

Proof theory: Part II

Gentzen's Hauptsatz

Michael Rathjen

Fischbachau, August 5th - August 9th, 2005

Gentzen's Hauptsatz

The most common logical calculi are Hilbert-style systems. They are specified by delineating a collection of schematic logical axioms and some inference rules. The choice of axioms and rules is more or less arbitrary, only subject to the desire to obtain a complete system. In model theory it is usually enough to know that there is a complete calculus for first order logic as this already entails the compactness theorem.

There are, however, proof calculi without this arbitrariness of axioms and rules. The **natural deduction calculus** and the **sequent calculus** were both invented by **Gentzen**. Both calculi are pretty illustrations of the symmetries of logic. In this course I shall focus on the sequent calculus since it is a central tool in ordinal analysis and allows for generalizations to infinitary logics. Gentzen's main theorem about the sequent calculus is the *Hauptsatz*, i.e. **cut elimination**.

A **sequent** is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sequences of formulae A_1, \dots, A_n and B_1, \dots, B_m , respectively. $\Sigma \Rightarrow \Delta$ is read, informally, as Γ yields Δ or, rather, the **conjunction** of the A_i yields the **disjunction** of the B_j .

In particular,

- If Γ is empty, the sequent asserts the disjunction of the B_j .
- If Δ is empty, it asserts the negation of the conjunction of the A_i .
- if Γ and Δ are both empty, it asserts the **impossible**, i.e. a **contradiction**.

We use upper case Greek letters $\Gamma, \Delta, \Lambda, \Theta, \Xi \dots$ to range over finite sequences of formulae.

Identity Axiom

$$A \Rightarrow A$$

where A is any formula. In point of fact, one could limit this axiom to the case of atomic formulae A .

CUT

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta} \text{Cut}$$

A is called the **cut formula** of the inference.

Structural Rules

$$\frac{\Gamma, A, B, \Lambda \Rightarrow \Delta}{\Gamma, B, A, \Lambda \Rightarrow \Delta} \mathcal{X}_l$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \mathcal{W}_l$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \mathcal{C}_l$$

Exchange, Weakening, Contraction

$$\frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda} \mathcal{X}_r$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \mathcal{W}_r$$

$$\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \mathcal{C}_r$$

LOGICAL INFERENCE

Negation

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \neg L$$

$$\frac{B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg B} \neg R$$

Implication

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Lambda \Rightarrow \Theta}{A \rightarrow B, \Gamma, \Lambda \Rightarrow \Delta, \Theta} \rightarrow L$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow R$$

Conjunction

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge L1$$

$$\frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge L2$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \wedge R$$

Disjunction

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \vee L$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \vee R1$$

$$\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee R2$$

Quantifiers

$$\frac{F(t), \Gamma \Rightarrow \Delta}{\forall x F(x), \Gamma \Rightarrow \Delta} \forall L$$

$$\frac{\Gamma \Rightarrow \Delta, F(a)}{\Gamma \Rightarrow \Delta, \forall x F(x)} \forall R$$

$$\frac{F(a), \Gamma \Rightarrow \Delta}{\exists x F(x), \Gamma \Rightarrow \Delta} \exists L$$

$$\frac{\Gamma \Rightarrow \Delta, F(t)}{\Gamma \Rightarrow \Delta, \exists x F(x)} \exists R$$

In $\forall L$ and $\exists R$, t is an arbitrary term. The variable a in $\forall R$ and $\exists L$ is an **eigenvariable** of the respective inference, i.e. a is not to occur in the **lower sequent**.

The formulae in a **logical inference** marked **blue** are called the **minor formulae** of that inference, while the **red** formula is the **principal formula** of that inference. The other formulae of an inference are called **side formulae**.

A **proof** (aka **deduction** or **derivation**) \mathcal{D} is a tree of sequents satisfying the following conditions:

- The topmost sequents of \mathcal{D} are identity axioms.
- Every sequent in \mathcal{D} except the lowest one is an upper sequent of an inference whose lower sequent is also in \mathcal{D} .

The INTUITIONISTIC case

The intuitionistic sequent calculus is obtained by requiring that all sequents be intuitionistic. A sequent $\Gamma \Rightarrow \Delta$ is said to be intuitionistic if Δ consists of at most one formula.

Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.

Our first example is a deduction of the law of excluded middle.

$$\begin{array}{c}
 \frac{A \Rightarrow A}{\Rightarrow A, \neg A} \neg R \\
 \frac{\Rightarrow A, \neg A}{\Rightarrow A, A \vee \neg A} \vee R \\
 \frac{\Rightarrow A, A \vee \neg A}{\Rightarrow A \vee \neg A, A} \mathcal{X}_r \\
 \frac{\Rightarrow A \vee \neg A, A \vee \neg A}{\Rightarrow A \vee \neg A} \vee R \\
 \frac{\Rightarrow A \vee \neg A, A \vee \neg A}{\Rightarrow A \vee \neg A} C_r
 \end{array}$$

Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.

