GIBBSIAN POINT PROCESSES

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Contents

Date: March 31, 2018.

1. The ideal gas

1.1. **Uniform distribution on energy shells. Boltzmann entropy.** For $m > 0$ and $n \in \mathbb{N}$, consider the function

$$H_n : (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n \to \mathbb{R}, \quad H_n(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i=1}^n \tfrac{1}{2m} |p_i|^2 = \tfrac{1}{2} |\boldsymbol{p}|^2. \tag{1.1}$$

H_n is the Hamiltonian for an ideal gas of n particles of mass m in $\mathbb{R}^3$. Particles have positions $x_1, \ldots, x_n$ and momenta $p_1, \ldots, p_n$. Hamilton functions are important in classical mechanics because they encode dynamics via an associated ordinary differential equation, given by

$$\dot{\boldsymbol{x}}(t) = \nabla_{\boldsymbol{p}} H_n(\boldsymbol{x}(t), \boldsymbol{p}(t)), \quad \dot{\boldsymbol{p}}(t) = -\nabla_{\boldsymbol{x}} H_n(\boldsymbol{x}(t), \boldsymbol{p}(t)), \tag{1.2}$$

which in our case become $\dot{\boldsymbol{x}}(t) = \frac{1}{m} \boldsymbol{p}(t)$, $\dot{\boldsymbol{p}}(t) = 0$, hence

$$p_j(t) = m\dot{x}_j(t), \quad m\ddot{x}_j(t) = 0 \quad (j = 1, \ldots, n). \tag{1.3}$$

The momentum is the mass times the velocity, and for the Hamiltonian defined above, the ODE consists of a system of n independent ODEs (no interaction between particles). From now on we choose $m = 1$. Fix $L, E, \varepsilon > 0$, set

$$\Lambda = [-\tfrac{L}{2}, \tfrac{L}{2}]^3 \tag{1.4}$$

and consider the energy shell

$$\Omega^{n\varepsilon}_{E,\Lambda,n} := \{(\boldsymbol{x}, \boldsymbol{p}) \in \Lambda^n \times (\mathbb{R}^3)^n \mid E - n\varepsilon \le H_n(\boldsymbol{x}, \boldsymbol{p}) \le E\}. \tag{1.5}$$

Let $\mathsf{P}^{n\varepsilon}_{E,\Lambda,n}$ be the uniform distribution on $\Omega^{n\varepsilon}_{E,\Lambda,n}$. We would like to know if $\mathsf{P}^{n\varepsilon}_{E,\Lambda,n}$ has a limit, in some sense, when $E = E_n$, $L = L_n$ and n all go to infinity in such a way that

$$\lim_{n\to\infty} \frac{E_n}{|\Lambda_n|} = u, \quad \lim_{n\to\infty} \frac{n}{|\Lambda_n|} = \rho, \quad |\Lambda_n| = L_n^3 \tag{1.6}$$

at fixed particle density $\rho > 0$ and energy density $u > 0$. We investigate first the asymptotics of the normalization constant $|\Omega^{n\varepsilon}_{E,\Lambda,n}|$, i.e., the Lebesgue volume of the energy shell. This is not strictly needed for the limiting behavior of the probability distributions but is of interest in its own.

Proposition 1.1. *Fix $u, \rho > 0$. Let $(E_n)_{n\in\mathbb{N}}$, $(L_n)_{n\in\mathbb{N}}$, and $\Lambda_n = [-L_n/2, L_n/2]^d$ be sequences in $\mathbb{R}_+$ that satisfy* (1.6). *Then for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \tfrac{1}{|\Lambda_n|} \log\left(\tfrac{1}{n!} |\Omega^{n\varepsilon}_{E_n,\Lambda_n,n}|\right) = s(u, \rho) \tag{1.7}$$

where

$$s(u, \rho) = -\rho(\log \rho - 1) + \rho\Big(\tfrac{3}{2} + \tfrac{3}{2} \log\left(\tfrac{4\pi u}{3\rho}\right)\Big) \tag{1.8}$$

The function $s(u, \rho) =$ is the *Boltzmann entropy* per unit volume of the ideal gas. Eq. (1.8) can be written more compactly as

$$s(u, \rho) = \rho\Big(\tfrac{5}{2} + \log\Big(\tfrac{1}{\rho}\left(\tfrac{4\pi u}{3\rho}\right)^{3/2}\Big)\Big) \tag{1.9}$$

which is a version of the *Sackur-Tetrode equation.*

Proof of Proposition 1.1. Clearly

$$\log\left(\tfrac{1}{n!}|\Omega^{n\varepsilon}_{E_n,L_n,n}|\right) = \log\left(\tfrac{1}{n!}L_n^{3n}\right) + \log\big|B_{3n}(\sqrt{2E_n}) \setminus B_{3n}(\sqrt{2E_n - n\varepsilon})\big|$$

where $B_{3n}(r)$ is the ball of radius r in $\mathbb{R}^{3n}$. Stirling's formula $n! \sim \sqrt{2\pi n}(n/\mathrm{e})^n$ yields

$$\tfrac{1}{n}\log\left(\tfrac{1}{n!}L_n^{3n}\right) = \tfrac{1}{n}\log\left(\tfrac{n^n}{n!}\right) + \log\left(\tfrac{L_n^3}{n}\right) \to 1 - \log\rho. \tag{1.10}$$

Volumes of balls are given in terms of the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ as

$$|B_{3n}(r)| = \frac{\pi^{3n/2}}{\Gamma(\frac{3n}{2}+1)}\, r^{3n}. \tag{1.11}$$

The Gamma function satisfies $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$ and Stirling's formula applies to non-integer input as well, i.e., $\Gamma(x+1) \sim \sqrt{2\pi x}(x/\mathrm{e})^x$ as $x \to \infty$, see Exercise 1.3. It follows that

$$\begin{aligned}
&\tfrac{1}{n}\log\big|B_{3n}(\sqrt{2E_n}) \setminus B_{3n}(\sqrt{2E_n - n\varepsilon})\big| \\
&\quad = \tfrac{1}{n}\log\left(\frac{\pi^{3n/2}}{\Gamma(\frac{3n}{2}+1)}\right) + \tfrac{1}{n}\log\left(\sqrt{2E_n}^{3n} - \sqrt{2(E_n - n\varepsilon)}^{3n}\right) \\
&\quad = 3\log\sqrt{\pi} + \frac{3}{2}\frac{1}{3n/2}\log\left(\frac{(3n/2)^{3n/2}}{\Gamma(\frac{3n}{2}+1)}\right) + \frac{1}{n}\log\left(\frac{\sqrt{2E_n}^{3n}}{\sqrt{3n/2}^{3n}}\right) \\
&\qquad + \frac{1}{n}\log\left(1 - \sqrt{1 - \tfrac{n\varepsilon}{E_n}}^{3n}\right) \\
&\quad \to \tfrac{3}{2}\left(\log\pi + 1 + \log\left(\tfrac{4}{3}\tfrac{u}{\rho}\right)\right) = \frac{3}{2} + \frac{3}{2}\log\left(\frac{4\pi u}{3\rho}\right)
\end{aligned}$$

and the proof is easily completed. □

Next we address the behavior of the probability measures. To that aim it is convenient to single out test functions F that are *local*, i.e., they depend only on the particles in some bounded Borel set $\Delta \subset \mathbb{R}^3$. We further assume that the test function does not depend on the labelling of the particle. Such a function F can be specified in the following form: let

$$N_\Delta(\boldsymbol{x}) := \#\{j \in \{1,\dots,n\} \mid x_j \in \Delta\} = \sum_{j=1}^n \delta_{x_j}(\Delta). \tag{1.12}$$

be the number of particles in Δ. Suppose we are given a scalar $f_0 \in \mathbb{R}$ and a family $(f_k)_{k\in\mathbb{N}_0}$ of functions $f_k : (\Delta \times \mathbb{R}^3)^k \to \mathbb{R}$ that are symmetric, i.e., $f_k(y_{\sigma(1)},\dots,y_{\sigma(k)}) = f_k(y_1,\dots,y_k)$ for all $\sigma \in \mathfrak{S}_k$ and $(y_1,\dots,y_k) \in (\Delta\times\mathbb{R}^3)^k$. We define

$$F(\boldsymbol{x},\boldsymbol{p}) = \begin{cases} f_0, & \text{if } N_\Delta(\boldsymbol{x}) = 0, \\ f_k\big(((x_j,p_j))_{j\in[n]:x_j\in\Delta}\big), & \text{if } N_\Delta(\boldsymbol{x}) = k \in \mathbb{N}. \end{cases} \tag{1.13}$$

Define

$$\beta = \frac{\partial s}{\partial u}(u,\rho), \quad \mu = -\frac{1}{\beta}\frac{\partial s}{\partial \rho}(u,\rho), \quad z = \exp(\beta\mu). \tag{1.14}$$

Equivalently,

$$\beta = \frac{3}{2}\frac{\rho}{u}, \quad z = \frac{\rho}{\sqrt{2\pi/\beta}^3}, \quad \mu = \frac{1}{\beta}\log z. \tag{1.15}$$

Proposition 1.2. *Fix $u, \rho > 0$ and let β, z, μ be as in* (1.15)*. Fix a bounded Borel set $\Delta \subset \mathbb{R}^3$. Let (E_n) and (L_n) be as in Proposition 1.1 and $\varepsilon > 0$. Then for families of bounded symmetric functions $(f_n)_{n\in\mathbb{N}_0}$, $f_n : (\Delta \times \mathbb{R}^3)^n \to \mathbb{R}$ and F as in* (1.13)*, such that F is bounded, we have*

$$\begin{aligned}\lim_{n\to\infty} \int_{\Omega^{n\varepsilon}_{E_n,L_n,n}} F \mathrm{d}\mathsf{P}^{n\varepsilon}_{E_n,L_n,n} & \\ &= \mathrm{e}^{-\rho|\Delta|} \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_{\Delta^k\times(\mathbb{R}^3)^k} f_k\left((x_j,p_j)_{j=1}^k\right) \mathrm{e}^{-\frac{1}{2}\beta|\boldsymbol{p}|^2} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p} \\ &= \mathrm{e}^{-\rho|\Delta|} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Delta^k\times(\mathbb{R}^3)^k} f_k\left((x_j,p_j)_{j=1}^k\right) \mathrm{e}^{-\beta[H_k(\boldsymbol{x},\boldsymbol{p})-\mu k]} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p}.\end{aligned}$$

In this sense the sequence of equidistributions on the energy shell $\Omega^{n\varepsilon}_{E_n,\Lambda_n,n}$ does indeed admit a limit. Moreover the limit does not depend on the shell's thickness ε.

Lemma 1.3. *Under the assumptions of Proposition 1.2, we have*

$$\lim_{n\to\infty} \mathsf{P}^{n\varepsilon}_{E_n,\Lambda_n,n}(N_\Delta = k) = \frac{(\rho|\Delta|)^k}{k!} \mathrm{e}^{-\rho|\Delta|} \tag{1.16}$$

for all $k \in \mathbb{N}_0$.

In particular, in the limit $n \to \infty$ the number N_Δ of particles in Δ converges to a Poisson random variable with parameter $\rho|\Delta|$.

Proof. To lighten notation, we abbreviate $\mathsf{P}^{n\varepsilon}_{E_n,\Lambda_n,n} = \mathsf{P}_n$. Let us first look at the distribution of the particle number N_Δ. Fix $k \in \{0, 1 \ldots, n\}$. Then

$$\begin{aligned}\mathsf{P}_n(N_\Delta = k) &= \sum_{\substack{J\subset[n]\\ \#J=k}} \frac{1}{|\Lambda_n|^n} \int_{\Lambda_n^n} \prod_{j\in J} \mathbb{1}_\Delta(x_j) \prod_{j\in[n]\setminus J} \mathbb{1}_{\Delta^c}(x_j)\mathrm{d}\boldsymbol{x} \\ &= \binom{n}{k} q_n^k (1-q_n)^{n-k} \qquad q_n := \frac{|\Delta|}{|\Lambda_n|} \\ &= \frac{1}{k!}\left[\prod_{\ell=0}^{k-1}\left(1-\frac{\ell}{n}\right)\right] (nq_n)^k (1-q_n)^{n-k} \to \frac{(\rho|\Delta|)^k}{k!}\mathrm{e}^{-\rho|\Delta|}. \end{aligned} \tag{1.17}$$

□

Lemma 1.4. *Under the assumptions of Proposition 1.2, we have for every $k \in \mathbb{N}$ and every bounded $h : (\mathbb{R}^3)^k \to \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{\int_{(\mathbb{R}^3)^n} h(p_1,\ldots,p_k) \mathbb{1}_{[E_n-n\varepsilon,E_n]}\left(\frac{1}{2}|\boldsymbol{p}|^2\right)\mathrm{d}\boldsymbol{p}}{\int_{(\mathbb{R}^3)^n} \mathbb{1}_{[E_n-n\varepsilon,E_n]}\left(\frac{1}{2}|\boldsymbol{p}|^2\right)\mathrm{d}\boldsymbol{p}} = \frac{1}{\sqrt{2\pi/\beta}^{3k}} \int_{(\mathbb{R}^3)^k} h(\boldsymbol{p})\mathrm{e}^{-\beta|\boldsymbol{p}|^2/2}\mathrm{d}\boldsymbol{p}. \tag{1.18}$$

Remark. The lemma is closely related to the following fact from probability: the finite-dimensional marginals of the uniform distribution on an n-dimensional ball of radius $\sqrt{n}$ converge weakly to standard Gaussians. See [21, Chapter 2.6].

Proof of Lemma 1.4. Set

$$\varphi_{nk}(p_1,\ldots,p_k) := \int_{(\mathbb{R}^3)^{n-k}} \mathbb{1}_{[E_n-n\varepsilon,E_n]}\Big(\frac{1}{2}\sum_{j=1}^k p_j^2 + \frac{1}{2}|\boldsymbol{q}|^2\Big)\mathrm{d}\boldsymbol{q}. \tag{1.19}$$

We have to evaluate

$$\frac{\int_{(\mathbb{R}^3)^k} h(\boldsymbol{p})\varphi_{nk}(\boldsymbol{p})\mathrm{d}\boldsymbol{p}}{\int_{(\mathbb{R}^3)^k} \varphi_{nk}(\boldsymbol{p})\mathrm{d}\boldsymbol{p}} \tag{1.20}$$

in the limit $n \to \infty$. Let ω_d be the surface area of $\mathbb{S}^{d-1} = \partial B_d(1) \subset \mathbb{R}^d$, the unit sphere in $\mathbb{R}^d$. Thus $\omega_2 = 2\pi$ and $\omega_3 = 4\pi$. We have, for every non-negative test function $v : [0, \infty) \to [0, \infty)$,

$$\int_{\mathbb{R}^d} v(|x|)\mathrm{d}x = \omega_d \int_0^\infty v(r)r^{d-1}\,\mathrm{d}r. \tag{1.21}$$

It follows that, for $|\boldsymbol{p}|^2 \le 2E_n - 2n\varepsilon$,

$$\begin{aligned}\varphi_{nk}(p_1, \dots, p_k) &= \omega_{3(n-k)} \int_0^\infty \mathbb{1}_{[E_n-n\varepsilon,E_n]}\Big(\tfrac{1}{2}\sum_{j=1}^k p_j^2 + \tfrac{1}{2}r^2\Big)r^{3(n-k)-1}\mathrm{d}r \\ &= \frac{\omega_{3(n-k)}}{3(n-k)}\Big((2E_n - |\boldsymbol{p}|^2)^{3[n-k]/2} - (2E_n - 2n\varepsilon - |\boldsymbol{p}|^2)^{3[n-k]/2}\Big).\end{aligned} \tag{1.22}$$

If $2E_n - 2n\varepsilon < |\boldsymbol{p}|^2 \le 2E_n$, the second term in the previous line has to be replaced with zero.

Now fix k and $\boldsymbol{p} = (p_1, \dots, p_k) \in (\mathbb{R}^3)^k$. Since $E_n \to \infty$, we have $|\boldsymbol{p}|^2 \le 2E_n$ for all sufficiently large n, and

$$\big(1 - \tfrac{|\boldsymbol{p}|^2}{2E_n}\big)^{3(n-k)/2} = \exp\Big(-\tfrac{3n}{2}(1 - \tfrac{k}{n})\log(1 - \tfrac{|\boldsymbol{p}|^2}{2E_n}\Big) \to \exp(-\beta|\boldsymbol{p}|^2/2). \tag{1.23}$$

Moreover

$$\big(1 - \tfrac{n\varepsilon}{2E_n} - \tfrac{|\boldsymbol{p}|^2}{2E_n}\big)_+^{3(n-k)/2} \le \big(1 - \varepsilon(1 + o(1))\tfrac{\rho}{u}\big)_+^{3(n-k)/2} \to 0. \tag{1.24}$$

Thus we may write

$$\varphi_{nk}(p_1, \dots, p_k) = \frac{\omega_{3(n-k)}}{3(n-k)}(2E_n)^{3(n-k)/2}\tilde{\varphi}_{nk}(p_1, \dots, p_k) \tag{1.25}$$

where

$$\lim_{n\to\infty} \tilde{\varphi}_{nk}(p_1, \dots, p_k) = \exp(-\beta|\boldsymbol{p}|^2/2) \tag{1.26}$$

as a pointwise limit of functions. Using the bound $\log(1 - t) \le -t$, valid for all $t \in (-1, \infty)$, we find that for some suitable $s > 0$, sufficiently large k, and all $\boldsymbol{p} \in (\mathbb{R}^3_+)^k$, we have

$$0 \le \tilde{\varphi}_{nk}(p_1, \dots, p_k) \le \exp(-s|\boldsymbol{p}|^2). \tag{1.27}$$

The monotone convergence theorem thus implies

$$\lim_{n\to\infty} \int_{(\mathbb{R}^3_+)^k} \tilde{\varphi}_{nk}(p_1, \dots, p_k)\mathrm{d}\boldsymbol{p} = \int_{(\mathbb{R}^3_+)^k} \mathrm{e}^{-\beta|\boldsymbol{p}|^2/2}\mathrm{d}\boldsymbol{p} = \sqrt{2\pi/\beta}^{3k}. \tag{1.28}$$

and, since h is bounded,

$$\lim_{n\to\infty} \int_{(\mathbb{R}^3)^k} h(p_1, \dots, p_k)\tilde{\varphi}_{nk}(p_1, \dots, p_k)\mathrm{d}\boldsymbol{p} = \int_{(\mathbb{R}^3_+)^k} h(\boldsymbol{p})\mathrm{e}^{-\beta|\boldsymbol{p}|^2/2}\mathrm{d}\boldsymbol{p}. \tag{1.29}$$

Taking the ratio of the two equations the proof of the lemma is completed. □

Proof of Proposition 1.2. It is enough to treat non-negative bounded functions (exploit linearity and $F = F_+ - F_-$). For F as in (1.13) with non-negative f_n and $k \in \mathbb{N}_0$, set $F_k := \mathbb{1}_{\{N_\Delta = k\}} F$. The monotone convergence theorem applied to the partial sums $\sum_{k=0}^m F_k$ yields

$$\mathsf{E}_n[F] = \mathsf{E}_n\left[\sum_{k=0}^\infty F_k\right] = \sum_{k=0}^\infty \mathsf{E}_n\left[F_k\right] \tag{1.30}$$

where E_n denotes expectation with respect to $\mathsf{P}_n = \mathsf{P}^{n\varepsilon}_{E_n,\Lambda_n,n}$. We evaluate first the limits of each individual summand. For $k=0$, we have

$$\lim_{n\to\infty} \mathsf{E}_n[F_0] = \lim_{n\to\infty} f_0 \mathsf{P}_n(N_\Delta = 0) = f_0 \mathrm{e}^{-\rho|\Delta|} =: I_0. \tag{1.31}$$

Next consider $k \in \mathbb{N}$. Define φ_{nk} as in the proof of Lemma 1.4. We have

$$\begin{aligned}\mathsf{E}_n[F_k] &= \frac{1}{|\Omega^{n\varepsilon}_{E_n,\Lambda_n,n}|}\int_{\Delta^k\times(\mathbb{R}^3)^k} f_k\big((x_j,p_j)_{j=1}^k\big)|\Lambda\setminus\Delta|^k \varphi_{nk}(\boldsymbol{p})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p}\\ &= \mathsf{P}_n(N_\Delta = k)\frac{1}{|\Delta|^k}\int_{\Delta^k}\frac{\int_{\mathbb{R}^3} f_k((x_j,p_j)_{j=1}^k)\varphi_{nk}(\boldsymbol{p})\mathrm{d}\boldsymbol{p}}{\int_{\mathbb{R}^3}\varphi_{nk}(\boldsymbol{p})\mathrm{d}\boldsymbol{p}}\mathrm{d}\boldsymbol{x}\end{aligned}$$

Lemma 1.3 and 1.4 yield

$$\lim_{n\to\infty}\mathsf{E}_n[F_k] = \frac{z^k}{k!}\int_{\Delta^k\times(\mathbb{R}^3)^k} f_k(\boldsymbol{x},\boldsymbol{p})\mathrm{e}^{-\beta|\boldsymbol{p}|^2/2}\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p} =: I_k. \tag{1.32}$$

It remains to check that we can exchange summation and limits in (1.30). Going back to the proof of Lemma 1.3, we see that

$$\big|\mathsf{E}_n\big[F_k\big]\big| \leq ||F||_\infty \mathsf{P}_n(N_\Delta = k) \leq ||F||_\infty \tfrac{1}{k!}\left(\tfrac{nq_n}{1-q_n}\right)^k \mathrm{e}^{n\log(1-q_n)} \tag{1.33}$$

where $q_n = |\Delta|/|\Lambda_n| \to 0$. Using $n\log(1-q_n) \leq -nq_n \to -\rho|\Delta|$ we conclude that for some $s,t>0$

$$\big|\mathsf{E}_n\big[F_k\big]\big| \leq ||F||_\infty \tfrac{s^k}{k!}\mathrm{e}^{-tk}. \tag{1.34}$$

The right-hand side is independent of n and the sum over k is finite. A straightforward $\varepsilon/3$-argument (or as an alternative, dominated convergence for the sum, treated as an integral with respect to counting measure) shows that

$$\lim_{n\to\infty}\sum_{k=0}^\infty \mathsf{E}_n[F_k] = \sum_{k=0}^\infty I_k. \tag{1.35}$$

Together with (1.30) this completes the proof. □

1.2. **Large deviations for Poisson and normal laws.** Proposition 1.2 shows how in the limit $n\to\infty$, starting from the uniform distribution on an energy shell, we end up with Poisson variables for particle numbers and Gaussian laws for momenta (or velocities). Armed with this knowledge we may ask for a probabilistic take on Proposition 1.1.

Let $N_{|\Lambda|} \sim \mathrm{Poi}(|\Lambda|)$ and $Z_j \in \mathcal{N}(0,1)$, $j\in\mathbb{N}$, be independent random variables defined on some common probability space $(\Omega,\mathcal{F},\mathbb{P})$. Then

$$\tfrac{1}{n!}|\Omega^{n\varepsilon}_{E,\Lambda,n}| = \mathrm{e}^{|\Lambda|}\sqrt{2\pi}^{3n}\,\mathbb{E}\Big[\mathbb{1}_{\{N_{|\Lambda|}=n\}}\mathbb{1}_{[E-n\varepsilon,E]}\big(\tfrac12\sum_{j=1}^{3n} Z_j^2\big)\,\mathrm{e}^{\frac12\sum_{j=1}^{3n} Z_j^2}\Big] \tag{1.36}$$

hence

$$\tfrac{1}{n!}|\Omega^{n\varepsilon}_{E,\Lambda,n}| \le \mathrm{e}^{|\Lambda|}\sqrt{2\pi}^{3n}\mathrm{e}^{E}\,\mathbb{P}\Big(N_{|\Lambda|}=n,\ E-n\varepsilon \le \tfrac{1}{2}\sum_{j=1}^{3n} Z_j^2 \le E\Big). \tag{1.37}$$

The upper bound becomes a lower bound if we replace E with $E-n\varepsilon$. So instead of analyzing directly the microcanonical partition function, we can investigate the asymptotic behavior of probabilities and expectations.

Proposition 1.5. *Let N_λ and $(Z_j)_{j\in\mathbb{N}}$ be independent random variables defined on some common probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $N_\lambda \sim \mathrm{Poi}(\lambda)$ and $Z_j \sim \mathcal{N}(0,1)$. Then for every $\rho>0$ and all $a,b\in(0,\infty)$ with $a<b$, we have*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\mathbb{P}(N_\lambda = \lfloor\lambda\rho\rfloor) = -\rho(\log\rho-1)-1$$

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\tfrac{1}{n}\sum_{j=1}^{n}Z_j^2\in[a,b]\Big) = -\inf_{\sigma^2\in[a,b]}\tfrac{1}{2}\big(\sigma^2-1-\log\sigma^2\big).$$

The proposition is proven at the end of this section. A heuristic way of rewriting it is

$$\mathbb{P}(N_\lambda\approx\rho\lambda)\approx\exp\big(-\lambda\big[\rho\log\rho-\rho+1\big]\big) \tag{1.38}$$

$$\mathbb{P}\Big(\sum_{j=1}^{n}Z_j^2\approx n\sigma^2\Big)\approx\exp\big(-n\tfrac{1}{2}\big[\sigma^2-1-\log\sigma^2\big]\big). \tag{1.39}$$

Asymptotic results of this type belong to the *theory of large deviations* [11, 15, 46]. Proposition 1.1 is recovered from (1.36) and Proposition 1.5 as follows. We have

$$\begin{aligned}\log\big(\tfrac{1}{n!}\Omega^{n\varepsilon}_{E,\Lambda,n}\big) &= |\Lambda| + \tfrac{3}{2}n\log(2\pi) + \log\mathbb{P}(N_{|\Lambda|}=n)\\ &\quad + \log\mathbb{E}\Big[\mathbb{1}_{[E-n\varepsilon,E]}\big(\tfrac{1}{2}\sum_{j=1}^{3n}Z_j^2\big)\,\mathrm{e}^{\frac{1}{2}\sum_{j=1}^{3n}Z_j^2}\Big].\end{aligned} \tag{1.40}$$

Heuristically,

$$\begin{aligned}\mathbb{E}\Big[\mathbb{1}_{[E-n\varepsilon,E]}\big(\tfrac{1}{2}\sum_{j=1}^{3n}Z_j^2\big)\,\mathrm{e}^{\frac{1}{2}\sum_{j=1}^{3n}Z_j^2}\Big] &\approx \int_{E/n-\varepsilon}^{E/n}\mathrm{e}^{n\bar u}\,\mathbb{P}\Big(\tfrac{1}{2}\sum_{j=1}^{3n}Z_j^2\approx n\bar u\Big)\mathrm{d}\bar u\\ &\approx \int_{E/n-\varepsilon}^{E/n}\exp\Big(n\bar u-\tfrac{3}{2}n\big[\tfrac{2\bar u}{3}-1-\log\big(\tfrac{2\bar u}{3}\big)\big]\Big)\mathrm{d}\bar u\\ &\approx \int_{E/n-\varepsilon}^{E/n}\exp\Big(\tfrac{3}{2}n\big[1+\log\big(\tfrac{2\bar u}{3}\big)\big]\Big)\mathrm{d}\bar u\\ &\approx \exp\big(\tfrac{3}{2}n\big[1+\log\big(\tfrac{2E}{3n}\big)\big]\big)\end{aligned} \tag{1.41}$$

(see Exercise 1.2 for the last step). Hence

$$\begin{aligned}\log\big(\tfrac{1}{n!}\Omega^{n\varepsilon}_{E,\Lambda,n}\big) &\approx |\Lambda| + \tfrac{3}{2}n\log(2\pi) - |\Lambda|\big[\rho\log\rho-\rho+1\big] + \tfrac{3}{2}n\big[1+\log\big(\tfrac{2E}{3n}\big)\big]\\ &\approx \tfrac{5}{2}n + n\log\Big(\tfrac{n}{|\Lambda|}\,\big(\tfrac{4\pi E}{3n}\big)^{3/2}\Big).\end{aligned}$$

The approximation (1.41) can be made rigorous with *Laplace integrals* from analysis or *Varadhan's lemma* from the theory of large deviations. In this way Proposition 1.5 could be used for an alternative proof of Proposition 1.1. The alternative proof is by no means shorter, its principal merit is to highlight connections between the Boltzmann entropy $s(u,\rho)$ and large deviations theory.

