Exercises for Stochastic Processes

Tutorial exercises:

T1. Show that normal and Poisson distributions are infinitely divisible.

T2. Determine all infinitely divisible distributions on $\mathbb{R}$ with finite support.

(Hint: Consider the Lindeberg-Feller central limit theorem.)

T3. Let N be a Poisson process with intensity $\lambda > 0$, $Y_1, Y_2, \ldots$ i.i.d. and $N, Y_1, Y_2, \ldots$ independent. Show "by hand" that the "compound Poisson process" given by

$$X_t := \sum_{n=1}^{N_t} Y_n$$

has stationary and independent increments.

T4. Is there a Lévy process X with $X_1 \sim \mathrm{Exp}(1)$?

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Homework exercises:

H1. Consider a Feller process with continuous paths whose generator, restricted to C_c^2 functions, is given by

$$\mathcal{L}f := \frac{c(x)}{2} f'' ,$$

where c is strictly positive and continuous.

(a) Let $\tau_{a,b}$ be the first hitting time of $\{a,b\}$ (with $a < b$). Show that

$$\mathbb{E}^x \tau_{a,b} = \int_a^b \frac{2}{c(u)} \frac{(x \wedge u - a)(b - x \vee u)}{b-a} \mathrm{d}u .$$

(Use a suitable martingale.)

(b) Let τ_a be its first hitting time of $a \in \mathbb{R}$. Show that, for $x > a$, $\mathbb{E}^x \tau_a$ is finite iff $\int_0^\infty \frac{\mathrm{d}x}{c(x)} < \infty$.

H2. Consider a Feller process on $\mathbb{R}$ whose generator, restricted to C_c^2 functions, is given by $\mathcal{L}f = cf''$, where $c \in C_b(\mathbb{R})$ is a nonnegative function. Show that such a process has a continuous modification.
(The ansatz you have seen for the Fisher-Wright diffusion goes through with some modifications.)

H3. Let X be a compound Poisson process (as in T3). Compute the characteristic function of X_t and find the corresponding Lévy-Khinchin triple.

H4. Let (a, σ, π) be a Lévy-Khinchin triple. Find a random variable with the corresponding characteristic function.

Deadline: Monday, 23.01.17