The second example is an intuitionistic deduction.

$$\begin{array}{c}
 \frac{F(a) \Rightarrow F(a)}{F(a) \Rightarrow \exists x F(x)} \exists R \\
 \frac{\frac{F(a) \Rightarrow \exists x F(x)}{\neg \exists x F(x), F(a) \Rightarrow} \neg L}{F(a), \neg \exists x F(x) \Rightarrow} \mathcal{A}_I \\
 \frac{\frac{F(a), \neg \exists x F(x) \Rightarrow}{\neg \exists x F(x) \Rightarrow \neg F(a)} \neg L}{\neg \exists x F(x) \Rightarrow \forall x \neg F(x)} \forall R \\
 \frac{\neg \exists x F(x) \Rightarrow \forall x \neg F(x)}{\Rightarrow \neg \exists x F(x) \rightarrow \forall x \neg F(x)} \rightarrow R
 \end{array}$$

Cut Elimination (Gentzen's Hauptsatz)

If a sequent $\Gamma \Rightarrow \Delta$ is provable,
then it is provable without cuts.

Here is an example of how to eliminate cuts of a special form:

$$\frac{\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \rightarrow R \quad \frac{\Lambda \Rightarrow \Theta, A \quad B, \Xi \Rightarrow \Phi}{A \rightarrow B, \Lambda, \Xi \Rightarrow \Theta, \Phi} \rightarrow L}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi} \text{Cut}$$

is replaced by

$$\frac{\frac{\Lambda \Rightarrow \Theta, A \quad A, \Gamma \Rightarrow \Delta, B}{\Lambda, \Gamma \Rightarrow \Theta, \Delta, B} \text{Cut} \quad B, \Xi \Rightarrow \Phi}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi} \text{Cut}$$

Remarks

- The proof of the cut elimination theorem is rather intricate as the process of removing cuts interferes with contraction.

The possibility of contraction accounts for the high cost of eliminating cuts. Let $|\mathcal{D}|$ be the height of the deduction \mathcal{D} . Also, let $\text{rank}(\mathcal{D})$ be supremum of the lengths of cut formulae occurring in \mathcal{D} . Turning \mathcal{D} into a cut-free deduction of the same end sequent results, in the worst case, in a deduction of height

$$\mathcal{H}(\text{rank}(\mathcal{D}), |\mathcal{D}|)$$

where

$$\mathcal{H}(0, n) = n \quad \mathcal{H}(k + 1, n) = 4^{\mathcal{H}(k, n)}.$$

- Cut-free proofs aren't suitable for the mathematical practice. The cut formulae in a proof usually carry the idea of the proof (lemmata). Removing cuts not only makes proofs longer but also renders them less understandable.

The **Hauptsatz** has an important corollary.

The **Subformula Property**

If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it has a deduction all of whose formulae are subformulae of the formulae of Γ and Δ .

Corollary

A contradiction, i.e. the empty sequent, is not deducible.

While mathematics is based on logic, it cannot be developed solely on the basis of **pure logic**. What is needed in addition are **axioms** that assert the **existence** of **mathematical objects** and their properties. Logic plus axioms gives rise to (formal) **theories** such as **Peano arithmetic** or the axioms of **Zermelo-Fraenkel set theory**.

What happens when we try to apply the procedure of cut elimination to theories? Well, axioms are poisonous to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory T when the cut formula is an axiom of T . However, sometimes the axioms of a theory are of **bounded syntactic complexity**. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of T .

This gives rise to

partial cut elimination.

This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as **atomic intuitionistic sequents** (also called **Horn clauses**), yielding the completeness of **Robinsons resolution method**.

Partial cut elimination also pays off in the case of **fragments** of **PA** and set theory with **restricted induction schemes**, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of Π_2^0 statements in such fragments.

Full arithmetic, i.e. **PA**, does not even allow for partial cut elimination since the induction axioms have unbounded complexity.

However, one can remove the obstacle against cut elimination in a drastic way by going **infinite**. The so-called **ω -rule** consists of the two types of **infinitary inferences**:

$$\frac{\Gamma \Rightarrow \Delta, F(0); \Gamma \Rightarrow \Delta, F(1); \dots ; \Gamma \Rightarrow \Delta, F(n); \dots}{\Gamma \Rightarrow \Delta, \forall x F(x)} \omega R$$

$$\frac{F(0), \Gamma \Rightarrow \Delta; F(1), \Gamma \Rightarrow \Delta; \dots ; F(n), \Gamma \Rightarrow \Delta; \dots}{\exists x F(x), \Gamma \Rightarrow \Delta} \omega L$$

The price to pay will be that deductions become infinite objects, i.e. **infinite well-founded trees**.