Proof of Proposition 1.5. Let $n = n_\lambda = \lfloor \rho\lambda \rfloor \to \infty$. Then by Stirling's formula,

$$\mathbb{P}(N_\lambda = n) = \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda} \sim \frac{1}{\sqrt{2\pi n}}\mathrm{e}^{n[1+\log(\lambda/n)-\lambda/n]} = \mathrm{e}^{\lambda(\rho-\rho\log\rho-1+o(1))} \tag{1.42}$$

We take the log, divide by λ, let $\lambda \to \infty$, and obtain the result for N_λ.

Next fix $\sigma^2 \in [a,b]$ and let $\widehat{Z}_j$, $j \in \mathbb{N}$ be i.i.d. random variables with law $\widehat{Z}_j \sim \mathcal{N}(0,\sigma^2)$. Let $\boldsymbol{x} \in \mathbb{R}^n$ and $c := \max_{\sigma^2\in[a,b]} |\frac{1}{2\sigma^2} - \frac{1}{2}|$. Then

$$\begin{aligned}
\mathbb{P}\Big(-\delta \le \tfrac{1}{n}\sum_{j=1}^n Z_j^2 - \sigma^2 \le \delta\Big) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}\sum_{j=1}^n |\boldsymbol{x}|^2}\mathbb{1}_{[n(\sigma^2-\delta),n(\sigma^2+\delta)]}(|\boldsymbol{x}|^2)\mathrm{d}\boldsymbol{x}\\
&\le \frac{1}{\sqrt{2\pi}^n}\mathrm{e}^{-\frac{1}{2}n\sigma^2+\frac{1}{2}n+nc\delta}\int_{\mathbb{R}^n}\mathrm{e}^{-\frac{1}{2\sigma^2}|\boldsymbol{x}|^2}\mathbb{1}_{[n(\sigma^2-\delta),n(\sigma^2+\delta)]}(|\boldsymbol{x}|^2)\mathrm{d}\boldsymbol{x}\\
&= \sigma^n\mathrm{e}^{-\frac{1}{2}n\sigma^2+\frac{1}{2}n+nc\delta}\,\mathbb{P}\Big(-\delta \le \tfrac{1}{n}\sum_{j=1}^n \widehat{Z}_j^2 - \sigma^2 \le \delta\Big).
\end{aligned}$$

The probability in the last line converges to one by the law of large numbers (notice $\mathbb{E}[\widehat{Z}_j^2] = \sigma^2$), hence

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}\Big(-\delta \le \tfrac{1}{n}\sum_{j=1}^n Z_j^2 - \sigma^2 \le \delta\Big) \le -\tfrac{1}{2}\big(\sigma^2 - 1 - \log\sigma^2\big) + c\delta. \tag{1.43}$$

A similar lower bound holds true for the liminf. Now let $a,b>0$ with $a<b$. Fix $\delta>0$ and $m\in\mathbb{N}$, $\sigma_1^2,\dots,\sigma_m^2 \in [a,b]$ such that $[a,b] \subset \cup_{j=1}^n[\sigma_j^2-\delta,\sigma_j^2+\delta]$. Then by (1.43), we have

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\tfrac{1}{n}\sum_{j=1}^n Z_j^2 \in [a,b]\Big) &\le \max_{j=1,\dots,m}\Big(-\tfrac{1}{2}\big(\sigma_j^2 - 1 - \log\sigma_j^2\big) + c\delta\Big)\\
&\le -\inf_{\sigma^2\in[a,b]}\tfrac{1}{2}\big(\sigma^2 - 1 - \log\sigma^2\big) + c\delta.
\end{aligned}$$

This holds true for every $\delta>0$, so we may let $\delta \searrow 0$ and find

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\tfrac{1}{n}\sum_{j=1}^n Z_j^2 \in [a,b]\Big) \le -\inf_{\sigma^2\in[a,b]}\tfrac{1}{2}\big(\sigma^2 - 1 - \log\sigma^2\big). \tag{1.44}$$

For the lower bound, let $\sigma^2 \in (a,b)$ and $\delta>0$ small enough so that $[\sigma^2-\delta,\sigma^2+\delta] \subset (a,b)$. Then

$$\begin{aligned}
\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\tfrac{1}{n}\sum_{j=1}^n Z_j^2 \in [a,b]\Big) &\ge \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big(\tfrac{1}{n}\sum_{j=1}^n Z_j^2 \in [\sigma^2-\delta,\sigma^2+\delta]\Big)\\
&\ge -\tfrac{1}{2}\big(\sigma^2 - 1 - \log\sigma^2\big) + c\delta.
\end{aligned}$$

We let $\delta \searrow 0$, take the infimum over the open interval (a,b), notice that it is equal to the minimum over the closed interval $[a,b]$ because of the continuity of $\sigma^2 - 1 - \log\sigma^2$, and obtain the desired statement. □

1.3. **Statistical ensembles and thermodynamic potentials.** In Section 1.1 we have chosen to work with the uniform distribution on an energy shell, but there are of course other choices of measures as well. Working with different probability measures corresponds, in the terminology of statistical mechanics, to working in different *statistical ensembles*. The uniform distribution corresponds to the *microcanonical ensemble*. The normalization constants are called *partition functions*, the functions characterizing the asymptotic behavior of the normalization constants are called *thermodynamic potentials*. Thus $\frac{1}{n!}|\Omega^{n\varepsilon}_{E,\Lambda,n}|$ is the *microcanonical partition function*, the associated thermodynamic potential is the Boltzmann entropy. The limiting procedure (as in (1.6)) is the *thermodynamic limit*.

Other common choices or measures are:

The canonical ensemble. This ensemble models the distribution of a system that might exchange energy with its environment so that the energy becomes random. Instead of taking a distribution on a fixed energy shell, therefore, it is a measure on all of $\Lambda^n \times (\mathbb{R}^3)^n$. The measure $\mathsf{P}^{\mathsf{can}}_{\beta,\Lambda,n}$ depends on an additional parameter $\beta > 0$ and has probability density

$$\frac{1}{Z_\Lambda(\beta,n)}\,\frac{1}{n!}\exp\bigl(-\beta H_n(\boldsymbol{x},\boldsymbol{p})\bigr) \tag{1.45}$$

where $Z_\Lambda(\beta,n)$ is the *canonical partition function*

$$Z_\Lambda(\beta,n) = \frac{1}{n!}\int_{\Lambda^n\times(\mathbb{R}^3)^n} \mathrm{e}^{-\beta H_n(\boldsymbol{x},\boldsymbol{p})}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p}. \tag{1.46}$$

The thermodynamic limit consists in letting $n\to\infty$ and $\Lambda = \Lambda_n \nearrow \mathbb{R}^3$ in such a way that $n/|\Lambda_n| \to \rho$ for some fixed particle density ρ. The parameter β is kept fixed. The associated thermodynamic potential is the *Helmholtz free energy* (per unit volume)

$$f(\beta,\rho) = -\lim_{n\to\infty}\frac{1}{\beta|\Lambda_n|}\log Z_{\Lambda_n}(\beta,n). \tag{1.47}$$

The grand-canonical ensemble. This ensemble corresponds to a system that might exchange both energy and matter with its environment. Both the energy and number of particles become random. Let us leave open for now how to prescribe a probability space and content ourselves with the corresponding partition functions. Given $\beta > 0$, $\mu \in \mathbb{R}$, and $z := \exp(\beta\mu)$, the *grand-canonical partition function* is

$$\Xi_\Lambda(\beta,z) = 1 + \sum_{n=1}^\infty \frac{z^n}{n!}\int_{\Lambda^n\times(\mathbb{R}^3)^n} \mathrm{e}^{-\beta H_n(\boldsymbol{x},\boldsymbol{p})}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p}. \tag{1.48}$$

The thermodynamic limit consists in letting $\Lambda \nearrow \mathbb{R}^3$ at fixed β,μ,z. The associated thermodynamic potential is the *pressure*

$$p(\beta,z) = \lim_{\Lambda\nearrow\mathbb{R}^3}\frac{1}{\beta|\Lambda|}\log\Xi_\Lambda(\beta,z). \tag{1.49}$$

The product $p(\beta,z)$ and the system's volume is also sometimes called *grand potential* or *Landau potential*. Thus we may view the pressure as a grand potential per unit volume.

Statistical ensemble	parameters	partition function	thermodynamic potential
Microcanonical	E, Λ, N	$\frac{1}{N!}\lvert\Omega_{E,\Lambda,N}^{N\varepsilon}\rvert$	Boltzmann entropy $s(u,\rho)$
Canonical	β, Λ, N	$Z_\Lambda(\beta, N)$	Helmholtz free energy $f(\beta,\rho)$
Grand-canonical	β, Λ, z	$\Xi_\Lambda(\beta, z)$	pressure $p(\beta, z)$
Isothermal-isobaric	β, p, N	$Q_N(\beta, p)$	Gibbs free energy $g(\beta, p)$

TABLE 1. Overview of some common statistical ensembles.

The isothermal-isobaric ensemble, also called *constant pressure ensemble* models a system that can exchange energy with its environment but not matter, however the volume is no longer fixed. The ensemble is particularly useful in dimension one and we write down the partition function in dimension one only. Given $\beta, p > 0$, we define

$$\begin{aligned} Q_n(\beta,p) &= \frac{1}{n!}\int_{\mathbb{R}_+^n\times\mathbb{R}^n} \mathrm{e}^{-\beta[H_n(\boldsymbol{x},\boldsymbol{p})+p\max_{i=1,\dots,n} x_i]}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p} \\ &= \int_{\mathbb{R}_+^n\times\mathbb{R}^n} \mathbb{1}_{\{0\le x_1\le\cdots\le x_n\}}\mathrm{e}^{-\beta[H_n(\boldsymbol{x},\boldsymbol{p})+px_n]}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{p}. \end{aligned} \tag{1.50}$$

Treating $V_n = \max(x_1,\dots,x_n)$ as a proxy for the system's volume (or length), we see that $Q_n(\beta,p)$ is a partition function for configurations on $[0,\infty)^n$ for which large volumes are penalized by a factor $\exp(-\beta p V_n)$. The thermodynamic limit consists in letting $n\to\infty$ at fixed β, p. The associated thermodynamic potential is the *Gibbs free energy* per particle

$$g(\beta,p) = -\lim_{n\to\infty}\frac{1}{\beta n}\log Q_n(\beta,p). \tag{1.51}$$

Note that we have used the same letter p for the parameter p in the constant pressure ensemble and the function $p(\beta,z)$ in (1.49). This is because they play the same role, physically, but the reader who prefers to do so may choose two different letters p and $\tilde{p}$.

The different ensembles are summarized in Table 1. Each depends on three parameters, chosen among pairs of dual variables (for each pair, pick one variable):

- the energy E and the inverse temperature β;
- the volume $|\Lambda|$ and the pressure p;
- the number of particles N and the activity z (or the chemical potential μ).

For the ideal gas, the thermodynamic potentials can be computed explicitly, and the following relations are easily checked:

$$\beta f(\beta,\rho) = \sup_{u>0}\bigl(\beta u - s(u,\rho)\bigr),\quad p(\beta,\mathrm{e}^{\beta\mu}) = \sup_{\rho>0}\bigl(\mu\rho - f(\beta,\rho)\bigr) \tag{1.52}$$

and in dimension 1,

$$\mu = g(\beta,p) \;\Leftrightarrow\; p = p(\beta,\mathrm{e}^{\beta\mu}). \tag{1.53}$$

Outlook. One of our tasks will be to generalize the previous considerations from the ideal gas to interacting particles, with Hamilton function of the type

$$H_n(\boldsymbol{x},\boldsymbol{p}) = \sum_{i=1}^n \tfrac{1}{2}|p_i|^2 + \sum_{1\le i<j\le n} v(|x_i-x_j|) \tag{1.54}$$

where $v : [0,\infty) \to \mathbb{R}\cup\{\infty\}$. The $\boldsymbol{p}$- and $\boldsymbol{x}$-dependent contributions are called *kinetic energy* and *potential energy*, respectively. The explicit computations that are possible for the ideal gas break down and in general we have to content ourselves with existence theorems on the limits. It turns out that the thermodynamic potentials are often well-defined and satisfy the relations (1.52), however the convergence of probability measures is much more delicate as there are situations where sequences of probability measures admit more than one accumulation point.

Another task is to formalize the convergence of probability measures from Proposition 1.2, using point processes and the notion of local convergence. The characterization of possible accumulation points of sequences of probability measures then leads us to the notion of infinite-volume Gibbs measures.

We will focus on the grand-canonical ensemble and the distribution of particle positions, forgetting about the momenta. Note that for the Hamilton function (1.54), the canonical partition function (1.46) is a product of an integral over $\boldsymbol{x}$ and an integral over $\boldsymbol{p}$; positions and velocities are independent with respect to the canonical Gibbs measure.

1.4. **Exercises.**

Exercise 1.1.

(a) Let $\ell, n \in \mathbb{N}$. Compute

$$k(n,\ell) := \#\{(n_1,\dots,n_\ell)\in\mathbb{N}_0^\ell \mid \sum_{i=1}^\ell n_i = n\}.$$

(b) For $h>0$ and $L>0$, let $V(n,L,h) := k(n,\lfloor L/h\rfloor)$. Fix $\rho>0$ and let $L_n\to\infty$ such that $n/L_n\to\rho>0$. Compare

$$\lim_{n\to\infty}\frac{1}{n}\log V(n,L_n,h)$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{L_n^n}{n!h^n}.$$

Exercise 1.2. Let $[a,b]\subset\mathbb{R}$ be a non-empty compact interval and $f:[a,b]\to\mathbb{R}$ a continuous function.

(a) Show that

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\log\left(\int_a^b \mathrm{e}^{\lambda f(t)}\,\mathrm{d}t\right) = \sup_{t\in[a,b]} f(t).$$

(b) Suppose that f is twice continuously differentiable, attains its maximum in a unique point $t_0\in(a,b)$, and $f''(t_0)\neq 0$. Show that

$$\int_a^b \mathrm{e}^{\lambda f(t)}\mathrm{d}t \sim \sqrt{\frac{2\pi}{\lambda|f''(t_0)|}}\,\mathrm{e}^{\lambda f(t_0)} \quad \text{as } \lambda\to\infty.$$

(c) Discuss what could go wrong if there is more than one maximizer, if $f''(t_0) = 0$, or if the domain of integration is not compact.

Exercise 1.3. Let $\Gamma(x) := \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$ $(x>0)$ be the Gamma function.

(a) Prove $\Gamma(x+1) \sim \sqrt{2\pi x}(x/\mathrm{e})^x$ as $x \to \infty$.
(b) Do you have an idea how to compute an expansion, e.g., as a power series of $1/x$, for correction terms?

Exercise 1.4. For $n \in \mathbb{N}$, let C_n be the hypercube $[-1,1]^n$ and $\mathcal{H}_n \subset \mathbb{R}^n$ the hyperplane defined by the equation $\sum_{j=1}^n x_j = 0$. For $\boldsymbol{x} \in \mathbb{R}^n$, let $\operatorname{dist}(\boldsymbol{x}, \mathcal{H}_n) := \inf_{\boldsymbol{y}\in\mathcal{H}_n} |\boldsymbol{x}-\boldsymbol{y}|$.

(a) Let $\operatorname{diam} C_n := \sup_{\boldsymbol{x},\boldsymbol{y}\in C_n} |\boldsymbol{x}-\boldsymbol{y}|$. Show that, for every $\varepsilon > 0$,
$$\lim_{n\to\infty} \frac{|\{\boldsymbol{x}\in C_n \mid \operatorname{dist}(\boldsymbol{x},\mathcal{H}_n) \leq \varepsilon \operatorname{diam}(C_n)\}|}{|C_n|} = 1.$$
Hint: there is a way of proving it using sums of i.i.d. random variables.
(b) What happens if in (a) the constant ε is replaced with a sequence ε_n such that $\varepsilon_n \searrow 0$ but $\varepsilon_n \gg 1/\sqrt{n}$ (i.e., $\sqrt{n}\varepsilon_n \to \infty$)? What happens if $\varepsilon_n \sim c/\sqrt{n}$ for some $c>0$?
(c) Let $B_n(1)$ be the n-dimensional closed unit ball centered at the origin and $\varepsilon_n \searrow 0$ with $\varepsilon_n \gg 1/\sqrt{n}$. Show that
$$\lim_{n\to\infty} \frac{|\{\boldsymbol{x}\in B_n(1) \mid \operatorname{dist}(\boldsymbol{x},\partial B_n(1)) \leq \varepsilon_n\}|}{|B_n(1)|} = 1.$$

2. Point processes

2.1. **Configuration space.** Let $(\mathbb{X}, \mathrm{dist})$ be a complete separable metric space and $\mathcal{X} = \mathcal{B}(\mathbb{X})$ the Borel-σ-algebra, i.e., the σ-algebra generated by the open sets, and λ a reference measure on $(\mathbb{X}, \mathcal{X})$. Write $\mathcal{X}_\mathsf{b}$ for the collection of bounded and measurable sets. We assume that $\lambda(B) < \infty$ for every $B \in \mathcal{X}_\mathsf{b}$.

For example, we could choose $\mathbb{X} = \mathbb{R}^d$ with the Euclidean distance, and λ as the Lebesgue measure. For lattice systems, we choose $\mathbb{X} = \mathbb{Z}^d$, with the Euclidean distance and find that $\mathcal{X}$ consists of the power set $\mathcal{P}(\mathbb{X})$, i.e., every set $B \subset \mathbb{Z}^d$ is measurable. The reference measure is chosen as $\lambda(B) = \#B$.

Our goal is to model configurations that consist of finite or countable collections $x_1, \dots, x_n$ or $(x_j)_{j\in\mathbb{N}}$ of points $x_j \in \mathbb{X}$. We assume that points do not accumulate, i.e., every bounded set contains at most finitely many points. The labelling of particles is considered irrelevant but multiplicities matter. Each such configuration is uniquely determined by a set with multiplicities (or "multi-set"), consisting of (1) the set $S \subset \mathbb{X}$ of particle locations, and (2) for each $x \in S$, the number of particles n_x occupying the location $x \in S$. More generally, we can count how many points there are in any given region B—with every point configuration, we can associate a counting measure.

Definition 2.1. *Let η be measure on $\mathbb{X}$. We call η a* finite counting measure *if $\eta(C) \in \mathbb{N}_0$ for all $C \in \mathcal{X}$, and a* locally finite counting measure *if $\eta(B) \in \mathbb{N}_0$ for all bounded sets $B \in \mathcal{X}_\mathsf{b}$.*[1] *The sets of finite and locally finite counting measures are denoted $\mathcal{N}_\mathsf{f}$ and $\mathcal{N}$, respectively.*

We further introduce families of maps from $\mathcal{N}$ to $\mathbb{N}_0 \cup \{\infty\}$ by

$$n_x(\eta) := \eta(\{x\}), \quad N_B(\eta) := \eta(B) \quad (x \in \mathbb{X},\ B \in \mathcal{X},\ \eta \in \mathcal{N}). \tag{2.1}$$

Clearly $n_x = N_{\{x\}}$. The next lemma checks that under our assumptions, every locally finite counting measure is associated with a point configuration (multi-set). Thus we may work with the space $\mathcal{N}$, which turns out to be preferrable to multi-sets for technical reasons.

Lemma 2.2. *Fix $\eta \in \mathcal{N}$. Set $S_\eta := \{x \in \mathbb{X} \mid n_x(\eta) \geq 1\}$. Then S_η is countable and*

$$\eta = \sum_{x\in S_\eta} n_x(\eta)\delta_x.$$

Proof. Consider first a finite counting measure $\eta \in \mathcal{N}_\mathsf{f}$. Let $C \in \mathcal{X}$. Then for every finite set $F \subset S_\eta$, we have

$$\eta(C) \geq \eta(C \cap S_\eta) \geq \eta(C \cap F) = \sum_{x\in C\cap F} \eta(\{x\}) = \sum_{x\in F} n_x(\eta)\delta_x(C),$$

hence

$$\eta(C) \geq \sup_{\substack{F\subset S_\eta:\\ \#F<\infty}} \sum_{x\in F} n_x(\eta)\delta_x(C). \tag{2.2}$$

[1] We follow the terminology of Last and Penrose [33]. In the theory of measures on topological spaces, often "locally finite" instead means that *compact* sets have finite measure. Daley and Vere-Jones [10] therefore say "boundedly finite" to avoid confusion. Of course for our preferred examples $\mathbb{X} = \mathbb{R}^d$ and $\mathbb{X} = \mathbb{Z}^d$, there is no difference.

Choosing $C = \mathbb{X}$ and using $n_x(\eta) \geq 1$ for all $x \in S_\eta$, we have

$$\eta(\mathbb{X}) \geq \sup_{\substack{F \subset S_\eta: \\ \#F < \infty}} \#F = \#S_\eta,$$

hence $\#S_\eta < \infty$. The inequality (2.2) shows $\eta \geq \sum_{x \in S_\eta} n_x(\eta)\delta_x$, it remains to prove the reverse inequality. For $x \in \mathbb{X}$ and $r \in \mathbb{N}$, let $B(x, r)$ be the closed ball of radius $r > 0$ centered at x. Then

$$\lim_{m \to \infty} \eta(B(x, \tfrac{1}{m})) = \eta(\{x\}) = n_x(\eta).$$

Since $\eta(B(x, 1/m)) \in \mathbb{N}_0$ for all $m \in \mathbb{N}$, it follows that the sequence is eventually constant and we have

$$\eta\big(B(x, \varepsilon_x) \setminus \{x\}\big) = 0$$

for some $\varepsilon_x > 0$. Since every compact set K can be covered by a finite union of balls $B(x, \varepsilon_x)$ with $x \in K$, we deduce

$$\eta(K) = \sum_{x \in S_\eta \cap K} n_x(\eta) = \sum_{x \in S_\eta} n_x(\eta)\delta_x(K).$$

As a finite Borel measure on a complete separable metric space, η satisfies

$$\eta(C) = \sup\{\eta(K) \mid K \subset C \text{ compact}\}$$

for all $C \in \mathcal{X}$ [3, Theorem 7.1.7], hence

$$\eta(C) = \sup\left\{ \sum_{x \in S_\eta} n_x(\eta)\delta_x(K) \,\Big|\, K \subset C \text{ compact} \right\} \leq \sum_{x \in S_\eta} n_x(\eta)\delta_x(C)$$

which completes the proof for finite η.

For locally finite η, let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of bounded sets with $\Lambda_n \nearrow \mathbb{X}$. Set $\eta_n(C) := \eta(C \cap [\Lambda_{n+1} \setminus \Lambda_n])$. Since each $C \in \mathcal{X}$ is the disjoint countable union of $C \cap [\Lambda_{n+1} \setminus \Lambda_n]$, the σ-additivity of the measure η yields

$$\eta = \sum_{n \in \mathbb{N}} \eta_n.$$

On the other hand each η_n is a finite counting measure, hence S_{η_n} is finite and $\eta_n = \sum_{x \in S_{\eta_n}} n_x(\eta_n)\delta_x$. Using $n_x(\eta_n) = n_x(\eta_n)\mathbb{1}_{\Lambda_{n+1} \setminus \Lambda_N}(x)$ we deduce

$$\eta = \sum_{n \in \mathbb{N}} \sum_{x \in S_\eta} n_x(\eta_n)\mathbb{1}_{\Lambda_{n+1} \setminus \Lambda_n}(x)\delta_x = \sum_{x \in S_\eta} n_x(\eta)\delta_x,$$

moreover $S_\eta = \cup_{n \in \mathbb{N}} S_{\eta_n}$ is a countable union of finite sets, hence countable. □

Definition 2.3. *$\mathfrak{N} := \sigma(N_B,\ B \in \mathcal{X})$ is the σ-algebra generated by the maps $N_B : \mathcal{N} \to \mathbb{N}_0 \cup \{\infty\}$, $\eta \mapsto N_B(\eta) = \eta(B)$.*

It is convenient not to deal with all counting variables N_B, but instead only with those associated with simple sets, e.g., rectangles $[a, b) \times [c, d)$ in $\mathbb{R}^2$.

Proposition 2.4. *Let $\mathcal{R} \subset \mathcal{X}_\mathsf{b}$ be a π-system ($A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$) such that $\sigma(\mathcal{R}) = \mathcal{X}$. Suppose in addition that $\mathbb{X} = \cup_{n \in \mathbb{N}} R_n$ for some increasing sequence $(R_n)_{n \in \mathbb{N}}$ in $\mathcal{R}$. Then $\mathfrak{N} = \sigma(N_B,\ B \in \mathcal{R})$.*

Proof. Set $\mathcal{F} = \sigma(N_B,\ B \in \mathcal{R})$ and let $\mathcal{D}$ be the collection of sets $C \subset \mathbb{X}$ such that N_C is $\mathcal{F}$-measurable; thus $\mathcal{R} \subset \mathcal{D}$. We want to check that $\mathcal{X} \subset \mathcal{D}$ using a Dynkin system theorem (see Appendix A). Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{D}$ and $A = \cup_{n\in\mathbb{N}} A_n$. By monotone convergence, $\eta(A_n) \nearrow \eta(A)$ for every $\eta \in \mathcal{N}$, i.e., $N_{A_n} \nearrow N_A$ pointwise on $\mathcal{N}$. Thus N_A is the pointwise limit of $\mathcal{F}$-measurable functions, hence $\mathcal{F}$-measurable. Therefore $A \in \mathcal{D}$ and $\mathcal{D}$ is closed with respect to limits of monotone increasing sequences. Applying this last bit to $\mathbb{X} = \cup_{n\in\mathbb{N}} R_n$ we see that $\mathbb{X} \in \mathcal{D}$.

The only missing piece for $\mathcal{D}$ to be a Dynkin system is that it is closed with respect to proper differences. One would like to use the identity $N_{B\setminus A} = N_B - N_A$ for $A \subset B$ but runs into the problem that N_B and N_A could be infinite. For that reason we proceed slightly differently and introduce, for $k \in \mathbb{N}$, the set

$$\mathcal{D}_k = \{A \subset \mathbb{X} \mid N_{A\cap R_k} \text{ is } \mathcal{F}\text{-measurable}\} = \{A \subset \mathbb{X} \mid A \cap R_k \in \mathcal{D}\}.$$

For $A, B \in \mathcal{D}_k$ with $A \subset B$, we have $N_{(B\setminus A)\cap R_k} = N_{B\cap R_k} - N_{A\cap R_k}$ where all quantities are finite because R_k is bounded. Then $N_{(B\setminus A)\cap R_k}$ is the difference of two $\mathcal{F}$-measurable maps, hence $\mathcal{F}$-measurable and $B \setminus A \in \mathcal{D}_k$. The set $\mathcal{D}_k$ is closed with respect to monotone increasing limits by an argument similar to $\mathcal{D}$. The inclusion $\mathbb{X} \in \mathcal{D}_k$ follows from $R_k \in \mathcal{R} \subset \mathcal{D}_k$. Thus $\mathcal{D}_k$ is a Dynkin system and $\mathcal{X} = \sigma(\mathcal{R}) \subset \mathcal{D}_k$. Consequently $N_{A\cap R_k}$ is $\mathcal{F}$-measurable for all $A \in \mathcal{X}$, and so is $N_A = \lim_{k\to\infty} N_{A\cap R_k}$. It follows that $\mathfrak{N} = \sigma(N_A, A \in \mathcal{X}) \subset \mathcal{F}$. The inclusion $\mathcal{F} \subset \mathfrak{N}$ is clearly true, hence $\mathcal{F} = \mathfrak{N}$. □

Corollary 2.5. *Let $\mathcal{R}$ be as in Proposition 2.4. Consider the collection $\mathcal{Z}$ of subsets of $\mathcal{N}$ that are of the form*

$$\{\eta \in \mathcal{N} \mid N_{R_1}(\eta) = k_1, \ldots, N_{R_m}(\eta) = k_m\}$$

with $m \in \mathbb{N}$, $R_1, \ldots, R_m \in \mathcal{R}$, $k_1, \ldots, k_m \in \mathbb{N}_0$. Then $\sigma(\mathcal{Z}) = \mathfrak{N}$.

