to show that this process can be iterated until some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row $(0, *)$. Show that in fact the bottom row is $(0, \pm 1)$, and since $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]^2 = -I$ it can be taken to be $(0, 1)$. Show that therefore $\alpha\gamma \in \Gamma$ and so $\alpha \in \Gamma$. Thus Γ is all of $\mathrm{SL}_2(\mathbf{Z})$.

1.1.2. (a) Show that $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)/|c\tau + d|^2$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.
(b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$.
(c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau + d)^2$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.

1.1.3. (a) Show that the set $\mathcal{M}_k(\mathrm{SL}_2(\mathbf{Z}))$ of modular forms of weight k forms a vector space over $\mathbf{C}$.
(b) If f is a modular form of weight k and g is a modular form of weight l, show that fg is a modular form of weight $k + l$.
(c) Show that $\mathcal{S}_k(\mathrm{SL}_2(\mathbf{Z}))$ is a vector subspace of $\mathcal{M}_k(\mathrm{SL}_2(\mathbf{Z}))$ and that $\mathcal{S}(\mathrm{SL}_2(\mathbf{Z}))$ is an ideal in $\mathcal{M}(\mathrm{SL}_2(\mathbf{Z}))$.

1.1.4. Let $k \geq 3$ be an integer and let $L' = \mathbf{Z}^2 - \{(0,0)\}$.
(a) Show that the series $\sum_{(c,d) \in L'} (\sup\{|c|, |d|\})^{-k}$ converges by considering the partial sums over expanding squares.
(b) Fix positive numbers A and B and let

$$\Omega = \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \leq A, \mathrm{Im}(\tau) \geq B\}.$$

Prove that there is a constant $C > 0$ such that $|\tau + \delta| > C \sup\{1, |\delta|\}$ for all $\tau \in \Omega$ and $\delta \in \mathbf{R}$. (Hints for this exercise are at the end of the book.)
(c) Use parts (a) and (b) to prove that the series defining $G_k(\tau)$ converges absolutely and uniformly for $\tau \in \Omega$. Conclude that G_k is holomorphic on $\mathcal{H}$.
(d) Show that for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, right multiplication by γ defines a bijection from L' to L'.
(e) Use the calculation from (c) to show that G_k is bounded on Ω. From the text and part (d), G_k is weakly modular so in particular $G_k(\tau + 1) = G_k(\tau)$. Show that therefore $G_k(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$.

1.1.5. Establish the two formulas for $\pi \cot \pi\tau$ in (1.1). (A hint for this exercise is at the end of the book.)

1.1.6. This exercise obtains formula (1.2) without using the cotangent. Let $f(\tau) = \sum_{d \in \mathbf{Z}} 1/(\tau + d)^k$ for $k \geq 2$ and $\tau \in \mathcal{H}$. Since f is holomorphic (by the method of Exercise 1.1.4) and $\mathbf{Z}$-periodic and since $\lim_{\mathrm{Im}(\tau) \to \infty} f(\tau) = 0$, there is a Fourier expansion $f(\tau) = \sum_{m=1}^{\infty} a_m q^m = g(q)$ as in the section, where $q = e^{2\pi i \tau}$ and

$$a_m = \frac{1}{2\pi i} \int_{\gamma} \frac{g(q)}{q^{m+1}} dq$$

is a path integral once counterclockwise over a circle about 0 in the punctured disk D'.
(a) Show that

$$a_m = \int_{\tau=0+iy}^{1+iy} f(\tau) e^{-2\pi i m \tau} d\tau = \int_{\tau=-\infty+iy}^{+\infty+iy} \tau^{-k} e^{-2\pi i m \tau} d\tau \quad \text{for any } y > 0.$$

(b) Let $g_m(\tau) = \tau^{-k} e^{-2\pi i m \tau}$, a meromorphic function on $\mathbf{C}$ with its only singularity at the origin. Show that

$$-2\pi i \mathrm{Res}_{\tau=0} g_m(\tau) = \frac{(-2\pi i)^k}{(k-1)!} m^{k-1}.$$

