N_{E/E_1}(\langle\langle\pi\rangle, \mu'_E, U_E^{(2)}, I_\Gamma(U_E^{(1)})\rangle) \\ &= \langle\langle\pi^p\rangle, \mu'_{E_1}, U_{E_1}^{(2)}, I_{\Gamma/\Delta}(U_{E_1}^{(1)})\rangle \subseteq \langle\langle\pi^p\rangle, \mu'_{E_1}, U_{E_1}^{(2)}\rangle \end{aligned}$$

and so $N_{L_0/E_1}(L_0^\times)$ cannot contain an element of $U_{E_1}^{(1)}$ that projects to a generator of $U_{E_1}^{(1)}/U_{E_1}^{(2)}$.

This completes the proof of Proposition 9.5.

9.2.3. Following Lemma 8.5 and Remark 8.6, the weakly ramified Galois extensions $L/\mathbb{Q}$ of smallest degree for which the projection of $\mathfrak{c}_{L/\mathbb{Q}}$ to $\mathrm{Cl}(G(L/\mathbb{Q}))$ might not vanish are non-abelian and of degree $\ell^2 p$ for an odd prime p and an odd prime ℓ that divides $p-1$. This motivates us to investigate such extensions numerically (in §10.2) and the next result lays the groundwork for such investigations by determining a family of local extensions that satisfies the required conditions.

Proposition 9.8. *Let ℓ and p be odd primes with ℓ dividing $p-1$. Then there exist exactly ℓ (non-isomorphic) weakly ramified non-abelian Galois extensions L of $\mathbb{Q}_p$ of degree $\ell^2 p$ with $G(E/\mathbb{Q}_p) \cong \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$, where $E := L^C$ and C is the Sylow-p-subgroup of $G(L/\mathbb{Q}_p)$.*

Proof. Let $L/\mathbb{Q}_p$ be an extension with the stated conditions and set $G := G(L/\mathbb{Q}_p)$. Then G has exactly one Sylow-p-subgroup which we label C. We set $E := L^C$ and note that $E/\mathbb{Q}_p$ is abelian and that $G(E/\mathbb{Q}_p) \simeq \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$.

As ℓ divides $p-1$ the ℓ-th roots of unity are contained in $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell \simeq \mathbb{Z}/\ell\mathbb{Z}\times\mathbb{Z}/\ell\mathbb{Z}$, so that E is the maximal abelian extension of $\mathbb{Q}_p$ of exponent ℓ. Explicitly, $E = E_1E_2$ where E_1 is the unramified extension of degree ℓ and $E_2 := \mathbb{Q}_p(\sqrt[\ell]{p})$. By local class field theory L corresponds to a subgroup X of $E^\times$ such that X is stable under the action of $\Gamma := G/C$, $|E^\times/X| = p$ and $U_E^{(2)} \subseteq X$ (by [30, Cor. 3 p. 228]).

Let H be a subgroup of Γ such that $|H| = \ell$. Since H is cyclic the extension L/E^H is abelian if and only if H acts trivially on $E^\times/X$. As a consequence $\Delta := G(E/E_1)$ acts non trivially on $E^\times/X$ since otherwise $G(L/E_1) = G_0(L/\mathbb{Q}_p)$ would be abelian and by [30, Cor. 2, p. 70] this contradicts $G_2(L/E_1) = 1$.

Since $p \nmid |\Gamma|$ the $\mathbb{F}_p[\Gamma]$-module $E^\times/X$ decomposes as $E^\times/X = \bigoplus_\phi e_\phi(E^\times/X)$ where ϕ runs over the $\mathbb{F}_p$-valued abelian characters of Γ and e_ϕ denotes the usual idempotent in $\mathbb{F}_p[\Gamma]$. In addition, since $|E^\times/X| = p$, exactly one of the components, for $\phi = \phi_0$ say, is non-trivial.

Since $H_0 := \ker(\phi_0)$ acts trivially on $e_{\phi_0}(E^\times/X)$ we deduce that $H_0 \neq \Delta$. In addition, for any subgroup H of Γ one has $T_H(E^\times/X) = (E^\times/X)^H$ and so, since $(E^\times/X)^\Gamma \subseteq (E^\times/X)^\Delta = 0$, we deduce $H_0 \neq \Gamma$.