The set $\mathcal{Z}$ plays a role analogous to cylinder sets in product spaces.

Proof. Let $R \in \mathcal{R}$ and $B \subset \mathbb{N}$. We have

$$N_R^{-1}(B) = \bigcup_{k\in B} \{\eta \in \mathcal{N} \mid N_R(\eta) = k\}.$$

The right-hand side is a countable union of elements in $\mathcal{Z}$, hence it is in $\sigma(\mathcal{Z})$. It follows that N_R is measurable with respect to $\sigma(\mathcal{Z})$. This holds for every $R \in \mathcal{R}$ and we deduce $\sigma(N_R,\ R \in \mathcal{R}) \subset \sigma(\mathcal{Z})$ so by Proposition 2.4, $\mathfrak{N} \subset \sigma(\mathcal{Z})$. The reverse inclusion is obvious and the claim follows. □

Corollary 2.6. *Pick $n \in \mathbb{N}$ and equip $\mathbb{X}^n$ with the product σ-algebra $\mathcal{X}^{\otimes n}$. The map*

$$\phi_n : \mathbb{X}^n \to \mathcal{N}, \quad (x_1, \ldots, x_n) \mapsto \sum_{j=1}^n \delta_{x_j}.$$

is measurable.

Proof. Let $B \in \mathcal{X}$ and $k \in \mathbb{N}_0$. Then $\varphi_n^{-1}(\{N_B = k\}) = \emptyset$ if $k = 0$ and

$$\varphi_n^{-1}(\{N_B = k\}) = \bigcup_{\substack{J\subset[n]\\ \#J=k}} \{\boldsymbol{x} \in \mathbb{X}^n \mid x_j \in B \text{ if and only if } j \in J\}$$

if $k \geq 1$. The right-hand side is a union of Cartesian products of B and $\mathbb{X} \setminus B$, hence a measurable subset of $\mathbb{X}^n$. Thus preimages of sets $\{N_B = k\}$ are measurable. Since sets of this form generate $\mathfrak{N}$, it follows that φ_n is measurable. $\square$

2.2. Probability measures.

Definition 2.7. *A* point process *on* $(\mathbb{X}, \mathcal{X})$ *is a random variable with values in* $(\mathcal{N}, \mathfrak{N})$*, i.e., a measurable map* Z *from some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *to* $(\mathcal{N}, \mathfrak{N})$.

Given a probability measure P on $(\mathcal{N}, \mathfrak{N})$, we can always find a point process with distribution P: set $(\Omega, \mathcal{F}, \mathbb{P}) := (\mathcal{N}, \mathfrak{N}, \mathsf{P})$ and $Z(\eta) := \eta$, then $\mathbb{P}(Z \in C) = \mathsf{P}(C)$, for all $C \in \mathfrak{N}$. By some abuse of language, probability measures P on $(\mathcal{N}, \mathfrak{N})$ are sometimes called point processes too.

Proposition 2.8. *Let* P, Q *be two probability measures on* $(\mathcal{N}, \mathfrak{N})$*, and* $\mathcal{R} \subset \mathcal{X}_\mathsf{b}$ *be a* π*-system with* $\sigma(\mathcal{R}) = \mathcal{X}$ *and* $\mathbb{X} = \bigcup_{n \in \mathbb{N}} R_n$ *for some increasing sequence in* $\mathcal{R}$*. Then* $\mathsf{P} = \mathsf{Q}$ *if and only if*

$$\mathsf{P}(N_{R_1} = k_1, \ldots, N_{R_n} = k_n) = \mathsf{Q}(N_{R_1} = k_1, \ldots, N_{R_n} = k_n)$$

for all $R_1, \ldots, R_n \in \mathcal{R}$ *and* $k_1, \ldots, k_n \in \mathbb{N}_0$.

The proposition implies that the distribution of a point process is uniquely determined by its *finite-dimensional distributions*.

Proof. Define $\mathcal{Z}$ as in Corollary 2.5. Thus $\mathsf{P} = \mathsf{Q}$ on $\mathcal{Z}$. Now $\mathcal{Z}$ is clearly closed under finite intersections, i.e., a π-systems. Since $\sigma(\mathcal{Z}) = \mathfrak{N}$ by Corollary 2.5 and probability measures are uniquely defined by their values on a generating π-system, it follows that $\mathsf{P} = \mathsf{Q}$. $\square$

Example 2.9 (Binomial point process). Let ν be a probability measure on $\mathbb{X}$, $m \in \mathbb{N}$, and $X_1, \ldots, X_m$ i.i.d. $\mathbb{X}$-valued random variables with distribution ν. Then $Z = \sum_{j=1}^m \delta_{X_j}$ is a point process (the measurability of $Z : \Omega \to \mathcal{N}$ follows from Corollary 2.6). Its distribution satisfies, for every $\Delta \in \mathcal{X}$ and $k \in \mathbb{N}_0$,

$$\mathsf{P}(N_\Delta = k) = \mathbb{P}\Big(\sum_{j=1}^m \mathbb{1}_{\{X_j \in \Delta\}} = k\Big) = \binom{m}{k} \nu(\Delta)^k (1 - \nu(\Delta))^{m-k}$$

P is the *binomial point process* with *sample size* m and *sampling distribution* ν. We had encountered a special case for the particle positions of the ideal gas in the microcanonical ensemble, see the proof of Lemma 1.3.

Example 2.10 (Ideal lattice gas). Take $\mathbb{X} = \mathbb{Z}^d$. Then $(\mathcal{N}, \mathfrak{N})$ can be identified with the product space $\mathbb{N}_0^{\mathbb{Z}^d}$, and any probability measure on $(\mathcal{N}, \mathfrak{N})$ is uniquely determined by the joint distributions of the occupation numbers n_x, $x \in \mathbb{Z}^d$. The *ideal lattice gas* at activity $z > 0$ corresponds to the unique measure P such that the occupation numbers n_x are i.i.d. Bernoulli-distributed with parameter $z/(1+z)$, i.e.,

$$\mathsf{P}(n_x = 1) = \frac{z}{1+z}, \quad \mathsf{P}(n_x = 0) = \frac{1}{1+z}.$$

We have for all $\ell \in \mathbb{N}$ and all $k_1, \ldots, k_n \in \{0,1\}^n$,

$$\mathsf{P}(n_1 = k_1, \ldots, n_\ell = k_\ell) = \frac{1}{(1+z)^\ell} \, z^{\#\{j \in [\ell] \mid k_j = 1\}} \, \mathbb{1}_{\{0,1\}^n}(k_1, \ldots, k_n).$$

With the convention $\exp(-\infty) = 0$, we can think of the indicator as a weight $\exp(-H)$ for an energy function H that is infinite if two particles occupy the same lattice site (*hard-core on-site repulsion*), and zero otherwise. Except for the hard-core interaction, this looks very much like the ideal gas that we encountered earlier in continuum systems.

Another important example, the *Poisson point process*, is introduced in Section 2.4.

2.3. **Observables.** It remains to understand measurable maps $F : \mathcal{N} \to \mathbb{R} \cup \{\infty\}$, which we call *observables*. We start with maps restricted to the space $\mathcal{N}_\mathsf{f}$ of finite configurations. Since $\mathcal{N}_\mathsf{f} = \{N_\mathbb{X} < \infty\}$, it is a measurable subset of $\mathcal{N}$. It comes equipped with the σ-algebra $\{A \in \mathfrak{N} \mid A \subset \mathcal{N}_\mathsf{f}\}$.

Proposition 2.11. *A map $F : \mathcal{N}_\mathsf{f} \to \mathbb{R} \cup \{\infty\}$ is measurable if and only if there exists a number $f_0 \in \mathbb{R} \cup \{\infty\}$ and a family $(f_n)_{n\in\mathbb{N}}$ of measurable, symmetric functions $f_n : \mathbb{X}^n \to \mathbb{R} \cup \{\infty\}$ such that $F(0) = f_0$ and*

$$F\Big(\sum_{j=1}^{n} \delta_{x_j}\Big) = f_n(x_1, \dots, x_n) \tag{2.3}$$

for all $n \in \mathbb{N}$ and $(x_1, \dots, x_n) \in \mathbb{X}^n$.

The proposition allows us to switch back and forth between symmetric functions of labelled points $x_1, \dots, x_n$ and functions of point configurations. As an example, let us look at a function that is a sum of pair contributions.

Example 2.12. Let $v : [0,\infty) \to \mathbb{R}\cup\{\infty\}$ be a pair potential. Set $U_0 := 0$, $U_1(x) \equiv 0$, and

$$U_n(x_1, \dots, x_n) := \sum_{1\le i<j\le n} v(|x_i - x_j|). \tag{2.4}$$

for $n \ge 2$. The associated map $U : \mathcal{N}_\mathsf{f} \to \mathbb{R} \cup \{\infty\}$ is given by

$$U(\eta) = \tfrac{1}{2} \sum_{x\in S_\eta} v(0) n_x(\eta)\big(n_x(\eta) - 1\big) + \tfrac{1}{2} \sum_{\substack{x,y\in S_\eta:\\ x\neq y}} v(|x-y|) n_x(\eta) n_y(\eta)$$

Proposition 2.11 guarantees that if v is measurable, then so is U.

Proof of Proposition 2.11. "$\Rightarrow$" Let $F : \mathcal{N}_\mathsf{f} \to \mathbb{R} \cup \{\infty\}$ be measurable. Define $f_0 := F(0)$ and for $n \in \mathbb{N}$, define $f_n(x_1, \dots, x_n)$ by (2.3). Each f_n is clearly symmetric. The measurability follows from Corollary 2.6 (write $f_n = F \circ \varphi_n$).

"$\Leftarrow$" Let $f_0 \in \mathbb{R} \cup \{\infty\}$ and f_n, $n \in \mathbb{N}$, be symmetric measurable functions from $\mathbb{X}^n$ to $\mathbb{R} \cup \{\infty\}$. We define $F : \mathcal{N}_\mathsf{f} \to \mathbb{R} \cup \{\infty\}$ by $F(0) := f_0$ and by (2.3). The function F is well-defined because every finite counting measure can be represented as a sum of Dirac measures by Lemma 2.2, and because $f_n(x_1, \dots, x_n)$ is independent of the chosen representation by the symmetry of f_n. It remains to check that F is measurable.

For $k \in \mathbb{N}_0$, set $F_k := \mathbb{1}_{\{N_\mathbb{X}=k\}} F$. Since $\sum_{k=0}^m F_k \to F$ as $m \to \infty$ and sums and pointwise limits of measurable functions are again measurable, it is enough to show that each F_k is measurable. For $k = 0$, we note that $F_0 = \mathbb{1}_{\{N_\mathbb{X}=k\}} f_0$ with constant f_0 so F_0 is measurable.

For $k \geq 1$, we proceed as follows. Let $\mathcal{D}_k$ be the class of measurable sets $B \subset \mathbb{X}^k$ such that the map $G_k : \mathcal{N}_\mathsf{f} \to \mathbb{R}$, uniquely defined by

$$G_k(\sum_{j=1}^k \delta_{x_j}) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathbb{1}_B(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \tag{2.5}$$

for all $x_1, \ldots, x_k \in \mathbb{X}^k$, and $G_k(\eta) = 0$ if $N_\mathbb{X}(\eta) \neq k$, is measurable. We use a Dynkin system argument to check that $\mathcal{D}_k = \mathcal{X}^{\otimes k}$.

For $B = \mathbb{X}^k$, the associated function G_k is $G_k = \mathbb{1}_{\{N_\mathbb{X}=k\}}$ which is measurable, so $\mathbb{X}^k \in \mathcal{D}_k$. If $(B_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{D}_k$, the function G_k associated with $B = \bigcup_{n\in\mathbb{N}} B_n$ is the sum of the functions associated with the B_n's, hence measurable. Thus $\mathcal{D}_k$ is closed with respect to disjoint countable unions. Similarly, if $A \subset B$ and $A, B \in \mathcal{D}_k$, the function G_k associated with $B \setminus A$ is the difference of the functions associated with B and A. Thus $\mathcal{D}_k$ is closed with respect to proper differences. Altogether we find that $\mathcal{D}_k$ is a Dynkin system.

Let $\mathcal{C}_k$ be the collection of Cartesian products of sets $B_1 \times \cdots \times B_k$ of measurable sets such that for all i, j, we have either $B_i \cap B_j = \emptyset$ or $B_i = B_j$. Cartesian products of this form are in $\mathcal{D}_k$: consider for example $k = 3$ and $B_1 = B_2$, $B_1 \cap B_3 = \emptyset$, then the function G associated with $B = B_1 \times B_2 \times B_3$ is proportional to $\mathbb{1}_{\{N_\mathbb{X}=3,\ N_{B_1}=2,\ N_{B_3}=1\}}$, which is measurable. Similar considerations apply to the general case and show $B_1 \times \cdots \times B_k \in \mathcal{D}_k$. Thus $\mathcal{C}_k \subset \mathcal{D}_k$. $\mathcal{C}_k$ is closed under pairwise intersections. Since every Cartesian product $A_1 \times \cdots \times A_k$ of measurable sets is a finite union of sets $B_1 \times \cdots \times B_k \in \mathcal{C}_k$, and Cartesian products generate $\mathcal{X}^{\otimes k}$, we have $\sigma(\mathcal{C}_k) = \mathcal{X}^{\otimes k}$.

Theorem A.1 shows $\mathcal{D}_k = \mathcal{X}^{\otimes k}$. Taking linear combinations of indicators and then pointwise monotone limits, we see that we can replace the indicator in (2.5) by any non-negative measurable function g_k and still obtain a measurable function G_k. In particular we can choose $g_k = f_k$ as any symmetric measurable function, which concludes the proof. □

Among the observables, a special role is played by *local* and *quasi-local* observables. For $\eta \in \mathcal{N}$ and $\Lambda \in \mathcal{X}$, define $\eta_\Lambda \in \mathcal{N}$ by

$$\eta_\Lambda(B) := \eta(B \cap \Lambda) \qquad (B \in \mathcal{X}).$$

Equivalently,

$$\eta_\Lambda := \sum_{x \in S_\eta \cap \Lambda} n_x(\eta)\delta_x.$$

The configuration η_Λ is obtained from η by keeping the points that are in Λ, discarding the rest.

Definition 2.13.

- *An observable F is* local *if for some bounded $\Lambda \in \mathcal{X}_\mathsf{b}$, we have*

$$\forall \eta, \gamma \in \mathcal{N}: \quad \eta_\Lambda = \gamma_\Lambda \;\Rightarrow\; F(\eta) = F(\gamma). \tag{2.6}$$

- *An observable is* quasilocal *if it is the uniform limit of a sequence of bounded local observables.*

Because of $(\eta_\Lambda)_\Lambda = \eta_\Lambda$, the condition (2.6) is equivalent to

$$\forall \eta \in \mathcal{N}:\ F(\eta) = F(\eta_\Lambda). \tag{2.7}$$

Proposition 2.11 with the observation $\eta_\Lambda \in \mathcal{N}_\mathsf{f}$ can be used to show that there is a one-to-one correspondence between local observables satisfying (2.6) and sequences $(f_n)_{n\in\mathbb{N}_0}$ of symmetric, measurable functions $f_n : \Lambda^n \to \mathbb{R}$.

Let $\mathcal{A}_\Lambda$ denote the set of bounded local observables that satisfy (2.6),

$$\mathcal{A}^{\mathrm{loc}} = \cup_{\Lambda\in\mathcal{X}_\mathsf{b}}\mathcal{A}_\Lambda$$

the set of local observables, and $\mathcal{A}$ the set of quasilocal observables. $\mathcal{A}$ is the closure of $\mathcal{A}^{\mathrm{loc}}$ with respect to the supremum norm $||\cdot||_\infty$ in the space $\mathscr{L}^\infty(\mathcal{N},\mathfrak{N})$ of bounded observables:

$$\mathcal{A}^{\mathrm{loc}} \subset \overline{\mathcal{A}^{\mathrm{loc}}}^{||\cdot||_\infty} = \mathcal{A} \subset \mathscr{L}^\infty(\mathcal{N},\mathfrak{N}). \tag{2.8}$$

We will often want to check that some property holds for all local observables, but prefer to check it for particularly simple subset of observables $\mathcal{H}$ only.

Proposition 2.14. *Let $\mathcal{M} \subset \mathscr{L}^\infty(\mathcal{N},\mathfrak{N})$ and $\mathcal{H} \subset \mathcal{M}$. Suppose that:*

- *$\mathcal{H}$ is closed with respect to products ($f,g \in \mathcal{H} \Rightarrow fg \in \mathcal{H}$) and $\sigma(\mathcal{H}) = \mathfrak{N}$.*
- *$\mathbf{1} \in \mathcal{M}$.*
- *$\mathcal{M}$ is a vector space.*
- *If $(f_n)_{n\in\mathbb{N}}$ is a monotone-increasing sequence of non-negative functions in $\mathcal{M}$ such that $f_n \nearrow f$ for some bounded function $f : \mathcal{N} \to \mathbb{R}$, then $f \in \mathcal{M}$.*

Then $\mathcal{M} = \mathscr{L}^\infty(\mathcal{N},\mathfrak{N})$.

A typical choice for $\mathcal{H}$ could be the collection of indicator functions of cylinder sets $\{N_{B_1} = k_1,\dots,N_{B_\ell} = k_\ell\}$ with $\ell \in \mathbb{N}$, $B_1,\dots,B_\ell \in \mathcal{X}_\mathsf{b}$, $k_1,\dots,k_\ell \in \mathbb{N}_0$. Other possible choices are $\mathcal{H} = \mathcal{A}^{\mathrm{loc}}$ or $\mathcal{H} = \mathcal{A}$; for these choices Proposition 2.14 characterizes the class of bounded observables as the smallest linear vector space $\mathcal{M}$ of bounded maps that contain the local observables and is closed with respect to pointwise monotone limits of uniformly bounded sequences. This information complements the inclusions (2.8).

Proof of Proposition 2.14. The proposition is a direct consequence of the *functional monotone class theorem*, see Theorem A.2. The latter ensures that $\mathcal{M}$ contains all bounded $\sigma(\mathcal{H})$-measurable functions. Since $\sigma(\mathcal{H}) = \mathfrak{N}$ by our set of assumptions, we have $\mathscr{L}^\infty(\mathcal{N},\mathfrak{N}) \subset \mathcal{M}$. We have assumed $\mathcal{M} \subset \mathscr{L}^\infty(\mathcal{N},\mathfrak{N})$ so we must have $\mathcal{M} = \mathscr{L}^\infty(\mathcal{N},\mathfrak{N})$. □

2.4. **Poisson point process.**

Theorem 2.15. *Let ν be locally finite measure on $\mathbb{X}$ (i.e., $\nu(B) < \infty$ for every $B \in \mathcal{X}_\mathsf{b}$). There exists a uniquely defined probability measure P on $(\mathcal{N},\mathfrak{N})$ such that:*

(i) $\mathsf{P}(N_B = k) = \frac{1}{k!}\nu(B)^k \exp(-\nu(B))$, *for every $k \in \mathbb{N}_0$ and $B \in \mathcal{X}_\mathsf{b}$.*
(ii) *For every $m \in \mathbb{N}$ and all pairwise disjoint sets $B_1,\dots,B_m \in \mathcal{X}$, the variables $N_{B_1},\dots,N_{B_m}$ are independent.*

Definition 2.16. *Let ν be a locally finite measure on $\mathbb{X}$. A* Poisson point process *with intensity measure ν is a point process whose distribution P satisfies the conditions (i) and (ii) from Theorem 2.15.*

For later purpose we note that if P is the distribution of a Poisson point process with intensity measure ν, then for all $\Delta \in \mathcal{X}_\mathsf{b}$,

$$\mathsf{E}[N_\Delta] = \nu(\Delta). \tag{2.9}$$

Before we come to the proof of Theorem 2.15, we give another characterization.

Theorem 2.17. *Let ν be a locally finite measure on $\mathbb{X}$ and P a probability measure on $\mathcal{N}$. Then P is the distribution of a Poisson point process with intensity measure ν if and only if for all $\Lambda \in \mathcal{X}_{\mathsf{b}}$ and all $F \in \mathscr{L}^\infty(\mathcal{N}, \mathfrak{N})$*

$$\mathsf{E}[F(\eta_\Lambda)] = \mathrm{e}^{-\nu(\Lambda)} \left(F(0) + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} F(\delta_{x_1} + \cdots + \delta_{x_n}) \mathrm{d}\nu^n(\boldsymbol{x}) \right). \tag{2.10}$$

The theorem is proven at the end of this section. It shows that expectations of local observables with respect to Poisson point processes can be computed by a fairly explicit formula, breaking things down to a series and integrals on $\mathbb{X}^n$. Moreover we recognize, for $\mathbb{X} = \mathbb{R}^d$ and $\nu = z\,\mathrm{Leb}$, the type of sum encountered in Proposition 1.2 for the ideal gas.

Proposition 2.18. *Let ν be a finite, non-zero measure on $\mathbb{X}$. Let M and $(X_j)_{j\in\mathbb{N}}$ be independent random variables, defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that M is integer-valued and has Poisson distribution with parameter $\nu(\mathbb{X})$ and the X_j's are i.i.d. with distribution $\mathbb{P}(X_j \in B) = \nu(B)/\nu(\mathbb{X})$. Then $Z := \sum_{j=1}^M \delta_{X_j}$ is a Poisson point process with intensity measure ν.*

Proof. First we note that $N_{\mathbb{X}}(Z) = M < \infty$ so Z takes values in $\mathcal{N}_{\mathsf{f}} \subset \mathcal{N}$; the measurability follows with the help of Corollary 2.6 and the decomposition

$$Z = \sum_{m=0}^\infty \mathbb{1}_{\{M=m\}} \sum_{j=1}^m \delta_{X_j}. \tag{2.11}$$

So Z is indeed a point process, i.e., a $\mathcal{N}$-valued random variable.

Let $m \in \mathbb{N}$ and $B_1, \ldots, B_m \in \mathcal{X}$ be disjoint sets. Adding $\mathbb{X} \setminus \cup_{j=1}^m B_j$ if necessary, we may assume without loss of generality that $\cup_{j=1}^m B_j = \mathbb{X}$. Pick $k_1, \ldots, k_m \in \mathbb{N}_0$. Set $k := k_1 + \cdots + k_m$. Then

$$\begin{aligned}
&\mathbb{P}\big(N_{B_1}(Z) = k_1, \ldots, N_{B_m}(Z) = k_m\big) \\
&\quad = \mathbb{P}\left(M = k, \sum_{j=1}^k \mathbb{1}_{\{X_j \in B_1\}} = k_1, \ldots, \sum_{j=1}^k \mathbb{1}_{\{X_j \in B_m\}} = k_m \right) \\
&\quad = \frac{\nu(\mathbb{X})^k}{k!} \mathrm{e}^{-\nu(\mathbb{X})} \binom{k}{k_1, \ldots, k_m} \left(\frac{\nu(B_1)}{\nu(\mathbb{X})} \right)^{k_1} \cdots \left(\frac{\nu(B_m)}{\nu(\mathbb{X})} \right)^{k_m} \\
&\quad = \prod_{j=1}^m \frac{\nu(B_j)^{k_j}}{k_j!} \mathrm{e}^{-\nu(B_j)}.
\end{aligned}$$

It follows that $N_{B_j}(Z), \ldots, N_{B_m}(Z)$ are independent Poisson variables with respective parameters $\nu(B_j)$, hence Z is a Poisson point process with intensity measure ν. □

Remark. Suppose that ν has bounded support, i.e., there exists some $\Lambda \in \mathcal{X}_{\mathsf{b}}$ such that $\nu(\mathbb{X} \setminus \Lambda) = 0$. Then we can choose the random variables X_j in the proof of Proposition 2.18 in such a way that $X_j(\omega) \in \Lambda$ for *all* $\omega \in \Omega$, which then implies $N_{\mathbb{X}\setminus\Lambda}(Z(\omega)) = 0$ for *all* ω [2]

[2] For general Poisson point process Z with intensity measure ν supported in Λ, we can only say that $N_{\mathbb{X}\setminus\Lambda}(Z) = 0$ $\mathbb{P}$-almost surely.

Proof of Theorem 2.15. Uniqueness. Conditions (i) and (ii) determine the distributions of $(N_{B_1}, \dots, N_{B_m})$ with $m \in \mathbb{N}$, $B_1, \dots, B_m \in \mathcal{X}_\mathsf{b}$, uniquely, so P is unique by Proposition 2.8.

Existence. For $n \in \mathbb{N}$, let $\Lambda_n := B(0,n)$ be the closed ball with radius n centered at the origin, and $\nu_n(A) := \nu(A \cap (\Lambda_n \setminus \Lambda_{n-1}))$, with the convention $\Lambda_0 := \emptyset$. Notice $\nu = \sum_{n=1}^\infty \nu_n$. Each ν_n is finite, so by Proposition 2.18, there is a Poisson point process Z_n with intensity measure ν_n. We may take $(Z_n)_{n\in\mathbb{N}}$ as independent variables defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover we may assume $N_{\mathbb{X}\setminus\Lambda_n}(Z_n(\omega)) = 0$ for all $\omega \in \Omega$.

We claim that $Z := \sum_{n=1}^\infty Z_n$ is a Poisson point process with intensity measure ν. For each $\omega \in \Omega$, $Z(\omega)$ is a measure on $\mathbb{X}$, we need to check that it is a locally finite counting measure. Let $B \in \mathcal{X}_\mathsf{b}$ and k large enough so that $B \subset \Lambda_k$. Then $N_{B_j}(Z(\omega)) = 0$ for all $\omega \in \Omega$ and $j \geq k+1$ and $N_B(Z(\omega)) = \sum_{j=1}^k N_B(Z_j(\omega)) \in \mathbb{N}_0$ is finite. It follows that $Z(\omega) \in \mathcal{N}$, for every $\omega \in \Omega$. The measurability of $Z : \Omega \to \mathcal{N}$ is left as an exercise.

Let $B \in \mathcal{X}_\mathsf{b}$ and $k \in \mathbb{N}$ with $B \subset \Lambda_k$. By the previous paragraph, $N_B(Z) = \sum_{j=1}^k N_B(Z_j)$, so $N_B(Z)$ is the sum of Poisson variables with parameters $\nu_j(B)$, $j = 1, \dots, k$, so it is itself a Poisson variable with parameter $\sum_{j=1}^k \nu_j(B) = \sum_{j=1}^\infty \nu_j(B) = \nu(B)$.

Finally let $m \in \mathbb{N}$ and $B_1, \dots, B_m \in \mathcal{X}$ be pairwise disjoint sets. Then $N_{B_j}(Z_n)$, $j = 1, \dots, m$, $n \in \mathbb{N}$, are independent. It follows that $\sum_n N_{B_1}(Z_n)$, ..., $\sum_n N_{B_m}(Z_n)$ are independent as well.

Consequently Z is a Poisson point process with intensity measure ν and its distribution P satisfies the conditions (i) and (ii). ☐

Proof of Theorem 2.17. "$\Leftarrow$" Suppose that P satisfies Eq. (2.10) for all bounded observables F and all $\Lambda \in \mathcal{X}_\mathsf{b}$. Choosing Λ, $F = \mathbb{1}_{\{N_\Lambda = k\}}$ we find that $N_\Lambda \sim \mathrm{Poi}(\nu(\Lambda))$. Let $m \in \mathbb{N}$ and $B_1, \dots, B_m \in \mathcal{X}_\mathsf{b}$ be pairwise disjoint. Let $\Lambda := \cup_{j=1}^m B_j \in \mathcal{X}_\mathsf{b}$. Let $k_1, \dots, k_m \in \mathbb{N}_0$ and $k = k_1 + \cdots + k_m$. If $k = 0$, then $k_1 = \cdots = k_m = 0$ and

$$\mathsf{P}(N_{B_1} = 0, \dots, N_{B_m} = 0) = \mathsf{P}(N_\Lambda = 0) = \mathrm{e}^{-\nu(\Lambda)} = \prod_{j=1}^m \mathrm{e}^{-\nu(B_j)} = \prod_{j=1}^m \mathsf{P}(N_{B_j} = 0).$$

If $k \geq 1$, then

$$\begin{aligned}
&\mathsf{P}(N_{B_1} = k_1, \dots, N_{B_m} = k_m) \\
&\quad = \frac{1}{k!} \mathrm{e}^{-\nu(\Lambda)} \int_{\mathbb{X}^k} \mathbb{1}_{\{N_{B_1} = k_1, \dots, N_{B_m} = k_m\}}(\delta_{x_1} + \cdots + \delta_{x_k}) \mathrm{d}\nu^k(\boldsymbol{x}) \\
&\quad = \frac{1}{k!} \mathrm{e}^{-\nu(\Lambda)} \binom{k}{k_1, \dots, k_m} \prod_{j=1}^m \nu(B_j)^{k_j} = \prod_{j=1}^m \mathsf{P}(N_{B_j} = k_j).
\end{aligned} \tag{2.12}$$

It follows that $N_{B_1}, \dots, N_{B_m}$ are independent if the B_j's are bounded. If $B_1, \dots, B_m \in \mathcal{X}$, let $\Lambda_n = B(0,n)$ for $n \in \mathbb{N}$, $\Lambda_0 = \emptyset$, $A_n = \Lambda_n \setminus \Lambda_{n-1}$. Then the variables $N_{B_j \cap A_n}$ with $j = 1, \dots, m$ and $n \in \mathbb{N}$, are independent; using $N_{B_j} = \sum_{n=1}^\infty N_{B_j \cap A_n}$, we deduce that $N_{B_1}, \dots, N_{B_m}$ are independent as well.

