Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that normal and Poisson distributions are infinitely divisible.

T2. Determine all infinitely divisible random variables with finite support.
(Hint: Consider the Lindeberg-Feller central limit theorem.)

T3. Let N be a Poisson process with intensity $\lambda > 0$, $Y_1, Y_2, \dots$ i.i.d. and $N, Y_1, Y_2, \dots$ independent. Show "by hand" that the "compound Poisson process" given by

$$X_t := \sum_{n=1}^{N_t} Y_n$$

has stationary and independent increments.

T4. Is there a Lévy process X with $X_1 \sim \text{Exp}(1)$?

Homework exercises:

Define
$$C_c^2(\mathbb{R}) := \{f \in C_0(\mathbb{R}) \; : \; f \text{ has compact support and } f', f'' \in C_0(\mathbb{R})\}.$$

H1. Consider a Feller process with continuous paths whose generator is given by

$$\mathcal{L}f := \frac{c(x)}{2} f'',$$

defined on $C_c^2(\mathbb{R})$, where c is strictly positive and continuous.

   (a) Let $\tau_{a,b}$ be the first hitting time of $\{a, b\}$, with $a < b$ and $a < x < b$. Show that

$$\mathbb{E}^x \tau_{a,b} = \int_a^b \frac{2}{c(u)} \frac{(x \wedge u - a)(b - x \vee u)}{b - a} \mathrm{d}u \,.$$

     (Apply Theorem 4.7 to a function $f \in C_c^2(\mathbb{R})$, with $f(x) = \int_a^x \int_a^z 2/c(u)\mathrm{d}u\mathrm{d}z$ to find a suitable martingale.)

   (b) Let τ_a be its first hitting time of $a \in \mathbb{R}$. Show that, for $x > a$, $\mathbb{E}^x \tau_a$ is finite if and only if $\int_0^\infty \frac{\mathrm{d}x}{c(x)} < \infty$.

H2. Consider a Feller process on $\mathbb{R}$ whose generator, restricted to C_c^2 functions, is given by $\mathcal{L}f = cf''$, where $c \in C_b(\mathbb{R})$ is a nonnegative function. Show that such a process has a continuous modification.
(The ansatz you have seen for the Fisher-Wright diffusion goes through with some modifications.)

H3. Let X be a compound Poisson process (as in T3). Compute the characteristic function of X_t and find the corresponding Lévy-Khinchin triple.

H4. Let (a, σ, π) be a Lévy-Khinchin triple. Find a random variable with the corresponding characteristic function.

🎄 Merry Christmas and a happy new year! 🎄

Deadline: Tuesday, 9.1.18