We claim that X contains $\langle \mu'_E, \sqrt[\ell]{p}, U_E^{(2)}, I_{H_0}(U_E^{(1)})\rangle$. To see this note $\mu'_E \subseteq X$ as $(\mu'_E)^p = \mu'_E$. Since $T_\Delta(E^\times/X) = 0$ we obtain $T_\Delta(\sqrt[\ell]{p}) = N_{E_2/\mathbb{Q}_p}(\sqrt[\ell]{p}) = p = (\sqrt[\ell]{p})^\ell \in X$. As $\ell \neq p$ it follows that $\sqrt[\ell]{p} \in X$. Finally, as L/E^{H_0} is abelian, X must contain $I_{H_0}(E^\times)$, and hence also $I_{H_0}(U_E^{(1)})$, as required.

We will show below that for any subgroup H of Γ with $|H| = \ell$ and $H \neq \Delta$ the subgroup

$$X(H) := \langle \mu'_E, \sqrt[\ell]{p}, U_E^{(2)}, I_H(U_E^{(1)})\rangle$$

is both stable under Γ and satisfies $|E^\times/X(H)| = p$.

This will show, in particular, that $X = X(H_0)$. Conversely, since each subgroup $X(H)$ corresponds by local class field theory (and [30, p. 70, Cor. 2]) to a weakly ramified extension $L/\mathbb{Q}_p$ as in the proposition, we will also have proved that the extensions L in the proposition correspond uniquely to the subgroups H of Γ with $|H| = \ell$ and $H \neq \Delta$.

It thus remains to show that for each subgroup H as above the subgroup $X(H)$ is stable under Γ and such that $|E^\times/X(H)| = p$.

Since $\gamma(\sqrt[\ell]{p}) \equiv \sqrt[\ell]{p} \pmod{\mu'_E}$ for all $\gamma \in \Gamma$ it is immediate that $X(H)$ is Γ-stable. The extension E/E^H is unramified and therefore

$$(U_E^{(1)}/U_E^{(2)})^H \simeq (U_E^{(1)}/U_E^{(2)})_H \simeq (\mathfrak{p}_E/\mathfrak{p}_E^2)_H \simeq \mathfrak{p}_{E^H}/\mathfrak{p}_{E^H}^2 \simeq \mathbb{Z}/p\mathbb{Z},$$

where the first isomorphism holds since each $U_E^{(n)}$ is H-cohomologically trivial, the second is canonical and the third is induced by the trace map tr_{E/E^H}. On the other hand, $(U_E^{(1)}/U_E^{(2)})_H = U_E^{(1)}/I_H(U_E^{(1)})U_E^{(2)}$ and so the decomposition $E^\times = \langle\sqrt[\ell]{p}\rangle \times \mu'_E \times U_E^{(1)}$ implies that the quotient $E^\times/X(H) \simeq U_E^{(1)}/I_H(U_E^{(1)})U_E^{(1)}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, as required. □

Remark 9.9. Assume the situation of Proposition 9.8. Then the extension $E/\mathbb{Q}_p$ has ℓ subextensions $F_1, \dots, F_\ell$ (corresponding to the subgroups H of Γ with $|H| = \ell$ and $H \neq \Delta$) that are ramified over $\mathbb{Q}_p$. For each such F_i there exists precisely one extension $L/\mathbb{Q}_p$ that satisfies the assumptions of Proposition 9.8 and is also such that L/F_i is abelian.

10. Numerical examples

In this section we investigate numerically, and thereby prove, Vinatier's conjecture for two infinite families of non-abelian weakly ramified Galois extensions of $\mathbb{Q}$.

At the same time we shall also explicitly compute both sides of the equality in Conjecture 7.2 for all weakly ramified non-abelian Galois extensions of $\mathbb{Q}_3$ of degree 27 and exponent 3, thereby verifying this conjecture, and hence also Breuning's local epsilon constant conjecture, in this case.

10.1. **Extensions of degree** 27.

10.1.1. We first compute explicitly a set $\mathcal{F}(3)$ as in Definition 9.6. To do this we have to find 3 weakly ramified Galois extensions L of $\mathbb{Q}$ of degree 27 with a unique 3-adic place w and such that $L_w/\mathbb{Q}_p$ runs over all extensions as in Proposition 9.5.

In the following p denotes 3. We shall also only consider Galois extensions $F/\mathbb{Q}$ that have a unique place w above p and so we write F_p in place of F_w.