(c) Establish (1.2) by integrating $g_m(\tau)$ clockwise about a large rectangular path and applying the Residue Theorem. Argue that the integral along the top side goes to a_m and the integrals along the other three sides go to 0.
(d) Let $h : \mathbf{R} \longrightarrow \mathbf{C}$ be a function such that the integral $\int_{-\infty}^{\infty} |h(x)| dx$ is finite and the sum $\sum_{d \in \mathbf{Z}} h(x + d)$ converges absolutely and uniformly on compact subsets and is infinitely differentiable. Then the *Poisson summation formula* says that

$$\sum_{d \in \mathbf{Z}} h(x + d) = \sum_{m \in \mathbf{Z}} \hat{h}(m) e^{2\pi i m x}$$

where $\hat{h}$ is the *Fourier transform* of h,

$$\hat{h}(x) = \int_{t=-\infty}^{\infty} h(t) e^{-2\pi i x t} dt.$$

We will not prove this, but the idea is that the left side sum symmetrizes h to a function of period 1 and the right side sum is the Fourier series of the left side since the mth Fourier coefficient is $\int_{t=0}^{1} \sum_{d \in \mathbf{Z}} h(t + d) e^{-2\pi i m t} dt = \hat{h}(m)$. Letting $h(x) = 1/\tau^k$ where $\tau = x + iy$ with $y > 0$, show that h meets the conditions for Poisson summation. Show that $\hat{h}(m) = e^{-2\pi m y} a_m$ with a_m from above for $m > 0$, and that $\hat{h}(m) = 0$ for $m \leq 0$. Establish formula (1.2) again, this time as a special case of Poisson summation. We will see more Poisson summation and Fourier analysis in connection with Eisenstein series in Chapter 4. (A hint for this exercise is at the end of the book.)

1.1.7. The *Bernoulli numbers* B_k are defined by the formal power series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Thus they are calculable in succession by matching coefficients in the power series identity

$$t = (e^t - 1) \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} B_k \right) \frac{t^n}{n!}$$

(i.e., the nth parenthesized sum is 1 if $n = 1$ and 0 otherwise) and they are rational. Since the expression

$$\frac{t}{e^t-1}+\frac{t}{2}=\frac{t}{2}\cdot\frac{e^t+1}{e^t-1}$$

is even, it follows that $B_1 = -1/2$ and $B_k = 0$ for all other odd k. The Bernoulli numbers will be motivated, discussed, and generalized in Chapter 4.

(a) Show that $B_2 = 1/6$, $B_4 = -1/30$, and $B_6 = 1/42$.

(b) Use the expressions for $\pi\cot\pi\tau$ from the section to show

$$1-2\sum_{k=1}^{\infty}\zeta(2k)\tau^{2k}=\pi\tau\cot\pi\tau=\pi i\tau+\sum_{k=0}^{\infty}B_k\frac{(2\pi i\tau)^k}{k!}.$$

Use these to show that for $k \geq 2$ even, the Riemann zeta function satisfies

$$2\zeta(k)=-\frac{(2\pi i)^k}{k!}B_k,$$

so in particular $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$. Also, this shows that the normalized Eisenstein series of weight k

$$E_k(\tau)=\frac{G_k(\tau)}{2\zeta(k)}=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

has rational coefficients with a common denominator.

(c) Equate coefficients in the relation $E_8(\tau) = E_4(\tau)^2$ to establish formula (1.3).

(d) Show that $a_0 = 0$ and $a_1 = (2\pi)^{12}$ in the Fourier expansion of the discriminant function Δ from the text.