The sequent-style version of Peano arithmetic with the ω -rule will be termed \mathbf{PA}_ω .

\mathbf{PA}_ω has no use for free variables. Thus free variables are discarded and thus all terms are closed. All formulae of this system are therefore closed, too.

The numerals are the terms \bar{n} , where $\bar{0} = 0$ and $\overline{n+1} = S\bar{n}$. We shall identify \bar{n} with the natural number n . All terms t of \mathbf{PA}_ω evaluate to a numeral \bar{n} .

\mathbf{PA}_ω has all the inference rules of the sequent calculus except for $\forall R$ and $\exists L$. In their stead, \mathbf{PA}_ω has the ωR and ωL inferences.

The **Axioms** of \mathbf{PA}_ω are the following:

-

$$\Rightarrow A$$

if A is a **true** atomic sentence.

-

$$B \Rightarrow$$

if B is a **false** atomic sentence.

-

$$F(s_1, \dots, s_n) \Rightarrow F(t_1, \dots, t_n)$$

if $F(s_1, \dots, s_n)$ is an atomic sentence and s_j and t_j evaluate to the same numeral.

With the aid of the ω -rule, induction becomes logically deducible in infinitary logic.

Theorem For every n there is a finite deduction \mathcal{D}_n of the sequent

$$F(0), \forall x [F(x) \rightarrow F(Sx)] \Rightarrow F(n).$$

Proof. Since $B, \Gamma \Rightarrow B$ is deducible for every formula B and sequence Γ , we obtain \mathcal{D}_0 .

Let $\Delta := F(0), \forall x [F(x) \rightarrow F(Sx)]$. From \mathcal{D}_n we obtain \mathcal{D}_{n+1} as follows:

$$\frac{\frac{\frac{\mathcal{D}_n}{\Delta \Rightarrow F(n)} \quad \frac{\mathcal{D}^*}{F(Sn), \Delta \Rightarrow F(Sn)}}{F(n) \rightarrow F(Sn), \Delta \Rightarrow F(S(n))} \rightarrow L}{\frac{\forall x [F(x) \rightarrow F(Sx)], \Delta \Rightarrow F(S(n))}{F(0), \forall x [F(x) \rightarrow F(Sx)] \Rightarrow F(S(n))} \forall L}{F(0), \forall x [F(x) \rightarrow F(Sx)] \Rightarrow F(S(n))} \text{Struc}$$

A final application of ωR yields an infinite deduction

$$\begin{array}{c}
 \begin{array}{c}
 \mathcal{D}_0 \qquad \qquad \qquad \mathcal{D}_n \\
 \Delta \Rightarrow F(0) \quad \cdots \quad \Delta \Rightarrow F(n) \quad \cdots
 \end{array} \\
 \hline
 F(0), \forall x [F(x) \rightarrow F(Sx)] \Rightarrow \forall x F(x) \quad \omega R \\
 \hline
 F(0) \wedge \forall x [F(x) \rightarrow F(Sx)], \forall x [F(x) \rightarrow F(Sx)] \Rightarrow \forall x F(x) \quad \wedge L \\
 \hline
 \forall x [F(x) \rightarrow F(Sx)], F(0) \wedge \forall x [F(x) \rightarrow F(Sx)] \Rightarrow \forall x F(x) \quad \wedge L \\
 \hline
 F(0) \wedge \forall x [F(x) \rightarrow F(Sx)], \text{same} \Rightarrow \forall x F(x) \quad C_I \\
 \hline
 F(0) \wedge \forall x [F(x) \rightarrow F(Sx)] \Rightarrow \forall x F(x) \\
 \hline
 \Rightarrow F(0) \wedge \forall x [F(x) \rightarrow F(Sx)] \rightarrow \forall x F(x) \quad \rightarrow R
 \end{array}$$

Cut Elimination for PA_ω

We want to measure the **height** and **cut rank** of a PA_ω deduction \mathcal{D} .

We will notate this by

$$\mathcal{D} \stackrel{\alpha}{\underset{k}{|}} \Gamma \Rightarrow \Delta.$$

The above relation is defined inductively following the buildup of the deduction \mathcal{D} .

For the **cut rank** we need the definition of the **length**, $|A|$ of a formula:

- $|A| = 0$ if A is atomic;
- $|\neg A_0| = |A_0| + 1$;
- $|A_0 \square A_1| = \max(|A_0|, |A_1|) + 1$ where $\square = \wedge, \vee, \rightarrow$;
- $|\exists x F(x)| = |\forall x F(x)| = |F(0)| + 1$.

