Übung 5

4.6.2020

Aufgabe 3

$$\left.\begin{array}{c} L \\ | \\ \mathbb{Q} \end{array}\right) G = \langle\sigma\rangle \quad 2$$

Berechnung von $H^q(G, \mathcal{O}_L)$ für $q = -1, 0$

Schreibe $L = \mathbb{Q}(\sqrt{d})$, d wie immer

$d \equiv 1 \pmod 4$

$$\mathcal{O}_L = \mathbb{Z} \oplus \mathbb{Z}\frac{1+\sqrt{d}}{2} = \mathbb{Z}\underbrace{\frac{1+\sqrt{d}}{2}}_{\theta} \oplus \mathbb{Z}\underbrace{\frac{1-\sqrt{d}}{2}}_{\sigma(\theta)}$$

$$\Rightarrow \quad \mathcal{O}_L \simeq \mathbb{Z}[G]$$

$$a\theta + b\sigma(\theta) \leftarrow\!\shortmid \; a + b\sigma \qquad \text{Iso. von } \mathbb{Z}[G]\text{-Moduln}$$

$$\Rightarrow \; H^q(G, \mathcal{O}_L) \cong H^q(G, \mathbb{Z}[G]) = 0, \; \forall q \in \mathbb{Z}.$$

$d \equiv 2, 3 \pmod 4$ $\quad \mathcal{O}_L = \mathbb{Z} \oplus \mathbb{Z}\sqrt{d}$

Zu H^{-1}: Sei $\alpha = a + b\sqrt{d}$, $a, b \in \mathbb{Z}$

$$N_G(\alpha) = 0 \Leftrightarrow Tr_{L|\mathbb{Q}}(\alpha) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow a = 0$$
$$\Leftrightarrow \alpha \in \mathbb{Z}\sqrt{d}$$

$$I_G \mathcal{O}_L = 2\mathbb{Z}\sqrt{d}$$

$$I_G = \mathbb{Z}(\sigma - 1), \; (\sigma - 1)\alpha = a - b\sqrt{d} - a - b\sqrt{d} = -2b\sqrt{d}$$

$$\Rightarrow \underline{\underline{H^{-1}(G, \mathcal{O}_L) \cong \mathbb{Z}/2\mathbb{Z}}}$$

Zu H^0:

$$\alpha \in \mathcal{O}_L^G \Leftrightarrow \sigma(\alpha) = \alpha \Leftrightarrow a - b\sqrt{d} = a + b\sqrt{d}$$
$$\Leftrightarrow b = 0 \Leftrightarrow \alpha \in \mathbb{Z}$$

$$N_G \alpha = Tr_{L|\mathbb{Q}}(\alpha) = 2a \Rightarrow N_G \mathcal{O}_L = 2\mathbb{Z}$$

$$\Rightarrow H^0(G, \mathcal{O}_L) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Beobachtung: Falls $d \equiv 2, 3 \pmod 4$, so ist $\mathcal{O}_L$ nicht frei über $\mathbb{Z}[G]$, weil sonst $\mathcal{O}_L$ c.t. wäre.

Berechnung von $H^q(G, \mathcal{O}_L^\times)$, $q = -1, 0$

Zu H^0

1. Fall: L i.q. $\Rightarrow N_G \mathcal{O}_L^\times = N_{L|\mathbb{Q}}(\mathcal{O}_L^\times)$

$$(\mathcal{O}_L^\times)^G = \mathbb{Z}^\times = \{\pm 1\} \qquad = \{1\}$$

$$\Rightarrow H^0(G, \mathcal{O}_L^\times) = \{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}$$

2. Fall: L r.q. $\Rightarrow \mathcal{O}_L = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$

$$\Rightarrow N_G \mathcal{O}_L^\times = \begin{cases} \{1\}, & \text{falls } N_{L|\mathbb{Q}}(\varepsilon) = +1 \\ \{\pm 1\}, & \text{falls } N_{L|\mathbb{Q}}(\varepsilon) = -1 \end{cases}$$

$$H^0(G,\mathcal{O}_L^\times) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{falls } N_{L|\mathbb{Q}}(\varepsilon)=1 \\ 0, & \text{falls } N_{L|\mathbb{Q}}(\varepsilon)=-1 \end{cases}$$