We let E_1 be the subextension of $\mathbb{Q}(\zeta_{p^2})$ of degree p and E_2 an abelian extension of degree p such that p is inert in E_2. We set $E := E_1E_2$ and let $\mathfrak{p}$ denote the unique prime ideal of $\mathcal{O}_E$ above p. We write Γ for the Galois group of $E/\mathbb{Q}$.

Set $Q_2 := \{\alpha \in (\mathcal{O}_E/\mathfrak{p}^2)^\times \mid \alpha \equiv 1 \pmod{\mathfrak{p}}\}$ and note that $(Q_2)_\Gamma \simeq (U_{E_p}^{(1)}/U_{E_p}^{(2)})_\Gamma \simeq \mathbb{Z}/p\mathbb{Z}_p$. Let $u \in \mathcal{O}_E$ be such that the class of u generates $(Q_2)_\Gamma$ and let $\pi \in \mathcal{O}_E$ be a uniformizing element for $\mathfrak{p}$.

By algorithmic global class field theory we compute ray class groups $\mathrm{cl}(q\mathfrak{p}^2)$ of conductor $q\mathfrak{p}^2$ for small positive integers q with $(q, p) = 1$ and search for subgroups $U \leq \mathrm{cl}(q\mathfrak{p}^2)$ of index p which are invariant under Γ and such that the corresponding extension L/E is ramified at $\mathfrak{p}$. Each such U corresponds to a Galois extension $L/\mathbb{Q}$ whose completion at p is one of the extensions of Proposition 9.5. As shown in the proof of Proposition 9.5 the local extensions

$L_p/\mathbb{Q}_p$ are in one-to-one correspondence with the elements πu^b for $b \in \{0, 1, 2, \dots, p-1\}$. More precisely, there is exactly one b such that $\mathrm{rec}_{L_p/\mathbb{Q}_p}(\pi u^b) = 1$. Thus we have to find extensions $L/\mathbb{Q}$ such that the resulting integers b range from 0 to $p-1$. In order to compute $\mathrm{rec}_{L_p/\mathbb{Q}_p}(\pi u^b)$ we compute $\xi \in \mathcal{O}_E$ such that $\xi \equiv \pi \pmod{q}$ and $\xi \equiv u^{-b} \pmod{\mathfrak{p}^2}$. Then class field theory shows that $\mathrm{rec}_{L_p/\mathbb{Q}_p}(\pi u^b) = \mathrm{rec}_{L/\mathbb{Q}}(\xi\mathcal{O}_E)$ which can be computed globally.

This approach is implemented in MAGMA. For E_2 we used the cubic subextension of $\mathbb{Q}(\zeta_{19})$ and found a set of 3 global extensions $L/\mathbb{Q}$ by taking $q \in \{4, 5, \dots, 30\}$. The minimal polynomials of the extensions are recorded in the file WeaklyRamifiedFields.data which can be downloaded from the first author's homepage.

10.1.2. Using results of [6] one can explicitly compute $\mathrm{Cl}(G)$ as an abstract group for each finite group G. In particular, for the two non-abelian groups of order 27 one finds in this way that $\mathrm{Cl}(G)$ is cyclic of order 9.

For each of the extensions $L/\mathbb{Q}$ computed in the last section we can use the algorithm described in [7, § 5] to compute the logarithm of $[\mathcal{A}_{L/\mathbb{Q}}]$ in $\mathrm{Cl}(G)$ with $G := G(L/\mathbb{Q})$. Since G is of odd order, $\mathcal{A}_{L/\mathbb{Q}}$ is a free G-module if and only if $[\mathcal{A}_{L/\mathbb{Q}}]$ is trivial.

In a little more detail, we first compute a normal basis element $\theta \in \mathcal{O}_L$ and the G-module $\mathcal{A}_\theta \subseteq \mathbb{Q}[G]$ such that $\mathcal{A}_\theta\theta = \mathcal{A}_{L/\mathbb{Q}}$. Then $\mathcal{A}_\theta \simeq \mathcal{A}_{L/\mathbb{Q}}$ and the element $[\mathcal{A}_\theta, \mathrm{id}, \mathbb{Z}[G]] \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ projects to the class of $\mathcal{A}_{L/\mathbb{Q}}$ in $\mathrm{Cl}(G)$. The algorithm in [7] now solves the discrete logarithm problem for $[\mathcal{A}_{\theta,\ell}, \mathrm{id}, \mathbb{Z}_\ell[G]]$ in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ for each of the primes ℓ dividing the generalized index $[\mathcal{A}_\theta : \mathbb{Z}[G]]$ and then uses the recipe described in [7, § 5] to compute the logarithm of $[\mathcal{A}_{L/\mathbb{Q}}]$ in $\mathrm{Cl}(G)$.