"$\Rightarrow$" Suppose that P satisfies the conditions (i) and (ii). For $F = \mathbb{1}_{\{N_{B_1} = k_1, \dots, N_{B_m} = k_m\}}$ with $B_1, \dots, B_m \in \mathcal{X}_\mathsf{b}$ and $k_1, \dots, k_m \in \mathbb{N}_0$, let $k = k_1 + \cdots + k_m$ and $\Lambda = \cup_{j=1}^m B_j$.

Then

$$\mathsf{E}[F(\eta_\Lambda)] = \mathsf{P}(N_{B_1} = k_1, \ldots, N_{B_m} = k_m)$$

which is equal to the right-hand side of Eq. (2.10) by a computation similar to (2.12). Thus Eq. (2.10) holds true for all indicator functions of sets of the given form. For general F the statement follows with a monotone class theorem. □

2.5. **Janossy densities.** Remember that $\mathbb{X}$ is equipped with a locally finite reference measure λ, for example, the Lebesgue measure on $\mathbb{X} = \mathbb{R}^d$ or $\lambda(B) = \#B$ on $\mathbb{X} = \mathbb{Z}^d$. Write λ^n for the product measure on $\mathbb{X}^n$. The *Janossy densities* play a role analogous to probability densities of real-valued random variables; they are called *system of density distributions* by Ruelle [49].

Definition 2.19. *Let* P *be a probability measure on* $(\mathcal{N}, \mathfrak{N})$*. A family of functions* $(j_{n,\Lambda})_{n\in\mathbb{N},\Lambda\in\mathcal{X}_\mathsf{b}}$ *is a system of* Janossy densities *of* P *with respect to* λ *if they are symmetric and for every non-negative test function* f *with* $f(0) = 0$*, we have*

$$\mathsf{E}\big[f(\eta_\Lambda)\big] = \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} f\Big(\sum_{j=1}^n \delta_{x_j}\Big) j_{n,\Lambda}(x_1, \ldots, x_n) \mathrm{d}\lambda^n(\boldsymbol{x}). \tag{2.13}$$

The Janossy densities, if they exist, are unique up to λ^n-null sets, and they determine the measure P uniquely.

Example 2.20 (Homogeneous Poisson point process). Let $\mathbb{X} = \mathbb{R}^d$ and λ the Lebesgue measure. Pick $z > 0$, and let P_z be the Poisson point process with intensity measure $z\,\lambda$. The functions $j_{n,\Lambda}$ given by

$$j_{n,\Lambda}(x_1, \ldots, x_n) = z^n \exp(-z|\Lambda|)$$

form a system of Janossy densities. This follows from Theorem 2.17.

Example 2.21 (Binomial process). Take $\mathbb{X} = \mathbb{R}^d$, λ the Lebesgue measure, and ν a probability measure on $\mathbb{R}^d$ with probability density $p(x)$. Fix $m \in \mathbb{N}$ and consider the binomial distribution P with sampling size m and sampling distribution ν. Then P admits the Janossy densities

$$j_{k,\Delta}(x_1, \ldots, x_k) = m(m-1)\cdots(m-k+1)(1-\nu(\Delta))^{m-k} \prod_{j=1}^k p(x_j),$$

see Exercise 2.6.

Notice that the Janossy density $j_{n,\Lambda}$ depends on the window Λ. For this reason it is often more convenient to work with other quantities, the *correlation functions* defined in Section 2.7.

2.6. **Intensity measure, one-particle density.** Real-valued random variables come with a bunch of associated quantities, e.g., expected values $\mathbb{E}[X]$, moments $\mathbb{E}[X^k]$ and generating functions, which have analogues for point processes. We start with the analogue of the expected value.

Lemma 2.22. *Let* $f : \mathbb{X} \to \mathbb{R}_+ \cup \{\infty\}$ *be a measurable function. Then the map* $\mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$, $\eta \mapsto \int_\mathbb{X} f \mathrm{d}\eta = \sum_{x\in S_\eta} n_x(\eta) f(x)$ *is measurable.*

Proof. Let $I_f(\eta) := \sum_{x\in S_\eta} n_x(\eta) f(x)$. If $f = \mathbb{1}_B$ for some $B \in \mathcal{X}$, then $I_f = N_B$ is measurable by the definition of the σ-algebra $\mathfrak{N}$. Taking linear combinations of indicator functions and then monotone limits, we see that I_f is measurable for all non-negative measurable f. □

Definition 2.23. *Let* P *be a probability measure on* $(\mathcal{N}, \mathfrak{N})$. *The* intensity measure *of* P *is the unique measure* μ *on* $(\mathbb{X}, \mathcal{X})$ *such that*

$$\mathsf{E}\left[\sum_{x\in S_\eta} n_x(\eta) f(x)\right] = \int_{\mathbb{X}} f \mathrm{d}\mu \tag{2.14}$$

for all measurable $f : \mathbb{X} \to \mathbb{R}_+ \cup \{\infty\}$. *If* μ *is absolutely continuous with respect to* λ, *the Radon-Nikodým derivative* $\rho_1 = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ *is called* one-particle density.

The intensity measure of a point process Z is the intensity measure of its distribution P, similarly for the one-particle density.

The existence and uniqueness of the intensity measure are easily checked. Choosing $f = \mathbb{1}_B$ we find that μ satisfies

$$\forall B \in \mathcal{X}:\ \mu(B) = \mathsf{E}[N_B], \tag{2.15}$$

which determines it uniquely. For the existence, one defines μ by Eq. (2.15) and checks that it is indeed a measure and that Eq. (2.14) holds true, see [33, Chapter 2.2]. In fact Eq. (2.15) is often taken as the definition of the intensity measure. Eq. (2.14) is *Campbell's formula*. It is usually written as

$$\mathsf{E}\Big[\int_{\mathbb{X}} f(x)\mathrm{d}\eta\Big] = \int_{\mathbb{X}} f \mathrm{d}\mu. \tag{2.16}$$

For Poisson point processes, Definition 2.23 is consistent with the earlier terminology because of (2.9).

Example 2.24. Let $\mathbb{X} = \mathbb{Z}^d$ and $\lambda(B) = \#B$. Then the one-point correlation function exists and is given by $\rho_1(x) = \mathsf{E}[n_x]$. If in addition $\mathsf{P}(n_x \geq 2) = 0$ for all $x \in \mathbb{Z}^d$, then $\rho_1(x) = \mathsf{P}(n_x = 1)$ is the probability that there is a point at x. However, $\sum_{x\in\mathbb{Z}^d} \rho_1(x) = \mathsf{E}[N_{\mathbb{X}}]$ which is in general different from 1—the one-particle density is not a probability density!

Example 2.25. Let $X_1, \ldots, X_n$ be $\mathbb{X}$-valued random variables. Let P be the distribution of $Z = \sum_{j=1}^n \delta_{X_j}$. Then for any non-negative measurable f, we have

$$\mathsf{E}\left[\sum_{x\in S_\eta} n_x(\eta) f(x)\right] = \mathbb{E}\left[\sum_{j=1}^n f(X_j)\right].$$

It follows that the intensity measure of Z is the sum of the distributions of the X_j's. If the joint distribution $(X_1, \ldots, X_n)$ has a Radon-Nikodým derivative $p(x_1, \ldots, x_n)$ with respect to λ^n, the one-particle density exists and is given by

$$\rho_1(x) = \sum_{j=1}^n \int_{\mathbb{X}^{n-1}} p(y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_n)\mathrm{d}\lambda^{n-1}(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n).$$

In particular, the one-particle density is a sum of one-dimensional marginals.

2.7. **Correlation functions.** The Campbell equation (2.14) allows us to express expectations of sums of one-particle functions in terms of the intensity measure or, if it exists, in terms of the one-particle density. We may ask for a similar representation of sums of functions associated with pairs or triplets of distinct particles, for example, the energy of a configuration given in terms of a sum of pair potentials as in (2.4). We need to formalize first what we mean by "sum over n-tuples of distinct particles."

Lemma 2.26. *Let $n \in \mathbb{N}$ and $f : \mathbb{X}^n \to \mathbb{R}_+ \cup \{\infty\}$ be measurable. Then there is a uniquely defined, measurable map $F : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ such that for all $k \in \mathbb{N}_0 \cup \{\infty\}$ and all $x_1, x_2, \ldots \in \mathbb{X}$ (not necessarily distinct),*

$$F\Big(\sum_{j=1}^k \delta_{x_j}\Big) = \sum_{(i_1,\ldots,i_n)}^{\neq} f(x_{i_1},\ldots,x_{i_n}) \tag{2.17}$$

where the sum is over n-tuples $(i_1,\ldots,i_n)$ with pairwise distinct entries in $\{1,\ldots,k\}$ if $k \in \mathbb{N}$, and pairwise distinct entries in $\mathbb{N}$ if $k = \infty$.

We follow the convention that sums over empty sets are zero. If $k \leq n-1$, then there is no way to choose n distinct indices out of $1,\ldots,n$ and the function F in (2.17) vanishes.

A more intrinsic way of writing the map F from Lemma 2.26 is the following. Given $\eta = \sum_{j=1}^k \delta_{x_j} \in \mathcal{N}$ the n-th *factorial measure* $\eta^{(n)}$ is the measure on $\mathcal{X}^{\otimes n}$ defined by

$$\eta^{(n)} = \sum_{(i_1,\ldots,i_n)}^{\neq} \delta_{(x_{i_1},\ldots,x_{i_n})}.$$

For example, for $n = 2$ and $A, B \in \mathcal{X}_\mathsf{b}$, we have $\eta^{(2)}(A \times B) = \eta(A)\eta(B) - \eta(A \cap B)$. Given $f : \mathbb{X}^n \to \mathbb{R} \cup \infty$, the map F from Lemma 2.26 is given by

$$F(\eta) = \int_{\mathbb{X}^n} f \mathrm{d}\eta^{(n)}. \tag{2.18}$$

Example 2.27. For $n = 2$ and $f : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+ \cup \{\infty\}$, we have

$$F(\eta) = \int_{\mathbb{X}^2} f \mathrm{d}\eta^{(2)} = \sum_{x \in S_\eta} n_x(\eta)(n_x(\eta) - 1) f(x,x) + \sum_{\substack{x,y \in S_\eta: \\ x \neq y}} n_x(\eta) n_y(\eta) f(x,y).$$

Proof. The function F, if it exists, is clearly unique. Let $m \in \mathbb{N} \cup \{\infty\}$ and $y_1, y_2, \ldots \in \mathbb{X}$ such that $\sum_{j=1}^m \delta_{y_j} = \sum_{j=1}^k \delta_{x_j}$. Then $m = k$ and there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ if $m = k = \infty$, or $\sigma \in \mathfrak{S}_k$ if $m = k < \infty$, such that $x_i = y_{\sigma(i)}$ for all i. Then

$$\sum_{(j_1,\ldots,j_n)}^{\neq} f(y_{j_1},\ldots,y_{j_n}) = \sum_{(i_1,\ldots,i_n)}^{\neq} f(y_{\sigma(i_1)},\ldots,y_{\sigma(i_n)}) = \sum_{(i_1,\ldots,i_n)}^{\neq} f(x_{i_1},\ldots,x_{i_n})$$

and it follows that F is well-defined.

The measurability for $n = 1$ has already been checked in Lemma 2.22. For $n = 2$, consider first indicator functions $f = \mathbb{1}_{A \times B}$ with $A, B \in \mathcal{X}$. If A and B are disjoint, then $F = N_A N_B$, which is measurable. If $A = B$, then $F = N_A(N_A - 1)$, which is again measurable. For general A, B, we note

$$\mathbb{1}_{A \times B} = \mathbb{1}_{(A \setminus B) \times (B \setminus A)} + \mathbb{1}_{(A \setminus B) \times (A \cap B)} + \mathbb{1}_{(A \cap B) \times (B \setminus A)} + \mathbb{1}_{(A \cap B) \times (A \cap B)}$$

hence the function F associated with $\mathbb{1}_{A\times B}$ is given by

$$\begin{aligned} F &= N_{A\setminus B}N_{B\setminus A} + N_{A\setminus B}N_{A\cap B} + N_{A\cap B}N_{B\setminus A} + N_{A\cap B}(N_{A\cap B}-1) \\ &= N_A N_B - N_{A\cap B}. \end{aligned}$$

which is again measurable. Note that the first line in the previous equation is always well-defined as a sum of non-negative terms, whereas the second line could be problematic if $N_{A\cap B}$ is infinite.

Now fix $\Lambda \in \mathcal{X}_\mathsf{b}$ and let $\mathcal{D}_\Lambda$ consist of those $D \in \mathcal{X}^{\otimes 2}$, $D \subset \Lambda^2$, for which the function F_D associated with $f = \mathbb{1}_D$ is measurable. We have just checked that $\mathcal{D}$ contains all Cartesian products $A\times B$ of measurable subsets $A, B \subset \Lambda$, hence in particular $\Lambda\times\Lambda$. If $C, D \in \mathcal{D}_\Lambda$, then $F_C(\eta) \leq F_D(\eta) \leq N_D(\eta)^2 < \infty$ because $D \subset \Lambda$ is bounded and η is locally finite. Therefore $F_{D\setminus C} = F_D - F_C$ is well-defined and measurable, thus $D\setminus C \in \mathcal{D}_\Lambda$. Finally let $(D_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{D}_\Lambda$. Given $\eta = \sum_j \delta_{x_j}$, let $\eta^{(2)} = \sum_{i\neq j}\delta_{(x_i,x_j)}$, then $F_{D_n} = \int_{\mathbb{X}^2} \mathbb{1}_{D_n}\mathrm{d}\eta^{(2)}$ and the σ-additivity of the measure $\eta^{(2)}$ shows $F_D = \sum_{n=1}^\infty F_{D_n}$, hence $D \in \mathcal{D}_\Lambda$. Thus $\mathcal{D}_\Lambda$ is a Dynkin system containing $\mathcal{C}_\Lambda$ of Cartesian products of measurable subsets of Λ. It follows that $\mathcal{D}_\Lambda$ contains all measurable subsets of $\Lambda\times\Lambda$.

Every $A \in \mathcal{X}$ is the countable union of disjoint bounded measurable sets (think $A_n = A\cap(B(0,n+1)\setminus B(0,n))$), so invoking again the σ-additivity of the measure $\eta^{(2)}$ from the previous paragraph, we find that the function F_A associated with $A \in \mathcal{X}$ is measurable. To conclude the case $n = 2$, we take linear combinations and monotone increasing limits.

The proof in the general case is similar and based on the obervation that the function F associated with indicators of Cartesian sets is a sum of of polynomials of the form

$$F = \prod_{k=1}^r N_{A_k}(N_{A_k}-1)\cdots(N_{A_k}-n_k+1) \tag{2.19}$$

with $A_1,\ldots,A_r \in \mathcal{X}$ pairwise disjoint, $n_1,\ldots,n_r \in \mathbb{N}$, and $n_1+\cdots+n_r = n$. □

Definition 2.28 (Factorial moment measures / correlation functions)**.** *Let* P *be a probability measure on* $(\mathcal{N},\mathfrak{N})$*, and* $n \in \mathbb{N}$*. The* n-th factorial moment measure *is the uniquely defined measure* α_n *on* $\mathbb{X}^n$ *such that for all measurable* $f : \mathbb{X}^n \to [0,\infty]$*, and* F *given by* (2.17)*, we have*

$$\mathsf{E}[F] = \int_{\mathbb{X}^n} f\mathrm{d}\alpha_n. \tag{2.20}$$

The n-point correlation function *or* n-th factorial moment density ρ_n*, if it exists, is the Radon-Nikodým derivative of* α_n *with respect to* λ^n*.*

For $n = 1$, we recover the intensity measure and one-point correlation function. We can rewrite Eq. (2.20) with the factorial measure $\eta^{(n)}$ as

$$\mathsf{E}\left[\int_{\mathbb{X}^n} f\mathrm{d}\eta^{(n)}\right] = \int_{\mathbb{X}^n} f\mathrm{d}\alpha_n = \int_{\mathbb{X}^n} f\rho_n\mathrm{d}\lambda^n.$$

Example 2.29. If $f = \mathbb{1}_{A\times\cdots\times A}$ is the indicator function of the n-th Cartesian product of $A \in \mathcal{X}$, then

$$F = N_A(N_A-1)\cdots(N_A-n+1)$$

and we find

$$\alpha_n(A^n) = \int_{A^n} \rho_n \mathrm{d}\lambda^n = \mathsf{E}\big[N_A(N_A-1)\cdots(N_A-n+1)\big],$$

whence the name *factorial moment measure.*

Example 2.30. Let $n=2$ and $f = \mathbb{1}_{A\times B}$ with disjoint $A, B \in \mathcal{X}$. Then $F = N_A N_B$ and we find

$$\alpha_2(A\times B) = \int_{A\times B} \rho_2(x,y)\mathrm{d}\lambda(x)\mathrm{d}\lambda(y) = \mathbb{E}[N_A N_B].$$

Expectations of products are closely related to covariances and correlation coefficients, which explains the name *correlation functions.*

Example 2.31. Take $\mathbb{X} = \mathbb{Z}^d$ and $\lambda(B) = \# B$. Then the correlation functions exist and can be expressed as mixed factorial moments of occupation numbers $n_x(\eta)$ similar to (2.19). For example,

$$\rho_2(x,y) = \begin{cases} \mathsf{E}[n_x(n_x-1)], & x = y,\\ \mathsf{E}[n_x n_y], & x\neq y.\end{cases}$$

Proposition 2.32. *Suppose that the probability measure* P *admits a system of Janossy densities* $(j_{n,\Lambda})_{n\in\mathbb{N},\Lambda\in\mathcal{X}_\mathsf{b}}$. *Then the correlation functions exist and for all* $n\in\mathbb{N}$, $\Lambda\in\mathcal{X}_\mathsf{b}$, λ^n*-almost all* $(x_1,\dots,x_n)\in\Lambda^n$, *we have*

$$\rho_n(x_1,\dots,x_n) = \sum_{m=n}^\infty \frac{1}{(m-n)!}\int_{\Lambda^{m-n}} j_{m,\Lambda}(x_1,\dots,x_m)\mathrm{d}\lambda(x_{n+1})\cdots\mathrm{d}\lambda(x_m) \tag{2.21}$$

where the summand for $m=n$ *is interpreted as* $j_{n,\Lambda}(x_1,\dots,x_n)$.

Notice that the Janossy densities on the right-hand side depend on Λ but the correlations functions on the left-hand side do not.

Proof. Fix $\Lambda\in\mathcal{X}_\mathsf{b}$, $n\in\mathbb{N}$, $f:\mathbb{X}^n\to\mathbb{R}_+$ measurable, and define F by (2.17). Assume that f vanishes on $\mathbb{X}\setminus\Lambda$. Then $F(\eta) = F(\eta_\Lambda)$ for all η, $F(\eta)=0$ for $N_\Lambda(\eta)\leq n-1$, and

$$\begin{aligned}\mathsf{E}[F] &= \sum_{m=n}^\infty \frac{1}{m!}\int_{\Lambda^m} F(\delta_{x_1}+\cdots+\delta_{x_n}) j_{m,\Lambda}(\boldsymbol{x})\mathrm{d}\lambda^k(\boldsymbol{x})\\ &= \sum_{m=n}^\infty \frac{1}{m!}\int_{\Lambda^m} \sum_{1\leq i_1,\dots,i_n\leq m}^{\neq} f(x_{i_1},\dots,x_{i_n}) j_{m,\Lambda}(x_1,\dots,x_m)\mathrm{d}\lambda^m(\boldsymbol{x}).\end{aligned}$$

The inner sum is a sum over $m(m-1)\cdots(m-n+1) = m!/(m-n)!$ terms. Since the Janossy densities are symmetric functions, the n-dimensional marginals are independent of the precise choice of indices $i_1,\dots,i_n$. Thus

$$\mathsf{E}[F] = \int_{\Lambda^n} f(x_1,\dots,x_n) u_{n,\Lambda}(x_1,\dots,x_n)\mathrm{d}\lambda^n(\boldsymbol{x})$$

with $u_{n,\Lambda}$ given by the right-hand side of (2.21). It follows that

$$\int_{\Lambda^n} f(x_1,\dots,x_n)\rho_n(x_1,\dots,x_n)\mathrm{d}\lambda^n(\boldsymbol{x}) = \int_{\Lambda^n} f(x_1,\dots,x_n) u_{n,\Lambda}(x_1,\dots,x_n)\mathrm{d}\lambda^n(\boldsymbol{x}). \tag{2.22}$$

This holds true for all non-negative measurable f, hence $\rho_n = u_{n,\Lambda}$, λ^n-almost everywhere in Λ^n. □

Example 2.33 (Homogeneous Poisson point process / ideal gas). Let $\mathbb{X} = \mathbb{R}^d$, $\lambda =$ Leb the Lebesgue measure, $z > 0$, and P the Poisson point process with intensity measure $z\,\mathrm{Leb}$. Remember $j_{n,\Lambda} = z^n \exp(-z\lambda(\Lambda)$

$$\begin{aligned}\rho_n(x_1,\ldots,x_n) &= \sum_{m=n}^{\infty} \frac{1}{(m-n)!} \int_{\mathbb{X}^{m-n}} z^m \mathrm{e}^{-z|\Lambda|} \mathrm{d}x_{m+1} \cdots \mathrm{d}x_n \\ &= z^n \mathrm{e}^{-z|\Lambda|} \sum_{m=n}^{\infty} \frac{1}{(m-n)!} \big(z|\Lambda|\big)^{m-n} = z^n.\end{aligned}$$

Proposition 2.32 shows that ρ_n inherits the symmetry of the Janossy densities.

Corollary 2.34. *Let* P *be a probability measure on* $(\mathcal{N},\mathfrak{N})$. *The correlation functions, if they exist, are symmetric functions, i.e.,*

$$\rho_n(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = \rho_n(x_1,\ldots,x_n) \tag{2.23}$$

for all $n \in \mathbb{N}$, $\sigma \in \mathfrak{S}_n$, *and* λ^n*-almost all* $(x_1,\ldots,x_n) \in \mathbb{X}^n$.

Finally we address signed observables. If in Lemma 2.26 the function $f(x_1,\ldots,x_n)$ takes both negative and positive values, we need to worry about the convergence of the sum in (2.17).

Proposition 2.35. *Let* P *be a probability measure on* $(\mathcal{N},\mathfrak{N})$. *Fix* $n \in \mathbb{N}$ *and a measurable* $f : \mathbb{X}^n \to \mathbb{R}$. *Suppose that* $\int_{\mathbb{X}^n} |f| \mathrm{d}\alpha_n < \infty$. *Then the right-hand side in* (2.17) *is absolutely convergent for* P*-almost all* $\eta = \sum_j \delta_{x_j} \in \mathcal{N}$. *Moreover the function* F *defined by* (2.17) *satisfies* $\mathsf{E}|F| < \infty$ *and* $\mathsf{E}[F] = \int_{\mathbb{X}^n} f \mathrm{d}\alpha_n$.

Proof. Write $f = f_+ - f_-$ with $f_\pm \geq 0$ the positive and negative parts of f. Let $F_\pm : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ be the functions associated to $f_\pm$ by Lemma 2.26. By the assumption of f, we have $\mathsf{E}[F_+] = \int_{\mathbb{X}^n} f_+ \mathrm{d}\alpha_n < \infty$, hence $F_+(\eta) < \infty$ for P-almost all η. A similar argument applies to F_-. Thus $F_+ + F_- < \infty$, P-almost surely. The proposition then follows from $F = F_+ - F_-$ and $|F| = F_+ + F_-$. □

2.8. **Generating functionals and Ruelle bound.** Let P be a probability measure on $(\mathcal{N},\mathfrak{N})$.

Definition 2.36. *For* $f : \mathbb{X} \to [0,\infty) \cup \{\infty\}$ *measurable, set*

$$\mathcal{L}_{\mathsf{P}}[f] := \mathsf{E}\Big[\exp\Big(-\sum_{x\in S_\eta} n_x(\eta) f(x)\Big)\Big] = \mathsf{E}\Big[\exp\Big(-\int_{\mathbb{X}} f \mathrm{d}\eta\Big)\Big]$$

with the convention $\exp(-\infty) = 0$. $\mathcal{L}_{\mathsf{P}}[\cdot]$ *is the (extended)* Laplace functional *of* P.

The qualificative "extended" is sometimes added to indicate that the domain of $\mathcal{L}_{\mathsf{P}}$ is chosen to include functions that may take the value ∞. If f takes some negative values but $\mathsf{E}[\exp(-\sum_{x\in\mathbb{X}} n_x(\eta) f(x))]$ exists, by a slight abuse of notation we use the same letter $\mathcal{L}_{\mathsf{P}}[f]$ in this situation as well.

Definition 2.37. *For* $h : \mathbb{X} \to [0,1]$ *measurable, set*

$$G_{\mathsf{P}}[h] := \mathsf{E}\Big[\prod_{x\in S_\eta} h(x)^{n_x(\eta)}\Big].$$

$G_{\mathsf{P}}[\cdot]$ *is the* probability generating functional *of* P.

Notice

$$G_{\mathsf{P}}[h] = \mathcal{L}_{\mathsf{P}}[-\log h] \tag{2.24}$$

with the convention $-\log 0 = \infty$.

Example 2.38. Let P be the distribution of a Poisson point process with intensity measure ν. Then for all non-negative f, we have

$$\mathcal{L}_{\mathsf{P}}[f] = \exp\Bigl(- \int_{\mathbb{X}} (1 - \mathrm{e}^{-f(x)}) \mathrm{d}\nu(x)\Bigr)$$

with the usual convention $\exp(-\infty) = 0$. Indeed, suppose first that f vanishes outside some bounded set $\Lambda \in \mathcal{X}_{\mathsf{b}}$. Then $\eta \mapsto \exp(-\int_{\mathbb{X}} f \mathrm{d}\eta)$ is in $\mathcal{A}_\Lambda$ and by Theorem 2.17, we have

$$\begin{aligned}\mathcal{L}_{\mathsf{P}}[f] &= \mathrm{e}^{-\nu(\Lambda)} \Bigl(1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} \mathrm{e}^{-\sum_{j=1}^n f(x_j)} \mathrm{d}\nu^n(\boldsymbol{x})\Bigr) \\ &= \exp\Bigl(- \int_{\Lambda} (1 - \mathrm{e}^{-f(x)}) \mathrm{d}\nu(x)\Bigr) = \exp\Bigl(- \int_{\mathbb{X}} (1 - \mathrm{e}^{-f(x)}) \mathrm{d}\nu(x)\Bigr)\end{aligned}$$

For general non-negative f, let $f_n := f \mathbb{1}_{B(0,n)}$. Then

$$\mathcal{L}_{\mathsf{P}}[f_n] = \exp\Bigl(- \int_{\mathbb{X}} (1 - \mathrm{e}^{-f_n(x)}) \mathrm{d}\nu(x)\Bigr). \tag{2.25}$$

Moreover $f_n \nearrow f$ so by monotone convergence, $\int_{\mathbb{X}} f_n \mathrm{d}\eta \nearrow \int_{\mathbb{X}} f \mathrm{d}\eta$ and by dominated convergence (note $\exp(-\int_{\mathbb{X}} f_n \mathrm{d}\eta) \le 1$), $\mathcal{L}_{\mathsf{P}}[f_n] \to \mathcal{L}_{\mathsf{P}}[f]$. We also have $0 \le 1 - \exp(-f_n) \nearrow 1 - \exp(-f)$ so $\int_{\mathbb{X}} (1 - \exp(-f_n)) \mathrm{d}\nu \to \int_{\mathbb{X}} (1 - \exp(-f)) \mathrm{d}\nu$ by monotone convergence. Passing to the limit in Eq. (2.25) we obtain the expression for $\mathcal{L}_{\mathsf{P}}[f]$.