Suppose the last inference of \mathcal{D} is of the form

$$\frac{\mathcal{D}_0 \quad \Gamma_0 \Rightarrow \Delta_0 \quad \cdots \quad \mathcal{D}_n \quad \Gamma_n \Rightarrow \Delta_n \quad \cdots \quad n < \tau}{\Gamma \Rightarrow \Delta} \quad l$$

where $\tau = 1, 2, \omega$ and the \mathcal{D}_n are the immediate subdeductions of \mathcal{D} . If

$$\mathcal{D}_n \mid_k^{\alpha_n} \Gamma_n \Rightarrow \Delta_n$$

and $\alpha_n < \alpha$ for all $n < \tau$ then

$$\mathcal{D} \mid_k^{\alpha} \Gamma \Rightarrow \Delta$$

providing that in the case of l being a **cut** with cut formula A we also have $|A| < k$.

We also just write $\mathbf{PA}_\omega \mid_k^\alpha \Gamma \Rightarrow \Delta$ if there exists a \mathbf{PA}_ω deduction $\mathcal{D} \mid_k^\alpha \Gamma \Rightarrow \Delta$.

Embedding Theorem If $\mathbf{PA} \vdash \Gamma \Rightarrow \Delta$ then

$$\mathbf{PA}_\omega \stackrel{\omega+m}{k} \Gamma \Rightarrow \Delta$$

for some $m, k < \omega$.

Reduction Lemma If $\mathbf{PA}_\omega \frac{\alpha}{k} \Gamma \Rightarrow \Delta, A$ and $\mathbf{PA}_\omega \frac{\beta}{k} A, \Lambda \Rightarrow \Theta$
with $k = |A|$, then

$$\mathbf{PA}_\omega \frac{\alpha\#\beta}{k} \Gamma, \Lambda \Rightarrow \Delta, \Theta.$$

Theorem If $\mathbf{PA}_\omega \mid_{k+1}^\alpha \Gamma \Rightarrow \Delta$, then $\mathbf{PA}_\omega \mid_k^{\omega^\alpha} \Gamma \Rightarrow \Delta$.

Cut Elimination Theorem If $\mathbf{PA}_\omega \mid_n^\alpha \Gamma \Rightarrow \Delta$, then

$$\mathbf{PA}_\omega \mid_0^{\omega^{\omega \cdots \omega^\alpha}} \Gamma \Rightarrow \Delta \quad \underbrace{\omega^{\omega \cdots \omega^\alpha}}_{n \text{ times}} .$$

Ockham's Razor

In what follows, we shall be solely dealing with classical logic. Therefore we can simplify the sequent calculus as follows:

- We get rid of the **structural rules** by using **sets of formulae** rather than **sequents of formulae**. This has the effect that **exchange** and **contraction** happen automatically:

$$\{C_1, \dots, C_r, A, A, D_1, \dots, D_s\} = \{D_1, \dots, D_s, A, C_1, \dots, C_r\}$$

We take care of **weakening** by adding all the formulae we may be interested in from the start; thus we have more liberal axioms:

$$A, \Gamma \Rightarrow \Delta, A$$

- Using the **De Morgan laws** of classical logic we can push **negations** in front of **atomic** formulae. Also, in classical logic \neg, \wedge, \vee forms a **complete** set of connectives. Thus we can simplify matters, by demanding that formulae are built up from **atomic** and **negated atomic formulae** (literals) by means of $\wedge, \vee, \forall, \exists$.

Negating a formula A then becomes a defined operation:

- $\neg\neg A := A$ if A is atomic;
 - $\neg(A \wedge B) = \neg A \vee \neg B$; $\neg(A \vee B) = \neg A \wedge \neg B$;
 - $\neg\forall x F(x) := \exists x \neg F(x)$; $\neg\exists x F(x) := \forall x \neg F(x)$.
- In classical logic we don't need the two sides of a sequent

$$A_1, \dots, A_r \Rightarrow \Delta$$

since it can be re-written as

$$\Rightarrow \neg A_1, \dots, \neg A_r, \Delta$$

In the **Tait-style** version of the **classical sequent calculus** $\Gamma, \Delta, \Lambda, \Theta, \dots$ range over finite sets of formulae in **negation normal form**. Γ, Δ stands for $\Gamma \cup \Delta$ and Δ, A is short for $\Delta \cup \{A\}$.

The **inferences** of the **Tait-calculus** are as follows:

(Axiom) $\Gamma, A, \neg A$

(\wedge)
$$\frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

(\vee)
$$\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \text{ if } i = 0 \text{ or } i = 1$$

(\forall)
$$\frac{\Gamma, F(a)}{\Gamma, \forall x F(x)}$$

(\exists)
$$\frac{\Gamma, F(t)}{\Gamma, \exists x F(x)}$$

(Cut)
$$\frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}$$

Of course, the variable a in (\forall) is an **eigenvariable**.