Exercises for Stochastic Processes

Tutorial exercises:

T1. Let B be a Brownian motion and $s, c > 0$. Show that

$$X_t := B_{s+t} - B_s$$

and

$$Y_t := \frac{B_{ct}}{\sqrt{c}}$$

(each defined for $t \geq 0$) also define Brownian motions!

T2. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded measurable function and X and Y are random variables such that X is $\mathfrak{G}$-measurable and Y is independent of $\mathfrak{G}$. Show that

$$\mathbb{E}(f(X,Y) \mid \mathfrak{G}) = g(X)$$

with

$$g(x) = \mathbb{E}(f(x,Y))\,!$$

T3. Let B be a Brownian motion. Show that $\frac{B_t}{t} \xrightarrow{t\to\infty} 0$ a.s.!

T4. Explain why

$$X_t := \int_0^t B_s \mathrm{d}s \quad (t \geq 0)$$

defines a Gaussian process! Compute its mean and covariance function!

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Homework exercises:

H1. Let B be a Brownian motion and $0 < s < t$. Compute $\mathbb{P}(B_s > 0, B_t > 0)$!

H2. (a) Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. with mean zero and finite variance and let $S_n := \sum_{k=1}^n X_k$. Show that a.s.

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} S_n = -\infty$$

and

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} S_n = \infty\,!$$

(Hint: First prove that $\mathbb{P}(|\liminf \frac{1}{\sqrt{n}} S_n|, |\limsup \frac{1}{\sqrt{n}} S_n| < C) < 1$ for all $C > 0$, by assuming the contrary and finding a contradiction with the central limit theorem. Then apply Kolmogorov's 0-1 law.)

(b) Let B be a Brownian motion.
Show that

$$\limsup_{t\uparrow\infty} \frac{B_t}{\sqrt{t}} = \limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \infty$$

and

$$\liminf_{t\uparrow\infty} \frac{B_t}{\sqrt{t}} = \liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty \quad \text{a.s.!}$$

H3. (a) Let B be a Brownian motion. Show that the process defined by

$$X_t := B_t - tB_1$$

for $t \in [0, 1]$ is Gaussian and compute its covariance function!

(b) Show that, for $0 < t_1 < \cdots < t_n < 1$ and real intervals $[a_1, b_1], \ldots, [a_n, b_n]$, the joint probabilities

$$\mathbb{P}\left(B_{t_1} \in [a_1, b_1], \ldots, B_{t_n} \in [a_n, b_n] \mid |B_1| \le \epsilon\right)$$

converge to

$$\mathbb{P}\left(X_{t_1} \in [a_1, b_1], \ldots, X_{t_n} \in [a_n, b_n]\right)$$

as $\epsilon \to 0$!

(Hint: First show that B_1 is independent of the vector $(X_{t_1}, \ldots, X_{t_n})$.)

Deadline: Monday, 7.11.16