The functionals are the analogues of the Laplace transform and probability generating function of integer-valued random variables. The analogy proves helpful in understanding the relation between the Laplace functional and the correlation functions. Let N be a random variable with values in $\mathbb{N}_0$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $t \ge 0$ and $s = \exp(-t) - 1$. A formal computation yields

$$\mathbb{E}\bigl[\mathrm{e}^{-tN}\bigr] = \mathbb{E}[(1+s)^N] = \mathbb{E}\Bigl[\sum_{k=0}^N \binom{N}{k} s^N\Bigr] = \sum_{k=0}^\infty \frac{s^k}{k!}\, \mathbb{E}\bigl[N(N-1)\cdots(N-k+1)\bigr]. \tag{2.26}$$

Eq. (2.26) shows how the Laplace transform might be expressed in terms of the factorial moments. However the computation is formal because the exchange of summation and expectations requires justification; in fact without assumptions on N the factorial moments might be infinite. A sufficient condition for (2.26) to hold true is that $\mathbb{E}[N(N-1)\cdots(N-k+1)] \le \xi^k$ for some $\xi > 0$ and all $k \in \mathbb{N}_0$. The following is an analogue of this condition.

Definition 2.39. *Let* P *be a probability measure on* $(\mathcal{N}, \mathfrak{N})$. *Then* P *satisfies* Ruelle's bound *if all its correlation functions* $\rho_n(x_1, \dots, x_n)$ *exist and satisfy the condition*

$$\rho_n(x_1, \dots, x_n) \le \xi^n \tag{$\mathscr{R}_\xi$}$$

for some $\xi > 0$, *all* $n \in \mathbb{N}$, *and* λ^n*-almost all* $(x_1, \dots, x_n) \in \mathbb{X}^n$.

Equivalently, if the factorial moment measures α_n satisfy

$$\alpha_n(B) \le \xi^n \lambda^n(B)$$

for all $n \in \mathbb{N}$ and all $B \in \mathcal{X}^{\otimes n}$.

Theorem 2.40. *Suppose that* P *satisfies Ruelle's bound. Then for all measurable* $f : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ *with*

$$\int_{\mathbb{X}} |\mathrm{e}^{-f(x)} - 1| \mathrm{d}\lambda(x) < \infty,$$

we have

$$\mathcal{L}_{\mathsf{P}}[f] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \prod_{j=1}^{n} (\mathrm{e}^{-f(x_j)} - 1) \rho_n(\boldsymbol{x}) \mathrm{d}\lambda^n(\boldsymbol{x})$$

with absolutely convergent integrals and sums.

Remark. From $\exp(-f) - 1 \ge -f$ we get $f_- \le |\exp(-f) - 1|$ and under the condition of the theorem, $\int_{\mathbb{X}} f_-(x)\rho_1(x)\mathrm{d}x < \infty$. Proposition 2.35 then shows $\int_{\mathbb{X}} f_- \mathrm{d}\eta < \infty$ for P-almost all η. Therefore $\int_{\mathbb{X}} f \mathrm{d}\eta$ is well-defined, but it can be infinite; similarly, $\exp(-\int_{\mathbb{X}} f \mathrm{d}\eta)$ is well-defined but can be zero.

Proof of Theorem 2.40. Let $\Lambda \in \mathcal{X}_{\mathsf{b}}$ with $f = 0$ on $\mathbb{X} \setminus \Lambda$. For $|\rho_n| \le \xi^n$, we have

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{X}^n} \prod_{j=1}^{n} \big|\mathrm{e}^{-f(x_j)} - 1\big| \rho_n(\boldsymbol{x}) \mathrm{d}\lambda^n(\boldsymbol{x}) \\ \le \exp\left(\xi \int_{\mathbb{X}} |\mathrm{e}^{-f(x)} - 1| \mathrm{d}\lambda(x) \right) < \infty \end{aligned} \tag{2.27}$$

which proves the absolute convergence. Write $\eta = \sum_{j=1}^{n} \delta_{x_j}$ with $n \in \mathbb{N}_0 \cup \{\infty\}$, $x_1, \ldots, x_n \in \mathbb{X}$. If $n < \infty$, we have

$$\Phi(\eta) := \mathrm{e}^{-\sum_{x \in S_\eta} n_x(\eta) f(x)} = \prod_{j=1}^{n} \mathrm{e}^{-f(x_j)} = 1 + \sum_{\substack{J \subset [n]: \\ J \ne \emptyset}} \prod_{j \in J} \big(\mathrm{e}^{-f(x_j)} - 1\big).$$

If $n = \infty$ we observe that $\eta \in \mathcal{N}$ can only have finitely many particles in Λ. Let $m \in \mathbb{N}_0$ be the number of those particles and assume without loss of generality that these are $x_1, \ldots, x_m$. Then

$$\Phi(\eta) = \prod_{j=1}^{m} \mathrm{e}^{-f(x_j)} = \sum_{J \subset [m]} \prod_{j \in J} \big(\mathrm{e}^{-f(x_j)} - 1\big) = \sum_{J \subset \mathbb{N}} \prod_{j \in J} \big(\mathrm{e}^{-f(x_j)} - 1\big)$$

with the convention that the product over the empty set is 1. Only finitely many $J \subset \mathbb{N}$ give a non-zero contribution to the sum. Instead of summing directly over subsets J, we can sum over integers k and then over subsets J with cardinality k, which gives

$$\Phi\Big(\sum_{j=1}^{n} \delta_{x_j}\Big) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{(i_1, \ldots, i_k)}^{\ne} \prod_{r=1}^{k} (\mathrm{e}^{-f(x_{i_r})} - 1). \tag{2.28}$$

The factorial $1/k!$ comes in because we sum over ordered k-tuples $(i_1, \ldots, i_k)$ instead of sets $J = \{i_1, \ldots, i_k\}$ of cardinality k. Only finitely many summands in (2.28) are non-zero. Let

$$G_k\Big(\sum_{j=1}^n \delta_{x_j}\Big) = \sum_{(i_1,\dots,i_k)}^{\neq} \prod_{r=1}^k (\mathrm{e}^{-f(x_{i_r})}-1) \tag{2.29}$$

so that $\Phi = 1+\sum_{k=1}^\infty \frac{1}{k!} G_k$. By Proposition 2.35 and Eq. (2.27), we have

$$\mathsf{E}[G_k] = \int_{\mathbb{X}^k} \prod_{j=1}^k (\mathrm{e}^{-f(x_j)}-1)\rho_k(\boldsymbol{x})\mathrm{d}\lambda^k(\boldsymbol{x})$$

with $\sum_{k=1}^\infty \frac{1}{k!}\mathsf{E}|G_k| < \infty$. Therefore we can exchange summation and expectation and find that

$$\mathsf{E}[\Phi] = \sum_{k=1}^\infty \frac{1}{k!}\mathsf{E}[G_k]$$

which concludes the proof if f has bounded support. If f has unbounded support, the bound (2.27) and Proposition 2.35 show that

$$\sum_{J\subset\mathbb{N}} \left| \prod_{j=1}^k (\mathrm{e}^{-f(x_j)}-1)\right| < \infty$$

for P-almost all $\eta = \sum_{j=1}^\infty \delta_{x_j} \in \mathcal{N}$. This together with Exercise 2.10 shows that the identity $\Phi = 1+\sum_{k=1}^\infty G_k/k!$ stays true, up to P-null sets, and the proof is concluded as before. □

Ruelle's bound $(\mathscr{R}_\xi)$ says that the correlation functions of P are bounded by those of a Poisson point process $\mathsf{P}_{\xi\lambda}$ with intensity measure $\xi\lambda$. Notice that as $k\to\infty$,

$$\mathsf{P}_{\xi\lambda}\big(N_B = k\big) = \frac{1}{k!}(\xi\lambda(B))^k \mathrm{e}^{-\xi\lambda(B)} \sim \frac{1}{\sqrt{2\pi k}}\,\mathrm{e}^{-k\log(k/\mathrm{e})+k\log(\lambda\xi(B))-\xi\lambda(B)}$$

by Stirling's formula. The next lemma provides a similar bound for $\mathsf{P}(N_B\geq k)$, which shows that $\mathsf{P}(N_B\geq k)$ goes to zero as $k\to\infty$ faster than any exponential: it is highly unlikely that many particles accumulate in B. This is the intuitive content of Ruelle's bound.

Lemma 2.41. *Let P satisfy condition $(\mathscr{R}_\xi)$. Then we have, for all $B\in\mathcal{X}_\mathsf{b}$ and all $k\in\mathbb{N}$,*

$$\mathsf{P}(N_B\geq k) \leq \exp\left(-\left(k\log\frac{k}{\xi\lambda(B)} - k + \xi\lambda(B)\right)\right).$$

Proof. We have

$$\begin{aligned}\mathsf{E}\big[\mathrm{e}^{tN_B}\big] &= \mathsf{E}\Big[1+\sum_{n=1}^\infty \frac{1}{n!} N_B(N_B-1)\cdots(N_B-n+1)(\mathrm{e}^t-1)^n\Big]\\ &\leq 1+\sum_{n=1}^\infty \frac{1}{n!}(\mathrm{e}^t-1)(\xi\lambda(B))^n = \exp\big(\xi(\mathrm{e}^t-1)\lambda(B)\big).\end{aligned}$$

In passing from the first to the second line, we have used that summation and expectations can because all terms are non-negative. (Alternatively, we may note that $\mathsf{E}[\exp(tN_B)] = \mathcal{L}_\mathsf{P}[-t\mathbb{1}_B]$ and use Theorem 2.40). By Markov's inequality, we have

$$\mathsf{P}\big(N_B\geq k\big) \leq \mathrm{e}^{-tk}\mathsf{E}\big[\mathrm{e}^{tN_B}\big] \leq \exp\big(\xi(\mathrm{e}^t-1)\lambda(B)-tk\big).$$

This holds true for all $t > 0$, so we may take the infimum over t on the right-hand side. We evaluate

$$\inf_{t>0}\big(\xi(\mathrm{e}^t - 1)\lambda(B) - tk\big) = k - \xi\lambda(B) - k\log\frac{k}{\xi\lambda(B)}$$

and obtain the required bound. □

2.9. **Moment problem. Inversion formulas.** Mathematical physics often deals with correlations functions only and disregards probability measures entirely. It is therefore reassuring to know that, under some conditions, the correlation functions determine the measure uniquely. This is analogous to the uniqueness part in the moment problem of probability—is the probability distribution of a real-valued random variable X uniquely determined by its moments $\mathbb{E}[X^n]$?[3]

Theorem 2.42. *Let* P, Q *be two probability measures on* $(\mathcal{N}, \mathfrak{N})$*. Suppose that* P *and* Q *have identical correlation functions and that* P *satisfies Ruelle's bound. Then* $\mathsf{P} = \mathsf{Q}$.

The theorem follows from Theorem 2.40 and the fact that the Laplace functional $\mathcal{L}_{\mathsf{P}}$ determines the probability measure P uniquely, compare Propositions 2.10 and 4.12 in [33]. We provide an alternative proof based on an inversion of the relation between correlation functions and Janossy densities from Proposition 2.32.

Theorem 2.43. *Assume that* P *satisfies Ruelle's condition. Then it admits a system of Janossy densities* $(j_{n,\Lambda})$ *and we have, for all non-empty* $\Lambda \in \mathcal{X}_{\mathsf{b}}$*,* $n \in \mathbb{N}$*, and* λ^n*-almost all* $(x_1, \dots, x_n) \in \mathbb{X}^n$*,*

$$j_{n,\Lambda}(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\Lambda^k} \rho_{n+k}(x_1, \dots, x_{n+k}) \mathrm{d}\lambda(x_{n+1}) \cdots \mathrm{d}\lambda(x_{n+k}). \tag{2.30}$$

Notice that Ruelle's condition is sufficient to guarantee the absolute convergence of the series in (2.30). Theorems 2.42 and 2.43 are proven at the end of the section.

For the proof of Theorem 2.43, we try to invert the mapping from Lemma 2.26.

Lemma 2.44. *Let* $F, G : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$*. The following two conditions are equivalent:*

(a) $G(0) = F(0)$ *and for all* $n \in \mathbb{N}$ *and* $x_1, \dots, x_n \in \mathbb{X}$,

$$G\big(\sum_{j=1}^n \delta_{x_j}\big) = \sum_{J \subset [n]} F\Big(\sum_{j \in J} \delta_{x_j}\Big). \tag{2.31}$$

(b) $F(0) = G(0)$ *and for all* $n \in \mathbb{N}$ *and* $x_1, \dots, x_n \in \mathbb{X}$,

$$F\big(\sum_{j=1}^n \delta_{x_j}\big) = \sum_{J \subset [n]} (-1)^{n - \# J} G\Big(\sum_{j \in J} \delta_{x_j}\Big). \tag{2.32}$$

Remark (Möbius inversion). Let $(M, \preceq)$ be a partially ordered set. Suppose that it has a minimal element and that for every $b \in M$, there are at most finitely many elements with $a \preceq b$. Then, given a weight function $w : M \to \mathbb{R}$ we can define a new weight by

$$\widehat{w}(b) = \sum_{a \preceq b} w(a).$$

[3] In general, the answer is no, see [9, Chapter 2.3.e] for counter-examples.

The mapping $w \mapsto \widehat{w}$ can be inverted as

$$w(b) = \sum_{a \preceq b} \widehat{w}(a) \mu(a,b)$$

for some function $\mu(a,b)$, called *Möbius function*, that depends on $(M, \preceq)$ only. Lemma 2.44 corresponds to the special case $(M,\preceq)$= the finite subsets of $\mathbb{N}$ ordered by inclusion (with minimal element the empty subset) and weights $w(I) = F(\sum_{i\in I} \delta_{x_i})$ for some given $x_1, x_2, \ldots \in \mathbb{X}$ and given F. The Möbius function in this example is $\mu(I,J) = (-1)^{\#(J\setminus I)}$.

Proof of Lemma 2.44. To simplify notation, let us write $f_0 = F(0)$, $f_n(x_1,\ldots,x_n) = F(\sum_{j=1}^n \delta_{x_j})$, and define $(g_n)_{n\in\mathbb{N}_0}$ in an analogous fashion.

"(a) $\Rightarrow$(b)": We evaluate

$$\begin{aligned}\sum_{J\subset[n]} (-1)^{n-\#J} g_{\#J}\big((x_j)_{j\in J}\big) &= (-1)^n \sum_{J\subset[n]} (-1)^{\#J} \sum_{I\subset J} f_n(\boldsymbol{x}_I)\\ &= (-1)^n \sum_{I\subset[n]} f_n(\boldsymbol{x}_I) \sum_{\substack{J\subset[n]:\\ J\supset I}} (-1)^{\#J}.\end{aligned}$$

If $I=[n]$, the only contribution to the inner sum is from $J=[n]$, which is $(-1)^n$. If $I \subsetneq [n]$, we may write (think $K = J\setminus I$)

$$\sum_{\substack{J\subset[n]:\\ J\supset I}} (-1)^{\#J} = (-1)^{\#I} \sum_{K\subset[n]\setminus I} (-1)^{\#K} = (-1)^{\#I} \prod_{k\in[n]\setminus I} (1+(-1)) = 0.$$

It follows that

$$f_n(x_1,\ldots,x_n) = \sum_{J\subset[n]} (-1)^{n-\#J} g_{\#J}\big((x_j)_{j\in J}\big)$$

which yields (b). The proof of the reverse implication "(b) $\Rightarrow$ (a)" is similar and therefore omitted. □

For $\Lambda \in \mathcal{X}_{\mathsf{b}}$, let $\mathcal{N}_\Lambda = \{\eta\in\mathcal{N} \mid N_{\mathbb{X}\setminus\Lambda}(\eta) = 0\}$. Notice $\mathcal{N}_\Lambda \subset \mathcal{N}_{\mathsf{f}}$.

Lemma 2.45. *Let $F, G : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$ be related by $F(0) = G(0)$ and Eqs.* (2.32) *and* (2.31)*. Let $\Lambda\in\mathcal{X}_{\mathsf{b}}$. The following statements are equivalent:*

(a) $F(\eta) = 0$ for all $\eta \in \mathcal{N}_{\mathsf{f}}\setminus\mathcal{N}_\Lambda$.
(b) $G(\eta) = G(\eta_\Lambda)$ for all $\eta\in\mathcal{N}_{\mathsf{f}}$.

Moreover if F satisfies condition (a) we may extend G to $\mathcal{N}$ by setting for $x_1, x_2, \ldots \in \mathbb{X}$,

$$G\Big(\sum_{j=1}^\infty \delta_{x_j}\Big) = \sum_{\substack{J\subset\mathbb{N}\\ \#J<\infty}} F\Big(\sum_{j\in J}\delta_{x_j}\Big). \tag{2.33}$$

The sum on the right-hand side is finite and the extended function G satisfies (b'): $G(\eta_\Lambda) = G(\eta)$ for all $\eta\in\mathcal{N}$.

Consequently the mapping $K : F \mapsto G$ is a bijection from the space of maps $F : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$ that are *locally supported*, meaning that they satisfy condition (a) of the previous lemma for some bounded $\Lambda \in \mathcal{X}_{\mathsf{b}}$, to the space of local observables. We call $G = KF$ the *K-transform* of F and note that Lemma 2.44(b) provides a formula for $K^{-1}G$.

Proof of Lemma 2.45. "(a)$\Rightarrow$ (b)": The only relevant contributions in the sum in (2.31) are from label sets $J \subset [n]$ such that $x_j \in \Lambda$ for all $j \in \Lambda$. It follows that $G(\eta_\Lambda) = G(\eta)$.

"(b)$\Rightarrow$ (a)" Let $\eta = \sum_{j=1}^n \delta_{x_j} \in \mathcal{N} \setminus \mathcal{N}_\Lambda$. Thus one of the x_j's is in $\mathbb{X} \setminus \Lambda$. We may assume without loss of generality $x_n \in \mathbb{X} \setminus \Lambda$. Then

$$F\Big(\sum_{j=1}^n \delta_{x_j}\Big) = \sum_{I \subset [n-1]} (-1)^{n-\#I} \Big(G\big(\sum_{j\in I} \delta_{x_j}\big) - G\big(\delta_{x_n} + \sum_{j \in I} \delta_{x_j}\big)\Big) = 0.$$

Thus we have proven the equivalence (a) $\Leftrightarrow$ (b). Now suppose F satisfies condition (a) and define G on $\mathcal{N} \setminus \mathcal{N}_{\mathsf{f}}$ by (2.33). The only relevant contributions are from sets J such that $x_j \in \Lambda$ for all $j \in J$. Since any configuration $\eta = \sum_{j=1}^\infty \delta_{x_j}$ has at most finitely many particles in Λ, this leaves us with a finite amount of non-zero summands. Condition (b') is checked as in the implication (a)$\Rightarrow$ (b). $\square$

Lemma 2.46. *Let $F : \mathcal{N}_{\mathsf{f}} \to [0,\infty) \cup \{\infty\}$. Define $G : \mathcal{N} \to [0,\infty)\cup\{\infty\}$ by* (2.31) *and* (2.33)*; thus $G = KF$. Then*

$$\mathsf{E}[G] = F(0) + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} F\Big(\sum_{j=1}^k \delta_{x_j}\Big) \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}). \tag{2.34}$$

The statement also holds true if $F : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$ is supported in $\mathcal{N}_\Lambda$ for some $\Lambda \in \mathcal{X}_{\mathsf{b}}$ and if it satisfies

$$\sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} \Big| F\Big(\sum_{j=1}^k \delta_{x_j}\Big)\Big| \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}) < \infty, \tag{2.35}$$

in which case we also have $\mathsf{E}|G| < \infty$.

Proof. Let $n \in \mathbb{N} \cup \{\infty\}$ and $x_1, x_2, \ldots \in \mathbb{X}$. For $k \in \mathbb{N}$, set $f_k(x_1,\ldots,x_k) = F(\delta_{x_1} + \cdots + \delta_{x_k})$. Eqs. (2.31) and (2.33) become (think $\#J = k$)

$$G\Big(\sum_{j=1}^n \delta_{x_j}\Big) = F(0) + \sum_{k=1}^n \frac{1}{k!} \sum_{(i_1,\ldots,i_k)}^{\neq} f_k(x_{i_1}, \ldots, x_{i_k}). \tag{2.36}$$

For non-negative F we can exchange summation and integration. The lemma follows from the definition of the correlation functions.

If F is real-valued, supported in $\mathcal{N}_\Lambda$, and satisfies condition (2.35), then for every $m \in \mathbb{N}$,

$$\mathsf{E}[G \mathbb{1}_{\{N_\Lambda \le m\}}] = \mathsf{E}[G(\eta_\Lambda) \mathbb{1}_{\{N_\Lambda \le m\}}] = F(0) + \sum_{k=1}^m \frac{1}{k!} \int_{\mathbb{X}^k} F(\delta_{x_1} + \cdots + \delta_{x_k}) \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x})$$

with absolutely convergent integrals. The passage to the limit $m \to \infty$ is justified with dominated convergence: we have

$$|G(\eta_\Lambda)| \mathbb{1}_{\{N_\Lambda \le m\}}(\eta) \le |G(\eta_\Lambda)| = |G(\eta)|.$$

Since $|G(\eta_\Lambda)|$ is given by an expression similar to (2.31) with F replaced with $|F|$, the first part of the lemma together with condition (2.35) shows $\mathsf{E}[|G(\eta_\Lambda)|] < \infty$. Thus $|G|$ is integrable and can be used as an m-independent majorizing function for the dominated convergence theorem. $\square$

Proof of Theorem 2.43. Let $G \in \mathcal{A}_\Lambda$ be a bounded local observable. Define F by (2.32). Then for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$,

$$|F(\sum_{j=1}^n \delta_{x_j})| \leq \Bigl(\sup_{\mathcal{N}} |G|\Bigr) \sum_{J \subset [n]} 1 = \Bigl(\sup_{\mathcal{N}} |G|\Bigr) 2^n \tag{2.37}$$

and $F(\eta) = 0$ for $\eta \in \mathcal{N} \setminus \mathcal{N}_\Lambda$. If P satisfies Ruelle's moment bound $(\mathscr{R}_\xi)$, then

$$\sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} |F(\sum_{j=1}^n \delta_{x_j})| \rho_n(\boldsymbol{x}) \mathrm{d}\lambda^n(\boldsymbol{x}) \leq \exp(2\xi\lambda(\Lambda)) < \infty.$$

Hence Lemma 2.46 applies and shows that $\mathsf{E}[G]$ is given by

$$\begin{aligned}\mathsf{E}[G] &= F(0) + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} F(\sum_{j=1}^n \delta_{x_j}) \rho_n(\boldsymbol{x}) \mathrm{d}\lambda^n(\boldsymbol{x}) \\ &= G(0) + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Lambda^n} \sum_{J \subset [n]} (-1)^{n - \# J} G\bigl(\sum_{j \in J} \delta_{x_j}\bigr) \rho_n(\boldsymbol{x}) \mathrm{d}\lambda^n(\boldsymbol{x}).\end{aligned}$$

The sums and integrals are all absolutely convergent, so we can exchange the order of operations. Exploiting the symmetry of the correlation functions, we deduce

$$\begin{aligned}\mathsf{E}[G] &= G(0) + \sum_{n=1}^\infty \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \int_{\Lambda^k} G\bigl(\sum_{j=1}^k \delta_{x_j}\bigr) \rho_n(\boldsymbol{x}) \mathrm{d}\lambda^n(\boldsymbol{x}) \\ &= G(0) \Bigl(1 + \sum_{n=1}^\infty \frac{(-1)^n}{n!} \mathsf{E}[N_\Lambda (N_\Lambda - 1) \cdots (N_\Lambda - n + 1)]\Bigr) \\ &\qquad + \sum_{k=1}^\infty \frac{1}{k!} \int_{\Lambda^k} G\bigl(\sum_{j=1}^k \delta_{x_j}\bigr) u_{k,\Lambda}(x_1, \dots, x_k) \mathrm{d}\lambda(x_1) \cdots \mathrm{d}\lambda(x_k).\end{aligned}$$

with

$$u_{k,\Lambda}(x_1, \dots, x_k) = \sum_{n=k}^\infty \frac{(-1)^{n-k}}{(n-k)!} \int_{\Lambda^{n-k}} \rho_n(\boldsymbol{x}) \mathrm{d}\lambda(x_{k+1}) \cdots \mathrm{d}\lambda(x_n).$$

Specializing to $G \in \mathcal{A}_\Lambda$ with $G(0) = 0$ and comparing with Definition 2.19, we deduce that $j_{k,\Lambda} := u_{k,\Lambda}$ forms a system of Janossy densities. (Note that it is enough to check (2.13) for bounded observables.) $\square$

Proof of Theorem 2.42. Let P and Q satisfy Ruelle's condition and have identical correlation functions. Theorem 2.43 shows that P and Q admit a system of Janossy densities, and that their Janossy densities coincide. By the definition of Janossy densities, this shows $\int_{\mathcal{N}} f \mathrm{d}\mathsf{P} = \int_{\mathcal{N}} f \mathrm{d}\mathsf{Q}$ for all $f \in \mathcal{A}^{\mathrm{loc}}$, hence $\mathsf{P} = \mathsf{Q}$. $\square$

Finally we address the question of the existence of a probability measure that has a given family of functions as correlation functions.

Theorem 2.47. *Let $(\rho_n)_{n \in \mathbb{N}}$ be a family of symmetric functions $\rho_n : \mathbb{X}^n \to \mathbb{R}_+$ with $\rho_n \leq \xi^n$ λ^n-a.e., for all $n \in \mathbb{N}$ and some $\xi > 0$. Then the ρ_n's are the correlation functions of some probability measure P on $\mathcal{N}$ if and only if for all $\Lambda \in \mathcal{X}_{\mathsf{b}}$, all $n \in \mathbb{N}$, and λ^n-almost all $(x_1, \dots, x_n) \in \Lambda^n$, we have*

$$\sum_{k=0}^\infty \frac{(-1)^k}{k!} \int_{\Lambda^k} \rho_{n+k}(x_1, \dots, x_{n+k}) \mathrm{d}\lambda(x_{n+1}) \cdots \mathrm{d}\lambda(x_{n+k}) \geq 0 \tag{2.38}$$

and for all $\Lambda \in \mathcal{X}_{\mathsf{b}}$,

$$1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\Lambda^k} \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}) \geq 0. \tag{2.39}$$

The non-negativity conditions are sometimes called *Lenard positivity*; they go back to [35]. It is instructive to work out an alternative formulation in terms of measures rather than densities: Given the family $(\rho_n)_{n\in\mathbb{N}}$, define a measure α on $\mathcal{N}_{\mathsf{f}}$ by the requirement

$$\int_{\mathcal{N}_{\mathsf{f}}} F \, \mathrm{d}\alpha = F(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} F(\delta_{x_1} + \cdots + \delta_{x_k}) \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}) \tag{2.40}$$

for all non-negative measurable $F : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}_+ \cup \{\infty\}$. If the ρ_n's are the correlation functions of some probability measure P, then α is called the *correlation measure* of P and Lemma 2.46 says that

$$\int_{\mathcal{N}} (KF) \mathrm{d}\mathsf{P} = \int_{\mathcal{N}_{\mathsf{f}}} F \, \mathrm{d}\alpha. \tag{2.41}$$

for every locally supported $F : \mathcal{N}_{\mathsf{f}} \to \mathbb{R} \cup \{\infty\}$ that is non-negative or satisfies $\int_{\mathcal{N}_{\mathsf{f}}} |F| \mathrm{d}\mathsf{P} < \infty$. Choosing $F = K^{-1}G$ with $G \in \mathcal{A}^{\mathrm{loc}}$, we get

$$\int_{\mathcal{N}} G \mathrm{d}\mathsf{P} = \int_{\mathcal{N}_{\mathsf{f}}} (K^{-1}G) \, \mathrm{d}\alpha$$

hence

$$G \geq 0 \ \Rightarrow \int_{\mathcal{N}_{\mathsf{f}}} (K^{-1}G) \, \mathrm{d}\alpha \geq 0. \tag{2.42}$$

Lenard positivity amounts to asking that the implication (2.42) holds true for all $G \in \mathcal{A}^{\mathrm{loc}}$.

