1. all f.t. /k

$$\begin{tikzcd}
\operatorname{Spec} F \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[r, "\alpha"', "\text{dominant}"] \arrow[ur, dashed, "\exists!\ \text{if}"] & X
\end{tikzcd}$$

Y_i $\cap$

this is proper

$\leftarrow$ normalization of X in L

$F = k(Z)$

$Z_i \subset$ integral normal

$U_i \subset$

$L = k(Y)$

$K = k(X)$

L/K fin.

integral

$$\begin{tikzcd}
& L \arrow[dr, dotted] & \\
K \arrow[ur, hook] \arrow[rr, hook, "\alpha^\#"'] & & F
\end{tikzcd}$$

Suppose first $Z = \operatorname{Spec} C$, C dvr.

Know that $Y \to X$ is finite hence proper.

$\therefore \exists!$ lift in this case.

We may assume that $Z = \operatorname{Spec} C$ is affine, s/c glue morphisms in open cover.

Suppose $U_1, \dots, U_n$ is an open affine cover of X.

$Y_n = X \times_{U_n} Y$ } open cover of Y

$Z_n = X \times_{U_n} Z$ } open cover of Z

By prev. reasoning, suffices to treat

$$\begin{tikzcd}
\eta \arrow[r] \arrow[d] & Y_n \arrow[d, "\text{finite}"] \\
Z_n \arrow[r] \arrow[ur, dashed, "\exists!"] & U_n
\end{tikzcd}$$

affine

$\leftarrow$ affine

Pick affine open cover of Z_n $\rightsquigarrow$ WMA $Z = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$.

$$\begin{tikzcd}
F & B \arrow[l, hook'] \\
\arrow[u] & \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

$$\begin{array}{ccc} \cup & \overset{\exists !}{\dashleftarrow} & \vert \\ C & \hookleftarrow & A \end{array}$$

This exists (uniquely) $\Longleftrightarrow$ $B \subset C$ as subsets of F

$$C = \bigcap_{\mathrm{ht}\, p = 1} C_p$$

$\Longleftrightarrow$ $B \subset C_p \quad \forall\, \mathrm{ht}(p) = 1$

$\Longleftrightarrow$ the same problem works for $Z' = \operatorname{Spec} \underset{\text{dvr}}{C_p}$ $\leftarrow$ already dealt with. $\square$

NB many properties of mor. (f.t., proper, open imm. ...) are stable under base change

But normalization is <u>not</u>. But under dense open imm.:

$$\begin{tikzcd} \tilde{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X} \arrow[d, "\text{normalization}"] \\ U \arrow[r, hook, "\text{open imm.}"'] & X \end{tikzcd}$$

Q. K field $\quad A_i \subset K$ i.c. $\forall i \in I$

$\mathrm{Frac}(A_i) = K$

1) Show $\bigcap_{i \in I} A_i \subset K$ is i.c. $\quad$ 2) Ex. when $\mathrm{Frac}(\bigcap_i A_i) \neq K$?

1) $x \in K$ integral over A

$\iff x^n + a_1 x^{n-1} + \dots + a_n = 0$ for some $a_j \in A$

$\Rightarrow a_j \in A_i \quad \forall i$

$\Rightarrow x$ integral over $A_i \quad \forall i$

$\Rightarrow x \in A_i \quad \forall i$

$\Rightarrow x \in A$.

2) $K = k(x) = k(\mathbb{A}^1)$

$k[x] \hookrightarrow k(x)$ is i.c.

$\overset{\shortparallel}{\mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)}$

Then $k[x^{-1}] \hookrightarrow k(x)$ is also i.c.

But $k[x] \cap k[x^{-1}] = k$ ...

Theory of complete non-singular curves $\longleftrightarrow$ fraction fields

C/k smooth complete curve

$K = k(C)$

$c \in C$ closed $\quad \mathcal{O}_{C,c} \subset K$ $\quad \sim$ dvr cfg. k

FACT: all dvrs of K ("centered on" cfg. k) arise in this way.

$\mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1) = \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \setminus \infty) \cap \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1 \setminus 0)$

$\qquad\qquad k[x] \qquad\qquad k[x^{-1}]$

3. $A \subset B$ dvrs. Show: $A = B$.

$\mathrm{Frac}(A) = \mathrm{Frac}(B) =: K$

$b \in B \setminus A \Rightarrow b^{-1} \in A \Rightarrow b \in B^\times$

Valuation of A: $K^\times \longrightarrow \mathbb{Z}$

$u \cdot t^n \longmapsto n$

Choose $t \in m_A$ s.t. $m_A = (t)$

Can write $x \in K^\times$ uniquely as $x = u \cdot t^n \quad n \in \mathbb{Z}$

$t \in B$.

Claim: $t \in \underbrace{B \setminus B^\times}_{= m_B}$ O/w $B \supset A[t, t^{-1}] = K$ $\cancel{X}$. $u \in A^\times$

Let $x = u \cdot t^n \in K$, $u \in A^\times$.

Suppose $x \in B$. Then $n \geq 0$: $t^n = x \cdot u^{-1} \in B$

So if $n < 0$ then $K = B$. $\cancel{X}$.

$\therefore x = u \cdot t^n \in A$.