$f: X \longrightarrow Y \in Sch$ $\qquad$ f^{-1} "is symm. mon." i.e. $f^{-1}(A \otimes B) \cong f^{-1}A \otimes f^{-1}B$

$$f^{-1}: Shv_Y \rightleftarrows Shv_X : f_*$$

$$(f_* F)(U) = F(f^{-1}U)$$

$$(f^{-1})^p F(U) = \operatorname{colim}_{f(U) \subset V} F(V)$$

$$f^{-1}F = a\,(f^{-1})^p(F)$$

(1) $F \in \mathcal{O}_X$-mod

$f_* F$ is naturally an $\mathcal{O}_Y$-mod.

$$\begin{array}{ccc} \mathcal{O}_Y(U) \otimes (f_*F)(U) & \longrightarrow & (f_*F)(U) \\ \downarrow f^\# \qquad \quad \| & & \| \\ \mathcal{O}_X(f^{-1}U) \otimes F(f^{-1}U) & \longrightarrow & F(f^{-1}U) \end{array}$$

$f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$

$f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

$\rightsquigarrow$ obtain $f_*: \mathcal{O}_X\text{-mod} \longrightarrow \mathcal{O}_Y\text{-mod}$.

Q: Is there a left adjoint?

$$\mathcal{O}_Y\text{-mod} \xrightarrow{f^*} \mathcal{O}_X\text{-mod}$$

$$F \longmapsto f^{-1}F \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

$f^{-1}\mathcal{O}_Y$-mod.

Problem: show that $f^* \dashv f_*$.

ADJUNCTIONS:

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

- $\mathrm{Hom}_{\mathcal{D}}(FX, Y) \underset{(*)}{\cong} \mathrm{Hom}_{\mathcal{C}}(X, GY)$

  $\nwarrow$ natural iso

- find: $\quad X \in \mathcal{C} \quad X \xrightarrow{u_X} GFX$ "unit"

  nat. transf. $\mathrm{id} \xrightarrow{u} GF$

  $u_X \in [X, GFX] \cong [FX, FX] \ni \mathrm{id}_X$

  $Y \in \mathcal{C} \quad FGY \xrightarrow{c_Y} Y$

  $c_Y \in [FGY, Y] \cong [GY, GY] \ni \mathrm{id}_Y$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FX, Y) \arrow[r, "G"] \arrow[dr] & \mathrm{Hom}_{\mathcal{C}}(GFX, GY) \arrow[d, "u_X^*"] \\
& \mathrm{Hom}_{\mathcal{C}}(X, GY)
\end{tikzcd}$$

this is the iso $(*)$

$\rightsquigarrow \quad F \dashv G \iff u: \mathrm{id} \to GF$

$\qquad\qquad\qquad\quad c: FG \to \mathrm{id}$

zig-zag identities:

$\forall X \in \mathcal{C} \quad FX \xrightarrow{Fu_X} FGFX \xrightarrow{c_{FX}} FX$ must be the identity

$\forall Y \in \mathcal{C}$ $\quad GY \xrightarrow{u_{GY}} GFGY \xrightarrow{Gc_Y} GY$ — "

$f^{-1} \dashv f_*$ $\quad 1 : \mathbb{Z} \xrightarrow{u} A$

$M \in \mathcal{O}_Y$-mod. $\quad u_M : M \longrightarrow f_* f^* M$

$$f_* f^* M = f_*\left(f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X\right)$$

$$M \longrightarrow f_*(f^{-1}M) \longrightarrow f_*\left(f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X\right) \quad ?$$

take f_* of $\quad f^{-1}M \to f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$

$$f^{-1}M = f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{O}_Y \xrightarrow{id \otimes f^\#} f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$$

$N \in \mathcal{O}_X$-mod. $\quad c_N : f^* f_* N \longrightarrow N$

$$f^* f_* N = f^{-1}(f_* N) \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X \longrightarrow N$$

have: $f^{-1} f_* N \xrightarrow{a} N$ map of ab. shv.

$$f^{-1} f_* N \longrightarrow f^* f_* N \xrightarrow{b} N \quad \text{a map of } \mathcal{O}_X\text{-mod}$$

$$f^* f_* N = f^{-1}(f_* N) \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$$

Can extend ~~as~~ canonically to get b

If a is a map of

Analogy: $A \to B$ comm. Rings

M A-mod $\quad N$ B-mod

$M \underset{A}{\otimes} B \longrightarrow N$ of B-mod

$M \longrightarrow N$ of A-mod

$f^{-1}\mathcal{O}_Y$-mod.

(1) Show that a is $f^{-1}\mathcal{O}_Y$-linear

(2) Check zigzag identities

$$f^{-1}f_* N \xrightarrow{a} N$$ a map $f^{-1}\mathcal{O}_Y$-linear map?

