Congruences for critical values of higher derivatives of twisted Hasse-Weil L-functions, III

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1. Introduction

Let A be an abelian variety defined over a number field k. The Birch and Swinnerton-Dyer conjecture for A over k (as extended to this setting by Tate) predicts a remarkable equality between the leading term $L^*(A,1)$ at $z=1$ of the Hasse-Weil L-series $L(A,z)$ of A over k (assuming that this function has a suitable meromorphic continuation) and the key algebraic invariants of A over k.

Nevertheless, there are various natural contexts in which it seems likely that this equality does not encompass the full extent of the interplay between the leading terms $L^*(A,\psi,1)$ at $z=1$ of the Hasse-Weil-Artin L-series $L(A,\psi,z)$, associated to A and to finite dimensional complex characters ψ of the absolute Galois group of k, and the algebraic invariants of A. For instance, building on a conjecture due to Deligne and Gross concerning the order of vanishing at $z=1$ of such functions one may, for a fixed character ψ, predict that suitable normalisations of $L^*(A,\psi,1)$ (are algebraic and) generate explicit fractional ideals inside any large enough number field. See [8, Prop. 7.3] for such explicit predictions.

However, even such conjectural formulas would not themselves account for any connections that might exist between the numbers $L^*(A,\psi,1)$ as ψ ranges over characters that are not necessarily in the same Galois orbit. In this direction, Mazur and Tate [17] have, in certain concrete settings and by building on the theory of modular symbols, predicted explicit congruence relations between the *values* $L(A,\psi,1)$ at $z=1$ of the functions $L(A,\psi,z)$. In addition, Darmon [12] has subsequently used the theory of Heegner points to formulate analogous predictions for the values $L'(A,\psi,1)$ at $z=1$ of the *first derivatives* $L'(A,\psi,z)$ of the functions $L(A,\psi,z)$.

Let F be a finite Galois extension of k. Then, in all cases, the congruence relations discussed in the previous paragraph, as ψ ranges over complex irreducible characters of

$\mathrm{Gal}(F/k)$, involve the evaluation at suitable points of $A(k)$ of the canonical $\mathrm{Gal}(F/k)$-valued height pairings that had been previously constructed by Mazur and Tate [16] by using the geometrical theory of biextensions.
In recent work [8] of Burns and the second author, a completely general framework for the conjectural theory of integral congruence relations between the leading terms $L^*(A,\psi,1)$ has been developed, thereby extending and refining the aforementioned conjectures of Mazur and Tate and of Darmon. This framework relies on the formulation of a completely general 'refined conjecture of Birch and Swinnerton-Dyer type' (or 'refined BSD conjecture' in the sequel) for A and F/k, which is then shown to encode general families of integral congruence relations involving the Mazur-Tate pairing.
Fix F as above and set $G := \mathrm{Gal}(F/k)$. We let A_F denote the base change of A through F/k and consider $M_F := h^1(A_F)(1)$ as a motive over k with a natural action of the semissimple $\mathbb{Q}$-algebra $\mathbb{Q}[G]$.
It is then also shown in [8] that the refined BSD conjecture is equivalent to the equivariant Tamagawa number conjecture (or 'eTNC' in the sequel) for the pair $(M_F, \mathbb{Z}[G])$, as formulated by Burns and Flach [5]. In addition, both of these conjectures decompose naturally into 'p-components', one for each rational prime number p, and each such component is itself of interest.
For example, if A has good ordinary reduction at p, then the compatibility results proved by Burns and Venjakob [10] show that this p-component is (under certain hypotheses) a consequence of the main conjecture of non-commutative Iwasawa theory for A, as formulated by Coates et al. [11].
Assume now that p is a fixed odd prime and F/k a cyclic p-extension of degree p^n for some natural number n. In this note we will continue the study of the p-component of the eTNC that we begun in [4]. We recall that the main result of loc.cit. was the computation of an equivariant regulator under certain not-too-stringent conditions. Through this computation it was then possible to give a reformulation of this p-component which both was of theoretical interest and also made it amenable to providing partial numerical evidence.
For simplicity of the exposition, let us in the rest of this introduction assume that A is an elliptic curve. The computation of the equivariant regulator in [4] relied on the fact that, under suitable hypotheses, a representation-theoretic result due to Yakovlev [23] implies that the p-completion $A(F)_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} A(F)$ of the Mordell-Weil group of A over F is a permutation $\mathbb{Z}_p[G]$-module. Explicitly speaking, this fact means that one may find points $P_{H,j} \in A(F)$, where H runs over all subgroups of G, with the property that $\mathbb{Z}_p[G/H] \cdot P_{H,j}$ is a free $\mathbb{Z}_p[G/H]$-module of rank one and also that

$$A(F)_p = \bigoplus_{H \leq G} \bigoplus_{j=1}^{m_H} \mathbb{Z}_p[G/H] \cdot P_{H,j}$$

(for some set non-negative integers $\{m_H : H \leq G\}$).
However, our formula for the equivariant regulator involved a choice of integral matrix Φ, with entries in $\mathbb{Z}_p[G]$, which depended upon a canonical extension class in the Yoneda 2-extension group $\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(\mathrm{Hom}_{\mathbb{Z}_p}(A(F)_p, \mathbb{Z}_p), A(F)_p)$ and was therefore not computable in any examples unless this group vanished. Given the above direct sum decomposition of $A(F)_p$, the vanishing of this group holds if and only if m_H is equal

to zero for each subgroup $H \neq 1$, or equivalently, if and only if $A(F)_p$ is a free $\mathbb{Z}_p[G]$-module of the form $\bigoplus_{j=1}^{m_1} \mathbb{Z}_p[G] \cdot P_{0,j}$.

This limitation of our previous methods is consistent with those occurring in all existing verifications of p-components of the eTNC for any elliptic curves. Indeed, in the settings of the theoretical verifications obtained by the first author in [3], by Burns, Wuthrich and the second author in [9], or of the recent extensions of these results by Burns and the second author in [8]; as well as of the numerical verifications carried out both in [9] and in our previous article [4]; a full verification of this conjecture was only ever achieved in situations which forced the $\mathbb{Z}_p[G]$-module $A(F)_p$ to be projective.

On the other hand, even for $n = 1$ (meaning that the extension F/k has degree p), the result [8, Thm. 9.11] shows that the p-component of the eTNC (or refined BSD conjecture) encodes a family of congruence relations between the leading terms $L^*(A, \psi, 1)$ and certain 'Mazur-Tate regulators' (coming from the evaluation of Mazur-Tate height pairings) which are non-trivial unless $A(F)_p$ is projective.

This observation is consistent with our previously encountered difficulties and also justifies why, from the point of view of our approach, it is only interesting to consider components of the general conjectures at primes which divide the degree of the extension.

In this note we use a result of Burns and the second author [8, Thm. 10.3] to obtain an alternative computation of the equivariant regulator which is much better suited for the purpose of verifying 'non-projective' instances of the refined BSD conjecture.

To be a little more precise we note that, under our hypotheses, and for any subgroup H of G, the Mazur-Tate pairing for A considered over the sub-extension F/F^H of F/k gives a well defined pairing

$$\langle\, ,\rangle^{\mathrm{MT}}_{F/F^H} : A(F^H)_p \otimes_{\mathbb{Z}_p} A(F^H)_p \to H \cong I_p(H)/I_p(H)^2.$$

Here $I_p(H)$ denotes the augmentation ideal in the group ring $\mathbb{Z}_p[H]$ and the isomorphism maps $g \in H$ to the class of $g - 1$.

Fix any choice of points as above and any generator σ of G. For any double-indices (H, j) and (J, i) with $H, J \neq 1$ and $H \leq J$, we let $\Psi_{(H,j),(J,i)}$ be any elements of $\mathbb{Z}_p[G]$ which, in

$$\mathbb{Z}_p[G/H] \otimes_{\mathbb{Z}_p} I_p(H)/I_p(H)^2,$$

satisfy the equality

$$\Psi_{(H,j),(J,i)} \cdot (\sigma^{|G/H|} - 1) = \sum_{\gamma \in G/H} \big(\gamma \otimes \langle P_{J,i}, \gamma(P_{H,j})\rangle^{\mathrm{MT}}_{F/F^H}\big).$$

For double-indices with $J < H$ we let $\Psi_{(H,j),(J,i)}$ be any elements of $\mathbb{Z}_p[G]$ which satisfy a straightforward variant of this equality.

We then show that the matrix Ψ obtained in this manner (by ordering all double-indices $(H, j), (J, i)$ lexicographically) is, independently of all of the above choices, a suitable replacement for the essentially inexplicit matrix Φ that occurred in the computation of the equivariant regulator in [4].

In the main result of this note, Theorem 2.1, we thus give a reformulation fo the p-component of the refined BSD conjecture in terms of Mazur-Tate regulators obtained from considering natural ψ-components of the matrix Ψ. This is in particular a suitable extension to general n of the result [8, Thm. 9.11] of Burns and the second author.

As an application we are now able to obtain the first numerical verifications of the p-component of the refined BSD conjecture in situations in which the p-completed Mordell-Weil group $A(F)_p$ is not a projective $\mathbb{Z}_p[G]$-module. We emphasise again that there exists no other theoretical or numerical verification for this conjecture in such situations.
We shall give a detailed description, which we feel may be of some independent interest, of the methods that are appropriate to the numerical computation of Mazur-Tate pairings. For some examples of the pairs $(A, F/k)$ for which we have verified the conjecture we refer the reader to the list given in [8, Ex. 9.14], although this list is not exhaustive. See also the webpage of the first author for more details.

2. Statement of the main result

In this section we state our standing hypotheses and, after defining all the relevant objects, state the main result of this article.

2.1. **The hypotheses.** We recall that A is an abelian variety of dimension d defined over the number field k. In addition, F/k is a cyclic field extension of degree p^n where p is an odd prime. We write A^t for the dual abelian variety.
As in [4, Sec. 2] we assume the following list of hypotheses.

(a) $p \nmid |A(k)_{\mathrm{tor}}| \cdot |A^t(k)_{\mathrm{tor}}|$;
(b) p does not divide the Tamagawa number of A at any place of k at which it has bad reduction;
(c) A has good reduction at all p-adic places of k;
(d) p is unramified in $F/\mathbb{Q}$;
(e) no place of bad reduction of A is ramified in F/k;
(f) if a place v of k ramifies in F/k then no point of order p of the reduction of A is defined over the residue field of v;
(g) $Ш(A_F)$ is finite,
(h) $Ш_p(A_{F^H})$ vanishes for all non-trivial subgroups H of G.
(i) The group $H^1(\mathrm{Gal}(k(A[p^n])/k), A[p^n])$ vanishes.

