Mathematisches Institut der Universität München
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WS 17/18
Sheet 7

Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that an irreducible Markov chain on a countable state space with one recurrent state is recurrent (i.e. all its states are).

T2. A stochastic process X is called **strictly stationary** if for all $n \in \mathbb{N}$ and all $t_1 < t_2 < \cdots < t_n$ the distribution of $(X_{t_1+s}, \ldots, X_{t_n+s})$ does not depend on s.

Let X be a Markov chain on S with transition probability $p_t(\cdot,\cdot)$ and starting distribution π. Show that X is strictly stationary if and only if π is invariant.

T3. Let X be a Markov chain on a finite set S with transition probability $p_t(\cdot,\cdot)$ and Q-matrix $q(\cdot,\cdot)$. Let π be a strictly positive measure on S. Show that π is reversible if and only if

$$\pi(x)q(x,y) = \pi(y)q(y,x) \quad \forall x, y \in S.$$

(N.B. this result also holds for countable S.)

T4. Let $(a_n)_{n\in\mathbb{N}}$ be a nonnegative subadditive sequence, i.e.,

$$0 \leq a_{n+m} \leq a_n + a_m \quad \forall n, m \in \mathbb{N}.$$

Show that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_n}{n}, n \in \mathbb{N}\right\}.$$

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Homework exercises:

H1. Show that an irreducible Markov chain on a countable state space with one positive recurrent state is positive recurrent (i.e. all its states are).

H2. (a) Show that recurrent Markov chains on a countable state space have a stationary (not necessarily normalizable) measure.

(b) Show that an irreducible Markov chain on a countable state space has a stationary distribution if and only if it is positive recurrent.

H3. Compute the transition function for the "linear birth chain", i.e. the (unique) continuous time Markov chain on $\mathbb{N}_0$ with Q-matrix given by

$$q(m,n) := \begin{cases} \rho m & \text{if } n = m+1, \\ -\rho m & \text{if } n = m, \end{cases}$$

(where $\rho > 0$ is an intensity parameter.)

H4. Let $p \in (0,1)\backslash\{\frac{1}{2}\}$. Find two invariant measures for the asymmetric random walk on $\mathbb{Z}$ defined by the Q-matrix

$$q(x,x-1) = 1-p, \quad q(x,x) = -1, \quad q(x,x+1) = p, \qquad \forall x \in \mathbb{Z},$$

such that both measures are not multiples of each other.

Hint: For the Poisson distribution we have the following estimate:

$$\mathbb{P}(\mathrm{POI}(\lambda) \geq k) \leq \exp\left(-k \log\left(\frac{k}{\lambda e}\right) + \lambda\right).$$

Deadline: Tuesday, 5.12.17