Algebraic Geometry 1

Exercises 8

Dr. Tom Bachmann Winter Semester 2020–21

Exercise 1. Let $\varphi : B \to A$ be a morphism of commutative rings. Suppose there exist $f_1, \dots, f_r \in B$ generating the unit ideal such that each $A_{\varphi(f_i)}$ is a finite type B_{f_i}-module. Show that A is a finite type B-module.

Exercise 2. Let $\varphi : B \to A$ be a morphism of commutative rings. Show that $\mathrm{Spec}\,(\varphi)$ is an open immersion if and only if there exists a family of elements $\{f_i \in B\}_{i \in I}$ such that $(\varphi(f_i))_{i \in I} = A$ and each map $B_{f_i} \to A_{\varphi(f_i)}$ is an isomorphism.

Exercise 3. Let X be a scheme. Show that the map

$$X \to \{\text{closed, irreducible subsets of } X\}, x \mapsto \overline{\{x\}}$$

is an injection. *[Hint: reduce to the affine case.]*

Exercise* 4. Let X be a scheme and $A = \mathcal{O}_X(X)$. Show that X is affine if and only if there exist $f_1, \dots, f_n \in A$ generating the unit ideal such that the open subschemes

$$X_{f_i} = \{x \in X \mid f_i(x) \neq 0 \in \kappa(x)\}$$

are affine. *[Hint: Show that $\mathcal{O}_X(X_{f_i}) = A_{f_i}$.]*