However, for an arbitrary choice of θ the algorithm will in general fail because of efficiency problems since this set of primes ℓ is often too large and contains primes ℓ which are much too big. We therefore first compute a maximal order $\mathcal{M}$ in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$ and an element $\delta \in \mathbb{Q}[G]$ such that $\mathcal{M}\mathcal{A}_\theta = \mathcal{M}\delta$. This is achieved by the method described in Step (1) to (5) of Algorithm 3.1 in [8]. We then set $\theta' := \delta(\theta)$ and start over again by computing $\mathcal{A}_{\theta'}$ such that $\mathcal{A}_{\theta'} \cdot \theta' = \mathcal{A}_{L/\mathbb{Q}}$. Then one has $\mathcal{M} \cdot \theta' = \mathcal{M}\mathcal{A}_\theta \cdot \theta = \mathcal{M}\mathcal{A}_{L/\mathbb{Q}} = \mathcal{M}\mathcal{A}_{\theta'} \cdot \theta'$.

Localizing at prime divisors ℓ of G we obtain $\mathbb{Z}_{(\ell)}[G] \cdot \theta' = \mathcal{A}_{\theta',(\ell)} \cdot \theta'$ and hence $\mathbb{Z}_{(\ell)}[G] = \mathcal{A}_{\theta',(\ell)}$. It follows that we only need to solve the discrete logarithm problem for $[\mathcal{A}_{\theta',\ell}, \mathrm{id}, \mathbb{Z}_\ell[G]]$ in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ for primes ℓ dividing $|G|$.

The computations show that for each of the 3 extensions computed in the previous section the G-module $\mathcal{A}_{L/\mathbb{Q}}$ is free. As a consequence of these computations and Theorem 9.7 we obtain the following result.

Theorem 10.1. *For all extensions in $\mathcal{L}(3)$ the G-module $\mathcal{A}_{L/K}$ is free. In particular, Vinatier's conjecture holds for all non-abelian extensions $L/\mathbb{Q}$ of degree 27.*

10.1.3. We now show how to compute $\mathfrak{a}_{L/\mathbb{Q}}$ for the extension $L/\mathbb{Q}$ in $\mathcal{F}(3)$ of exponent 3. By Theorems 7.5 and 8.1 we have $\mathfrak{a}_{L/\mathbb{Q}} = \mathfrak{a}_{L_p/\mathbb{Q}_p}$ and both $\mathfrak{a}_{L_p/\mathbb{Q}_p}$ and the right hand side of the equality in Conjecture 7.2 can be computed by adapting the methods of [4]. In the following we indicate where special care has to be taken to improve the performance of the general implementation used to obtain the results of [4]

For the computation of $[\mathcal{A}_{L/\mathbb{Q}}, \kappa_L, H_L]$ we choose a normal basis element θ and write

$$[\mathcal{A}_{L/\mathbb{Q}}, \kappa_L, H_L] = [\mathbb{Z}[G] \cdot \theta, \kappa_L, H_L] + [p\mathcal{A}_{L/\mathbb{Q}}, \mathrm{id}, \mathbb{Z}[G] \cdot \theta] + \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(p)).$$

For computational reasons we proceed as in §10.1.2 and choose θ such that $\mathcal{M}\cdot\theta = \mathcal{M}\mathcal{A}_{L/\mathbb{Q}}$. The second and the third term are straightforward to compute and the first term is given by norm resolvents (see, for example, [4, (13)]).

For the computation of $\delta_{G,p}(j_p((\psi_{2,*}-1)(\tau'_{L_p/\mathbb{Q}_p}))$ we first digress to describe the character theory of non-abelian groups of order p^3.

The centre $Z = Z(G)$ of any such group G is equal to the commutator subgroup of G and the quotient G/Z is isomorphic to $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ so that G has p^2 linear characters of order dividing p.

It is also easy to see that G has normal subgroups A that are isomorphic to $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$ and contain Z. We fix such a subgroup A and for each non-trivial character λ of Z we choose a character $\psi\in\widehat{A}$ which restricts to give λ on Z. Then it can be shown that $\mathrm{ind}_A^G(\psi)$ depends only on λ and does not depend on the choice of ψ. In addition, it is an irreducible character of G of degree p. Since $(p-1)p^2+p^2=p^3$ we have found all irreducible characters of G.