Proof of Theorem 2.47. "$\Rightarrow$" If the ρ_n's are the correlation functions of some probability measure P, then Eq. (2.38) is the formula for the Janossy-densities, which must be non-negative (up to λ^n-null sets). The expression (2.39) is treated with the help of Exercise 2.8.

"$\Leftarrow$" Let $j_{n,\Lambda}(x_1, \dots, x_n)$ and $j_{0,\Lambda}$ be the expressions from (2.38) and (2.39), respectively. The functions $j_{n,\Lambda}$ are well-defined, symmetric, and non-negative, but we do not yet know that they are the Janossy densities of some measure P. For $\Lambda \in \mathcal{X}_{\mathsf{b}}$ and $A \in \mathfrak{N}$ with $A \subset \mathcal{N}_\Lambda$, set

$$\mathsf{P}_\Lambda(A) := \mathbb{1}_A(0) j_{0,\Lambda} + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \mathbb{1}_A(\delta_{x_1} + \cdots + \delta_{x_k}) j_{k,\Lambda}(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}).$$

One easily checks that P_Λ defines a measure on $\mathcal{N}_\Lambda$. We claim that it is in fact a probability measure. Proceeding as in the proof of Theorem 2.43, one checks that

$$\mathsf{P}_\Lambda(A) = \varphi(\mathbb{1}_A \circ \pi_\Lambda)$$

where $\pi_\Lambda(\eta) = \eta_\Lambda$ and $\varphi : \mathcal{A}^{\mathrm{loc}} \to \mathbb{R}$ is given by

$$\varphi(g) = \int_{\mathcal{N}_{\mathsf{f}}} (K^{-1}g) \mathrm{d}\alpha = (K^{-1}g)(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} (K^{-1}g)(\delta_{x_1} + \cdots + \delta_{x_k}) \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}).$$

In particular, $\mathsf{P}_\Lambda(\mathcal{N}_\Lambda) = \varphi(K^{-1}\mathbf{1})$. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{X}$, we have

$$(K^{-1}\mathbf{1})\big(\sum_{j=1}^n \delta_{x_j}\big) = \sum_{I \subset [n]} (-1)^{n-\#I} = (-1)^n (1 + (-1))^{\#I} = 0.$$

By definition of the K-transform, we also have $K^{-1}\mathbf{1}(0) = \mathbf{1}(0) = 1$. Hence $K^{-1}\mathbf{1} = \mathbb{1}_{\{N_\mathbb{X}=0\}}$ and

$$\mathsf{P}_\Lambda(\mathcal{N}_\Lambda) = \int_{\mathcal{N}_\mathsf{f}} (K^{-1}\mathbf{1})\mathrm{d}\alpha = 1.$$

It follows that P_Λ is a probability measure. The family $(\mathsf{P}_\Lambda)_{\Lambda \in \mathcal{X}_\mathsf{b}}$ is a *consistent family*: let $\Delta \subset \Lambda$ and $p : \mathcal{N}_\Lambda \to \mathcal{N}_\Delta$, $\eta \mapsto \eta_\Delta$, then we have, for measurable $A \subset \mathcal{N}_\Delta$ and $\eta \in \mathcal{N}$,

$$\mathbb{1}_{p^{-1}(A)}(\eta_\Lambda) = \mathbb{1}_A(p(\eta_\Lambda)) = \mathbb{1}_A\big((\eta_\Lambda)_\Delta\big) = \mathbb{1}_A(\eta_\Delta)$$

hence

$$\mathsf{P}_\Lambda(p^{-1}(A)) = \mathsf{P}_\Delta(A).$$

By a variant of Kolmogorov's extension theorem,[4] there exists a uniquely defined probability measure P on $\mathcal{N}$ such that $\mathsf{P} \circ \pi_\Lambda^{-1} = \mathsf{P}_\Lambda$ for all $\Lambda \in \mathcal{X}_\mathsf{b}$. The measure P has the system of Janossy densities $(j_{n,\Lambda})_{n \in \mathbb{N}, \Lambda \in \mathcal{X}_\mathsf{b}}$.

To conclude, it remains to check that the ρ_n's are indeed the correlation functions of P. This can be done with Proposition 2.32. Alternatively, we observe that

$$\mathsf{E}[\mathbb{1}_B] = \mathsf{P}(B) = \varphi(\mathbb{1}_B) = \int_{\mathcal{N}_\mathsf{f}} \big(K^{-1}\mathbb{1}_B\big)\mathrm{d}\alpha$$

for all events $B \in \mathfrak{N}$ that are local, i.e., $\mathbb{1}_B \in \mathcal{A}^{\mathrm{loc}}$. The identity extends to all local observables $g = Kf$ with $\int_{\mathcal{N}_\mathsf{f}} |f|\mathrm{d}\alpha < \infty$, which gives $\mathsf{E}[Kf] = \int_{\mathcal{N}_\mathsf{f}} f\mathrm{d}\alpha$. On the other hand $\mathsf{E}[Kf]$ is determined by the correlation functions by Lemma 2.46. Comparing the two expressions one deduces that the ρ_n's are indeed the correlation functions of P. □

Remark. Another way to organize the proof of the implication "$\Leftarrow$", especially if we take (2.42) as a starting point (the reader should check that the latter readily follows from (2.38) and (2.39)), is as follows. Let $\mathfrak{N}_\Lambda$ be the σ-algebra generated by the counting variables N_B with measurable $B \subset \Lambda$. It follows from Exercise 2.2 that $B \in \mathfrak{N}_\Lambda$ if and only if $\mathbb{1}_B \in \mathcal{A}_\Lambda$. Let

$$\mathcal{Z} = \{B \in \mathfrak{N} \mid \mathbb{1}_B \in \mathcal{A}^{\mathrm{loc}}\} = \bigcup_{\Lambda \in \mathcal{X}_\mathsf{b}} \mathfrak{N}_\Lambda$$

be the collection of *local events*. Note that $\mathcal{Z}$ is an *algebra*: it contains $\mathcal{N}$ and is closed with respect to complements and finite unions. For $B \in \mathcal{Z}$, set

$$\mathsf{P}(B) = \int_{\mathcal{N}_\mathsf{f}} (K^{-1}\mathbb{1}_B)\mathrm{d}\alpha. \tag{2.43}$$

$K^{-1}\mathbb{1}_B$ is well-defined because $\mathbb{1}_B$ is local, and $\int_{\mathcal{N}_\mathsf{f}} |K^{-1}\mathbb{1}_B|\mathrm{d}\alpha < \infty$ because of Ruelle's bound and because $K^{-1}\mathbb{1}_B$ is locally supported. We have $\mathsf{P}(B) \geq 0$ because of Eq. (2.42), $\mathsf{P}(\mathcal{N}) = 1$ because of $\mathbb{1}_\mathcal{N} = \mathbf{1}$ and $K^{-1}\mathbf{1} = \mathbb{1}_{\{N_\mathbb{X}=0\}}$, and P is clearly

[4] See [42, Proposition 1.3]. Note that the space $(\mathcal{N}, \mathfrak{N})$ is a *standard Borel space*: there exists a metric d on $\mathcal{N}$ such that (i) $(\mathcal{N}, d)$ is a complete, separable metric space, and (ii) $\mathfrak{N}$ is precisely the associated Borel-σ-algebra. See for example [8, Appendix A.2.6]. Extension theorems for inverse limits of standard Borel spaces are discussed in [39, Chapter 3].

finitely additive. If we knew that P is continuous in the sense that $\lim_{n\to\infty} \mathsf{P}(A_n) = 0$ for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{Z}$ with $\cap_{n\in\mathbb{N}} A_n = \varnothing$, then we would conclude that P is in fact countably additive and extends to a probability measure P on $\sigma(\mathscr{Z}) = \mathscr{N}$ by Carathéodory's extension theorem. Eq. (2.43) then shows that α ist the correlation measure of P, or equivalently, the ρ_n's are indeed the correlation functions of P.

Now, for each $\Lambda \in \mathcal{X}_{\mathsf{b}}$, the restriction of P to the σ-algebra $\mathfrak{N}_\Lambda$ is countably additive and continuous: indeed if $A_n \in \mathfrak{N}_\Lambda$ and $A_n \searrow \varnothing$, then $K^{-1}\mathbb{1}_{A_n} \searrow 0$ and $|K^{-1}\mathbb{1}_{A_n}| \leq \mathbb{1}_{\{N_{\mathbb{X}\setminus\Lambda}=0\}} 2^{N_\Lambda}$, which is integrable against α, and one concludes with dominated convergence.

So the missing link is to go from continuity (or σ-additivity) on $\mathfrak{N}_\Lambda$ to continuity on $\mathfrak{N}$. This is exactly the non-trivial step taken care of in the proof of Kolmogorov-type extension theorems [39, Chapter 3]. The step requires additional properties from the measure spaces, which are inherited from the completeness and separability of the metric space $(\mathbb{X}, \mathrm{dist})$.

2.10. **Local convergence.**

Definition 2.48 (Local convergence)**.** *A sequence $(\mathsf{P}_n)_{n\in\mathbb{N}}$ of probability measures on $(\mathscr{N}, \mathfrak{N})$* converges locally *to a probability measure P, written $\mathsf{P}_n \xrightarrow{\mathrm{loc}} \mathsf{P}$, if*

$$\forall f \in \mathcal{A}^{\mathrm{loc}}: \quad \lim_{n\to\infty} \int_{\mathscr{N}} f \mathrm{d}\mathsf{P}_n = \int_{\mathscr{N}} f \mathrm{d}\mathsf{P}. \tag{2.44}$$

A straightforward $\varepsilon/3$-argument shows that if $\mathsf{P}_n \xrightarrow{\mathrm{loc}} \mathsf{P}$, then the convergence in Eq. (2.44) extends to all quasi-local observables $f \in \mathcal{A}$.

Example 2.49 (Binomial and Poisson point process). Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{X}_{\mathsf{b}}$ such for every $B \in \mathcal{X}_{\mathsf{b}}$ and all sufficiently large n, we have $B \subset \Lambda_n$; for example, $\Lambda_n = B(0, n)$. Let Z_n be a binomial point process with sample size n and sampling distribution $\nu_n(B) = \lambda(B \cap \Lambda_n)/\lambda(\Lambda_n)$. Fix $\rho > 0$ and suppose that $n/\lambda(\Lambda_n) \to \rho$. Proceeding as in the proof of Proposition 1.2, one can show that Z_n converges locally to a Poisson point process with intensity measure $\rho\,\lambda$.

Proposition 2.50. *Let $(\mathsf{P}_n)_{n\in\mathbb{N}}$ be a sequence of probability measures that satisfy the Ruelle bound $(\mathscr{R}_\xi)$ for some n-independent $\xi > 0$. Suppose that $(\mathsf{P}_n)_{n\in\mathbb{N}}$ converges locally to some probability measure P as $n \to \infty$. Then P satisfies $(\mathscr{R}_\xi)$ as well.*

Proof. Let $\alpha_k^{(n)}$ and α_k be the factorial moment measures of P_n and P, respectively. For $B \in \mathcal{X}^{\otimes k}$, let $f : \mathscr{N} \to [0, \infty) \cup \{\infty\}$ be the observable defined by the requirement

$$f\Big(\sum_{j=1}^{\ell} \delta_{x_j}\Big) = \sum_{(i_1,\dots,i_k)}^{\neq} \mathbb{1}_B(x_{i_1}, \dots, x_{i_k})$$

for all $\ell \in \mathbb{N}_0 \cup \{\infty\}$ and $x_1, x_2, \ldots \in \mathbb{X}$, so that $\mathsf{E}[f] = \alpha_k(B)$ and $\mathsf{E}_n[f] = \alpha_k^{(n)}(B)$. Suppose that $B \subset \Lambda^k$ for some $\Lambda \in \mathcal{X}_{\mathsf{b}}$. Then f is local but in general unbounded, so the convergence (2.44) applies, a priori, to truncated versions of f only. For $m \in \mathbb{N}$, we have

$$\mathsf{E}\big[f\mathbb{1}_{\{N_\Lambda \leq m\}}\big] = \lim_{n\to\infty} \mathsf{E}_n\big[f\mathbb{1}_{\{N_\Lambda \leq m\}}\big] \leq \limsup_{n\to\infty} \alpha_k^{(n)}(B) \leq \xi^k \lambda^k(B).$$

By monotone convergence, $\mathsf{E}\big[f\mathbb{1}_{\{N_\Lambda\le m\}}\big]$ converges to $\mathsf{E}[f]=\alpha_k(B)$ as $m\to\infty$. Thus $\alpha_k(B)\le \xi^k\lambda^k(B)$. The lemma follows. □

The next theorem states that under Ruelle's condition $(\mathscr{R}_\xi)$, local convergence of probability measures is equivalent to a suitably defined weak convergence of the correlation functions.

Theorem 2.51. *Let* P *and* P_n, $n\in\mathbb{N}$, *be probability measures on* $(\mathcal{N},\mathfrak{N})$ *that satisfy condition* $(\mathscr{R}_\xi)$. *Let* $\rho_k^{(n)}$ *and* ρ_k *be their respective correlation functions. Then* $(\mathsf{P}_n)_{n\in\mathbb{N}}$ *converges locally to* P *if and only if, for all* $k\in\mathbb{N}$ *and all* $f\in L^1(\mathbb{X}^k,\lambda^k)$, *we have*

$$\lim_{n\to\infty}\int_{\mathbb{X}^k} f\rho_k^{(n)}\mathrm{d}\lambda^k=\int_{\mathbb{X}^k} f\rho_k\mathrm{d}\lambda^k. \tag{2.45}$$

Remark. It is not needed to explicitly ask that P satisfies Ruelle's bound. Indeed if $(\mathsf{P}_n)_{n\in\mathbb{N}}$ converges locally to P, then P inherits Ruelle's bound by Proposition 2.50. Conversely, if we merely know that the correlation functions ρ_k of P exist and (2.45) holds true for all $f\in L^1(\mathbb{X}^k,\lambda^k)$, then $\rho_k^{(n)}\le\xi^k$ implies

$$\int_{\mathbb{X}^k} f\rho_k\mathrm{d}\lambda^k=\lim_{n\to\infty}\int_{\mathbb{X}^k} f\rho_k^{(n)}\mathrm{d}\lambda^k\le\int_{\mathbb{X}^k} f\xi^k\mathrm{d}\lambda^k$$

for all non-negative $f\in L^1(\mathbb{X}^k,\lambda^k)$, hence $\rho_k\le\xi^k$ λ^k-a.e.

Proof of Theorem 2.51. "⇒" Suppose that (P_n) converges to P locally. Fix $k\in\mathbb{N}$ and $f\in L^1(\mathbb{X}^k,\lambda^k)$. Let $(\Lambda_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{X}_\mathsf{b}$ such that $\Lambda_m\nearrow\mathbb{R}^d$. Set $f_m:=f\mathbb{1}_{\{|f|\le m\}}\mathbb{1}_{\Lambda_m}$ and let F_m be the associated local observable from Lemma 2.26,

$$F_m\Big(\sum_{j=1}^{\ell}\delta_{x_j}\Big)=\sum_{(i_1,\dots,i_k)}^{\neq} f_m(x_{i_1},\dots,x_{i_k}).$$

Similarly, let F be the observable associated with f; by Proposition 2.35, it is defined up to P_n-null sets and P-null sets. F_m is local and satisfies

$$|F_m|\le mN_\Lambda(N_\Lambda-1)\cdots(N_\Lambda-k+1)\le mN_\Lambda^k.$$

We claim that $\lim_{n\to\infty}\mathsf{E}_n[F_m]=\mathsf{E}[F_m]$ even though F_m is not bounded. Indeed for every $p\in\mathbb{N}$, we have

$$\begin{aligned}|\mathsf{E}_n[F_m]-\mathsf{E}[F_m]|\le{}&\big|\mathsf{E}_n[F_m\mathbb{1}_{\{N_\Lambda\le p\}}]-\mathsf{E}_n[F_m\mathbb{1}_{\{N_\Lambda\le p\}}]\big|\\&+m\mathsf{E}_n\big[N_\Lambda^k\mathbb{1}_{\{N_\Lambda>p\}}\big]+m\mathsf{E}\big[N_\Lambda^k\mathbb{1}_{\{N_\Lambda>p\}}\big].\end{aligned} \tag{2.46}$$

By the Cauchy-Schwarz inequality,

$$\mathsf{E}\big[N_\Lambda^k\mathbb{1}_{\{N_\Lambda>p\}}\big]\le\big(\mathsf{E}_n\big[N_\Lambda^{2k}\big]\big)^{1/2}\big(\mathsf{P}_n(N_\Lambda>p)\big)^{1/2}.$$

N_Λ^{2k} can be written as a linear combination of factorial moments $N_\Lambda(N_\Lambda-1)\cdots(N_\Lambda-r+1)$, Ruelle's bound therefore implies that $\mathsf{E}_n[N_\Lambda^k]\le q(\xi)$ for some n-independent polynomial $q(\xi)$. By Lemma 2.41, we have $(\mathsf{P}_n(N_\Lambda>p)\le\delta_p(\xi)$ for some δ_p that depends on ξ and p only and satisfies $\lim_{p\to\infty}\delta_p(\xi)=0$. Similar bounds apply if E is replaced with E_n. Consequently given $\varepsilon>0$, we can find $p\in\mathbb{N}$ such that

$$m\mathsf{E}_n\big[N_\Lambda^k\mathbb{1}_{\{N_\Lambda>p\}}\big]\le\varepsilon/3,\quad m\mathsf{E}_n\big[N_\Lambda^k\mathbb{1}_{\{N_\Lambda>p\}}\big]\le\varepsilon/3.$$

In (2.46) $F_m \mathbb{1}_{\{N_\Lambda \le p\}}$ is local and bounded so given ε and p, we can find n_0 such that for all $n \ge n_0$,

$$\big|\mathsf{E}_n[F_m \mathbb{1}_{\{N_\Lambda \le p\}}] - \mathsf{E}_n[F_m \mathbb{1}_{\{N_\Lambda \le p\}}]\big| \le \varepsilon/3.$$

Altogether for all $n \ge n_0$, we have $|\mathsf{E}_n[F_m] - \mathsf{E}[F_m]| \le \varepsilon$. This proves $\mathsf{E}_n[F_m] \to \mathsf{E}[F_m]$ as $n \to \infty$ at fixed m.

Next we observe

$$\mathsf{E}_n\big[|F - F_m|\big] = \int_{\mathbb{X}^k} |f - f_m| \rho_k^{(n)} \mathrm{d}\lambda^k \le \int_{\mathbb{X}^k} |f - f_m| \xi^k \mathrm{d}\lambda^k =: \varepsilon_m$$

where ε_m is independent of n and goes to zero as $m \to \infty$. A similar bound applies to $\mathsf{E}[|F - F_m|]$, and an $\varepsilon/3$-argument then shows $\mathsf{E}_n[F] \to \mathsf{E}[F]$ which proves (2.45).

"$\Leftarrow$" Let $G \in \mathcal{A}_\Lambda$ for some $\Lambda \in \mathcal{X}_\mathsf{b}$. Define $F = K^{-1}G$. Then $|F| \le 2^{N_\Lambda} \sup |G|$ and F vanishes on $\mathcal{N}_\mathsf{f} \setminus \mathcal{N}_\Lambda$. It follows that

$$|F(0)| + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} |F(\delta_{x_1} + \cdots + \delta_{x_k}| \rho_k(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x})$$
$$\le \sum_{k=0}^\infty \frac{1}{k!} (2\xi\lambda(\Lambda)))^k \sup |G| \le \mathrm{e}^{2\xi\lambda(\Lambda)} < \infty.$$

Lemma 2.46 yields

$$\mathsf{E}_n[G] = \mathsf{E}_n[KF] = F(0) + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} F\Big(\sum_{j=1}^k \delta_{x_j}\Big) \rho_k^{(n)}(\boldsymbol{x}) \mathrm{d}\lambda^k(\boldsymbol{x}). \tag{2.47}$$

A similar formula holds for $\mathsf{E}[G]$. By (2.45), each integral in the sum in (2.47) converges to $\int_{\mathbb{X}^k} f_k \rho_k \mathrm{d}\lambda^k$ with $f_k(\boldsymbol{x}) = F(\delta_{x_1} + \cdots + \delta_{x_k})$. An $\varepsilon/3$-argument based on

$$\Big|\mathsf{E}_n[G] - \mathsf{E}[G]\Big| \le \Big|\sum_{k=1}^p \frac{1}{k!}\Big| \int_{\mathbb{X}^k} f_k(\rho_k^{(n)} - \rho_k) \mathrm{d}\lambda^k(\boldsymbol{x})\Big| + 2 \sup |G| \sum_{k=p+1}^\infty \frac{1}{k!} (2\xi\lambda(\Lambda))^k$$

shows $\mathsf{E}_n[G] \to \mathsf{E}[G]$. □

The theorem has the following corollary (use dominated convergence):

Corollary 2.52. *Let P_n, P, $\rho_k^{(n)}$, ρ_k be as in Theorem 2.51. Suppose that*

$$\lim_{n\to\infty} \rho_k^{(n)}(x_1, \ldots, x_k) = \rho_k(x_1, \ldots, x_k) \tag{2.48}$$

for all $k \in \mathbb{N}$ and λ^k-almost all $(x_1, \ldots, x_k) \in \mathbb{X}^k$. Then $\mathsf{P}_n \xrightarrow{\text{loc}} \mathsf{P}$.

Finally we show that the set of probability measures that satisfy $(\mathscr{R}_\xi)$ for a given $\xi > 0$ is sequentially compact.

Theorem 2.53. *Let P_n, $n \in \mathbb{N}$ be probability measures on $(\mathcal{N}, \mathfrak{N})$ that satisfy condition $(\mathscr{R}_\xi)$ for some n-independent $\xi > 0$. Then $(\mathsf{P}_n)_{n\in\mathbb{N}}$ has a locally convergent subsequence.*

The proof is based on diagonal sequences and the Banach-Alaoglu theorem from functional analysis, which says that the unit ball in the dual of a Banach space is compact in the weak*-topology. Applied to the Banach space $L^1(\mathbb{X}^k, \mathcal{X}^{\otimes k}, \lambda^k)$ with its dual $L^\infty(\mathbb{X}^k, \mathcal{X}^{\otimes k}, \lambda^k)$, this shows that every bounded sequence (φ_n) in

$L^\infty(\mathbb{X}^k, \mathcal{X}^{\otimes k}, \lambda^k)$ has a subsequence $(\varphi_{n_j})_{j\in\mathbb{N}}$ that converges in the weak* sense, i.e.,

$$\int_{\mathbb{X}^k} g\varphi_{n_j} \mathrm{d}\lambda^k \to \int_{\mathbb{X}^k} g\varphi \mathrm{d}\lambda^k$$

for some bounded measurable $\varphi : \mathbb{X}^k \to \mathbb{R}$ and all $g \in L^1(\mathbb{X}^k, \mathcal{X}^{\otimes k}, \lambda^k)$.

Proof. Let $\rho_k^{(n)}$ be the correlation functions of P. Since $\rho_k^{(n)} \leq \xi^k$, the Banach-Alaoglu theorem in the form recalled above applies. For $k = 1$, it yields the existence of a subsequence $(\rho_1^{(n_j)})_{j\in\mathbb{N}}$ and function $\rho_1 : \mathbb{X} \to \mathbb{R}_+$ with $||\rho_1||_\infty \leq \xi$ such that

$$\lim_{j\to\infty} \int_{\mathbb{X}} f\rho_1^{(n_j)} \mathrm{d}\lambda = \int_{\mathbb{X}} f\rho_1 \mathrm{d}\lambda$$

for every $f \in L^1(\mathbb{X}, \lambda)$. Taking successive subsequences, we see that there are limit functions $\rho_k : \mathbb{X}^k \to \mathbb{R}_+$ with $||\rho_k||_\infty \leq \xi^k$ and injective maps $\psi_k : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{j\to\infty} \int_{\mathbb{X}^k} f\rho_k^{(\psi_k\circ\cdots\circ\psi_1(j))} \mathrm{d}\lambda^k = \int_{\mathbb{X}^k} f\rho_k \mathrm{d}\lambda^k$$

for all $f \in L^1(\mathbb{X}^k, \lambda^k)$. Set $m_j := \psi_j \circ \cdots \circ \psi_1(j)$, then we have for all $k \in \mathbb{N}$ and all $f \in L^1(\mathbb{X}^k, \lambda^k)$,

$$\lim_{j\to\infty} \int_{\mathbb{X}^k} f\rho_k^{(m_j)} \mathrm{d}\lambda^k = \int_{\mathbb{X}^k} f\rho_k \mathrm{d}\lambda^k. \tag{2.49}$$

In view of Theorem 2.51, it remains to check that there is a probability measure P such that the functions ρ_k are the correlation functions of P. Let α_n and α be the measures on $\mathcal{N}_{\mathsf{f}}$ associated with the families $(\rho_k^{(n)})_{k\in\mathbb{N}}$ and $(\rho_k)_{k\in\mathbb{N}}$ via (2.40). As noted in (2.42), we have $\int_{\mathcal{N}_{\mathsf{f}}} (K^{-1}g)\mathrm{d}\alpha_n \geq 0$ for all $n \in \mathbb{N}$ and all non-negative $g \in \mathcal{A}^{\mathrm{loc}}$. It follows that

$$\int_{\mathcal{N}_{\mathsf{f}}} K^{-1}g\mathrm{d}\alpha = \lim_{n\to\infty} \int_{\mathcal{N}_{\mathsf{f}}} K^{-1}g\mathrm{d}\alpha_n \geq 0.$$

Proceeding as in the proof of Theorem 2.43 or Theorem 2.47, one checks that for all $g \in \mathcal{A}_\Lambda$, we have

$$\int_{\mathcal{N}_{\mathsf{f}}} K^{-1}g\mathrm{d}\alpha = g(0)j_{0,\Lambda} + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} g(\delta_{x_1} + \cdots + \delta_{x_k}) j_{k,\Lambda}(\boldsymbol{x})\mathrm{d}\lambda^k(\boldsymbol{x}) \tag{2.50}$$

where $j_{k,\Lambda}(x_1, \ldots, x_k)$ and $j_{0,\Lambda}$ given by (2.38) and (2.39). For (2.50) to be non-negative for all non-negative $g \in \mathcal{A}_\Lambda$, it is necessary that $j_{k,\Lambda} \geq 0$ for all $k \in \mathbb{N}_0$, λ^k-a.e. Consequently the ρ_k's satify the criteria of Theorem 2.47 and are the correlation functions of some probability measure P. □

2.11. Summary.

- Configurations are modelled by locally finite counting measures η on $(\mathbb{X}, \mathcal{X})$. The space $\mathcal{N}$ of configurations is equipped with the smallest σ-algebra $\mathfrak{N}$ generated by sets of the form $\{N_B = n\}$ with $B \in \mathcal{X}$.
- Probability measures on $(\mathcal{N}, \mathfrak{N})$ are uniquely determined by the joint distributions of the counting variables N_B. The system of Janossy densities, if it exists, plays a role analogous to the probability density of a real-valued random variable.