Remark 2.1. The hypotheses (a)-(h) recover those in place throughout [4]. The full list of hypotheses also recovers those that are in place in [8, Thm. 9.9, Thm.9.11]. We refer the reader to [4, Rem. 2.1] or [8, Rem. 6.1] for a further discussion of these hypotheses.
The hypothesis (i) recovers Hypothesis 10.1 from [8] in our setting and will hence allow us to apply Theorem 10.3 of loc. cit.. We recall that it is widely satisfied. For instance, it holds whenever multiplication-by-'-1' belongs to the image of the canonical Galois representation $G_k \to \mathrm{Aut}_{\mathbb{F}_p}(A[p])$. In particular, if A is an elliptic curve then this hypothesis excludes only finitely many places by a result of Serre.

In the sequel, for any abelian group M we set $M_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} M$. We also set $\widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^\times)$.
Our main result will concern an 'equivariant regulator' $\mathrm{Reg}_{A,F/k,j}$ in $\mathbb{C}_p[G]^\times/\mathbb{Z}_p[G]^\times$ for each isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$, which we now proceed to define. Throughout the construction of this element, we will always use j to implicitly identify $\widehat{G}$ with $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}_p^\times)$.

2.2. **Néron-Tate regulators.** For $0 \le r \le n$ we denote by J_r the subgroup of G of order p^{n-r} and set $F_r := F^{J_r}$ and $\Gamma_r := G/J_r$. Clearly, $[F_r : k] = |\Gamma_r| = p^r$.
Our hypotheses allow us to apply a result of Yakovlev [23] in order to restrict the Galois structure of the Mordell-Weil groups $A(F)_p$ and $A^t(F)_p$ as follows. For any natural number m we write $[m]$ for the set $\{1, \dots, m\}$.
By [4, Prop. 2.2 and (2.1)] we then fix, as we may, non-negative integers $\{m_r : 0 \le r \le n\}$ and subsets

$$\begin{aligned}\mathcal{P}_{(r)} &= \{P_{(r,j)} : j \in [m_r]\} \subseteq A(F_r),\\ \mathcal{P}^t_{(r)} &= \{P^t_{(r,j)} : j \in [m_r]\} \subseteq A^t(F_r)\end{aligned}$$

such that the $\mathbb{Z}_p[\Gamma_r]$-modules generated by each of the points $P_{(r,j)}$ and $P^t_{(r,j)}$ are free of rank one and there are direct sum decompositions of $\mathbb{Z}_p[G]$-modules

$$A(F)_p = \bigoplus_{r=0}^{n}\bigoplus_{j=1}^{m_r} \mathbb{Z}_p[\Gamma_r] \cdot P_{(r,j)} \quad \text{and} \quad A^t(F)_p = \bigoplus_{r=0}^{n}\bigoplus_{j=1}^{m_r} \mathbb{Z}_p[\Gamma_r] \cdot P^t_{(r,j)}. \tag{1}$$

We set

$$\mathcal{P} := \bigcup_{r=0}^{n} \mathcal{P}_{(r)}, \quad \mathcal{P}^t := \bigcup_{r=0}^{n} \mathcal{P}^t_{(r)}.$$

By ordering our fixed choice of points $\mathcal{P}$ and $\mathcal{P}^t$ lexicographically, we obtain a 'regulator matrix'

$$R^{\mathrm{NT}}_{A,F/k}(\mathcal{P}, \mathcal{P}^t) := \left(\sum_{g \in G} \langle g(P^t_{(r,j)}), P_{(s,i)}\rangle_{A_F} \cdot g^{-1}\right)_{(r,j),(s,i)} \in M_N(\mathbb{R}[G])$$

with $N := \sum_{r=0}^{n} m_r$, and where $\langle -, - \rangle_{A_F}$ denotes the Néron-Tate height pairing for A over F.
For any matrix $M = (m_{(r,j),(s,i)})$ where the indices $\{(r,j) : 0 \le r \le n, j \in [m_r]\}$ are ordered lexicographically and for any $0 \le t \le n$, we set

$$M_t := (m_{(r,j),(s,i)})_{r,s \ge t}$$

and, given any matrix N with entries in $\mathbb{C}[G]$, resp. $\mathbb{C}_p[G]$, and any $\psi \in \widehat{G}$, we write $\psi(N)$ for the matrix with entries in $\mathbb{C}$, resp. $\mathbb{C}_p$, obtained after extending ψ to a function on $\mathbb{C}[G]$, resp. $\mathbb{C}_p[G]$, by linearity and then evaluating ψ at each entry of N.
For any $\psi \in \widehat{G}$ we define an integer t_ψ between 0 and n by the equality $\ker(\psi) = H_{t_\psi}$ and then also set

$$m_\psi := \sum_{r=t_\psi}^{n} (n-r) m_r$$

and

$$\mathrm{Reg}^{\mathrm{NT}}_\psi(\mathcal{P}, \mathcal{P}^t) := p^{-2m_\psi} \det\left(\psi\left(R^{\mathrm{NT}}_{A,F/k}(\mathcal{P}, \mathcal{P}^t)_{t_\psi}\right)\right). \tag{2}$$

Each regulator term $\mathrm{Reg}^{\mathrm{NT}}_\psi(\mathcal{P}, \mathcal{P}^t)$ coincides with the element $\lambda_\psi(\mathcal{P}, \mathcal{P}^t)$ defined in [4, Def. 2.4].

We finally fix a generator σ of G and then define a non-zero complex number

$$\delta_\psi := \prod_{r=0}^{t_\psi - 1} \left(\psi(\sigma)^{p^r} - 1\right)^{m_r}. \tag{3}$$

2.3. **Mazur-Tate regulators.** For any $0 \le r \le n$ we recall that, under our given hypotheses, [8, Prop. 6.3(ii)] implies that every element of $A^t(F_r)_p$ and $A(F_r)_p$ is 'locally-normed'. In particular, the construction of Mazur and Tate using the theory of biextensions gives well defined canonical height pairings

$$\langle\ ,\ \rangle^{\mathrm{MT}}_{F/F_r} : A^t(F_r)_p \otimes_{\mathbb{Z}_p} A(F_r)_p \to J_r \cong I_p(J_r)/I_p(J_r)^2. \tag{4}$$

Here $I_p(J_r)$ denotes the augmentation ideal in the group ring $\mathbb{Z}_p[J_r]$ and the isomorphism maps $g \in J_r$ to the class of $g - 1$.
In the sequel we also write ρ_r for the canonical projection $\mathbb{Z}_p[G] \to \mathbb{Z}_p[\Gamma_r]$.
For any indices $0 \le r, s \le n-1$, any $j \in [m_r]$ and any $i \in [m_s]$, we set $\ell := \max(r, s)$ and then fix any elements $\Psi_{(r,j),(s,i)}$ of $\mathbb{Z}_p[G]$ which satisfy the equality

$$\rho_r(\Psi_{(r,j),(s,i)}) \otimes (\sigma^{p^\ell} - 1) = \sum_{\gamma \in \Gamma_r} \left(\gamma \otimes \langle P^t_{(s,i)}, \gamma(P_{(r,j)})\rangle^{\mathrm{MT}}_{F/F_\ell}\right) \tag{5}$$

in

$$\mathbb{Z}_p[\Gamma_r] \otimes_{\mathbb{Z}_p} I_p(J_\ell)/I_p(J_\ell)^2.$$

It will be clear from Proposition 3.1 below that such elements always exist.

Remark 2.2. Let $\mathcal{I}_p(J_r)$ denote the ideal of $\mathbb{Z}_p[G]$ generated by $I_p(J_r)$. Then the inclusion $I_p(J_\ell) \subset I_p(J_r)$ induces a canonical inclusion

$$\mathbb{Z}_p[\Gamma_r] \otimes_{\mathbb{Z}_p} I_p(J_\ell)/I_p(J_\ell)^2 \hookrightarrow \mathbb{Z}_p[\Gamma_r] \otimes_{\mathbb{Z}_p} I_p(J_r)/I_p(J_r)^2 \cong \mathcal{I}_p(J_r)/\mathcal{I}_p(J_r)^2. \tag{6}$$

Here the isomorphism is canonical (see [6, Prop. 4.9]). It is then clear the left-hand side of the equality (5) coincides with the class of $\Psi_{(r,j),(s,i)} \cdot (\sigma^{p^\ell} - 1)$ in the quotient $\mathcal{I}_p(J_r)/\mathcal{I}_p(J_r)^2$.

Using these choices we construct a matrix

$$\Psi(\mathcal{P}, \mathcal{P}^t) := \left(\begin{array}{c|c} \left(\Psi_{(r,j),(s,i)}\right)_{r,s<n} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \ \ \dots \ \ 0 & I_{m_n} \end{array}\right) \tag{7}$$

with entries in $\mathbb{Z}_p[G]$. Here I_{m_n} denotes the identity $m_n \times m_n$ matrix.
If we finally set, for any $\psi \in \widehat{G}$,

$$\varepsilon_\psi(\Psi(\mathcal{P}, \mathcal{P}^t)) := \det\left(\psi\left(\Psi(\mathcal{P}, \mathcal{P}^t)_{t_\psi}\right)\right),$$

then Lemma 3.2 below will show that the element

$$\sum_{\psi \in \widehat{G}} \varepsilon_\psi(\Psi(\mathcal{P}, \mathcal{P}^t)) \cdot e_\psi \tag{8}$$

of $\mathbb{C}_p[G]$ belongs to $\mathbb{C}_p[G]^\times$ (so in particular each term $\varepsilon_\psi(\Psi(\mathcal{P}, \mathcal{P}^t))$ is non-zero) and is independent, up to multiplication by an element of $\mathbb{Z}_p[G]^\times$, of the choices made in (5).

