Algebraic Geometry 1

Exercises 3

Dr. Tom Bachmann Winter Semester 2020–21

Throughout let k be an algebraically closed field. Recall that $S = k[X_0, \ldots, X_n]$ and $S_+ = (X_0, \ldots, X_n)$.

Exercise 1.

(1) Let I be a homogeneous ideal in S. Show that also $\sqrt{I}$ is homogeneous.
(2) Let I be a homogeneous ideal such that $\sqrt{I} = S_+$. Show that $I_n = S_n$ for n sufficiently large.

Exercise 2. Describe all the maximal homogeneous proper ideals of S.

Exercise 3. Suppose $char(k) \neq 2$. Consider the subsets

$$\{XY = 1\}, \{X^2 + Y^2 = 1\}, \{Y - X^2 = 0\}, \subset \mathbb{A}^2(k).$$

Let Z_1, Z_2 denote two of these subsets, and write $\overline{Z_1}, \overline{Z_2}$ for their closures in $\mathbb{P}^2(k)$. Show that there exists $A \in GL_3(k)$ such that $A\overline{Z_1} = \overline{Z_2}$.

Exercise 4. Let $f \in k[X_1, \ldots, X_n]$ have degree d, and write

$$f^h(X_0, X_1, \ldots, X_n) = \sum_{i=0}^{d} X_0^{d-i} f_i(X_1, \ldots, X_n)$$

for its homogenization (where f_i denotes the homogeneous component of degree i, so that in particular $f_d \neq 0$). Show that the closure of $Z(f)$ in $\mathbb{P}^n(k)$ is given by $Z^h(f^h)$.