Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that the same characteristic exponent ψ in the Lévy-Khinchin formula cannot be represented via several different choices of σ (i.e. that the "Brownian part" of a Lévy process is uniquely determined).

(Hint: Show that $\lim_{\theta\to\infty} \operatorname{Re}\left(\frac{\psi(\theta)}{\theta^2}\right) = -\frac{\sigma^2}{2}$)

T2. Let $S = X^V$, for a topological space X and arbitrary set V. Show that the product topology is the smallest topology on S which makes one-dimensional projections continuous.

T3. Let $S = \{0,1\}^{\mathbb{Z}^d}$. Let $\alpha : \mathbb{Z}^d \to (0,\infty)$ with $\sum_v \alpha(v) < \infty$. The metric

$$\rho(\eta,\xi) := \sum_{v\in\mathbb{Z}^d} \alpha(v)|\eta(v) - \xi(v)|$$

on S generates the topology

$$T_\rho = \{A \subseteq S \,:\, \forall\eta \in A\ \exists r \in \mathbb{R} \text{ such that } \rho(\eta,\xi) < r \Rightarrow \xi \in A\}.$$

Show that T_ρ is equal to the product topology.

T4. Show that a sequence (η_n) in $\{0,1\}^{\mathbb{Z}^d}$ converges w.r.t. the product topology if and only if it converges pointwise.

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Homework exercises:

H1. Alter the proof of Theorem 4.13 to show that X is infinitely divisible if and only if

$$\mathbb{E}e^{i\vartheta X} = e^{\psi(\vartheta)}$$

with

$$\psi(\vartheta) = ia\vartheta + \int \left(e^{i\vartheta x} - 1 - \frac{i\vartheta x}{1+x^2} \right) \frac{1+x^2}{x^2} \nu(\mathrm{d}x)\,,$$

for some $a \in \mathbb{R}$ and ν a finite measure.

H2. Let $\psi(\theta)$ be a characteristic exponent satisfying the Lévy-khinchin formula and let ξ_t be a random variable with characteristic function $\exp\big(t\psi(\theta)\big)$. Show that

$$T_t f(x) := \mathbb{E}[f(x+\xi_t)]$$

is a probability semigroup.

H3. Let X_t be a Lévy process with Lévy-Khinchin triple (a, σ^2, π). Show that the generator of X_t is given by

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \big(f(x+y) - f(x) - yf'(x)\mathbb{1}_{\{|y|<1\}} \big) \pi(\mathrm{d}y).$$

H4. Let X be a compact space and I a countable set. Let $S = X^I$ and associate with this the product topology. Show that S is compact.

Deadline: Monday, 30.01.17