# Exercises for Stochastic Processes

**Tutorial exercises:**

T1. Show that, for any square-integrable martingale $(M_t)$ and $r < s < t$,

$$\mathbb{E}\left[(M_t - M_s)^2 \mid \mathfrak{F}_r\right] = \mathbb{E}\left[M_t^2 - M_s^2 \mid \mathfrak{F}_r\right]!$$

T2. Show that Gaussian processes $(X_t)_{t\geq 0}$ that are martingales have independent increments!

Let $B$ be a standard Brownian motion and $(\mathfrak{F}_t)$ the usual right continuous filtration.

T3. Show that

$$M_t := B_t$$

and

$$N_t := B_t^2 - t$$

define martingales!

T4. Define

$$\tau := \inf\{t \geq 0 \mid B_t \in \{-a, b\}\}$$

for $-a < 0 < b$. Show that $\mathbb{P}(B_\tau = b) = \frac{a}{a+b}$ and $\mathbb{E}(\tau) = ab$!

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**Homework exercises:**

Let $B$ be a standard Brownian motion and $(\mathfrak{F}_t)$ the usual right continuous filtration.

H1. (a) Show that, for $\sigma \geq 0$, the process

$$\left(e^{\sigma B(t)-\frac{\sigma^2 t}{2}}\right)_{t\geq 0}$$

is a martingale!

(b) Show that the following processes are martingales:

- $(B_t^2 - t)$
- $(B_t^3 - 3tB_t)$
- $(B_t^4 - 6tB_t^2 + 3t^2)$
- ...

and find the general formula for the above sequence.

(Hint: use (a))

H2. Let $\tau$ be a stopping time with finite mean. Show that

$$\mathbb{E}(\tau^2) \leq 4\mathbb{E}(B_\tau^4) \leq 120\mathbb{E}(\tau^2)\,!$$

H3. With $\tau := \inf\{t \geq 0 \mid B_t = a + bt\}$ for $a, b > 0$, show that $\mathbb{P}(\tau < \infty) = e^{-2ab}$!
(Hint: First show using H1(a) that

$$\mathbb{E}[e^{-\lambda\tau}\mathbb{1}_{\{\tau<\infty\}}] = \exp\Big(-a\big(b + \sqrt{b^2 + 2\lambda}\big)\Big),$$

for all $\lambda > 0$.)

H4. Find $\mathbb{E}(\tau^2)$ for $\tau := \inf\{t \geq 0 \mid B(t) \in \{a, b\}\}$ and $a < 0 < b$ !

**Deadline:** Monday, 21.11.16