A history of ordinal representations: 1908 – 1985

A set A with a total ordering $\prec$ is a **wellordering** if every non-empty subset X of A has a $\prec$-least element, i.e.,

$$(\exists u \in X)(\forall y \in X)[u \prec y \vee u = y].$$

An **ordinal** is a transitive set wellordered by $\in$.

Fact: Every wellordering $(A, \prec)$ is order isomorphic to an ordinal $(\alpha, \in)$.

Ordinals are traditionally denoted by lower case Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ and the relation $\in$ on ordinals is notated by $<$.

Cantor 1897

For every ordinal $\beta \neq 0$ there is a unique $k < \omega$ and unique sequences $\beta_0 > \cdots > \beta_k$ and $0 < n_0, \ldots, n_k < \omega$ such that

$$\beta = \omega^{\beta_0} \cdot n_0 + \cdots + \omega^{\beta_k} \cdot n_k .$$

Let ε_0 = least ordinal $\alpha > 0$ such that $\alpha = \omega^\alpha$.

Every ordinal $\alpha < \beta < \varepsilon_0$ can be represented by smaller ordinals.

Hardy 1904

He wanted to "construct" a subset of the continuum of size $\aleph_1$, the first uncountable ordinal.

Idea: Associate with each countable ordinal an increasing sequence of natural numbers and then to correlate a decimal expansion with each such sequence.

The sequences

(1) $1, 2, 3, \dots$

(2) $2, 3, 4, \dots$

(3) $3, 4, 5, \dots$

represent the ordinals 1, 2, 3.

A representation of ω is obtained by diagonalizing:

(ω) $1, 3, 5, \dots$

Hardy used two processes on sequences of natural numbers:

- Removing the first element to represent the successor
- Diagonalizing at limits.

This gives explicit representations for all ordinals $< \omega^2$

In general, if

(β) $\quad b_1, b_2, b_3$

represents β then

$(\beta+1)$ $\quad b_2, b_3, b_4, \dots$

represents $\beta+1$.

If $\lambda = \lim \lambda_n$ with

(λ_n) $\quad b_{n1}, b_{n2}, b_{n3}, \dots$ representing λ_n,

then (λ) $\quad b_{11}, b_{22}, b_{33}, \dots$

represents λ.