1. Find $f: X \to Y$ mor. of f.t. k-schemes
+ M a coh. sheaf on X s.t. $f_* M$ is not coherent.

Take $X = \mathbb{A}^1_k$, $Y = \operatorname{Spec} k$, $M = \mathcal{O}_X$.
Then $f_* M \in \operatorname{QCoh}(Y) \cong k\text{-v.s.}$ is given by $k[t]$,
which has ∞ dimension & hence is not coherent.

2. X noeth. top. space.
(1) F a presheaf on X satisfying sheaf cond^n for finite coverings. Show also holds for infinite coverings.

Since X is hereditarily quasi-compact, suffices to check for covering of X. Thus let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of X. $\exists$ sub cover $U_1, \dots, U_n$.
Let $s_\alpha \in F(U_\alpha)$ be a compatible family.

1) If $s, s' \in F(X)$ are two gluings, then $s|_{U_i} = s'|_{U_i}$ & hence $s = s'$, by sheaf cond^n for finite cover.

2) Let $s \in F(X)$ be the gluing obtained from sheaf cond^n for finite cover. $\{U_\alpha \cap U_i\}_{i=1,\dots,n}$ is an open covering of U_α. Now $s|_{U_\alpha \cap U_i} = s|_{U_i}|_{U_\alpha \cap U_i} = s_i|_{U_\alpha \cap U_i}$

$= s_\alpha|_{U_\alpha \cap U_j}$

$\therefore$ By 1) applied to U, deduce that $s|_{U_j} = s_\alpha$.

(2) Deduce that if $\{M_i\}_{i\in I}$ are sheaves of ab. gps, then $U \longmapsto \bigoplus_i M_i(U)$ is also a sheaf.

Need only check sheaf cond^n for finite cover $U_1, \dots, U_n$.

I.e. need exactness of

$$0 \longrightarrow \bigoplus_i M_i(X) \longrightarrow \bigoplus_{i,j} M_i(U_j) \longrightarrow \bigoplus_{i,j,j'} M_i(U_j \cap U_{j'}).$$

This is just $\bigoplus_i$ of the exact sequences for the M_i.

NB: point is that sheaf cond^n involves products.
Infinite sums commute with finite products of ab. gps (finite products being finite coproducts in Ab), but not with general infinite products.

3. (1) $X = \operatorname{Spec} A$, A a dvr. (a) Describe $\mathcal{O}_X$-mod.

$X = \{\eta, x\}$, η gen. pt., x closed pt

open subsets of X:

$X, \{\eta\}, \emptyset$.

Presheaf : $\begin{array}{c} M(X) \\ \downarrow \\ M(\eta) \\ \downarrow \\ M(\emptyset) \end{array}$ arbitrary sets (ab. gps) & maps between them.

Sheaf : $M(\emptyset) = *$

O_X-mod : $M(X)$ an A-module

$M(\eta)$ a $\mathrm{Frac}(A)$-module

$M(X) \to M(\eta)$ A-linear.

(b) Give example of O_X-mod. which is not q. coh.

Take $M(X) = A$ $M(\eta) = 0$ $(= M(*))$

q. coh : $M(X) \otimes_A \mathrm{Frac}(A) \cong M(\eta)$ — does not hold.

(2) Repeat with A replaced by a Ded. domain.

$X = \{\eta, \text{(closed points)}\}$

gen. pt.

open subsets of X :

$X \setminus \{x_1, \dots, x_n\}$, $\emptyset$.

Sheaf of O_X-mod.: $O_X(X \setminus \{x_1, \dots, x_n\})$ - module

$M(X \setminus \{x_1, \dots, x_n\})$

+ restriction maps for inclusions

+ satisfy sheaf condition.

Sheaf which is not χcoh: Pick $x \in X$ closed,

$i : \operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow X$. NB: $\mathcal{O}_{X,x} = A_p$ is a dvr.

Take i_* (example from (1)).

I.e.: $M(U) = \begin{cases} A_p : & x \in U \\ 0 : & \text{else}. \end{cases}$

This is a sheaf b/c i_* pres. sheaves (or check directly).

Not χ.coh. b/c $M_\eta = 0$

$$\neq M(X) \otimes_A \operatorname{Frac}(A)$$

$$= \operatorname{Frac}(A).$$