= associated sheaf of

$$\Big(U \longmapsto \operatorname{colim}_{f(U)\subset V} \overbrace{N(f^{-1}V)}^{M(U)}\Big) \longrightarrow (U \mapsto N(U)) \qquad \text{NB: } f^{-1}V \supset U$$

- STP module map before sheafifying

- restriction

$$\begin{tikzcd} N(f^{-1}V) \arrow[r] \arrow[d] & N(U) \\ N(f^{-1}V') \arrow[ur] & \end{tikzcd}$$

$f(U)\subset V' \subset V$

$\rightsquigarrow M(U) \longrightarrow N(U)$

One may show this map is a

$$P(U) = \operatorname{colim}_{f(U)\subset V} \mathcal{O}_Y(V)$$

$f^{-1}f_*N$ is $f^{-1}\mathcal{O}_Y$-mod

need to check that

$$\begin{tikzcd} P(U)\otimes M(U) \arrow[r] \arrow[d, "a\otimes a"] & M(U) \arrow[d, "a"] \\ \mathcal{O}_X(U)\otimes N(U) \arrow[r] & N(U) \end{tikzcd}$$

$\mathcal{O}_X(f^{-1}V)\otimes N(f^{-1}V)$ N is an $\mathcal{O}_X$-mod

PNR $f(U)\subset V \rightsquigarrow \mathcal{O}_Y(V) \otimes N(f^{-1}V) \longrightarrow N(f^{-1}V)$

want: $\downarrow$ res? $\circlearrowleft$ $\downarrow$ res

$$\mathcal{O}_X(U) \otimes N(U) \xrightarrow[m]{} N(U)$$

Yes: s/c N is an $\mathcal{O}_X$-module (i.e. mult^n & rest^n commute)

2. $M, N \in \mathcal{O}_Y$-mod

$$f^*M \underset{\mathcal{O}_X}{\otimes} f^*N \xrightarrow{\sim} f^*(M \underset{\mathcal{O}_Y}{\otimes} N).$$

$$\| $$

$$\left(f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X\right) \underset{\mathcal{O}_X}{\otimes} \left(f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X\right)$$

$$\cong$$

$$\left(f^{-1}M \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}N\right) \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$$

$$f^{-1}(M \underset{\mathcal{O}_Y}{\otimes} N) \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X$$

$f^{-1}(M \underset{A}{\otimes} N) \cong f^{-1}M \underset{f^{-1}A}{\otimes} f^{-1}N$

for A-mod M, N //

3. ("projection formula") $M \in \mathcal{O}_X$-mod, $N \in \mathcal{O}_Y$-mod

α

$$f_*(f^*(N) \otimes_{\mathcal{O}_X} M) \underset{\uparrow}{\overset{\alpha}{\leftarrow}} N \otimes_{\mathcal{O}_Y} f_* M$$

when?

(1) Construct α.

$$N \otimes_{\mathcal{O}_Y} f_* M \longrightarrow f_*(f^* N \otimes M)$$

$\leftrightsquigarrow$ $f^*(N \otimes f_* M) \longrightarrow f^* N \otimes M$

is

$f^* N \otimes f^* f_* M$

$$\begin{tikzcd} f^* N \otimes f^* f_* M \arrow[r, "\mathrm{id} \otimes c"] & f^* N \otimes M \end{tikzcd}$$

$f^* f_* M \xrightarrow{c} M$

$\leftrightsquigarrow$ $f_* M \xrightarrow{\mathrm{id}} f_* M$

$$\begin{aligned} N \otimes f_* M &\xrightarrow{u} f_* f^*(N \otimes f_* M) \\ &\simeq f_*(f^* N \otimes f^* f_* M) \\ &\xrightarrow{f_*(\mathrm{id} \otimes c)} f_*(f^* N \otimes M) \end{aligned}$$

(2) If N is finite loc. free, show that α is iso.

- may check locally

i.e. WMA $N = \mathcal{O}_Y^n$

$N \otimes f_* M = (f_* M)^n$ ═══ b/c f_*, f^* additive

$$f_*(f^!N \otimes M) = f_*(\mathcal{O}_X^n \otimes M) = f_*(M^n)$$

Need to check that α is the canonical map.

Given sheaves F, G, and a map $\alpha : F \to G$,

then α is iso $\Leftrightarrow$ it is so locally.

But even if F, G are $\cong$ locally, they need not so.

E.g. if $X = U \cup V$

$\alpha_1 : F|_U \xrightarrow{\cong} G|_U$

$\alpha_2 : F|_V \xrightarrow{\cong} G|_V$

there need not exist

$\alpha : F \to G$

s.t. $\alpha|_U = \alpha_1$, $\alpha|_V = \alpha_2$

(holds iff $\alpha_1|_{U \cap V} = \alpha_2|_{U \cap V}$)