Mathematisches Institut der Universität München
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WS 17/18
Sheet 13

Exercises for Stochastic Processes

Tutorial exercises:

T1. (Mecke Formula) Complete the proof of Theorem 6.3 in the general setting, where $m \geq 1$ and λ is not necessarily finite.

Homework exercises:

H1. (Laplace Functional) Let $N = \{N_B\}$ be a point process with locally finite intensity measure λ. Denote by $L_N(u)$ the Laplace functional of $\{N_B\}$. Show that $\{N_B\}$ is a Poisson point process with intensity measure λ if and only if

$$L_N(u) = \exp\big(- \int \big(1 - e^{-u(x)}\big)\, \lambda(\mathrm{d}x)\big).$$

H2. Give an example of two Poisson point processes $\{N_B\}$ and $\{N'_B\}$ with $N_B \leq N'_B$ for every bounded Borel set $B \subset \mathbb{R}^d$, and for which $\{N'_B - N_B\}$ is not a Poisson point process.

H3. (Matérn II process) Let $\{(X_n, U_n)\}_{n\geq 1}$ be a Poisson point process in $\mathbb{R}^d \times [0,1]$ with Lebesgue intensity measure. Define

$$Y = \Big\{ X_n : U_n \leq \min_{m:\, |X_n - X_m| < 1} U_m \Big\}$$

as the thinning consisting of all points X_n whose mark U_n is smaller than the mark of all other points in a 1-environment. Compute the intensity measure of Y and show that Y is not a Poisson point process.

H4. (Displacement Theorem) Let λ be a locally finite Borel measure on $\mathbb{R}^d$ and μ be a probability measure on $\mathbb{R}^d$. The *convolution* $\lambda * \mu$ is the measure on $\mathbb{R}^d$ defined by

$$(\lambda * \mu)(B) = \iint \mathbb{1}_B(x+y)\lambda(\mathrm{d}x)\mu(\mathrm{d}y).$$

Let $X = \{X_n\}_{n\geq 1}$ be a Poisson point process with intensity measure λ and let $\{Y_n\}_{n\geq 1}$ be a sequence of random variables with distribution μ, independent of X.

(a) Show that $\lambda * \mu$ is a locally finite Borel measure.

(b) Show that $\{X_n + Y_n\}_{n\geq 1}$ is a Poisson point process with intensity measure $\lambda * \mu$.

Deadline: Tuesday, 30.01.18