Exercises for Stochastic Processes

Tutorial exercises:

T1. Let X be an irreducible and recurrent Markov chain. Show that every non-negative harmonic function for X is constant.

T2. Let X be a Feller process on S, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in C_0(S)$ and $t_1, \ldots, t_n > 0$. Show that

$$x \mapsto \mathbb{E}^x \prod_{k=1}^{n} f_k(X_{t_k})$$

is continuous.

T3. Let B be a one-dimensional Brownian motion.

 (a) Show that $T_t f(x) := \mathbb{E}^x f(B_t)$ for $f \in C_0(\mathbb{R})$ defines a probability semigroup.

 (b) What would go wrong if $C_0(\mathbb{R})$ were replaced by $C_b(\mathbb{R})$?

T4. Show that $\mathcal{L} f := f'$ defined on $\mathcal{D}(\mathcal{L}) := \{f \in C_0(\mathbb{R}) \mid f' \in C_0(\mathbb{R})\}$ is a probability generator. What is the corresponding probability semigroup and Feller process?

 (Hint for property (G1): consider the Stone-Weierstrass theorem)

Homework exercises:

H1. Let $T_t f(x) := \sum_{y \in S} p_t(x, y) f(y)$ be the semigroup associated to a transition function p on an a state space $S \subset \mathbb{Z}$.

   (a) Show that, if S is finite, then $\{T_t\}$ is a probability semigroup.

   (b) Show that, if S is infinite, then $\{T_t\}$ is a probability semigroup iff

   $$\lim_{|x| \to \infty} p_t(x, y) = 0 \text{ for all } y \in S, \, t > 0\,.$$

H2. Show that, for a Q-matrix Q on a finte state space S,

   $$\mathcal{L}f(x) := \sum_y q(x, y) f(y)$$

   defines a probability generator on $C_0(S)$.

H3. Consider the Q-matrix on $S = \mathbb{N}_0$ for a "pure death process" given by

   $$q(0, 1) = 1, \, q(0, 0) = -1,$$
   $$q(k, k - 1) = \delta_k, \, q(k, k) = -\delta_k \text{ for } k \in \mathbb{N}\,,$$

   with $\delta_k > 0$. Define
   $$\mathcal{L}f(x) := \sum_y q(x, y) f(y).$$

   Show that this operator satisfies conditions (G1), (G2) and (G4) of the definition of a probability generator. For what values of the sequence (δ_k) is condition (G3) satisfied?

H4. Show that
   $$\mathcal{L}f := f'''$$

   defined on
   $$\mathcal{D} := \{f \in C_0(\mathbb{R}) \mid f', f'', f''' \in C_0(\mathbb{R})\}$$

   is no probability generator.

Deadline: Monday, 19.12.16