Algebraic Geometry 1

Exercises 9

Dr. Tom Bachmann — Winter Semester 2020–21

Exercise 1. Let $f : X \to Y$ be a dominant morphism of finite type with X, Y integral and $K(X) \simeq K(Y)$ (via f). Show that there exists an open subset U of X such that $f(U) \subset Y$ is open and $f : U \to f(U)$ is an isomorphism.

Exercise 2. Let X be a scheme. Show that X is noetherian if and only if X is quasi-compact and for every affine open subscheme $\operatorname{Spec} A \subset X$, A is noetherian.

Exercise 3. Let X be a scheme. Given $x \in X$, let $\Omega_x X := m_x/m_x^2$, where $m_x \subset \mathcal{O}_{X,x}$ is the corresponding maximal ideal. Show that $\Omega_x X$ is a $\kappa(x)$-vector space. Let T_x denote its $\kappa(x)$-linear dual space.

Now suppose that X is a scheme over a field k. Exhibit a bijection

$$\mathrm{Hom}_{k\text{-}Sch}(\operatorname{Spec} k[\epsilon]/\epsilon^2, X) \simeq \coprod_{x \in X(k)} T_x X.$$

(Here $X(k)$ denotes the set of points $x \in X$ with $\kappa(x) = k$.)

Exercise 4. Let $\mathbb{P}^1_{\mathbb{Z}}$, $\bar{\mathbb{A}}^1_{\mathbb{Z}}$ denote respectively the projective line and the affine line with the origin doubled, over $\mathbb{Z}$.

(1) Show that $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(\mathbb{P}^1_{\mathbb{Z}}) = \mathbb{Z}$.
(2) Show that $\mathbb{P}^1_{\mathbb{Z}}$ is not isomorphic to $\bar{\mathbb{A}}^1_{\mathbb{Z}}$.
(3) Show that neither of $\mathbb{P}^1_{\mathbb{Z}}$, $\bar{\mathbb{A}}^1_{\mathbb{Z}}$ is affine.