2.4. **The equivariant regulator.** The claims made in the previous paragraph imply that, for any isomorphism $j\colon \mathbb{C} \to \mathbb{C}_p$, the element of $\mathbb{C}_p[G]^\times$ defined by

$$\mathrm{Reg}_{A,F/k,j} := (-1)^{N-m_n} \cdot \sum_{\psi \in \widehat{G}} \frac{j(\mathrm{Reg}_\psi^{\mathrm{NT}}(\mathcal{P}, \mathcal{P}^t) \cdot \delta_\psi)}{\varepsilon_\psi(\Psi(\mathcal{P}, \mathcal{P}^t))} \cdot e_\psi \tag{9}$$

depends upon the choices of $\mathcal{P}, \mathcal{P}^t, \sigma$ and the matrix $\Psi(\mathcal{P}, \mathcal{P}^t)$ only modulo $\mathbb{Z}_p[G]^\times$. Here again we have set $N := \sum_{r=0}^n m_r$, so that $N - m_n = \sum_{r=0}^{n-1} m_r$.

Definition 2.3. For a fixed isomorphism $j\colon \mathbb{C} \to \mathbb{C}_p$ the class of $\mathrm{Reg}_{A,F/k,j}$ in $\mathbb{C}_p[G]^\times/\mathbb{Z}_p[G]^\times$ is called the (p-primary) equivariant regulator associated to A and F/k.

By abuse of terminology we will often refer to any lift $\mathrm{Reg}_{A,F/k,j} \in \mathbb{C}_p[G]^\times$ itself as the equivariant regulator.

Remark 2.4. The results of Proposition 3.1 and Lemma 3.2 combine to show that $\mathrm{Reg}_{A,F/k,j}$ is precisely the element

$$\sum_{\psi \in \hat{G}} \lambda_\psi(\mathcal{P}, \mathcal{P}^t) \cdot \varepsilon_\psi(\Phi) \cdot \delta_\psi \cdot e_\psi \pmod{\mathbb{Z}_p[G]^\times}$$

which occurs in [4, Th. 2.9]. The main new insight is that the elements $\varepsilon_\psi(\Phi) \in \mathbb{C}_p^\times$ which depended upon an essentially unknown matrix $\Phi \in M_N(\mathbb{Z}_p[G])$ can be explicitly determined by (5).

2.5. **Statement of the result.** The refined Birch and Swinnerton-Dyer conjecture is an equality between analytic and algebraic invariants associated with A/k and F/k. We now briefly describe the analytic part refering the reader to [4, Sec. 2] or [8] for further details.
For each $\psi \in \widehat{G}$ we set

$$\mathcal{L}_\psi^* = \mathcal{L}_{A,F/k,\psi}^* := \frac{L_{S_{\mathrm{r}}}^*(A, \check{\psi}, 1)\tau^*(\mathbb{Q}, \psi)^d}{\Omega_A^\psi \cdot w_\psi^d} \in \mathbb{C}^\times,$$

where

- $L_{S_r}^*(A, \psi, 1)$ is the leading term in the Taylor expansion at $z = 1$ of the ψ-twisted Hasse-Weil-Artin L-function $L_{S_r}(A, \psi, z)$ of A, truncated by removing the Euler factors corresponding to the set S_{r} of primes of k which ramify in F/k;
- $\tau^*(\mathbb{Q}, \psi)$ is a suitably modified global Galois-Gauss sum;
- $\Omega_A^\psi \cdot w_\psi^d$ is a (suitably normalised) product of periods.

We finally set

$$\mathcal{L}^* = \mathcal{L}_{A,F/k}^* := \sum_{\psi \in \widehat{G}} \mathcal{L}_{A,F/k,\psi}^* e_\psi \in \mathbb{C}[G]^\times$$

and note that the element $\mathcal{L}^*$ defined immediately above [8, Th. 6.5] specialises precisely to our definition.
Without any further mention we will always assume that the functions $L_{S_r}(A, \psi, z)$ have analytic continuation to $z = 1$, so that the above term is well-defined. We also

recall that Deligne and Gross have then predicted that these functions should vanish at $z = 1$ exactly to order equal to the multiplicity with which the character ψ occurs in the representation $\mathbb{C} \otimes_{\mathbb{Z}} A^t(F)$ of G.
We now formulate the main result of this manuscript. For any $j : \mathbb{C} \cong \mathbb{C}_p$ we write j_* for the associated map $\mathbb{C}[G]^\times \to \mathbb{C}_p[G]^\times$.

Theorem 2.1. *Assume that the hypotheses (a)-(i) are valid. Let $\mathcal{P}$ and $\mathcal{P}^t$ be any choice of points such that (1) holds. Assume also that $\text{Ш}_p(A_F) = 0$.*
Then the p-component of the refined Birch and Swinnerton-Dyer conjecture is valid if and only if, for any $j : \mathbb{C} \cong \mathbb{C}_p$, the element

$$\frac{j_*\left(\mathcal{L}^*_{A,F/k}\right)}{\mathrm{Reg}_{A,F/k,j}} \tag{10}$$

belongs to $\mathbb{Z}_p[G]^\times$.

Remark 2.5. One may rephrase the condition (10) in terms of explicit congruence relations in the augmentation filtration, as occurring in [15, Conj. 3.11].

The proof of Theorem 2.1 will occupy the next section.

3. The proof of Theorem 2.1

Under our listed hypotheses, Theorem 6.5 and Remark 6.6 in [8] reformulate the p-component of the refined BSD conjecture (denoted by $\mathrm{BSD}_p(A_{F/k})$ throughout loc. cit.) as an equality of the form

$$\delta_{G,p}\left(j_*\left(\mathcal{L}^*_{A,F/k}\right)\right) = \chi_{G,p}(\mathrm{SC}_p(A_{F/k}), h^j_{A,F})$$

in the relative algebraic K-group $K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$. Here

$$\delta_{G,p} : \mathbb{C}_p[G]^\times \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$$

is the canonical 'extended boundary homomorphism' considered by Burns and Flach in [5] while the right-hand side is the refined Euler characteristic of the pair $(\mathrm{SC}_p(A_{F/k}), h^j_{A,F})$. We thus also recall that $\mathrm{SC}_p(A_{F/k})$ is the 'classical p-adic Selmer complex' for A and F/k, as defined in Definition 2.3 of [8], while $h^j_{A,F}$ is the canonical trivialisation of this complex that is induced by the Néron-Tate height (see below).
At the outset we set $C := \mathrm{SC}_p(A_{F/k})$ and note that, under the hypotheses of Theorem 2.1, it is proved in [8, Prop. 6.3] that C is a perfect complex of $\mathbb{Z}_p[G]$-modules. In addition, there are canonical identifications

$$r_i : H^i(C) \cong \begin{cases} A^t(F)_p, & i = 1, \\ A(F)^*_p, & i = 2 \\ 0, & i \neq 1, 2. \end{cases} \tag{11}$$

The trivialisation

$$h^j_{A,F} : \mathbb{C}_p \otimes_{\mathbb{Z}_p} H^1(C) \cong \mathbb{C}_p \otimes_{\mathbb{Z}_p} H^2(C)$$

is then the canonical isomorphism induced by the Néron-Tate height pairing via (11) and via j.

We now rephrase the validity of the p-component of the refined BSD conjecture as the vanishing of the element

$$\xi := \delta_{G,p}\left(j_*\left(\mathcal{L}^*_{A,F/k}\right)\right) - \chi_{G,p}(C, h^j_{A,F})$$

of $K_0(\mathbb{Z}_p[G], \mathbb{C}_p[G])$.

The first key step is to explicitly compute the second term that occurs in the definition of ξ and to do this we use the canonical identifications (11) to identify the complex C with a unique element $\delta_{A,F,p} = \delta_{C,r_1,r_2}$ of the Yoneda Ext-group $\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(A(F)^*_p, A^t(F)_p)$.

For each pair (r,j) we define a dual element $P^*_{(r,j)}$ of $A(F)^*_p$ by setting, for any other triple (s,i) and $\tau \in G$,

$$P^*_{(r,j)}(\tau P_{(s,i)}) := \begin{cases} 1, & \text{if } r = s, j = i \text{ and } \tau \in J_r, \\ 0, & \text{otherwise .} \end{cases}$$

By [4, Lem. 4.1] one then has

$$A(F)^*_p = \bigoplus_{(r,j)} \mathbb{Z}_p[\Gamma_r] P^*_{(r,j)}$$

with each summand isomorphic to $\mathbb{Z}_p[\Gamma_r]$. We fix a free $\mathbb{Z}_p[G]$-module

$$X := \bigoplus_{(r,j)} \mathbb{Z}_p[G] b_{(r,j)}$$

of rank $N = \sum_r m_r$ and consider the exact sequence

$$0 \to A^t(F)_p \xrightarrow{\iota} X \xrightarrow{\Theta} X \xrightarrow{\pi} A(F)^*_p \to 0, \tag{12}$$

where we set

$$\pi(b_{(r,j)}) := P^*_{(r,j)}, \quad \Theta(b_{(r,j)}) := (\sigma^{p^r} - 1) b_{(r,j)} \text{ and } \iota(P^t_{(r,j)}) := \mathrm{Tr}_{J_r}\, b_{(r,j)}.$$

This sequence defines a canonical isomorphism

$$\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(A(F)^*_p, A^t(F)_p) \cong \mathrm{End}_{\mathbb{Z}_p[G]}(A^t(F)_p)/\iota_*(\mathrm{Hom}_{\mathbb{Z}_p[G]}(X, A^t(F)_p)) \tag{13}$$

where ι_* denotes composition with ι.

To describe the isomorphism (13), for a given $\phi \in \mathrm{End}_{\mathbb{Z}_p[G]}(A^t(F)_p)$ we consider the push-out commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^t(F)_p & \xrightarrow{\iota} & X & \xrightarrow{\Theta} & X & \xrightarrow{\pi} & A(F)^*_p & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\downarrow & & \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & A^t(F)_p & \longrightarrow & X(\phi) & \longrightarrow & X & \xrightarrow{\pi} & A(F)^*_p & \longrightarrow & 0. \end{array} \tag{14}$$

In this diagram $X(\phi)$ is defined as the push-out of ι and ϕ and all the unlabeled arrows are the canonical maps induced by the push-out construction. Then the pre-image of ϕ under (13) is represented by the bottom row of this diagram.

Then, since C belongs to $D^{\mathrm{perf}}(\mathbb{Z}_p[G])$, the results of [4, Lem. 4.2 and Lem. 4.3] imply that there exists an automorphism ϕ of the $\mathbb{Z}_p[G]$-module $A^t(F)_p$ that represents the image of $\delta_{A,F,p}$ under (13) and also fixes the element $P^t_{(n,j)}$ for every j in $[m_n]$.