In particular, for $p=3$ the characters of G comprise the trivial character, 8 linear characters of order 3 and two characters of degree 3.

Returning now to the computation of local Galois Gauss sums we essentially proceed as described in [3, § 2.5] but for reasons of efficiency must take care in the 'Brauer induction step' of loc. cit.

The computation of $\tau(\mathbb{Q}_p,\chi)$ for abelian characters is clear. Let now $\chi = \mathrm{ind}_A^G(\psi)$ be one of the characters of degree p. We set $M := L^A$ and $N := L^{\ker(\psi)}$ and use

$$\begin{aligned}\tau(\mathbb{Q}_p,\chi) = \tau(\mathbb{Q}_p,\mathrm{ind}_A^G(\psi-1_A))\tau(\mathbb{Q}_p,\mathrm{ind}_A^G(1_A)) &= \tau(M_p,\psi-1_A)\tau(\mathbb{Q}_p,\mathrm{ind}_A^G(1_A))\\ &= \tau(M_p,\psi)\prod_{\substack{\varphi\in\widehat{G}\\ \varphi|_A=1_A}}\tau(\mathbb{Q}_p,\varphi).\end{aligned}$$

The problematic part is the computation of $\tau(M_p,\psi)$. To explain why we write $\mathfrak{f}(\psi)$ for the conductor of ψ and choose $c\in M_p$ such that $\mathfrak{f}(\psi)\mathfrak{D}_{M_p/\mathbb{Q}_p} = c\mathcal{O}_{M_p}$. Then

$$\tau(M_p,\psi) = \sum_x \psi\left(\mathrm{rec}_{N_p/M_p}(x/c)\right)\psi_{\mathrm{add}}\left(x/c\right)$$

where ψ_{add} denotes the standard additive character and x runs over a set of representatives of $\mathcal{O}_{M_p}^\times$ modulo $U_{M_p}^{(2)}$. From an algorithmic point of view it is now important to choose the subgroup A such that $L^A/\mathbb{Q}$ is totally ramified (e.g. we may take $A = G(L/E_1)$) because then $\mathcal{O}_{M_p}^\times/U_{M_p}^{(2)}$ has order 6 as compared to order 702 if $M/\mathbb{Q}$ were the unique unramified subextension of $L/\mathbb{Q}$.

From the explicit description of the unramified characteristic in (8) it is now easy to compute $\tau'(\mathbb{Q}_p,\chi) = \tau(\mathbb{Q}_p,\chi)y(\mathbb{Q}_p,\chi)^{-1}$ for all $\chi\in\widehat{G}$ and based on this it is straightforward by the methods of [7] to compute the term $\delta_{G,p}(j_p((\psi_{2,*}(\tau'_{L_p/\mathbb{Q}_p}))))$.

Our computations show that for the extension $L/\mathbb{Q}$ in $\mathcal{F}(3)$ of exponent 3 we have

$$\mathfrak{a}_{L_p/\mathbb{Q}_p} = -\mathfrak{c}_{L_p/\mathbb{Q}_p} \neq \mathfrak{c}_{L_p/\mathbb{Q}_p} \tag{35}$$

with the twisted unramified characteristic defined in (23). Combining this with Theorems 7.5 and 8.1 we obtain the following result.

Proposition 10.2. *If L/K belongs to $\mathcal{L}(3)$ and is such that G_w is of order 27 and exponent 3 for each wildly ramified place w of L, then $\mathfrak{a}_{L/K} = -\mathfrak{c}_{L/K}$ where $\mathfrak{c}_{L/K}$ is as defined in (29).*

Remark 10.3. The equality $\mathfrak{a}_{L/K} = -\mathfrak{c}_{L/K}$ in Proposition 10.2 combines with the results of Theorems 5.1(iv) and 10.1 to imply that the image of $\mathfrak{c}_{L/K}$ in $\mathrm{Cl}(G)$ vanishes. Under the stated conditions, this fact also follows directly from Lemma 8.5.

Remark 10.4. Proposition 10.2 combines with Corollary 8.3 to give a rather striking fact: for *every* extension of number fields L/K for which we have been able to compute $\mathfrak{a}_{L/K}$ explicitly one has $\mathfrak{a}_{L/K} \in \{\mathfrak{c}_{L/K}, -\mathfrak{c}_{L/K}\}$. In this context, the observation in Remark 8.7 is perhaps also of relevance.

