# Algebraic Geometry 1

## Exercises Tutorium 8

Dr. Tom Bachmann — Winter Semester 2020–21

**Exercise 1.** Let $A$ be a commutative ring and $f \in A$. Show that the locally ringed space $\operatorname{Spec} A \setminus V(f)$ is isomorphic to the locally ringed space $\operatorname{Spec} A_f$.

**Exercise 2.** Call a scheme $X$ *reduced* if for every $U \subset X$ open, $\mathcal{O}_X(U)$ is reduced (i.e. has no nilpotent elements). Show that the following are equivalent: (1) $X$ is reduced, (2) there exists an affine open cover $X = U_1 \cup \dots$ such that $\mathcal{O}_X(U_i)$ is reduced for every $i$, (3) for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

**Exercise 3.** For a scheme $X$, denote by $\mathcal{O}_{X_{red}}$ the sheaf on $X$ given by $U \mapsto \mathcal{O}_X(U)_{red}$, the maximal reduced quotient of $\mathcal{O}_X(U)$. Show that $X_{red} := (X, \mathcal{O}_{X_{red}})$ is a scheme and $X_{red} \to X$ is a morphism of schemes.

Show that $X_{red}$ satisfies the following universal property: given a morphism of schemes $f : Y \to X$ with $Y$ reduced, there exists a unique morphism $g : Y \to X_{red}$ such that $Y \xrightarrow{g} X_{red} \to X$ is $f$.

**Exercise 4.** For a locally ringed space $X$, make sense of the sheaf $\mathcal{F}$ of functions $f$ on $X$ with values $f(x) \in \kappa(x)$, and construct a homomorphism $\alpha : \mathcal{O}_X \to \mathcal{F}$. If $X$ is a reduced scheme, show that $\alpha$ is injective. [*Hint*: You may wish to recall that the intersection of all prime ideals in a commutative ring equals the nilradical.]

**Exercise 5.** Let $X$ be a reduced scheme. Given $Z \subset X$ closed, denote by $I_Z \subset \mathcal{O}_X$ the subsheaf consisting of those regular functions vanishing along $Z$. Show that this induces an injection from the set of closed subsets of $X$ to the set of subsheaves of $\mathcal{O}_X$. Can you characterize its image?