It follows that the exact sequence

$$0 \to A^t(F)_p \xrightarrow{\iota\circ\phi^{-1}} X \xrightarrow{\Theta} X \xrightarrow{\pi} A(F)_p^* \to 0 \tag{15}$$

is a representative of the extension class $\delta_{A,F,p}$.
In particular, for any choice of $\mathbb{C}_p[G]$-equivariant splittings

$$s_1 : \mathbb{C}_p \cdot X \to \mathbb{C}_p \cdot A^t(F)_p \oplus \mathbb{C}_p \cdot \mathrm{im}(\Theta) \tag{16}$$

and

$$s_2 : \mathbb{C}_p \cdot X \to \mathbb{C}_p \cdot A(F)_p^* \oplus \mathbb{C}_p \cdot \mathrm{im}(\Theta) \tag{17}$$

of the scalar extensions of the canonical exact sequences

$$0 \to A^t(F)_p \xrightarrow{\iota} X \xrightarrow{\Theta} \mathrm{im}(\Theta) \to 0$$

and

$$0 \to \mathrm{im}(\Theta) \to X \xrightarrow{\pi} A(F)_p^* \to 0,$$

an explicit computation of the refined Euler characteristic occurring in the definition of ξ implies that

(18)

$$\begin{aligned}
&-\chi_{G,p}(C, h^j_{A,F})\\
=&-\delta_{G,p}(\det_{\mathbb{C}_p[G]}(s_2^{-1} \circ (h^j_{A,F} \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ ((\mathbb{C}_p\cdot\phi) \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ s_1))\\
=&-\delta_{G,p}(\det_{\mathbb{C}_p[G]}(s_2^{-1} \circ (h^j_{A,F} \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ s_1 \circ s_1^{-1} \circ ((\mathbb{C}_p\cdot\phi) \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ s_1))\\
=&\,\delta_{G,p}(\det_{\mathbb{C}_p[G]}(s_1^{-1} \circ (h^{j,-1}_{A,F} \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ s_2))\\
&\qquad\qquad + \delta_{G,p}(\det_{\mathbb{Q}_p[G]}(s_1^{-1} \circ ((\mathbb{Q}_p\cdot\phi^{-1}) \oplus \mathrm{id}_{\mathbb{Q}_p\cdot\mathrm{im}(\Theta)}) \circ s_1))\\
=&\,\delta_{G,p}(\det_{\mathbb{C}_p[G]}(s_1^{-1} \circ (h^{j,-1}_{A,F} \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ s_2))\\
&\qquad\qquad + \delta_{G,p}(\det_{\mathbb{Q}_p[G]}((\mathbb{Q}_p\cdot\phi^{-1}) \oplus \mathrm{id}_{\mathbb{Q}_p\cdot\mathrm{im}(\Theta)})).
\end{aligned}$$

In addition, specialising the explicit computation of [4, Prop. 4.4] to the case $\Phi = \mathrm{id}_{A^t(F)_p}$ shows that

$$\det_{\mathbb{C}_p[G]}(s_1^{-1} \circ (h^{j,-1}_{A,F} \oplus \mathrm{id}_{\mathbb{C}_p\cdot\mathrm{im}(\Theta)}) \circ s_2) = \left(\sum_{\psi\in\widehat{G}} j(\mathrm{Reg}^{\mathrm{NT}}_\psi(\mathcal{P},\mathcal{P}^t)\cdot\delta_\psi)\cdot e_\psi\right)^{-1}. \tag{19}$$

To compute the second term that occurs in the final equality of (18) we fix, as we may, any elements $\Lambda_{(r,j),(s,i)}$ of $\mathbb{Z}_p[G]$ with the property that

$$\phi^{-1}(P^t_{(s,i)}) = \sum_{(r,j)} \Lambda_{(r,j),(s,i)} P^t_{(r,j)}. \tag{20}$$

We thus obtain an invertible matrix

$$\Lambda = \Lambda(\mathcal{P},\mathcal{P}^t) := \left(\Lambda_{(r,j),(s,i)}\right)_{(r,j),(s,i)}$$

with entries $\Lambda_{(r,j),(s,i)}$ in $\mathbb{Z}_p[G]$ uniquely determined modulo the kernel of the canonical projection $\rho_r : \mathbb{Z}_p[G] \to \mathbb{Z}_p[\Gamma_r]$, which also has the form (7).

Then the chosen properties of the representative ϕ of $\delta_{A,F,p}$ fixed above imply that

$$\det_{\mathbb{Q}_p[G]}((\mathbb{Q}_p \cdot \phi^{-1}) \oplus \mathrm{id}_{\mathbb{Q}_p\cdot \mathrm{im}(\Theta)}) = \sum_{\psi \in \widehat{G}} \varepsilon_\psi(\Lambda) \cdot e_\psi. \tag{21}$$

The equalities (18), (19) and (21) now combine with the definition of ξ to imply that $\xi = \delta_{G,p}(\mathcal{L})$ with

$$\mathcal{L} := j_*(\mathcal{L}^*_{A,F/k}) \cdot \left(\sum_{\psi\in\widehat{G}} j(\mathrm{Reg}^{\mathrm{NT}}_\psi(\mathcal{P},\mathcal{P}^t)\cdot \delta_\psi)\cdot e_\psi \right)^{-1} \cdot \left(\sum_{\psi\in\widehat{G}} \varepsilon_\psi(\Lambda)\cdot e_\psi \right).$$

It thus follows that ξ vanishes if and only if $\mathcal{L}$ belongs to $\mathbb{Z}_p[G]^\times = \ker(\delta_{G,p})$ and hence the proof of Theorem 2.1 is completed by the proposition below.

Proposition 3.1. *For any $0 \le r, s \le n-1$ we set $\ell := \max(r,s)$. Then for any $j \in [m_r]$, any $i \in [m_s]$ and any elements $\Lambda_{(r,j),(s,i)}$ of $\mathbb{Z}_p[G]$ satisfying (20) one has*

$$\rho_r(\Lambda_{(r,j),(s,i)}) \otimes (\sigma^{p^\ell} - 1) = -\sum_{\gamma\in\Gamma_r} \left(\gamma \otimes \langle P^t_{(s,i)}, \gamma(P_{(r,j)})\rangle^{\mathrm{MT}}_{F/F_\ell}\right)$$

in $\mathbb{Z}_p[\Gamma_r] \otimes_{\mathbb{Z}_p} I_p(J_\ell)/I_p(J_\ell)^2$.

Proof. It is proved in [8, Thm. 8.3] that

$$\langle\,,\rangle^{\mathrm{MT}}_{F/F_\ell} \colon A^t(F_\ell)_p \otimes_{\mathbb{Z}_p} A(F_\ell)_p \to I_p(J_\ell)/I_p(J_\ell)^2$$

coincides with the inverse of the pairing induced by

$$\beta_{A,F/F_\ell,p}\colon A^t(F_\ell)_p \to I_p(J_\ell)/I_p(J_\ell)^2 \otimes_{\mathbb{Z}_p} (A(F)^*_p)_{J_\ell} \to I_p(J_\ell)/I_p(J_\ell)^2 \otimes_{\mathbb{Z}_p} A(F_\ell)^*_p,$$

where the first arrow is the Bockstein homomorphism associated to the complex of $\mathbb{Z}_p[J_\ell]$-modules $\mathrm{SC}_p(A_{F/F_\ell})$ together with the canonical identifications (11), and the second arrow is induced by restriction to $A(F_\ell)_p$.

Now, it is immediately clear from their definitions in [8, Def. 2.3] that the complexes $\mathrm{SC}_p(A_{F/F_\ell})$ and $\mathrm{SC}_p(A_{F/k})$ are canonically isomorphic in $D(\mathbb{Z}_p[J_\ell])$ and furthermore that this isomorphism is compatible with the identifications (11). The complex $\mathrm{SC}_p(A_{F/F_\ell})$ may therefore be represented by the (restriction of scalars of) the exact sequence (15).

Now $\beta_{A,F/F_\ell,p}$ may be computed, through the representative (15), as the connecting homomorphism which arises when applying the snake lemma to the following commutative diagram (in which both rows and the third column are exact and the first column is a complex)

$$\begin{tikzcd}
 & & & A^t(F_\ell)_p \arrow[d, "(\iota\circ\phi^{-1})^{J_\ell}"] & \\
0 \arrow[r] & I_p(J_\ell)\otimes_{\mathbb{Z}_p[J_\ell]} X \arrow[r, "\subseteq"] \arrow[d, "\mathrm{id}\otimes_{\mathbb{Z}_p[J_\ell]}\Theta"] & X \arrow[r, "\mathrm{Tr}_{J_\ell}"] \arrow[d, "\Theta"] & X^{J_\ell} \arrow[r] \arrow[d, "\Theta^{J_\ell}"] & 0 \\
0 \arrow[r] & I_p(J_\ell)\otimes_{\mathbb{Z}_p[J_\ell]} X \arrow[r, "\subseteq"] \arrow[d, "(\mathrm{id}\otimes_{\mathbb{Z}_p[J_\ell]}\pi)_{J_\ell}"] & X \arrow[r, "\mathrm{Tr}_{J_\ell}"] & X^{J_\ell} \arrow[r] & 0 \\
 & I_p(J_\ell)/I_p(J_\ell)^2\otimes_{\mathbb{Z}_p}(A(F)_p^*)_{J_\ell}. & & &
\end{tikzcd}$$

From (20) and the definition of ι we immediately derive

$$\iota(\phi^{-1}(P^t_{(s,i)})) = \iota\left(\sum_{(u,h)}\Lambda_{(u,h);(s,i)}P^t_{(u,h)}\right) = \sum_{(u,h)}\Lambda_{(u,h);(s,i)}\,\mathrm{Tr}_{J_u}(b_{(u,h)}).$$