- Among the observables (measurable maps), we have singled out local observables and quasi-local observables. There is a one-to-one correspondence between bounded local observables $f \in \mathcal{A}_\Lambda$ and families $(f_n)_{n\in\mathbb{N}_0}$ of measurable symmetric functions $f_n : \Lambda^n \to \mathbb{R}$ with $\sup_n ||f_n||_\infty < \infty$.
- The intensity measure and factorial moment measure (or the one-particle density and the correlation functions) are analogous to the expected value and factorial moments of a random variable. If a probability measure P satisfies Ruelle's moment bound $(\mathscr{R}_\xi)$, then it is uniquely determined by its correlation functions.
- The Poisson point processes form an important class of probability measures, for which the correlation functions and Laplace functional can be computed explicitly.
- We work with the notion of local convergence of probability measures. If a sequence $(\mathsf{P}_n)_{n\in\mathbb{N}}$ satisfies Ruelle's bound $(\mathscr{R}_\xi)$ for some n-independent $\xi > 0$, then:
  - It has a locally convergent subsequence, and every accumulation point satisfies $(\mathscr{R}_\xi)$ too.
  - Local convergence of $(\mathsf{P}_n)_{n\in\mathbb{N}}$ is equivalent to a suitably defined weak convergence of the correlation functions.

2.12. **Exercises.**

Exercise 2.1. Let $\mathbb{X} = \mathbb{R}$. Consider the set $\mathcal{F}$ of sets that are either countable or have countable complement. Define $\eta(A) = 0$ if A is countable, and 1 if A is not countable. Show that (a) $\mathcal{F}$ is a σ-algebra, (b) η is a measure, (c) η cannot be written as $\sum_{x\in S} n_x\delta_x$ with S countable and $n_x \in \mathbb{N}$.

Exercise 2.2. Fix a non-empty set $\Lambda \in \mathcal{X}_\mathsf{b}$ and let $\mathfrak{N}_\Lambda := \sigma(N_B : B \in \mathcal{X} \text{ and } B \subset \Lambda)$. Let $F : \mathcal{N} \to \mathbb{R}$ be $\mathfrak{N}$-measurable. Consider the following conditions:

(i) $F(\eta) = F(\eta_\Lambda)$ for all $\eta \in \mathcal{N}$, where $\eta_\Lambda(B) := \eta(B \cap \Lambda)$.
(ii) F is $\mathfrak{N}_\Lambda$-measurable.

The purpose of this exercise is to check that (i) and (ii) are equivalent.

(a) Let $\varphi_\Lambda : \mathcal{N} \to \mathcal{N}$, $\eta \mapsto \eta_\Lambda$. Show that for every $A \in \mathfrak{N}$, $\varphi_\Lambda^{-1}(A) \in \mathfrak{N}_\Lambda$.
(b) Show that $\mathfrak{N}_\Lambda$ is the smallest σ-algebra among the σ-algebras $\mathcal{F}$ such that φ_Λ, as a map from $(\mathcal{N}, \mathcal{F})$ to $(\mathcal{N}, \mathfrak{N})$, is measurable.
(c) Prove (i) $\Leftrightarrow$ (ii).

Hint: you can use the factorization theorem [33, Theorem A.3].

Exercise 2.3. Let $f : \mathbb{X} \to [0, \infty)$ be measurable and $F(\eta) := \sum_{x\in S_\eta} f(x)n_x(\eta)$.

(a) Provide a sufficient condition on f so that F is local. (You don't need to prove the measurability of F.)
(b) For $\mathbb{X} = \mathbb{R}^d$, Euclidean distance, $\Lambda_n := B(0, n)$, prove or disprove the following claim: if $\lim_{|x|\to\infty} f(x) = 0$, then F_n defined by $F_n(\eta) := F(\eta_{\Lambda_n})$ converges uniformly to F.

Exercise 2.4.

(a) Let $A \in \mathfrak{N}$. Show that if $\mathbb{1}_A$ is quasi-local, then it is local.
(b) Assume that $\mathbb{X}$ is unbounded. Show that $\mathbb{1}_{\{N_\mathbb{X}\geq 1\}} \in \mathscr{L}^\infty(\mathcal{N}, \mathfrak{N}) \setminus \mathcal{A}$, where $\mathcal{A}$ is the set of bounded quasi-local observables.

Exercise 2.5. Let $Z_j : \Omega \to \mathcal{N}$, $j \in \mathbb{N}$, be a family of point processes defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\sum_{j=1}^\infty N_B(Z_j(\omega)) < \infty$ for all $B \in \mathcal{X}_\mathsf{b}$ and $\omega \in \Omega$. Show that $Z := \sum_{j=1}^\infty Z_j$ is a a point process.

Exercise 2.6. Check the expression for the Janossy densities of the binomial point process from Example 2.21.

Exercise 2.7. Let P be a probability measure on $\mathcal{N}$ with correlation functions $\rho_n(x_1, \ldots, x_n)$.

(a) Show that if $\rho_2(x, y) = \rho_1(x)\rho_1(y)$ for all $x, y \in \mathbb{X}$ with $x \neq y$, then for all disjoint $A, B \in \mathcal{X}_\mathsf{b}$, the variables N_A and N_B are uncorrelated.

(b) Suppose that P is the distribution of a Poisson point process. Prove or disprove: for all $n \in \mathbb{N}$ and λ^n-almost all $(x_1, \ldots, x_n) \in \mathbb{X}^n$, we have $\rho_n(x_1, \ldots, x_n) = \rho_1(x_1) \cdots \rho_n(x_n)$.

(c) Let $A, B \in \mathcal{X}_\mathsf{b}$ be two disjoint sets. Suppose that the random variables η_A and η_B are independent. Show that for all $m, n \in \mathbb{N}$ and λ^{m+n}-almost all $(x_1, \ldots, x_{m+n}) \in A^m \times B^n$,

$$\rho_{m+n}(x_1, \ldots, x_{m+n}) = \rho_m(x_1, \ldots, x_m)\rho_n(x_{m+1}, \ldots, x_{m+n}).$$

Exercise 2.8. Let P be a probability distribution on $\mathcal{N}$. Suppose that the reference measure λ is atom-free, i.e., $\lambda(\{x\}) = 0$ for all $x \in \mathbb{X}$, and that the two-point correlation function $\rho_2(x, y)$ exists. Show that $\mathsf{P}(\exists x \in \mathbb{X} : \, n_x(\eta) \geq 2) = 0$.

Exercise 2.9. Let P be a probability distribution on $\mathcal{N}$ and $B \in \mathcal{X}_\mathsf{b}$. Suppose that P satisfies Ruelle's bound $(\mathscr{R}_\xi)$. Show that the avoidance probability $\mathsf{P}(N_B = 0)$ can be expressed as a series involving the correlation functions ρ_n.

Exercise 2.10. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers.

(a) Suppose that $\sum_{J \subset \mathbb{N}, \# J < \infty} \prod_{j \in J} |a_j| < \infty$. Show that

$$\lim_{n \to \infty} \prod_{j=1}^n (1 + a_j) = 1 + \sum_{\substack{J \subset \mathbb{N}: \\ \# J < \infty}} \prod_{j \in J} a_j.$$

(b) Prove or disprove: in (a) it is actually enough to assume $\sum_{j=1}^\infty |a_j| < \infty$.

Exercise 2.11. Let P be a probability measure on $\mathcal{N}$ and Q the distribution of the Poisson point process on $\mathbb{X}$ with intensity measure λ. Let P_Λ, Q_Λ be the images of P, Q under the projection $\mathcal{N} \to \mathcal{N}_\Lambda$, $\eta \mapsto \eta_\Lambda$.

(a) P admits a system of Janossy densities $(j_{n,\Lambda})$ if and only if each P_Λ, $\Lambda \in \mathcal{X}_\mathsf{b}$, is absolutely continuous with respect to Q_Λ.

(b) Suppose that P admits a system of Janossy densities. Let $(\rho_k)_{k \in \mathbb{N}}$ be the correlation functions of P. Find functions $\varphi_\Lambda : \mathcal{N}_\Lambda \to \mathbb{R}_+$ such that

$$\rho_k(x_1, \ldots, x_k) = \int_{\mathcal{N}_\Lambda} \varphi_\Lambda(\delta_{x_1} + \cdots + \delta_{x_k} + \eta) \mathrm{d}\mathsf{Q}_\Lambda(\eta),$$

for all $\Lambda \in \mathcal{X}_\mathsf{b}$, $k \in \mathbb{N}$, and $x_1, \ldots, x_k \in \Lambda$.

3. Gibbs measures in finite volume

The grand-canonical Gibbs measure in a finite volume $\Lambda \in \mathcal{X}_{\mathsf{b}}$, $\Lambda \neq \varnothing$ (with empty boundary conditions), is a probability measure on $\mathcal{N}_\Lambda = \{N_{\mathbb{X}\setminus\Lambda} = 0\}$ that depends on two parameters $\beta, z > 0$ and an energy function H. The quantities enter in the combination $z^{N_\Lambda}\exp(-\beta H)$.

In statistical mechanics, β, z, H are physical quantities, called inverse temperature, activity, and (potential) energy, and the grand-canonical Gibbs measure models a system that can exchange particles and energy with its environment.

In stochastic geometry and spatial statistics, the Gibbs measure is one of several models used to model random configurations. The function H is a way to encode dependencies and deviations from the Poisson distribution—large values of H correspond to unlikely configurations, $H = \infty$ to forbidden configurations. The parameter β controls how strongly the preferences encoded in H affect the probability measure (not at all for $\beta = 0$, very much for large β). The parameter z controls the number of particles per unit volume.

3.1. Energy functions and interaction potentials.

Remember that by Proposition 2.11 there is a one-to-one correspondence between measurable maps $H : \mathcal{N}_{\mathsf{f}} \to \mathbb{R} \cup \{\infty\}$ and families $(H_n)_{n\in\mathbb{N}_0}$ of measurable symmetric functions $H_n : \mathbb{X}^n \to \mathbb{R} \cup \{\infty\}$. We formulate definitions in terms of $H(\eta)$ but the reader who prefers to do so might reformulate them in terms of $H_n(x_1, \ldots, x_n)$.

Definition 3.1. *An* energy function H *is a measurable map* $H : \mathcal{N}_{\mathsf{f}} \to \mathbb{R} \cup \{\infty\}$ *with the following properties:*

(i) $H(0) = 0$.
(ii) *If* $H(\eta) < \infty$ *and* $x \in S_\eta$, *then* $H(\eta - \delta_x) < \infty$.
(iii) *For some* $B \geq 0$ *and all* $\eta \in \mathcal{N}_{\mathsf{f}}$, *we have*

$$H(\eta) \geq -B N_{\mathbb{X}}(\eta). \tag{3.1}$$

Condition (i) says that empty configuration have zero energy. It is natural for statistical mechanics but could be relaxed to $H(0) < \infty$, a *non-degeneracy* condition. Condition (ii) says that if a configuration has finite energy, then any configuration obtained by removing a particle has finite energy as well. This property is sometimes called *hereditarity*. Condition (iii) is called *stability*.

Example 3.2 (Pair potentials). Take $\mathbb{X} = \mathbb{R}^d$. Let $v : [0, \infty) \to \mathbb{R} \cup \{\infty\}$. Suppose that either $v(r) \geq 0$ for all $r \geq 0$, or

- v has a hard core: for some $r_{\mathrm{hc}} > 0$ and all $r < r_{\mathrm{hc}}$, $v(r) = \infty$.
- v is lower regular: there exists a monotone-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^\infty r^{d-1}\psi(r)\mathrm{d}r < \infty$ such that $v(r) \geq -\psi(r)$ for all $r \geq 0$.

Define $H_0 = 0$, $H_1(x_1) \equiv 0$, and $H_n : \mathbb{X}^n \to \mathbb{R} \cup \{\infty\}$ by

$$H_n(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} v(|x_i - x_j|).$$

Then the associated map $H : \mathcal{N}_{\mathsf{f}} \to \mathbb{R} \cup \{\infty\}$ is an energy function. Conditions (i) and (ii) are trivial. For the stability, let $n \geq 2$ and $x_1, \ldots, x_n \in \mathbb{X}$. If $H_n(x_1, \ldots, x_n) < \infty$, then $|x_i - x_j| \geq r_{\mathrm{hc}}$ for all $i \neq j$ and we have, for every fixed

i and some constant $C > 0$,

$$\sum_{\substack{j\in[n]:\\ j\neq i}} v(|x_i - x_j|) \geq -\sum_{n=0}^{\infty} \varphi(n)\, \#\{j \mid n \leq |x_j - x_i| \leq n+1\}$$

$$\geq -C \sum_{n=0}^{\infty} \varphi(n) \frac{(n+1)^{d-1}}{r_{\mathrm{hc}}^d} =: -2B.$$

Our assumptions on φ imply that $B < \infty$. It follows that

$$H_n(x_1,\ldots,x_n) = \frac{1}{2}\sum_{i=1}^n \sum_{\substack{j\in[n]:\\ j\neq i}} v(|x_i - x_j|) \geq -Bn.$$

For $H_n(x_1,\ldots,x_n) = \infty$ or $n = 1$, the inequality holds true as well, and we have checked that H is stable.

Example 3.3 (Widom-Rowlinson model). Define $H_0 = 0$ and for $n \in \mathbb{N}$, $x_1,\ldots,x_n \in \mathbb{X}$,

$$H_n(x_1,\ldots,x_n) = |\bigcup_{j=1}^n B(x_j,1)| - n|B(0,1)|. \tag{3.2}$$

Then the associated map $H : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}\cup\{\infty\}$ is an energy function. By the inclusion-exclusion principle, we have

$$H_n(x_1,\ldots,x_n) = \sum_{\substack{I\subset[n]\\ \#I\geq 2}} (-1)^{\#I-1} \big|\bigcap_{j=1}^n B(x_j,1)\big|. \tag{3.3}$$

Thus the energy of the Widom-Rowlinson model is not a sum of pair potentials, but instead of multi-body potentials. In fact any energy function can be written in this way.

Proposition 3.4. *Let $H : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}\cup\{\infty\}$ be an energy function. Then there exists a measurable function $V : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}\cup\{\infty\}$ with $V(0) = 0$ such that for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in \mathbb{X}$, we have*

$$H\Big(\sum_{j=1}^n \delta_{x_j}\Big) = \sum_{I\subset[n]} V\Big(\sum_{i\in I}\delta_{x_i}\Big). \tag{3.4}$$

If V and W satisfy (3.4) *with $V(0) = W(0) = 0$, then then $V(\eta) = W(\eta)$ for every finite-energy configuration η.*

We call V a *multi-body potential* or *interaction potential* compatible with H. If H is everywhere finite, the interaction potential is uniquely determined, otherwise there are some ambiguities due to the convention that $\infty + x = \infty$ for all $x \in \mathbb{R}$.

Proof. The proposition is a variant of Lemma 2.44. Set $V(0) = 0$. For $\eta = \delta_{x_1} + \cdots + \delta_{x_n}$ with $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in \mathbb{X}$, we distinguish two cases. If $H\big(\sum_{i\in I}\delta_{x_i}\big) = \infty$ for some $I \subsetneq [n]$, we set $V(\eta) = 0$. If $H\big(\sum_{i\in I}\delta_{x_i}\big) < \infty$ for all $I \subsetneq [n]$, we set

$$V\Big(\sum_{j=1}^n \delta_{x_j}\Big) := \sum_{I\subset[n]} (-1)^{n-\#I} H\Big(\sum_{i\in I}\delta_{x_i}\Big). \tag{3.5}$$

If $H(\delta_{x_1} + \cdots + \delta_{x_n}) < \infty$, then (3.4) is proven as in Lemma 2.44. If $H(\delta_{x_1} + \cdots + \delta_{x_n}) = \infty$, there is at least one subset $I \subset [n]$ such that $H(\sum_{i \in I} \delta_{x_i}) = \infty$ and $H(\sum_{i \in J} \delta_{x_i}) < \infty$ for all $J \subsetneq I$. Then $V(\sum_{i \in I} \delta_{x_i}) = \infty$ and (3.4) holds true as well.

Next let $\eta = \sum_{j=1}^{n} \delta_{x_j} \in \mathcal{N}_{\mathsf{f}}$ be a finite-energy configuration and V, W two multi-body potentials compatible with H. Proceeding as in Lemma 2.44, one checks that both $V(\eta)$ and $W(\eta)$ are equal to the right-hand side of (3.5), which shows $V(\eta) = W(\eta)$. $\square$

A particularly simple class of energy functions and interaction potentials is the following.

Definition 3.5.

(a) *An interaction potential V has* finite range *if for some $R \geq 0$, we have $V(\eta) = 0$ whenever $\operatorname{diam}(S_\eta) > R$.*
(b) *An energy function has finite range if it has a compatible interaction potential V with finite range.*

For example, the energy of the Widom-Rowlinson model has finite range. For rotationally invariant pair potentials $v(r)$ as in Example 3.2, if v has bounded support $\operatorname{supp} v \subset [0, R]$, then the interaction has finite range.

From now on we assume that we are given an interaction potential V and that H is the energy function associated with it (note that the stability condition imposes conditions on V).

3.2. **Boundary conditions.** Next we introduce a notion of energy in a bounded volume with boundary conditions, which is needed to deal with Gibbs measures in infinite volume. Fix $\Delta \in \mathcal{X}_{\mathsf{b}}$. We want to define a function

$$H_\Delta : \mathcal{N}_\Delta \times \mathcal{N} \to \mathbb{R} \cup \{\infty\}, \quad (\eta, \gamma) \mapsto H_\Delta(\eta \mid \gamma),$$

with $H_\Delta(\eta \mid \gamma)$ the energy of η given the boundary condition γ. If γ is a finite configuration with $H(\gamma_{\mathbb{X}\setminus\Delta}) < \infty$, we define

$$H_\Delta(\eta \mid \gamma) := H(\eta + \gamma_{\mathbb{X}\setminus\Delta}) - H(\gamma_{\mathbb{X}\setminus\Delta}). \tag{3.6}$$

Clearly

$$H_\Delta(0 \mid \gamma) = 0, \quad H_\Delta(\eta \mid 0) = H(\eta), \quad H_\Lambda(\eta \mid \gamma) = H_\Lambda(\eta \mid \gamma_{\mathbb{X}\setminus\Delta}).$$

Furthermore,

$$H_\Delta\Big(\sum_{i=1}^{n} \delta_{x_i} \mid \sum_{i=n+1}^{n+k} \delta_{y_j}\Big) = \sum_{\substack{I \subset [n+k] \\ I \cap [n] \neq \emptyset}} V\Big(\sum_{i \in I \cap [n]} \delta_{x_i} + \sum_{i \in I \setminus [n]} \delta_{y_i}\Big) \tag{3.7}$$

for all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $x_1, \ldots, x_n \in \Delta$, and $y_{n+1}, \ldots, y_{n+k} \in \mathbb{X} \setminus \Delta$ with $H(\delta_{y_{n+1}} + \cdots + \delta_{y_n}) < \infty$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. Thus $H_\Delta(\eta \mid \gamma)$ collects all interactions of points in η and $\gamma_{\mathbb{X}\setminus\Delta}$ that involve at least one particle from η and discards interactions between particles of $\gamma_{\mathbb{X}\setminus\Delta}$.

Eq. (3.6) allows us to extend the definition of the conditional energy as follows. Let $\eta \in \mathcal{N}_\Delta$, $\gamma \in \mathcal{N}$. Write

$$\eta = \sum_{j=1}^n \delta_{x_j}, \quad \gamma_{\mathbb{X}\setminus\Delta} = \sum_{j=n+1}^{n+k} \delta_{y_j}$$

with $n \in \mathbb{N}_0$, $k \in \mathbb{N}_0 \cup \{\infty\}$ and the convention $\eta = 0$ if $n = 0$ and $\gamma = 0$ if $k = 0$. Write $\sum'$ for the sum over subsets I such that $I \subset [n+k]$ if $k \in \mathbb{N}$, and $I \subset \mathbb{N}$, $\#I < \infty$ if $k = \infty$. We set

$$H_\Delta(\eta \mid \gamma) = \sum_{I\cap[n]\neq\emptyset}{}' V\Big(\sum_{i\in I\cap[n]} \delta_{x_i} + \sum_{i \in I\setminus[n]} \delta_{y_i}\Big) \tag{3.8}$$

if the sum is absolutely convergent, and ∞ otherwise. The extended function $H_\Delta : \mathcal{N}_\Delta \times \mathcal{N} \to \mathbb{R} \cup \{\infty\}$ satisfies (3.2) as well.

Suppose that V has finite range $R > 0$. Fix $\Delta \in \mathcal{X}_\mathsf{b}$ and let

$$\Delta_R = \{x \in \mathbb{X} \mid \mathrm{dist}(x, \Delta) \leq R\}.$$

Then for all $\eta \in \mathcal{N}_\Delta$ and $\gamma \in N$, we have

$$H_\Delta(\eta \mid \gamma) = H_\Delta(\eta \mid \gamma_{\Delta_R}).$$

3.3. **Grand-canonical Gibbs measure.** Remember that we are given a locally finite reference measure λ on $\mathbb{X}$ (e.g., Lebesgue measure on $\mathbb{X} = \mathbb{R}^d$).

Definition 3.6. *Fix $\beta, z > 0$ and a non-empty set $\Lambda \in \mathcal{X}_\mathsf{b}$. Let $\gamma \in \mathcal{N}$. The* grand-canonical partition function *with boundary condition γ is*

$$\Xi_{\Lambda|\gamma} = \Xi_{\Lambda|\gamma}(\beta, z) = 1 + \sum_{n=1}^\infty \frac{z^n}{n!} \int_{\mathbb{X}^n} \exp\Big(-\beta H_\Lambda\big(\sum_{j=1}^n \delta_{x_j} \mid \gamma\big)\Big) \mathrm{d}\lambda^n(\boldsymbol{x}). \tag{3.9}$$

For $\gamma = 0$ (empty boundary conditions), we write Ξ_Λ instead of $\Xi_{\Lambda|0}$. The parameters β and z are called inverse temperature *and* activity*, respectively.*

Integrals such as (3.9) can be cumbersome to write. A more compact form is possible if we introduce a new measure $\tilde{\lambda}$ on $\mathcal{N}_\mathsf{f}$, defined by

$$\int_{\mathcal{N}_\mathsf{f}} f(\eta) \mathrm{d}\tilde{\lambda}(\eta) = f(0) + \sum_{n=1}^\infty \frac{1}{n!} \int_{\mathbb{X}^n} f\big(\delta_{x_1} + \cdots + \delta_{x_n}\big) \mathrm{d}\lambda^n(\boldsymbol{x}) \tag{3.10}$$

for all non-negative f. When $\mathbb{X} = \mathbb{R}^d$ and $\lambda = \text{Leb}$, the measure $\tilde{\lambda}$ is often called *Poisson-Lebesgue measure*. Notice that it is not a probability measure—in general, we have $\tilde{\lambda}(\mathcal{N}_\mathsf{f}) = \infty$ whenever $\lambda(\mathbb{X}) = \infty$. The grand-canonical partition function can now be rewritten as

$$\Xi_{\Lambda|\gamma} = \int_{\mathcal{N}_\Lambda} z^{N_\Lambda(\eta)} \mathrm{e}^{-\beta H_\Lambda(\eta|\gamma)} \mathrm{d}\tilde{\lambda}(\eta).$$

Notice that for the empty boundary condition,

$$1 \leq \Xi_\Lambda \leq \exp\big(z \mathrm{e}^{\beta B} \lambda(\Lambda)\big) < \infty.$$

Definition 3.7. *Fix $\beta, z > 0$ and a non-empty set $\Lambda \in \mathcal{X}_{\mathsf{b}}$. Let $\gamma \in \mathcal{N}$ with $\Xi_{\Lambda|\gamma} < \infty$. The* grand-canonical Gibbs measure *is the probability measure $\mathsf{P}_{\Lambda|\gamma}$ on $\mathcal{N}_\Lambda$ uniquely defined by the requirement that*

$$\int_{\mathcal{N}_\Lambda} f \mathrm{d}\mathsf{P}_{\Lambda|\gamma} = \frac{1}{\Xi_{\Lambda|\gamma}} \int_{\mathcal{N}_\Lambda} f(\eta) z^{N_\Lambda(\eta)} \mathrm{e}^{-\beta H_\Lambda(\eta|\gamma)} \mathrm{d}\tilde{\lambda}(\eta)$$

for all measurable $f : \mathcal{N}_\Lambda \to [0, \infty)$.

Equivalently,

$$\int_{\mathcal{N}_\Lambda} f \mathrm{d}\mathsf{P}_{\Lambda|\gamma}$$
$$= \frac{1}{\Xi_{\Lambda|\gamma}} \left(f(0) + \sum_{n=1}^\infty \frac{z^n}{n!} \int_{\Lambda^n} f_n\big(\delta_{x_1} + \cdots + \delta_{x_n}\big) \exp\Big(-\beta H_\Lambda(\sum_{j=1}^n \delta_{x_j} \mid \gamma)\Big) \mathrm{d}\lambda^n(\boldsymbol{x}) \right).$$

Remark. Let Q_Λ be the distribution of the Poisson point process on Λ with intensity measure λ (more precisely, λ restricted to Λ). The Gibbs measure $\mathsf{P}_{\Lambda|\gamma}$ is absolutely continuous with respect to Q_Λ, with Radon-Nikodým derivative

$$\frac{\mathrm{d}\mathsf{P}_{\Lambda|\gamma}}{\mathrm{d}\mathsf{Q}_\Lambda}(\eta) = \frac{\mathrm{e}^{\lambda(\Lambda)}}{\Xi_{\Lambda|\gamma}} z^{N_\Lambda(\eta)} \mathrm{e}^{-\beta H_\Lambda(\eta|\gamma)} =: u(\eta).$$

The density $u(\eta)$ has the property that if $u(\eta) = 0$, then $u(\zeta) = 0$ whenever $\zeta \geq \eta$. Such densities are called hereditary. In the probabilistic literature, the existence of a hereditary density is sometimes adopted as a definition of Gibbs point processes in finite volume.

3.4. **The pressure and its derivatives.** The grand-canonical partition function $\Xi_{\Lambda|\gamma}(\beta, z)$ enters the stage as a rather modest normalization constant. However a much better probabilistic analogy is to view it as a generating function. Indeed, as we explain in this section, taking derivatives with respect to β and z one obtains information on expected values of the energy and particle numbers, variances, etc.

We assume throughout this section that we are given an energy function and a fixed finite reference volume $\Lambda \in \mathcal{X}_{\mathsf{b}}$. For simplicity we restrict to empty boundary conditions. The finite volume *pressure* is

$$p_\Lambda(\beta, z) := \frac{1}{\beta \lambda(\Lambda)} \log \Xi_\Lambda(\beta, z). \tag{3.11}$$

Sometimes it is more convenient to work with the chemical potential μ instead of the activity $z = \exp(\beta\mu)$, so we also define

$$\bar{p}_\Lambda(\beta, \mu) := p_\Lambda(\beta, \mathrm{e}^{\beta\mu}). \tag{3.12}$$

Notice that

$$\frac{\partial \bar{p}_\Lambda}{\partial \mu}(\beta, \mu) = z \frac{\partial \beta p_\Lambda}{\partial z}(\beta, z)\Big|_{z=\exp(\beta\mu)}. \tag{3.13}$$

Write $\langle f \rangle = \int_{\mathcal{N}_\Lambda} f \mathrm{d}\mathsf{P}_\Lambda$ for the expected value with respect to the finite volume Gibbs measure.

Proposition 3.8. *Suppose that* $\mathbb{X} = \mathbb{R}^d$ *and* $\lambda = \mathrm{Leb}$. *Then*

$$\frac{\partial}{\partial\beta}\beta p_\Lambda(\beta,z) = -\Big\langle \frac{H}{|\Lambda|}\Big\rangle, \qquad \frac{\partial^2}{\partial\beta^2}\,\beta p_\Lambda(\beta,z) = \Big\langle \frac{(H-\langle H\rangle)^2}{|\Lambda|}\Big\rangle$$

$$z\frac{\partial}{\partial z}\beta p_\Lambda(\beta,z) = \Big\langle \frac{N_\Lambda}{|\Lambda|}\Big\rangle, \qquad \Big(z\frac{\partial}{\partial z}\Big)^2\beta p_\Lambda(\beta,z) = \Big\langle \frac{(N_\Lambda-\langle N_\Lambda\rangle)^2}{|\Lambda|}\Big\rangle.$$

For general $\mathbb{X}$ *and* λ*, the formulas hold true if we replace* $|\Lambda|$ *with* $\lambda(\Lambda)$.

