# Exercises for Stochastic Processes

**Tutorial exercises:**

T1. Show that the same characteristic exponent $\psi$ in the Lévy-Khinchin formula cannot be represented via several different choices of $\sigma$ (i.e. that the "Brownian part" of a Lévy process is uniquely determined).

$\Big($Hint: Show that $\lim_{\theta\to\infty} \mathrm{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}\Big)$

T2. Let $S = X^V$, for a topological space $X$ and arbitrary set $V$. Show that the product topology is the smallest topology on $S$ which makes one-dimensional projections continuous.

T3. Let $S = \{0,1\}^{\mathbb{Z}^d}$. Let $\alpha : \mathbb{Z}^d \to (0,\infty)$ with $\sum_v \alpha(v) < \infty$. The metric

$$\rho(\eta,\xi) := \sum_{v\in\mathbb{Z}^d} \alpha(v)|\eta(v) - \xi(v)|$$

on $S$ generates the topology

$$T_\rho = \{A \subseteq S \;:\; \forall \eta \in A\ \exists r > 0 \text{ such that } B_r(\eta) \subset A\},$$

where

$$B_r(\eta) := \{\xi \in S \;:\; \rho(\eta,\xi) < r\}.$$

Show that $T_\rho$ is equal to the product topology.

T4. Show that a sequence $(\eta_n)$ in $\{0,1\}^{\mathbb{Z}^d}$ converges w.r.t. the product topology if and only if it converges pointwise.

→ Page 2

**Homework exercises:**

H1. Alter the proof of Theorem 4.12 to show that $X$ is infinitely divisible if and only if

$$\mathbb{E}e^{i\vartheta X} = e^{\psi(\vartheta)}$$

with

$$\psi(\vartheta) = ia\vartheta + \int_{\mathbb{R}} \left( e^{i\vartheta x} - 1 - \frac{i\vartheta x}{1+x^2} \right) \frac{1+x^2}{x^2} \nu(\mathrm{d}x)\,,$$

for some $a \in \mathbb{R}$ and $\nu$ a finite measure.

(The integrand should evaluated at 0 by continuous extension.)

H2. Let $\psi(\theta)$ be a characteristic exponent satisfying the Lévy-Khinchin formula and let $\xi_t$ be a random variable with characteristic function $\exp\big(t\psi(\theta)\big)$. Show that

$$T_t f(x) := \mathbb{E}[f(x+\xi_t)]$$

is a probability semigroup.

H3. Let $X_t$ be a Lévy process with Lévy-Khinchin triple $(a, \sigma^2, \pi)$. Show that the generator of $X_t$ satisfies

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \big( f(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|<1\}} \big) \pi(\mathrm{d}y),$$

for functions $f \in C_c^\infty(\mathbb{R})$, i.e., infinitely continuously differentiable functions with compact support.

H4. Let $X$ be a compact space and $I$ a countable set. Let $S = X^I$ and associate with this the product topology. Show that $S$ is compact.

**Deadline:** Tuesday, 16.01.18