Since $P^*_{u,h}(\gamma(P_{r,j})) = 0$ whenever the pair (u,h) is different from (r,j), we thus find for any $\gamma\in\Gamma_r$ that

$$\begin{aligned}
-\langle P^t_{(s,i)},\gamma(P_{(r,j)})\rangle^{\mathrm{MT}}_{F/F_\ell} &= \left((\mathrm{id}\otimes_{\mathbb{Z}_p[J_\ell]}\pi)_{J_\ell}\left(\Theta\left(\mathrm{Tr}_{J_r/J_\ell}(\Lambda_{(r,j);(s,i)}b_{(r,j)})\right)\right)\right)(\gamma(P_{(r,j)}))\\
&=\left((\mathrm{id}\otimes_{\mathbb{Z}_p[J_\ell]}\pi)_{J_\ell}\left((\sigma^{p^r}-1)\,\mathrm{Tr}_{J_r/J_\ell}(\Lambda_{(r,j);(s,i)}b_{(r,j)})\right)\right)(\gamma(P_{(r,j)}))\\
&=\left((\mathrm{id}\otimes_{\mathbb{Z}_p[J_\ell]}\pi)_{J_\ell}\left((\sigma^{p^\ell}-1)(\Lambda_{(r,j);(s,i)}b_{(r,j)})\right)\right)(\gamma(P_{(r,j)}))\\
&=\left(\left((\sigma^{p^\ell}-1)+I_p(J_\ell)^2\right)\otimes(\Lambda_{(r,j);(s,i)}P^*_{(r,j)})\right)(\gamma(P_{(r,j)}))\\
&=(\sigma^{p^\ell}-1)((\Lambda_{(r,j);(s,i)}P^*_{(r,j)})(\gamma(P_{(r,j)})))+I_p(J_\ell)^2.
\end{aligned}$$

Here the first equality uses the fact that $\mathrm{Tr}_{J_r}(\Lambda_{(r,j);(s,i)}b_{(r,j)}) = \mathrm{Tr}_{J_\ell}(\mathrm{Tr}_{J_r/J_\ell}(\Lambda_{(r,j);(s,i)}b_{(r,j)}))$.

Now, if we write $\rho_r(\Lambda_{(r,j),(s,i)}) = \sum_{\gamma\in\Gamma_r}a_\gamma\gamma$ in $\mathbb{Z}_p[\Gamma_r]$, then

$$(\Lambda_{(r,j);(s,i)}P^*_{(r,j)})(\gamma(P_{(r,j)})) = a_\gamma.$$

It therefore follows that the right-hand side of the claimed equality is equal to

$$\sum_{\gamma\in\Gamma_r}(\gamma\otimes(a_\gamma(\sigma^{p^\ell}-1))) = \sum_{\gamma\in\Gamma_r}((a_\gamma\gamma)\otimes(\sigma^{p^\ell}-1)) = \rho_r(\Lambda_{(r,j),(s,i)})\otimes(\sigma^{p^\ell}-1),$$

as required.

□

We finally justify that all constructions of the statement of our main result were indeed well-defined.

Lemma 3.2. *For any elements $\Psi_{(r,j),(s,i)}$ of $\mathbb{Z}_p[G]$ satisfying the equalities (5), the sum (8) belongs to $\mathbb{C}_p[G]^\times$ and is independent, up to multiplication by an element of $\mathbb{Z}_p[G]^\times$, of the choices made.*

Proof. By Proposition 3.1, any collection of elements $-\Lambda_{(r,j),(s,i)}$ (for $r, s < n$) satisfying (20) also constitutes an appropriate choice satisfying the equalities (5). In addition, the sum

$$\sum_{\psi \in \widehat{G}} \varepsilon_\psi(-\Lambda) \cdot e_\psi$$

clearly belongs to $\mathbb{C}_p[G]^\times$. (In fact, by the argument of [4, Lem. 4.8], it also belongs to $\mathcal{M}^\times$, where $\mathcal{M}$ denotes the (unique) maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$.)
It is therefore enough to set $\Psi'_{(r,j),(s,i)} := -\Lambda_{(r,j),(s,i)}$ as well as $\Psi' := -\Lambda$, fix any collection of elements $\Psi_{(r,j),(s,i)}$ satisfying the equalities (5), and prove that the sum

$$\sum_{\psi \in \widehat{G}} \frac{\varepsilon_\psi(\Psi)}{\varepsilon_\psi(\Psi')} \cdot e_\psi$$

belongs to $\mathbb{Z}_p[G]^\times$.
The main step involved in proving this assertion is given by the following intermediate result.

Lemma 3.3. *The endomorphism ψ of $A^t(F)_p$ which maps a point $P^t_{(s,i)}$ to the sum*

$$\sum_{(r,j)} (\Psi_{(r,j),(s,i)} - \Psi'_{(r,j),(s,i)}) \cdot P^t_{(r,j)}$$

factors through the map $\iota : A^t(F)_p \to X$ occurring in the exact sequence (12).
In addition, if we define an endomorphism γ of $A^t(F)_p$ by setting

$$\gamma(P^t_{(s,i)}) := \sum_{(r,j)} \Psi_{(r,j),(s,i)} P^t_{(r,j)},$$

then γ is bijective.

Proof. By Lemma 3.4 below, for each pair $(r,j),(s,i)$ with $r, s < n$, there exist elements $\lambda_{(r,j),(s,i)}$ and $\mu_{(r,j),(s,i)}$ in $\mathbb{Z}_p[G]$ with the property that
(22)

$$\Psi_{(r,j),(s,i)} - \Psi'_{(r,j),(s,i)} = \begin{cases} (\sigma^{p^r} - 1) \cdot \lambda_{(r,j),(s,i)} + \mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)}, & \text{if } r \geq s, \\ \big((\sigma^{p^r} - 1) \cdot \lambda_{(r,j),(s,i)} + \mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)}\big) p^{r-s}, & \text{if } r < s. \end{cases}$$

For $r = n$ or $n = r$ we may simply take $\lambda_{(r,j),(s,i)}$ and $\mu_{(r,j),(s,i)}$ to be equal to 0 and still have these equalities.
Now since σ^{p^r} acts trivially on $P^t_{(r,j)}$ one thus has

$$\begin{aligned} \psi(P^t_{(s,i)}) &= \sum_{r \geq s} \mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)} + \sum_{r<s} p^{r-s} \, \mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)} \\ &= \sum_{r \geq s} \mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)} + \sum_{r<s} \mathrm{Tr}_{J_s} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)}. \end{aligned} \tag{23}$$

In addition, J_s acts trivially both on $P^t_{(s,i)}$ and on the second summand of the above expression and therefore it must also act trivially on each term $\mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)}$ with $r \geq s$. Now, since $\mathbb{Z}_p[G] \cdot P^t_{(r,j)} = \mathbb{Z}_p[G/J_r] \cdot P^t_{(r,j)}$ is a free $\mathbb{Z}_p[G/J_r]$-module of rank one, this condition necessarily implies that Tr_{J_s/J_r} divides $\mu_{(r,j),(s,i)}$. We may therefore, for every indices with $r \geq s$, write $\mu_{(r,j),(s,i)} = \mathrm{Tr}_{J_s/J_r} \cdot \tilde{\mu}_{(r,j),(s,i)}$ for some element $\tilde{\mu}_{(r,j),(s,i)}$ of $\mathbb{Z}_p[G]$.

We now define a homomorphism $\alpha : X \to A^t(F)_p$ by setting

$$\alpha(b_{(s,i)}) := \sum_{r \geq s} \tilde{\mu}_{(r,j),(s,i)} \cdot P^t_{(r,j)} + \sum_{r<s} \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)}$$

and claim that $\psi = \alpha \circ \iota$, as required to prove the first claim of the lemma.

Indeed, one has

$$\begin{aligned}(\alpha \circ \iota)(P_{(s,i)}) =& \alpha(\mathrm{Tr}_{J_s} \cdot P_{(s,i)}) = \mathrm{Tr}_{J_s} \cdot \alpha(P_{(s,i)}) \\ =& \sum_{r \geq s} \mathrm{Tr}_{J_s} \cdot \tilde{\mu}_{(r,j),(s,i)} \cdot P^t_{(r,j)} + \sum_{r<s} \mathrm{Tr}_{J_s} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)} \\ =& \sum_{r \geq s} \mathrm{Tr}_{J_r} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)} + \sum_{r<s} \mathrm{Tr}_{J_s} \cdot \mu_{(r,j),(s,i)} \cdot P^t_{(r,j)},\end{aligned}$$

which is equal to $\psi(P_{(s,i)})$ by (23). This proves the first claim of the lemma.

To prove the second claim we write $\epsilon : \mathbb{Z}_p[G] \to \mathbb{Z}_p$ for the canonical augmentation map and define a canonical ring homomorphism $\epsilon_{\mathbb{F}_p}$ as the composition

$$\mathbb{Z}_p[G] \xrightarrow{\epsilon} \mathbb{Z}_p \to \mathbb{F}_p.$$

Now, the equalities (22) imply that

$$\epsilon(\Psi'_{(r,j),(s,i)}) = \begin{cases} \epsilon(\Psi_{(r,j),(s,i)}) - p^{n-r}\epsilon(\mu_{(r,j),(s,i)}), & \text{if } r \geq s, \\ \epsilon(\Psi_{(r,j),(s,i)}) - p^{n-s}\epsilon(\mu_{(r,j),(s,i)}), & \text{if } r < s, \end{cases}$$

and hence also that if $r, s < n$ then $\epsilon_{\mathbb{F}_p}(\Psi'_{(r,j),(s,i)}) = \epsilon_{\mathbb{F}_p}(\Psi_{(r,j),(s,i)})$ (this equality is trivial for $r = n$ or $s = n$).

The ring $\mathbb{Z}_p[G]$ is local with maximal ideal equal to $\ker(\epsilon_{\mathbb{F}_p})$. In particular an element x of $\mathbb{Z}_p[G]$ is a unit if and only if $\epsilon_{\mathbb{F}_p}(x) \neq 0$.

One then knows that $\epsilon_{\mathbb{F}_p}(\det(\Psi)) = \det(\epsilon_{\mathbb{F}_p}(\Psi)) = \det(\epsilon_{\mathbb{F}_p}(\Psi')) = \epsilon_{\mathbb{F}_p}(\det(\Psi')) \neq 0$. It thus follows that $\det(\Psi) \in \mathbb{Z}_p[G]^\times$ and therefore that γ is bijective, as required. □

Now, if we denote by $[f]$ the pre-image in $\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(A(F)^*_p, A^t(F)_p)$ under the isomorphism (13) of the class of an endomorphism f of $A^t(F)_p$, then Lemma 3.3 implies that $[\gamma] - [-\phi^{-1}] = [\psi] = 0$ and hence also that $[\gamma] = [-\phi^{-1}]$.