Zu H^{-1}

1. Fall: L i.q. $\Rightarrow {}_{N_G}\mathcal{O}_L^\times = \mathcal{O}_L^\times$

$$I_G\mathcal{O}_L^\times = \begin{cases} \{1\} & d \neq -1,-3 \\ \{\pm 1\} & d=-1 \\ \langle \zeta_3 \rangle & d=-3 \end{cases}$$

$d=-1 \Rightarrow \mathcal{O}_L^\times = \{\pm 1, \pm i\}$

$(\sigma-1)(\pm i) = \mp i / \pm i = -1$

$\Rightarrow I_G\mathcal{O}_L^\times = \{\pm 1\}$

$d=-3 \Rightarrow \mathcal{O}_L^\times = \langle -\zeta_3 \rangle$

$\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$

$\quad | \; \rangle \langle \sigma_2 \rangle$

$\mathbb{Q}$

$(\sigma-1)(\pm\zeta_3) = \pm\zeta_3^2 / \pm\zeta_3 = \zeta_3$

$(\sigma-1)(\pm\zeta_3^2) = \pm\zeta_3^4 / \pm\zeta_3^2 = \zeta_3^2$

$\Rightarrow I_G\mathcal{O}_L^\times = \langle \zeta_3 \rangle$

Insgesamt: $H^{-1}(G,\mathcal{O}_L^\times) = \mathcal{O}_L^\times / I_G\mathcal{O}_L^\times \simeq \underline{\underline{\mathbb{Z}/2\mathbb{Z}}}$

2. Fall: L r.q.

$$N_G \mathcal{O}_L^\times = \begin{cases} \{\pm 1\} \times \langle \varepsilon \rangle = \mathcal{O}_L^\times, & N_{L|\mathbb{Q}}(\varepsilon) = 1 \\ \{\pm 1\} \times \langle \varepsilon^2 \rangle & N_{L|\mathbb{Q}}(\varepsilon) = -1 \end{cases}$$

$$(\sigma - 1)(\pm \varepsilon) = \frac{\pm \sigma(\varepsilon)}{\pm \varepsilon} = \sigma(\varepsilon)/\varepsilon$$

$$= \begin{cases} 1/\varepsilon^2, & N_{L|\mathbb{Q}}(\varepsilon) = 1 \\ -1/\varepsilon^2, & N_{L|\mathbb{Q}}(\varepsilon) = -1 \end{cases}$$

$$\left[\begin{array}{l} N_{L|\mathbb{Q}}(\varepsilon) = 1 \iff \varepsilon\, \sigma(\varepsilon) = 1 \iff \sigma(\varepsilon) = \frac{1}{\varepsilon} \\ N_{L|\mathbb{Q}}(\varepsilon) = 1 \qquad\qquad\qquad\qquad \iff \sigma(\varepsilon) = -\frac{1}{\varepsilon} \end{array}\right.$$

$$\Longrightarrow I_G \mathcal{O}_L^\times = \begin{cases} \langle \varepsilon^2 \rangle, & N_{L|\mathbb{Q}}(\varepsilon) = 1 \\ \langle -1/\varepsilon^2 \rangle = \langle -\varepsilon^2 \rangle, & N_{L|\mathbb{Q}}(\varepsilon) = -1 \end{cases}$$

$$\Longrightarrow H^{-1}(G, \mathcal{O}_L^\times) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & N_{L|\mathbb{Q}}(\varepsilon) = 1 \\ \mathbb{Z}/2\mathbb{Z}, & \text{sonst} \end{cases}$$

