Algebraic Geometry 1

Exercises Tutorium 6

Dr. Tom Bachmann Winter Semester 2020–21

Exercise 1. Let $\mathcal{C}$ be a category and $\alpha : c \to d \in \mathcal{C}$. Show that the following are equivalent.

(1) α is an epimorphism.
(2) The following is a pushout square in $\mathcal{C}$

$$\begin{tikzcd} c \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & d \arrow[d, equal] \\ d \arrow[r, equal] & d. \end{tikzcd}$$

(3) The pushout $d \amalg_c d$ exists and the two maps $d \to d \amalg_c d$ are equal.

Exercise 2. Let X be a topological space and $\alpha : F \to G, \beta : F \to H \in PSh(X)$. Show that the presheaf

$$U \mapsto G(U) \amalg_{F(U)} H(U)$$

defines a pushout of G and H along F in $PSh(X)$. Deduce that a morphism $\alpha : F \to G$ is an epimorphism if and only if $\alpha(U) : F(U) \to G(U)$ is an epimorphism for every U.

Exercise 3. Let X be a topological space and $\alpha : F \to G, \beta : F \to H \in Shv(X)$. Show that the sheaf associated with the presheaf $G \amalg_F^p H$ (where $\amalg^p$ denotes pushout in $PSh(X)$) defines a pushout of G and H along F in $Shv(X)$. Deduce that for $x \in X$, the functor

$$Shv(X) \to Set, F \mapsto F_x$$

preserves pushouts.

Hint: You may use that "colimits commute".

Exercise 4. Let X be a topological space and $\alpha : F \to G \in Shv(X)$. Show that the following are equivalent.

(1) α is an epimorphism.
(2) $\alpha_x : F_x \to G_x$ is an epimorphism for every $x \in X$.
(3) For $U \subset X$ open, $s \in G(U)$ there exists an open covering $\{U_i\}$ of U and $s_i \in F(U_i)$ such that $\alpha(s_i) = s|_{U_i}$.

Exercise 5. *Extra problem*: Give an example of a morphism of sheaves which is an epimorphism, but not an epimorphism when viewed as a morphism of presheaves.

Exercise 6. *Extra problem*: Give an example showing that the forgetful functor from sheaves to presheaves does not preserve binary coproducts (in general).