In addition, since both γ and $-\phi^{-1}$ are bijective, this class in $\mathrm{Ext}^2_{\mathbb{Z}_p[G]}(A(F)^*_p, A^t(F)_p)$ may be represented by both of the exact sequences obtained by replacing ϕ^{-1} by γ^{-1} or by $-\phi$ in the exact sequence (15).

By the general result [7, Lem. 4.7] there exist automorphisms κ^1 and κ^2 of X with the property that the (exact) diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^t(F)_p & \xrightarrow{\iota\circ\gamma^{-1}} & X & \xrightarrow{\Theta} & X & \xrightarrow{\pi} & A(F)^*_p & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\kappa^1} & & \downarrow{\scriptstyle\kappa^2} & & \| & & \\ 0 & \longrightarrow & A^t(F)_p & \xrightarrow{\iota\circ(-\phi)} & X & \xrightarrow{\Theta} & X & \xrightarrow{\pi} & A(F)^*_p & \longrightarrow & 0 \end{array} \tag{24}$$

is commutative.

Now, one may compute (by, for instance, the argument of [4, Prop. 4.4]) the desired sum as

$$\sum_{\psi\in\widehat{G}}\frac{\varepsilon_\psi(\Psi)}{\varepsilon_\psi(\Psi')}\cdot e_\psi=\frac{\det_{\mathbb{Q}_p[G]}(\langle\gamma,\Theta,s_1,s_2\rangle)}{\det_{\mathbb{Q}_p[G]}(\langle-\phi^{-1},\Theta,s_1,s_2\rangle)}$$

where, for any ($\mathbb{Q}_p[G]$-equivariant) splittings s_1 and s_2 as in (16) and (17), and any endomorphism β of $A^t(F)_p$, we have written $\langle\beta,\Theta,s_1,s_2\rangle$ for the composite isomorphism

$$\mathbb{Q}_p\cdot X\xrightarrow{s_1}\mathbb{Q}_p\cdot A^t(F)_p\oplus\mathbb{Q}_p\cdot\mathrm{im}(\Theta)\xrightarrow{\beta\oplus\mathrm{id}}\mathbb{Q}_p\cdot A^t(F)_p\oplus\mathbb{Q}_p\cdot\mathrm{im}(\Theta)$$
$$\to\mathbb{Q}_p\cdot A(F)_p^*\oplus\mathbb{Q}_p\cdot\mathrm{im}(\Theta)\xrightarrow{s_2^{-1}}\mathbb{Q}_p\cdot X,$$

with the unlabeled arrow simply mapping a point $P^t_{(s,i)}$ to $P^*_{(s,i)}$.
It is then straightforward to deduce from the commutativity of the diagram (24) that

$$\sum_{\psi\in\widehat{G}}\frac{\varepsilon_\psi(\Psi)}{\varepsilon_\psi(\Psi')}\cdot e_\psi=\det_{\mathbb{Z}_p[G]}(\kappa_1)\cdot\det_{\mathbb{Z}_p[G]}(\kappa_2)^{-1}$$

and, since both of the determinants on the right hand side are by construction elements of $\mathbb{Z}_p[G]^\times$, this concludes the proof of Lemma 3.2.

□

The following general result was used in the proof of Lemma 3.3 and is straightforward to prove.

Lemma 3.4. *For any indices $0\le r\le\ell\le n$ and any elements Ψ and Ψ' of $\mathbb{Z}_p[G]$, one has that*

$$\rho_r(\Psi)\otimes(\sigma^{p^\ell}-1)=\rho_r(\Psi')\otimes(\sigma^{p^\ell}-1)$$

in $\mathbb{Z}_p[\Gamma_r]\otimes_{\mathbb{Z}_p}I_p(J_\ell)/I_p(J_\ell)^2$ if and only if $p^{\ell-r}(\Psi-\Psi')$ belongs to the ideal $(\sigma^{p^r}-1)\cdot\mathbb{Z}_p[G]+\mathrm{Tr}_{J_r}\cdot\mathbb{Z}_p[G]$ of $\mathbb{Z}_p[G]$.

4. Computation of the Mazur-Tate pairing

In this section we explain how one may numerically compute the Mazur-Tate pairing. This computation can be reduced to the computation of local Tate duality pairings which, in turn, may in simple situations be computed by the evaluation of Hilbert symbols thanks to recent results of Fisher and Newton [13] or of Visse [22].

4.1. **The general strategy.** We continue to assume the hypotheses of §2.1. In particular, [8, Prop. 6.3(ii)] implies that every element of $A^t(k)_p$ and $A(k)_p$ is 'locally-normed'.
Under this condition, Bertolini and Darmon have defined in [1, §3.4.1] and [2, §2.2] a pairing

$$\langle\ ,\ \rangle_1\colon A^t(k)_p\otimes_{\mathbb{Z}_p}A(k)_p\longrightarrow G.$$

Although the definition of this pairing is only given in the case that A is an elliptic curve, it extends naturally to our more general setting.

The results of Bertolini and Darmon in [2, Thm. 2.8 and Rem. 2.10] and of Tan in [21, Prop. 3.1] combine to directly show that the pairing $\langle\ ,\ \rangle_1$ coincides with the Mazur-Tate pairing $\langle\ ,\ \rangle^{\mathrm{MT}}_{F/k}$. We are therefore left with the task to describe the explicit computation of $\langle\ ,\ \rangle_1$.
Let B be either A or its dual A^t. For a finite set S of non archimedean places of k we define

$$\mathrm{Sel}^{(p^n)}_S(B/F) \leq H^1(F, B[p^n])$$

to be the kernel of the localisation map

$$H^1(F, B[p^n]) \longrightarrow \prod_{w \notin S(F)} H^1(F_w, B).$$

(with the product running over all non-archimedean places of F that do not belong to the set $S(F)$ of places that lie above a place in S).
By Kummer theory we then have

$$\begin{aligned} & \mathrm{Sel}^{(p^n)}_S(B/F) \\ = \ & \{\xi \in H^1(F, B[p^n]) \mid \mathrm{res}_w(\xi) \in \delta_w(B(F_w)/p^n B(F_w)) \text{ for all } w \notin S(F)\}. \end{aligned}$$

Here $\mathrm{res}_w : H^1(F, B[p^n]) \to H^1(F_w, B[p^n])$ denotes the canonical localisation map and $\delta_w : B(F_w)/p^n B(F_w) \to H^1(F_w, B[p^n])$ denotes the canonical Kummer map. In particular, when S is taken to be the empty set, one recovers the usual Selmer group $\mathrm{Sel}^{(p^n)}(B/F)$ associated with multiplication by p^n.
We set $Z := \mathbb{Z}/p^n\mathbb{Z}$ and $R := Z[G]$ and define additional R-modules

$$B_S(\mathbb{A}_F)/p^n := \prod_{w \in S(F)} B(F_w)/p^n B(F_w)$$

and

$$H^1_S(\mathbb{A}_F, B[p^n]) := \prod_{w \in S(F)} H^1(F_w, B[p^n]).$$

We recall that by local Tate duality we have a perfect equivariant pairing

$$\langle\ ,\ \rangle \colon H^1_S(\mathbb{A}_F, A^t[p^n]) \times H^1_S(\mathbb{A}_F, A[p^n]) \longrightarrow R$$

explicitly given by

$$\langle x, y\rangle = \sum_{g \in G} \langle x^g, y\rangle_S\, g^{-1} \tag{25}$$

where

$$\langle x, z\rangle_S = \sum_{w \in S(F)} \langle x_w, z_w\rangle_{F_w, p^n}.$$

Here, for $L = F_w$ and any $w \in S(F)$, we have written

$$\langle\ ,\ \rangle_{L,p^n} \colon H^1(L, A^t[p^n]) \times H^1(L, A[p^n]) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

for the local Tate duality pairing obtained by combining the cup product, the Weil pairing and the invariant map as follows:

$$\begin{aligned} H^1(L, A^t[p^n]) \times H^1(L, A[p^n]) &\xrightarrow{\cup} H^2(L, A^t[p^n] \times A[p^n]) \\ &\longrightarrow H^2(L, \mu_{p^n}) \\ &\xrightarrow{\mathrm{inv}_L} \mathbb{Q}_p/\mathbb{Z}_p. \end{aligned}$$

We now recall the explicit definition of $\langle P, Q\rangle_1$ for $P \in A^t(k)$ and $Q \in A(k)$. Let Σ be an admissible set of primes as in [1, Def. 2.22] or [2, Def. 1.5]. We set

$$X = X_\Sigma := \mathrm{Sel}_\Sigma^{(p^n)}(A^t/F) \text{ and } Y = Y_\Sigma := A_\Sigma(\mathbb{A}_F)/p^n.$$

The reader should be aware that this notation is consistent with [8] but differs from the notation used by Bertolini and Darmon in [1] or [2] where X and Y are interchanged. We write δ for the canonical global Kummer map and let $\tilde{x} \in X^G$ denote the image of P under the canonical composition

$$A^t(k) \longrightarrow (A^t(F)/p^n A^t(F))^G \xrightarrow{\delta^G} \mathrm{Sel}_\Sigma^{(p^m)}(A^t/F)^G = X^G.$$

We also let $\tilde{y} \in Y^G$ be the image of Q under the canonical (diagonal) localisation map

$$A(k) \longrightarrow (A(F)/p^n A(F))^G \longrightarrow (A_\Sigma(\mathbb{A}_F)/p^n)^G = Y^G.$$

By [1, §3.1] the R-modules X and Y are G-cohomologically trivial. We therefore find elements $x \in X$ and $y = (y_w)_w \in Y$ such that

$$\mathrm{Tr}_G(x) = \tilde{x}, \quad \mathrm{Tr}_G(y) = \tilde{y}.$$

We next consider the canonical (diagonal) localisation map

$$X = \mathrm{Sel}_\Sigma^{(p^n)}(A^t/F) \subseteq H^1(F, A^t[p^n]) \xrightarrow{\oplus\,\mathrm{res}_w} \oplus_{w\in\Sigma(F)} H^1(F_w, A^t[p^n])$$

and the product of local Kummer maps

$$Y = \prod_{w\in\Sigma(F)} A(F_w)/p^n A(F_w) \xrightarrow{\oplus\delta_w} \bigoplus_{w\in\Sigma(F)} H^1(F_w, A[p^n]).$$

Then by definition (see also the proof of [2, Th. 2.7] or the paragraph preceeding equation (109) in [8]),

$$\langle P, Q\rangle_1 \equiv \langle x, y\rangle \pmod{I_p(G)^2}$$

with $\langle\ ,\ \rangle$ as in (25), or explicitly,

$$\langle P, Q\rangle_1 \equiv \sum_{g\in G}\left(\sum_{w\in\Sigma(F)} \langle \mathrm{res}_w(x^g), \delta_w(y_w)\rangle_{F_w,p^n}\right) g^{-1}. \tag{26}$$

Noting the sign involved in the definition (9) of the equivariant regulator, we are in fact interested in computing the inverse $-\langle P, Q\rangle_1$. We obtain the expression

$$-\langle P, Q\rangle_1 \equiv \sum_{g\in G}\left(\sum_{w\in\Sigma(F)} \langle \mathrm{res}_w(x^g), \delta_w(y_w)\rangle_{F_w,p^n}\right) g, \tag{27}$$

which is convenient for the explicit evaluations which we will describe in the next subsection.