Remark 10.5. By adapting the methods implemented for [4] one can also compute the right hand side of the equality in Conjecture 7.2 for the extension $L/\mathbb{Q}$ in $\mathcal{F}(3)$ of exponent 3. These computations show that

$$\mathcal{E}_{L_p/\mathbb{Q}_p} - J_{2,L_p/\mathbb{Q}_p} + \mathfrak{c}_{L_p/\mathbb{Q}_p} - M_{L_p/\mathbb{Q}_p} = -\mathfrak{c}_{L_p/\mathbb{Q}_p},$$

and thus prove Conjecture 7.2, and hence also Breuning's conjecture, for these extensions. Combining this fact with [10, Th. 4.1] and [2, Cor. 7.6] we see that the central conjecture of [2] is valid for all L/K in $\mathcal{L}(3)$ for which G_w has order 27 and exponent 3 for each wildly ramified place w of L.

10.2. **Extensions of degree** 63.

10.2.1. Let ℓ and p be odd primes with ℓ dividing $p-1$. We now sketch how to compute a set of Galois extensions $L/\mathbb{Q}$ of degree $\ell^2 p$ such that $L/\mathbb{Q}$ is at most tamely ramified outside p and the extensions $L_w/\mathbb{Q}_p$ cover the set of local extensions of Proposition 9.8 (where as usual w denotes the unique place of L above p).

We use a simple heuristic approach which is motivated by the proof of Proposition 9.8 and which works well for $\ell = 3$ and $p = 7$.

We fix a cyclic extension $E_1/\mathbb{Q}$ of degree ℓ such that p is inert and ℓ is unramified. By local class field theory we know that there are up to isomorphism exactly ℓ totally ramified cyclic extensions $N/\mathbb{Q}_p$ of degree ℓ. We compute a set $\mathcal{F}$ of cyclic extensions $F/\mathbb{Q}$ of degree ℓ such that ℓ is unramified and such that the completions $F_w/\mathbb{Q}_p$ range over the set of totally ramified cyclic extensions $N/\mathbb{Q}_p$ of degree ℓ.

For each field $E_2 \in \mathcal{F}$ we set $E := E_1E_2$ and let $\mathfrak{p}$ denote the unique prime ideal of $\mathcal{O}_E$ above p. We then compute ray class groups $\mathrm{cl}(q\mathfrak{p}^2)$ of conductor $q\mathfrak{p}^2$ for small positive integers q with $(q, \ell p) = 1$ and search for subgroups U of $\mathrm{cl}(q\mathfrak{p}^2)$ of index p that are invariant under $G(E/\mathbb{Q})$ and such that the corresponding extension is both wildly and weakly ramified above $\mathfrak{p}$. A quick search using MAGMA produces these extensions. We also record the minimal polynomials in the file WeaklyRamifiedFields.data which can be downloaded from the first author's homepage.

10.2.2. We now fix $\ell := 3$ and $p := 7$ and apply class group methods to verify Vinatier's conjecture for the three extensions $L/\mathbb{Q}$ described in the previous section. The principal approach is exactly the same as described in §10.1.2.

For the locally free class group of a non-abelian group G of order 63 such that $G/C \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (where C denotes the Sylow-7-subgroup) one finds that $\mathrm{Cl}(G)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

Our computations show that for each of the three extensions L computed in the previous subsection Vinatier's conjecture is valid. Taken in conjunction with Theorem 7.5 and Lemma 9.4 this fact implies the following result.

Theorem 10.6. *Let L/K be a weakly ramified odd degree Galois extension of number fields for which at each wildly ramified place v of K one has $K_v = \mathbb{Q}_7$, $|G_w| = 63$ and G_w/C_w is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, where w denote a fixed place of L above v and C_w is the Sylow-7-subgroup of G_w. Then $\mathcal{A}_{L/K}$ is a free $G(L/K)$-module.*

Remark 10.7. One can also use numerical methods to show that for each of the extensions L/K considered in Theorem 10.6 the projection of $\mathfrak{c}_{L/K}$ to $\mathrm{Cl}(G)$ vanishes. Such computations are again consistent with there being a close link between $\mathfrak{a}_{L/K}$ and $\mathfrak{c}_{L/K}$.

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