1.1.8. Recall that μ_3 denotes the complex cube root of unity $e^{2\pi i/3}$. Show that $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right](\mu_3) = \mu_3 + 1$ so that by periodicity $g_2(\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right](\mu_3)) = g_2(\mu_3)$. Show that by modularity also $g_2(\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right](\mu_3)) = \mu_3^4 g_2(\mu_3)$ and therefore $g_2(\mu_3) = 0$. Conclude that $g_3(\mu_3) \neq 0$ and $j(\mu_3) = 0$. Argue similarly to show that $g_3(i) = 0$, $g_2(i) \neq 0$, and $j(i) = 1728$.

1.1.9. This exercise shows that the modular invariant $j : \mathcal{H} \longrightarrow \mathbf{C}$ is a surjection. Suppose that $c \in \mathbf{C}$ and $j(\tau) \neq c$ for all $\tau \in \mathcal{H}$. Consider the integral

$$\frac{1}{2\pi i}\int_{\gamma}\frac{j'(\tau)d\tau}{j(\tau)-c}$$

where γ is the contour shown in Figure 1.1 containing an arc of the unit circle from $(-1+i\sqrt{3})/2$ to $(1+i\sqrt{3})/2$, two vertical segments up to any height greater than 1, and a horizontal segment. By the Argument Principle the integral is 0. Use the fact that j is invariant under $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ to show that the integrals over the two vertical segments cancel. Use the fact that j is invariant under $\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]$ to show that the integrals over the two halves of the circular arc cancel. For the integral over the remaining piece of γ make the change of coordinates $q = e^{2\pi i\tau}$, remembering that $j'(\tau)$ denotes derivative with respect to τ and that $j(\tau) = 1/q + \cdots$, and compute that it equals 1. This contradiction shows that $j(\tau) = c$ for some $\tau \in \mathcal{H}$ and j surjects.

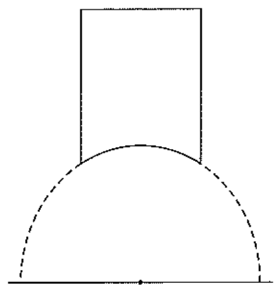

Figure 1.1. A contour

1.2 Congruence subgroups

Section 1.1 stated that if a meromorphic function $f : \mathcal{H} \longrightarrow \mathbf{C}$ satisfies

$$f(\gamma(\tau))=(c\tau+d)^k f(\tau)\quad\text{for } \gamma=\begin{bmatrix}1&1\\0&1\end{bmatrix}\text{ and }\gamma=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

then f is weakly modular, i.e,

$$f(\gamma(\tau))=(c\tau+d)^k f(\tau)\quad\text{for all }\gamma=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{SL}_2(\mathbf{Z}).$$

Replacing the modular group $\mathrm{SL}_2(\mathbf{Z})$ in this last condition by a subgroup Γ generalizes the notion of weak modularity, allowing more examples of weakly modular functions.

For example, a subgroup arises from the *four squares problem* in number theory, to find the number of ways (if any) that a given nonnegative integer n can be expressed as the sum of four integer squares. To address this, define more generally for nonnegative integers n and k the *representation number* of n by k squares,

$$r(n,k)=\#\{v\in\mathbf{Z}^k : n=v_1^2+\cdots+v_k^2\}.$$

Note that if $i + j = k$ then $r(n,k) = \sum_{l+m=n} r(l,i)r(m,j)$, summing over nonnegative values of l and m that add to n (Exercise 1.2.1). This looks like the rule $c_n = \sum_{l+m=n} a_l b_m$ relating the coefficients in the formal product of two power series,

$$\left(\sum_{l=0}^{\infty}a_l q^l\right)\left(\sum_{m=0}^{\infty}b_m q^m\right)=\sum_{n=0}^{\infty}c_n q^n.$$

So consider the *generating function* of the representation numbers, meaning the power series with nth coefficient $r(n,k)$,

$$\theta(\tau,k)=\sum_{n=0}^{\infty}r(n,k)q^n,\quad q=e^{2\pi i\tau},\ \tau\in\mathcal{H}.$$