4.2. **Algorithmic evaluation of the pairing.** In this subsection we describe how we algorithmically evaluate the right hand side of (27). From an algorithmic point of view we are mainly interested in the case of elliptic curves and therefore henceforth assume that $E := A = A^t$ is an elliptic curve defined over k. Furthermore, we assume that F/k is cyclic of degree p where as before p is an odd prime satisfying our running hypotheses.

We fix an algebraic closure $\bar{F}$ of F and let $W \subseteq E[p] \setminus \{0\}$ be a $G_F := \mathrm{Gal}(\bar{F}/F)$-invariant spanning set for $E[p]$. We write $\bar{A} := \mathrm{Map}(W, \bar{F})$ for the set of maps from W to $\bar{F}$. Then the Galois group G_F acts on $\bar{A}$ by conjugation,

$$(\sigma a)(P) = \sigma(a(\sigma^{-1}(P))) \text{ for all } \sigma \in G_F, a \in \bar{A}, P \in W.$$

We denote by $A = \mathrm{Map}_{G_F}(W, \bar{F}) := \mathrm{Map}(W, \bar{F})^{G_F}$ the set of G_F-invariant maps in $\bar{A}$. Then A is a finite dimensional étale F-algebra. Explicitly, let $P_1, \ldots, P_s \in W$ be a set of G_F-orbit representatives of W and set $H_i := \{\sigma \in G_F \mid \sigma(P_i) = P_i\}$ and $L_i := \bar{F}^{H_i}$. Then each L_i/F is a finite separable field extension and we have a canonical isomorphism

$$A \longrightarrow \prod_{i=1}^{s} L_i, \quad a \mapsto (a(P_i))_{1 \le i \le s}.$$

The Weil pairing e_p defines a map

$$\omega\colon E[p] \longrightarrow \mathrm{Map}(W, \mu_p(\bar{F})) = \mu_p(\bar{A}), \quad P \mapsto e_p(P, \_\,).$$

This induces a map

$$\bar{\omega}\colon H^1(F, E[p]) \longrightarrow H^1(F, \mu_p(\bar{A})) \tag{28}$$

which is known to be injective if $p \nmid |\mathrm{Gal}(F(E[p])/F)|$ or $p \nmid |W|$ (see [19, Prop. 4.3]). In addition, by an immediate generalization of Hilbert's Theorem 90, we have a Kummer isomorphism

$$\kappa\colon H^1(F, \mu_p(\bar{A})) \longrightarrow A^\times/A^{\times p} \tag{29}$$

which combined with $\bar{\omega}$ defines an embedding (assuming for example that $p \nmid |W|$)

$$H^1(F, E[p]) \hookrightarrow A^\times/A^{\times p}.$$

For any finite set V of places of F we write $H^1(F, E[p]; V)$ for the group of cohomology classes that are unramified outside V. Henceforth we fix a finite set V of places of F containing the p-adic places and all places w such that the Tamagawa number c_w of E at w is divisible by p. Then for a fixed set Σ of places of k that is admissible (as in §4.1) one has that

$$\begin{aligned}
& \mathrm{Sel}^{(p)}(E/F) \\
= \ & \{\xi \in H^1(F, E[p]; V \cup \Sigma(F)) \mid \mathrm{res}_w(\xi) \in \delta_w(E(F_w)/pE(F_w)) \text{ for all } w \in V \cup \Sigma(F)\}, \\
& \mathrm{Sel}^{(p)}_{\Sigma}(E/F) \\
= \ & \{\xi \in H^1(F, E[p]; V \cup \Sigma(F)) \mid \mathrm{res}_w(\xi) \in \delta_w(E(F_w)/pE(F_w)) \text{ for all } w \in V \setminus \Sigma(F)\}.
\end{aligned}$$

Roughly speaking, the algorithm of Schaefer and Stoll [19] in a first step computes $H^1(F, E[p]; V \cup \Sigma(F))$ as a subset of $A^\times/A^{\times p}$ and then in a second step checks the local conditions.

To describe the first step we let L/F be a finite extension. For any finite set T of places of F we set

$$L(T,p) := \{a \in L^\times/L^{\times p} \mid \mathrm{ord}_s(a) \in p\mathbb{Z} \text{ for all } s \notin T(L)\}.$$

If $A \simeq \prod_{i=1}^s L_i$, then we define

$$A(T,p) := \prod_{i=1}^s L_i(T,p).$$

Then, using the embedding $H^1(F, E[p]) \hookrightarrow A^\times/A^{\times p}$, one has by [19, Cor. 5.9].

$$H^1(F, E[p]; V \cup \Sigma(F)) = H^1(F, E[p]) \cap A(V \cup \Sigma(F), p).$$

The work of Schaefer and Stoll now describes how to compute the group $H^1(F, E[p])$ inside $A^\times/A^{\times p}$ and by intersecting with the finite group $A(V \cup \Sigma(F), p)$ we obtain $H^1(F, E[p]; V \cup \Sigma(F))$ as a subgroup of $A^\times/A^{\times p}$.

Testing the local conditions we finally obtain both $\mathrm{Sel}^{(p)}(E/F)$ and $\mathrm{Sel}^{(p)}_\Sigma(E/F)$ as subgroups of $A^\times/A^{\times p}$, or even better, as subgroups of the finite group $A(V \cup \Sigma(F), p)$. Note however, that the computation of $A(V \cup \Sigma(F), p)$ requires the computation of ideal class groups and units in all of the fields L_i, $i = 1, .., s$. In the generic case G_F acts transitively on $E[p] \setminus \{0\}$ so that we are forced to use $W = E[p] \setminus \{0\}$. In this case we fix $S_0 \in W$ and set $L := F(S_0)$. Then $[L : F] = p^2 - 1$ and $A \simeq L$.

For any place w in $\Sigma(F)$ we fix an embedding $\iota_w : F \to F_w$ and consider the following commutative diagram

$$\begin{array}{ccccccc} E(F)/pE(F) & \xrightarrow{\delta} & H^1(F, E[p]) & \xrightarrow{\bar\omega} & H^1(F, \mu_p(\bar A)) & \xrightarrow{\kappa} & A^\times/A^{\times p} \\ \downarrow{\scriptstyle \iota_w} & & \downarrow{\scriptstyle \mathrm{res}_w} & & \downarrow{\scriptstyle \mathrm{res}_w} & & \downarrow{\scriptstyle \iota_{w,*}} \\ E(F_w)/pE(F_w) & \xrightarrow{\delta_w} & H^1(F_w, E[p]) & \xrightarrow{\bar\omega_w} & H^1(F_w, \mu_p(\bar A_w)) & \xrightarrow{\kappa_w} & A_w^\times/A_w^{\times p}, \end{array} \tag{30}$$

where $A_w := F_w \otimes_F A$ and $\iota_{w,*}$ is induced by composition with ι_w while $\bar A_w = \bar F_w \otimes_F A$, the maps $\bar\omega_w$ and κ_w are the local analogues of (28) and (29), respectively, and res_w is the localisation map. Note that there is also a canonical isomorphism

$$A_w = F_w \otimes_F A \xrightarrow{\simeq} \mathrm{Map}_{G_{F_w}}(W, \bar F_w), \quad a \otimes z \mapsto (S \mapsto \iota_w(a(S))z)\,.$$

The local Tate pairing

$$\langle\ ,\ \rangle_{F_w,p} \colon H^1(F_w, E[p]) \times H^1(F_w, E[p]) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

considered in §4.1 induces a pairing $\langle\ ,\ \rangle_{A_w}$ on the image of $\kappa_w \circ \bar\omega_w$. In what follows we will describe the computation of $\langle\ ,\ \rangle_{A_w}$.

We write q_{A_w} for the unique quadratic form such that for all $a, b \in (\kappa_w \circ \bar\omega_w)\big(H^1(F_w, E[p])\big)$ one has

$$\langle a, b\rangle_{A_w} = q_{A_w}(ab) - q_{A_w}(a) - q_{A_w}(b).$$

Let $\{\ ,\ \}_{F_w,p} \colon F_w^\times/F_w^{\times p} \times F_w^\times/F_w^{\times p} \longrightarrow \mu_p$ denote the Hilbert symbol. We henceforth assume that $E[p]$ is contained in $E(F_w)$ (which is always the case for places $w \in \Sigma(F)$ for an admissible set Σ of places of k). In this particular case one can shift the problem of computing the local Tate pairing to the computation of Hilbert symbols which is easy for places w which are prime to p by [18, Ch. V, Prop. 3.4].