$\{\pm 1\}$ ist ein VS von $\dfrac{\{\pm 1\} \times \langle \varepsilon^2 \rangle}{\langle -\varepsilon^2 \rangle} = \dfrac{\{\pm 1\} \times \langle -\varepsilon^2 \rangle}{\langle -\varepsilon^2 \rangle}$

$$H^q(G,\mu_L) = H^q(G,\mathcal{O}_L^\times) \quad \text{für } L \text{ i.q.}$$

<u>L r.q.</u>

$$H^q(G,\mu_2) = H^q(G,\{\pm 1\})$$

$$H^0(G,\mu_2) = \{\pm 1\}/\{1\} \simeq \mathbb{Z}/2\mathbb{Z}$$

$$H^{-1}(G,\mu_2) = \{\pm 1\}/\{1\} \simeq \mathbb{Z}/2\mathbb{Z}$$

<u>Berechnung von $H^q(G,I_L)$:</u> Für jeden Zkp. L gilt:

$$I_L \simeq \bigoplus_{\mathfrak{p}\neq 0} \mathbb{Z}$$

kein Iso. von G-Moduln

$$\mathfrak{a} \longmapsto \left(v_{\mathfrak{p}}(\mathfrak{a}) \right)_{\mathfrak{p}}$$

falsch

$$\Rightarrow H^q(G,I_L) \simeq \bigoplus_{\mathfrak{p}\neq 0} H^q(G,\mathbb{Z})$$

$$= \begin{cases} 0 & q=-1 \\ \mathbb{Z}/|G|\mathbb{Z} & q=0 \end{cases}$$

__Korrektur:__ L sei quadratisch

$$I_L \cong \bigoplus_{\substack{p \text{ Pz.} \\ p=\mathfrak{p}\bar{\mathfrak{p}}}} \mathbb{Z}[G]\cdot\mathfrak{p} \;\oplus\; \bigoplus_{\substack{p \text{ Pz.} \\ p=\mathfrak{p}}} \mathbb{Z}\cdot\mathfrak{p} \;\oplus\; \bigoplus_{\substack{p \text{ Pz.} \\ p=\mathfrak{p}^2}} \mathbb{Z}\cdot\mathfrak{p}$$

Iso. von $\mathbb{Z}[G]$-Moduln

$$\Rightarrow H^q(G, I_L) \simeq \bigoplus_{p=\mathfrak{p}\bar{\mathfrak{p}}} H^q(G, \mathbb{Z}[G]) \;\oplus\; \bigoplus_{\substack{p \text{ nicht} \\ \text{zerlegt}}} H^q(G, \mathbb{Z})$$

$$\simeq \bigoplus_{\substack{p \text{ nicht} \\ \text{zerlegt}}} H^q(G, \mathbb{Z})$$

bekannt für $q = -1, 0$

__Allgemein:__

$$\begin{array}{lcl} L & & \mathfrak{P}_1 \cdots \mathfrak{P}_r \\ \mid &)G & \\ K & & \\ \mid & & \mathfrak{p} \\ \mathbb{Q} & & \end{array}$$

$G/D_{\mathfrak{P}_1}\,\mathfrak{P}_1 = \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$ $D_{\mathfrak{P}_1}$ Zerlegungsgruppe

$$\Longrightarrow I_L \simeq \bigoplus_{\mathfrak{p} \neq 0} \mathbb{Z}[G/D_{\hat{\mathfrak{p}}}]\,\hat{\mathfrak{p}} \quad \text{als}$$

$\mathbb{Z}[G]$-Moduln

$$\begin{array}{ccl} L & \hat{\mathfrak{p}} & \text{fixiert} \\ | & | & \\ K & \mathfrak{p} & \end{array}$$

$$\Longrightarrow H^q(G, I_L) \cong \bigoplus_{\mathfrak{p} \neq 0} H^q(G, \mathbb{Z}[G/D_{\hat{\mathfrak{p}}}])$$

Aufgabe 4

Zeige: $H^{-1}(G, L^\times) = 0$

$L \,|\, K$ G **zyklisch**, $G = \langle \sigma \rangle$

Beweis: Sei $\alpha \in L^\times$:

$$\alpha \in \ker(N_G) \Leftrightarrow N_{L|K}(\alpha) = 1 \overset{\text{HS90}}{\Leftrightarrow}$$

$$\exists \beta \in L^\times : \alpha = \sigma(\beta)/\beta \Leftrightarrow \alpha \in I_G L^\times$$

$$\Longrightarrow H^{-1}(G, L^\times) = \ker(N_G) / I_G L^\times = 0.$$