Fix generators $S, T \in E(F_w)$ of $E[p]$ and define

$$\xi_{S,T} \colon \mu_p \longrightarrow \mathbb{Z}/p\mathbb{Z} \quad \text{by} \quad e_p(T,S) \mapsto 1 + p\mathbb{Z}.$$

Then, by [22, Th. 3.7 and 3.11] or by [13] in the case $p = 3$, we have

$$q_{A_w}(a) = \xi_{S,T}\left(\{a(S), a(T)\}_{F_w,p}\right)$$

for all $a \in (\kappa_w \circ \bar{\omega}_w)\left(H^1(F_w, E[p])\right)$. As a consequence we obtain

$$\langle a, b\rangle_{A_w} = \xi_{S,T}\left(\frac{\{a(S), b(T)\}_{F_w,p}}{\{a(T), b(S)\}_{F_w,p}}\right). \tag{31}$$

We are now in position to describe the explicit computation of $-\langle P, Q\rangle_1$ using the formula in (27). Let $\tilde{y}$ denote the image of Q in $Y_\Sigma = \oplus_{w \in \Sigma(F)} E(F_w)/pE(F_w)$. Assuming that the point Q is not divisible by p ensures that $\tilde{y} \neq 0$ since admissibility guarantees that the composition

$$E(F)/pE(F) \hookrightarrow \mathrm{Sel}^{(p)}(E/F) \hookrightarrow Y_\Sigma.$$

is injective. For each $v \in \Sigma$ we fix a place $\hat{v}$ of F lying over v. For each $w \in \Sigma(F)$ we define

$$y_w := \begin{cases} \tilde{y}_{\hat{v}}, & \text{if } w \mid v \text{ and } w = \hat{v}, \\ 0, & \text{otherwise }. \end{cases}$$

Then $y := (y_w)_{w \in \Sigma(F)}$ satisfies $\mathrm{Tr}_G(y) = \tilde{y}$ and the formula in (27) simplifies to

$$-\langle P, Q\rangle_1 \equiv \sum_{g \in G}\left(\sum_{v \in \Sigma}\langle \mathrm{res}_{\hat{v}}(x^g), \delta_{\hat{v}}(y_{\hat{v}})\rangle_{F_{\hat{v}},p}\right) g, \tag{32}$$

Next we explain how to compute $x \in X_\Sigma = \mathrm{Sel}_\Sigma^{(p)}(E/F)$ with $\mathrm{Tr}_G(x) = \tilde{x}$. The generalized Selmer group X_Σ is computed as a subgroup of $A^\times/A^{\times p}$ which carries an action of G induced by conjugation,

$$(\sigma a)(S) = \hat{\sigma}(a(\hat{\sigma}^{-1}(S)))$$

for all $\sigma \in G, a \in A, S \in W$ and an arbitrary lift $\hat{\sigma} \in G_k$ of σ. We can easily compute the $\mathbb{F}_p$-representation of G induced by X_Σ and then represent Tr_G as a linear map $X_\Sigma \longrightarrow X_\Sigma$ and thus compute a preimage x of $\tilde{x} = \delta(P)$.
In the same way we can compute the elements x^g occuring in (32). Note that as an element of $A^\times/A^{\times p}$ the element $\tilde{x}$ is given by $(\kappa \circ \bar{\omega} \circ \delta)(P)$. Furthermore, the explicit computation of $H := \kappa \circ \bar{\omega} \circ \delta$ and $H_w := \kappa_w \circ \bar{\omega}_w \circ \delta_w$ is explained in [20, Ch. 2] and [19, Ch. 3].
For each $v \in \Sigma$ we now compute the completion $F_{\hat{v}}$ together with an embedding $\iota_v \colon F \longrightarrow F_{\hat{v}}$ and define $a_{P,g,\hat{v}} := \mathrm{res}_{\hat{v}}(x^g) = \iota_v \circ x^g$.
In order to compute $\delta_{\hat{v}}(y_{\hat{v}})$ we use the commutativity of diagram (30)and define

$$a_{Q,\hat{v}} := \kappa_{\hat{v}}(\bar{\omega}_{\hat{v}}(\delta_{\hat{v}}(\iota_{\hat{v}}(Q)))) = \iota_{\hat{v}} \circ (\kappa(\bar{\omega}(\delta(Q))) = \iota_{\hat{v}} \circ H(Q)$$

and finally obtain

$$\langle \mathrm{res}_{\hat{v}}(x^g), \delta_{\hat{v}}(y_{\hat{v}})\rangle_{F_{\hat{v}},p} = \xi_{S,T}\left(\frac{\{a_{P,g,\hat{v}}(S), a_{Q,\hat{v}}(T)\}_{F_w,p}}{\{a_{P,g,\hat{v}}(T), a_{Q,\hat{v}}(S)\}_{F_w,p}}\right).$$

4.3. **Comments on the implementation.** Let us assume that F/k is cyclic of order p and let E/k be an elliptic curve. Assume that $W = E[p] \setminus \{0\}$ and that G_F acts transitively on W.
We choose $\lambda \in \mathbb{Q}$ such that the elements

$$w_S := y_S + \lambda x_S, \quad S = (x_S, y_S) \in W,$$

are pairwise distinct. Since G_F acts transitively on W the polynomial

$$f(x) := \prod_{S \in W} (x - w_S) \in F[x]$$

is irreducible. It is easily seen that $F(S_0) = F(w_{S_0})$. We therefore may and will use $L := F(w_{S_0})$. This has the following useful consequence for the computation of $\iota_w \circ a$ for $a \in A = \mathrm{Map}_{G_F}(W, \bar{F})$.
Assume that $a \in A$ corresponds to $\bar{h} \in F[x]/(f(x))$ under the composite

$$A \xrightarrow{\alpha} L \xrightarrow{\beta^{-1}} F[x]/(f(x)), \quad \alpha(a) := a(S_0), \quad \beta(\bar{h}) := h(w_{S_0}).$$

Assume that $\tilde{S} \in E(F_w)[p]$ and let $S \in W$ be such that $\iota_w(S_0) = \tilde{S}$. Then a straightforward computation shows

$$\begin{aligned} & a_w(\tilde{S}) = \iota_w(a(S)) = \iota_w(a(\tau(S_0))) = \iota_w(\tau(a(S_0))) = \iota_w(\tau(h(w_{S_0}))) \\ = \quad & \iota_w(\tau(h(w_{S_0}))) = \iota_w(h(\tau(w_{S_0}))) = \iota_w(h(w_S)) = (\iota_w h)(\iota_w(w_S)) = (\iota_w h)(w_{\tilde{S}}). \end{aligned}$$

A further useful consequence is a particularly simple description of the action of G on A. Recall that $\sigma \in G$ acts on A in the following way: choose a lift $\hat{\sigma} \in G_k$ of σ, then $(\sigma a)(S) = \hat{\sigma}(a(\hat{\sigma}^{-1}(S)))$. If $\bar{h} \in F[x]/(f(x))$ corresponds to $a \in A$, then it is not difficult to show that σa corresponds to $\overline{\sigma h}$. Indeed, $a(S_0) = h(w_{S_0})$ and if $\tau(S_0) = \hat{\sigma}^{-1}(S_0)$ with $\tau \in G_F$, then

$$(\sigma a)(S_0) = \hat{\sigma}(a(\hat{\sigma}^{-1}(S_0))) = \hat{\sigma}(\tau(h(w_{S_0}))) = \hat{\sigma}(h(\tau(w_{S_0}))) = (\sigma h)(w_{S_0}).$$

We finally remark that if we wish to work with $p = 3$ and assume that E is given in special Weierstrass form

$$E \colon y^2 = x^3 + ax + b$$

with $a \neq 0$ then the y-coordinates of all 3-torsion points are pairwise distinct, so that we can use $w_S = y_S$ for all $S \in W$. In addition, in those cases there is a closed formula for f in terms of a and b, see [14, Sec. 7.1].

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References

[1] M. Bertolini, H. Darmon, Derived heights and generalized Mazur-Tate regulators, Duke Math Journal **76** No. 1 (1994) pp. 75-111.

[2] M. Bertolini, H. Darmon, Derived p-adic heights, American Journal of Math **117** (1995) pp. 1517-1554.

[3] W. Bley, The equivariant Tamagawa number conjecture and modular symbols, Math. Ann. **356** (2013), 179-190.

[4] W. Bley, D. Macias Castillo, Congruences for critical values of higher derivatives of twisted Hasse-Weil L-functions, J. reine u. angew. Math. **722** (2017) 105-136.

[5] D. Burns, M. Flach, Tamagawa numbers for motives with (non-commutative) coefficients, Doc. Math. **6** (2001) 501-570.

[6] D. Burns, M. Kurihara, T. Sano, On zeta elements for $\mathbb{G}_m$, Documenta Math. **21** (2016) 555-626.

[7] D. Burns, D. Macias Castillo, Organising matrices for arithmetic complexes, Int. Math. Res. Notices **2014** 10 (2014) 2814-2883.

[8] D. Burns, D. Macias Castillo, On refined conjectures of Birch and Swinnerton-Dyer type for Artin-Hasse-Weil L-series, submitted for publication.

[9] D. Burns, D. Macias Castillo, C. Wuthrich, On Mordell-Weil groups and congruences between derivatives of twisted Hasse-Weil L-functions, J. reine angew. Math. **734** (2017) 187-228.

[10] D. Burns, O. Venjakob, On descent theory and main conjectures in non-commutative Iwasawa theory, J. Inst. Math. Jussieu **10** (2011) 59-118.

[11] J. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob, The GL_2-main conjecture for elliptic curves without complex multiplication, Publ. IHES **101** (2005) 163-208.

[12] H. Darmon, A refined conjecture of Mazur-Tate type for Heegner points, Invent. Math. **110** (1992) 123-146.

[13] T. Fisher, R. Newton, Computing the Cassels-Tate pairing on the 3-Selmer group of an elliptic curve, ArXiv e-prints, June 2013

[14] C. Geishauser, Computation of 2-extensions of dual Selmer groups, Master thesis, LMU 2018.

[15] D. Macias Castillo, Congruences for critical values of higher derivatives of twisted Hasse-Weil L-functions, II, to appear in Acta Arith..

[16] B. Mazur, J. Tate, Canonical height pairings via biextensions, In: 'Arithmetic and Geometry' vol. 1, Prog. Math. **35** (1983) 195-237.

[17] B. Mazur, J. Tate, Refined Conjectures of the Birch and Swinnerton-Dyer Type, Duke Math. J. **54** (1987) 711-750.

[18] J. Neukirch, Algebraic Number Theory, Springer-Verlag, 1999.

[19] E.F. Schaefer, M. Stoll, How to do a p-descent on an ellitpic curve, Transactions of the AMS, **356** (2003) 1209-1231.

[20] E.F. Schaefer, , Computing a Selmer group of a Jacobian using functions on the curve, ArXiv e-prints, July 2015.

[21] K.-S. Tan, p-adic pairings, Contemp. Math. **165** (1994) 111-121.

[22] E. Visse, Calculating the Tate local pairing for any odd prime number, ArXiv e-prints, October 2016.

[23] A. V. Yakovlev, Homological definability of p-adic representations of groups with cyclic Sylow p-subgroup, An. St. Univ. Ovidius Constanţa **4** (1996) 206-221.

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