Example 3.9. For the ideal gas ($H \equiv 0$), we have $\beta p_\Lambda(\beta,z) = z$ and $\langle \frac{N_\Lambda}{|\Lambda|}\rangle = z$, hence $\beta p_\Lambda(\beta,z) = \langle \frac{N_\Lambda}{|\Lambda|}\rangle$. The reader with knowledge in thermodynamics should recognize the *ideal gas law* $pV = Nk_{\mathsf{B}}T$.

Proof of Proposition 3.8. We treat the case $\mathbb{X} = \mathbb{R}^d$ and the β-derivatives only, the general case and the z-derivatives are similar. Let us check first that H has finite expectation and variance. Fix $\varepsilon \in (0,\beta/2)$. In view of $\exp(\varepsilon|H|) \le \exp(\varepsilon H) + \exp(-\varepsilon H)$, we get

$$\int_{\mathcal{N}_\Lambda} \mathrm{e}^{\varepsilon|H|} z^{N_\Lambda}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda \le \Xi_\Lambda(\beta+\varepsilon,z) + \Xi_\Lambda(\beta-\varepsilon,z) < \infty.$$

With the inequality $x\exp(-x) \le 1/\mathrm{e}$ for all $x > 0$, we get

$$|H|^k \le \frac{1}{(\delta \mathrm{e})^k}\,\mathrm{e}^{\delta k|H|}$$

for all $k \in \mathbb{N}$ and $\delta > 0$. Choosing $\delta = \delta(k,\varepsilon,\beta)$ small enough, we find that

$$\int_{\mathcal{N}_\Lambda} |H|^k \mathrm{e}^{\varepsilon|H|} z^{N_\Lambda}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda < \infty.$$

for all $k \in \mathbb{N}_0$. In particular, all moments of $|H|$ with respect to P_Λ are finite. Next we note that we can exchange differentiation and integration. Indeed for $t \neq 0$ with $|t| \le \varepsilon$, we have

$$\frac{1}{t}\Big(\Xi_\Lambda(\beta+t,z) - \Xi_\Lambda(\beta,z)\Big) = \int_{\mathcal{N}_\Lambda}\frac{1}{t}(\mathrm{e}^{-tH}-1)z^{N_\Lambda}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda.$$

The integrand on the right-hand side converges pointwise to $-H\exp(-\beta H)$ as $t \to 0$. Furthermore

$$\Big|\frac{1}{t}(\mathrm{e}^{-tH}-1)\Big| = \frac{1}{|t|}\Big|\int_0^t H\mathrm{e}^{-sH}\mathrm{d}s\Big| \le |H|\mathrm{e}^{\varepsilon|H|}.$$

Dominated convergence shows that we can pass to the limit $t \to 0$ and we get

$$\frac{\partial}{\partial\beta}\Xi_\Lambda = -\int_{\mathcal{N}_\Lambda} H z^{N_\Lambda(\eta)}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda.$$

Similar arguments work for the second derivative. It follows that

$$\frac{\partial}{\partial\beta}\beta p_\Lambda(\beta,z) = \frac{1}{|\Lambda|}\frac{1}{\Xi_\Lambda}\frac{\partial}{\partial\beta}\Xi_\Lambda = -\frac{1}{\Xi_\Lambda}\int_{\mathcal{N}_\Lambda}\frac{H}{|\Lambda|}z^{N_\Lambda}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda = -\Big\langle\frac{H}{|\Lambda|}\Big\rangle$$

and

$$\frac{\partial}{\partial\beta}\beta p_\Lambda(\beta,z) = \frac{1}{\Xi_\Lambda}\int_{\mathcal{N}_\Lambda}\frac{H^2}{|\Lambda|}z^{N_\Lambda}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda - \frac{1}{|\Lambda|\Xi_\Lambda^2}\Big(\int_{\mathcal{N}_\Lambda} H z^{N_\Lambda}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda\Big)^2$$

$$= \frac{\langle H^2\rangle - \langle H\rangle^2}{|\Lambda|} = \Big\langle\frac{(H-\langle H\rangle)^2}{|\Lambda|}\Big\rangle. \qquad \square$$

Proposition 3.8 brings in the variances of H and N_Λ. Since variances are always non-negative, we obtain as a first corollary a statement on the monotonicity of the energy and particle densities. Let us make the β, z, Λ-dependence of $\langle f\rangle$ explicit and write $\langle f\rangle_{\beta,z,\Lambda}$ instead.

Corollary 3.10.

(a) The energy density $\langle \frac{H}{|\Lambda|}\rangle_{\beta,z,\Lambda}$ *is a monotone increasing function of* $1/\beta$.

(b) The particle density $\langle \frac{N_\Lambda}{|\Lambda|}\rangle_{\beta,z,\Lambda}$ *is a monotone increasing function of* z.

Thus if the temperature increases, the energy increases; and if the activity (or the chemical potential) increases, then the particle density increases.

Functions of a real variable defined on some interval that have a non-negative second derivative are convex, so we have a second corollary on convexity (which one could have checked directly using Hölder's or Jensen's inequality).

Corollary 3.11.

(a) The map $\mathbb{R} \ni \mu \mapsto \bar{p}_\Lambda(\beta,\mu)$ *is convex, for every* $\beta > 0$.

(b) The map $(0,\infty) \ni \beta \mapsto \beta p_\Lambda(\beta, z)$ *is convex, for every* $z > 0$.

Remark. The pressure is analogous to the *cumulant generating function*

$$\varphi(t) = \log \mathbb{E}[\exp(tX)]$$

of real-valued random variable X: Suppose that $\mathbb{E}[\exp(tX)] < \infty$ for all t in some neighborhood $(-\varepsilon, \varepsilon)$ of the origin, then

$$\varphi'(0) = \mathbb{E}[X], \quad \varphi''(0) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big].$$

Derivatives at $t \neq 0$ give the expectation and variance of *tilted* random variables $\hat{X}_t$ with law $\mathbb{P}(\hat{X}_t \in B) = \mathbb{E}[\mathbb{1}_B(X)\exp(tX)]/\mathbb{E}[\exp(tX)]$. The cumulant generating function $\varphi(t)$ is convex.

3.5. **Correlation functions.** Fix $\beta, z > 0$, $\Lambda \in \mathcal{X}_{\mathsf{b}}$, and $\gamma \in \mathcal{N}$ with $\Xi_{\Lambda|\gamma} < \infty$. Proceeding as in Proposition 2.32, we see that for all $k \in \mathbb{N}$ and λ^k-almost all $(x_1, \dots, x_k) \in \Lambda^k$,

$$\rho_k(x_1,\dots,x_k) = \frac{1}{\Xi_{\Lambda|\gamma}} \sum_{n=0}^\infty \frac{z^n}{n!} \int_{\Lambda^n} z^k \mathrm{e}^{-\beta H_\Lambda(\delta_{x_1} + \dots + \delta_{x_{k+n}} | \gamma)} \mathrm{d}\lambda(x_{k+1}) \cdots \mathrm{d}\lambda(x_{k+n}). \tag{3.14}$$

In statistical mechanics Eq. (3.14) is usually adopted as the definition of the correlation function, thus removing indeterminacies on null sets from the definition as a Radon-Nikodým derivative. We adopt this modified definition in the remainder of this section.

We would like to know whether Ruelle's bound ($\mathscr{R}_\xi$) holds true. To that purpose we note that $\rho_k(x_1,\dots,x_k)$ can be written as an expectation of a certain function with respect to the Gibbs measure $\mathsf{P}_{\Lambda|\gamma}$: For $\eta = \sum_{j=1}^k \delta_{x_j} \in \mathcal{N}_{\mathsf{f}}$ and $\zeta = \sum_{j=k+1}^{k+m} \delta_{x_j} \in \mathcal{N}$ we define, in analogy with $H_\Delta(\eta \mid \gamma)$,

$$H\Big(\sum_{j=1}^k \delta_{x_j} \Big| \sum_{j=k+1}^{k+m} \delta_{x_j}\Big) = \sum_{I \cap [k] \neq \varnothing}{}' V\Big(\sum_{i\in I} \delta_{x_i}\Big)$$

if the sum is absolutely convergent, and ∞ otherwise. Again the prime refers to summation over finite subsets of $[k+m]$ if m is finite and $\mathbb{N}$ if m is infinite. If η, ζ are finite configuations with $H(\zeta) < \infty$, then

$$H(\eta \mid \zeta) = H(\eta + \zeta) - H(\zeta).$$

Notice $H_\Delta(\eta \mid \zeta) = H(\eta \mid \zeta_{\mathbb{X}\setminus\Delta})$. We observe that for all $\chi, \eta \in \mathcal{N}_\Lambda$ and all $\gamma \in N$, we have

$$H(\chi + \eta \mid \gamma_{\mathbb{X}\setminus\Lambda}) = H(\chi \mid \eta + \gamma_{\mathbb{X}\setminus\Lambda}) + H(\eta \mid \gamma_{\mathbb{X}\setminus\Lambda})$$

or equivalently,

$$H_\Lambda(\chi + \eta \mid \gamma) = H(\chi \mid \eta + \gamma_{\mathbb{X}\setminus\Lambda}) + H_\Lambda(\eta \mid \gamma).$$

The identity applied to $\chi = \delta_{x_1} + \cdots + \delta_{x_k}$ and $\eta = \delta_{x_{k+1}} + \cdots + \delta_{x_n}$ in (3.14) yields the following lemma.

Lemma 3.12. *Let $\gamma \in \mathcal{N}$ with $\Xi_{\Lambda|\gamma} < \infty$. The k-point correlation function satisfies*

$$\rho_k(x_1, \ldots, x_k) = \int_{\mathcal{N}_\Lambda} z^k \mathrm{e}^{-\beta H(\delta_{x_1} + \cdots + \delta_{x_k} | \eta + \gamma_{\mathbb{X}\setminus\Lambda})} \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\eta) \tag{3.15}$$

for all $(x_1, \ldots, x_k) \in \Lambda^k$.

For $H = 0$, we recover the expression $\rho_k = z^k$ from the ideal gas (Poisson point process with intensity measure $z\lambda$). Eq. (3.15) replaces this explicit formula when there are non-zero interactions.

Definition 3.13. *An energy function H is* locally stable *if for some $C \geq 0$, all $x \in \mathbb{X}$, and all finite-energy configurations $\eta \in \mathcal{N}_\mathsf{f}$, we have*

$$H(\eta + \delta_x) - H(\eta) \geq -C.$$

An interaction potential V is locally stable if the associated energy function H is locally stable.

For example, the sum of pair potentials from Example 3.2 is locally stable. The energy of the Widom-Rowlinson model is strongly stable as well, since for all $n \geq 1$ and $x_1, \ldots, x_n$ we have

$$\Big(|\cup_{j=1}^n B(x_j, 1)| - n|B(0,1)|\Big) - \Big(|\cup_{j=1}^{n-1} B(x_j, 1)| - (n-1)|B(0,1)|\Big) \geq -|B(0,1)|.$$

The telescope identity

$$H\big(\delta_{x_1} + \cdots + \delta_{x_n}\big) = \sum_{k=1}^n \big[H(\delta_{x_1} + \cdots + \delta_{x_k}) - H(\delta_{x_1} + \cdots + \delta_{x_{k-1}})\big] \tag{3.16}$$

shows that local stability implies the stability condition (iii) from Definition 3.1. More generally, we have the following:

Lemma 3.14. *Suppose that V is locally stable with constant C. Let $\gamma \in \mathcal{N}$ be such that $H(\zeta) < \infty$ for all $\zeta \in \mathcal{N}_\mathsf{f}$ with $\zeta \leq \gamma$. Then*

$$H(\eta \mid \gamma) \geq -C N_\mathbb{X}(\eta)$$

for all $\eta \in \mathcal{N}_\mathsf{f}$.

Proof. If $H(\eta \mid \gamma) = \infty$, there is nothing to prove. If $H(\eta \mid \gamma) < \infty$, set $\Lambda_n := B(0,n)$ and observe

$$H(\eta \mid \gamma) = \lim_{n\to\infty} H(\eta \mid \gamma_{\Lambda_n}).$$

Since γ_{Λ_n} is a finite configuration and smaller than γ, the assumption on γ guarantees $H(\gamma_n) < \infty$. A telescope sum analogous to (3.16) shows that

$$H(\eta \mid \gamma_{\Lambda_n}) = H(\eta + \gamma_{\Lambda_n}) - H(\gamma_{\Lambda_n}) \geq -CN_{\mathbb{X}}(\eta).$$

Passing to the limit $n \to \infty$, we obtain the required inequality. □

Proposition 3.15. *Suppose that V is locally stable with constant C. Let $\gamma \in \mathcal{N}$ be such that $H(\zeta) < \infty$ for all $\zeta \in \mathcal{N}_\mathsf{f}$ with $\zeta \leq \gamma_{\mathbb{X}\setminus\Delta}$. Set $\xi = z\exp(\beta C)$. Then $\Xi_{\Lambda|\gamma} < \infty$ and for all $k \in \mathbb{N}$, we have $\rho_k \leq \xi^k$ on Λ^k.*

Proof. Lemma 3.14 implies that

$$H_\Lambda(\eta \mid \gamma) = H(\eta \mid \gamma_{\mathbb{X}\setminus\Lambda}) \geq -CN_{\mathbb{X}}(\eta)$$

for all $\eta \in \mathcal{N}_\Lambda$. It follows that $\Xi_{\Lambda|\gamma} \leq \exp(z\mathrm{e}^{\beta C}\lambda(\Lambda)) < \infty$.

The only relevant contributions to the representations (3.14) and (3.15) of the correlation function $\rho_k(x_1,\dots,x_k)$ come from configurations $\eta \in \mathcal{N}_\Lambda$ such that $H(\delta_{x_1} + \cdots + \delta_{x_k} + \eta \mid \gamma_{\mathbb{X}\setminus\Lambda}) < \infty$. For such configurations, we have $H(\zeta) < \infty$ for all finite configurations ζ with $\zeta \leq \eta + \gamma_{\mathbb{X}\setminus\Lambda}$, hence

$$H(\delta_{x_1} + \cdots + \delta_{x_k} \mid \eta + \gamma_{\mathbb{X}\setminus\Lambda}) = -Ck$$

by Lemma 3.14. We insert the inequality into (3.15) and find that $\rho_k(x_1,\dots,x_k) \leq z^k\exp(\beta Ck)$. □

Remark. Ruelle's bound can be proven for a broader class of potentials or energies, for example, *superstable potentials*, see [49]. Superstability estimates allow for pair potentials $v(r)$ that do not have a hard core.

3.6. **Summary.**

- The key ingredient to the definition of a Gibbs measure is an *energy function* H. Energy functions are required to be *stable* and *hereditary*, and the vacuum $\eta = 0$ has zero energy. Every energy function can be written as a sum of pair or multi-body interactions, the associated interaction potential V is essentially unique.
- The grand-canonical Gibbs measure in a finite volume Λ depends, in addition to H, on two positive parameters $\beta, z > 0$. For empty boundary conditions β, z, H enter the definition of the Gibbs measure in the combination $z^{N_\Lambda}\exp(-\beta H)$.
- The pressure is proportional to the logarithm of a normalization constant. It is a function of β, z whose partial derivatives encode information on the expected value and variance of the energy and particle number, and it has some convexity properties.
- We have introduced additional functions derived from the energy H:
  - $H(\eta \mid \gamma)$ is the sum of the energy of η and the interactions of points in η with points in γ.
  - $H_\Lambda(\eta \mid \gamma) = H(\eta \mid \gamma_{\mathbb{X}\setminus\Lambda})$ enters the definition of the Gibbs measure with boundary condition γ.

- For the ideal gas ($H=0$), the k-point correlation function is equal to z^k. For general H, it is equal to z^k times the expected value of the exponential of an interaction term. Under additional assumptions on the energy, the correlation functions satisfy Ruelle's moment bound ($\mathscr{R}_\xi$).

3.7. **Exercises.**

Exercise 3.1. Fix $z_1, z_2 > 0$ and $\Lambda \in \mathcal{X}_\mathsf{b}$. Let $Z_1, Z_2 : \Omega \to \mathcal{N}_\Lambda$ be independent Poisson processes on Λ, defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with respective intensity measures $z_1\lambda$, $z_2\lambda$ (or more precisely $z_j\lambda_\Lambda$ with λ_Λ the restriction of λ to Λ). We may think of points of Z_1 as blue and points of Z_2 as red. Consider the event C that no two points of different colors can have distance ≤ 1. Find a function $H : \mathcal{N}_\Lambda \to \mathbb{R}$ and parameters $\beta, z > 0$ such that

$$\mathbb{P}(Z_1 \in A \mid C) = \frac{1}{\Xi} \int_{\mathcal{N}_\Lambda} \mathbb{1}_A z^{N_\Lambda} \mathrm{e}^{-\beta H} \mathrm{d}\tilde{\lambda}.$$

for some normalization constant Ξ.

Exercise 3.2. Let $V, H : \mathcal{N}_\mathsf{f} \to \mathbb{R} \cup \{\infty\}$ with $V(0) = 0$ and $H(\delta_{x_1} + \cdots + \delta_{x_n}) = \sum_{I \subset [n]} V(\sum_{i \in I} \delta_{x_i})$. Suppose that for some $C \geq 0$ and all $n \in \mathbb{N}$, $x_1, \ldots, x_n$, and $i_0 \in [n]$, we have

$$\sum_{\substack{I \subset [n] \\ I \ni i_0}} \frac{1}{\# I}\, V\Big(\sum_{i \in I} \delta_{x_i}\Big) \geq -C.$$

Show that H is an energy function.

Exercise 3.3. Fix $\Lambda \in \mathcal{X}_\mathsf{b}$. Let Q_Λ be the distribution of a Poisson point process on Λ with intensity measure λ (more precisely, λ restricted to Λ). Let P_Λ be a probability measure on $\mathcal{N}_\Lambda$. Suppose that P_Λ is absolutely continuous with respect to Q_Λ, and that it has a Radon-Nikodým derivative $u = \mathrm{d}\mathsf{P}_\Lambda / \mathrm{d}Q_\Lambda$ such that

$$u(\eta) = 0 \Rightarrow \forall x \in \Lambda : \; u(\eta + \delta_x) = 0.$$

Show that there exists a function $H : \mathcal{N}_\Lambda \to \mathbb{R} \cup \{\infty\}$ that satisfies conditions (i) and (ii) from the definition of energy functions and

$$u(\eta) = c \exp(-H(\eta))$$

for some $c > 0$ and all $\eta \in \mathcal{N}_\Lambda$.

Exercise 3.4. Take $\mathbb{X} = \mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be an absolutely integrable function with

$$\int_{\mathbb{R}^d} \varphi(x) \mathrm{d}x < 0.$$

Let $H_n(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} \varphi(x_i - x_j)$. Show that we can choose $\Lambda \in \mathcal{X}_\mathsf{b}$ so that for all $\beta, z > 0$,

$$1 + \sum_{n=1}^\infty \frac{z^n}{n!} \int_{\Lambda^n} \mathrm{e}^{-\beta H_n(\boldsymbol{x})} \mathrm{d}\boldsymbol{x} = \infty.$$

Hint: use Jensen's inequality.

4. The infinite-volume limit of the pressure

Here we show that the limit $\lim_{n\to\infty} p_{\Lambda_n}(\beta,z)$ along so-called *van Hove sequences* $(\Lambda_n)_{n\in\mathbb{N}}$ exists, and deduce a number of consequences. We specialize to $\mathbb{X}=\mathbb{R}^d$, $\lambda=\mathrm{Leb}$, and pair potentials $v(x-y)$ that are either non-negative or are lower regular with a hard core as in Example 3.2. As noted earlier, one of the reasons to be interested in the pressure is its analogy with cumulant generating functions. In fact we have, for every $t\in\mathbb{R}$,

$$\frac{1}{\beta|\Lambda|}\log \mathsf{E}_\Lambda^{(\beta,z)}\big[\mathrm{e}^{\beta t N_\Lambda}\big] = \bar p_\Lambda(\beta,\mu+t)-\bar p_\Lambda(\beta,\mu)$$

where $z=\exp(\beta\mu)$. The existence of the infinite-volume limit of the pressure hence translates into pointwise convergence of rescaled cumulant generating functions. We start with a few probabilistic consequences.

4.1. **An intermezzo on real-valued random variables.** Let $(X_n)_{n\in\mathbb{N}}$ be sequence of real-valued random variables defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, and $(s_n)_{n\in\mathbb{N}}$ a sequence of positive numbers with $s_n\to\infty$ as $n\to\infty$; think $s_n=\beta|\Lambda_n|$ and X_n a variable whose law is the distribution of $N_{\Lambda_n}/|\Lambda_n|$ under $\mathsf{P}_{\Lambda_n}^{(\beta,z)}$, where $(\Lambda_n)_{n\in\mathbb{N}}$ is a sequence of bounded Borel sets with $|\Lambda_n|\to\infty$. Suppose that the pointwise limit

$$\varphi(t)=\lim_{n\to\infty}\frac{1}{s_n}\log\mathbb{E}\big[\mathrm{e}^{t s_n X_n}\big]\in\mathbb{R}\cup\{\infty\}$$

exists for all $t\in\mathbb{R}$. Then $\varphi:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ is a convex function with $\varphi(0)=0$. Therefore its *effective domain* $D:=\{t\in\mathbb{R}\mid\varphi(t)<\infty\}$ is non-empty and convex; let us assume that it has non-empty interior and that $0\in\mathrm{Int}\,(D)$. By general facts on convex functions, φ is continuous in $\mathrm{Int}(D)$ and the left and right derivatives

$$\varphi'(t\pm)=\lim_{h\searrow 0}\frac{\varphi(t\pm h)-\varphi(t)}{\pm h}$$

exist for all $t\in\mathrm{Int}\,(D)$. Moreover they have the monotonicity property

$$\varphi'(a-)\le\varphi'(a+)\le\varphi'(b-)\le\varphi'(b+)$$

for all $a,b\in\mathrm{Int}(D)$ with $a<b$.

Proposition 4.1. *Suppose that the pointwise limit $\varphi(t)$ exists on $\mathbb{R}$ and is finite in some neighborhood of the origin. Then:*

(a) If φ is differentiable in 0, then $X_n\to\varphi'(0)$ in probability.
(b) More generally, set $x_\pm:=\varphi'(0\pm)$. We have for every $\delta>0$,

$$\lim_{n\to\infty}\mathbb{P}\big(X_n\in[x_--\delta,x_++\delta]\big)=1.$$

Proposition 4.2. *Suppose that the pointwise limit $\varphi(t)$ exists on $\mathbb{R}$ and is finite in some neighborhood of the origin. Then:*

(a) If φ is differentiable in 0, then $\lim_{n\to\infty}\mathbb{E}[X_n]=\varphi'(0)$.
(b) More generally, set $x_\pm=\varphi'(0+)$. Then

$$x_-\le\liminf_{n\to\infty}\mathbb{E}[X_n]\le\limsup_{n\to\infty}\mathbb{E}[X_n]\le x_+.$$

Proof of Proposition 4.1 . Fix $\delta > 0$. Set $x_\pm := \varphi'(0\pm)$ and

$$\varphi^*(x) := \sup_{t\in\mathbb{R}}\bigl(tx - \varphi(t)\bigr) \qquad (x \in \mathbb{R}).$$

(The function $\varphi^* : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is the *Legendre transform* of φ.) We show

$$\limsup_{n\to\infty} \frac{1}{s_n} \log \mathbb{P}\Bigl(X_n \geq x_+ + \delta\Bigr) \leq -\varphi^*(x_+ + \delta) < 0, \tag{4.1}$$

$$\limsup_{n\to\infty} \frac{1}{s_n} \log \mathbb{P}\Bigl(X_n \leq x_- - \delta\Bigr) \leq -\varphi^*(x_- - \delta) < 0. \tag{4.2}$$

We have, for every $t > 0$,

$$\mathbb{P}\Bigl(X_n \geq x_+ + \delta\Bigr) \leq \mathrm{e}^{-s_n t[x_+ + \delta]}\mathbb{E}\Bigl[\mathrm{e}^{s_n t X_n}\Bigr]$$

hence

$$\limsup_{n\to\infty} \frac{1}{s_n} \log \mathbb{P}\Bigl(X_n \geq x_+ + \delta\Bigr) \leq -\sup_{t>0}\left(t(x_+ + \delta) - \varphi(t)\right).$$

Similarly,

$$\limsup_{n\to\infty} \frac{1}{s_n} \log \mathbb{P}\Bigl(X_n \leq x_- - \delta\Bigr) \leq \sup_{t<0}\left(t(x_+ + \delta) - \varphi(t)\right).$$

Because of the convexity of φ, we have $\varphi(t) \geq \varphi(0) + tx_\pm = tx_\pm$ for all t, hence for $t < 0$,

$$t(x_+ + \delta) - \varphi(t) \leq t\delta < 0 = 0(x_+ + \delta) - \varphi(0).$$

Therefore

$$\sup_{t>0}\left(t(x_+ + \delta) - \varphi(t)\right) = \sup_{t\in\mathbb{R}}\left(t(x_+ + \delta) - \varphi(t)\right) = \varphi^*(x_+ + \delta).$$

Since

$$x_+ + \delta = \delta + \lim_{t\searrow 0} \frac{\varphi(t) - \varphi(0)}{t}$$

and $\varphi(0) = 0$, there exists some $t > 0$

$$\varphi^*(x_+ + \delta) \geq t_0(x_+ + \delta) - \varphi(t_0) > 0.$$

Thus we have proven (4.1). The proof of (4.2) is similar. Part (b) now follows and part (a) is an immediate consequence of part (a). □

Proof of Proposition 4.2. Let $\varphi_n(t) := s_n^{-1} \log \mathbb{E}[\exp(\beta t s_n X_n)]$. Pick $\varepsilon > 0$ so that $\varphi < \infty$ on $[-\varepsilon, \varepsilon]$. Taking n large enough, we may assume that $\varphi_n(\pm\varepsilon) < \infty$ hence for all $t \in [-\varepsilon, \varepsilon]$,

$$\mathbb{E}[\mathrm{e}^{s_n t|X_n|}] \leq \mathbb{E}[\mathrm{e}^{s_n \varepsilon X_n}] + \mathbb{E}[\mathrm{e}^{-s_n \varepsilon X_n}] = \varphi_n(\varepsilon) + \varphi_n(-\varepsilon) < \infty.$$

It follows that $\varphi_n < \infty$ on $(-\varepsilon, \varepsilon)$ and, by an argument similar to the proof of Proposition 3.8,

$$\varphi_n'(t) = \frac{1}{s_n} \frac{\mathbb{E}[s_n X_n \exp(s_n t X_n)]}{\mathbb{E}[\exp(s_n t X_n)]}$$

for all $t \in (-\varepsilon, \varepsilon)$. In particular $\varphi_n'(0) = \mathbb{E}[X_n]$. Moreover φ_n is convex, hence

$$\varphi_n(t) \geq \varphi_n(0) + \varphi_n'(0)t.$$

It follows that for all $t \in (0, \varepsilon)$,

$$\frac{\varphi(t) - \varphi(0)}{t} = \lim_{n\to\infty} \frac{\varphi_n'(t) - \varphi_n'(0)}{t} \geq \limsup_{n\to\infty} \varphi_n'(0).$$

Passing to the limit $t \searrow 0$ we get

$$\varphi'(0+) \geq \limsup_{n\to\infty} \varphi_n'(0) = \limsup_{n\to\infty} \mathbb{E}[X_n].$$

Similarly, for every $t \in (-\varepsilon, 0)$,

$$\frac{\varphi(t)-\varphi(0)}{t} = \lim_{n\to\infty} \frac{\varphi_n'(t)-\varphi_n'(0)}{t} \leq \liminf_{n\to\infty} \varphi_n'(0)$$

hence $\varphi'(0-) \leq \liminf_{n\to\infty} \mathbb{E}[X_n]$. This proves the first part of the lemma. If φ is differentiable in 0, then $\varphi'(0-) = \varphi'(0+) = \varphi'(0)$ and the previous inequalities yield $\lim_{n\to\infty} \mathbb{E}[X_n] = \varphi'(0)$. □

Remark. A close look at the proof shows: if $f : I \to \mathbb{R}$ is the pointwise limit of a sequence of convex, differentiable functions $f_n : I \to \mathbb{R}$ in some open interval I and if f is differentiable, then $f' = \lim_{n\to\infty} f_n'$.

Next we mention a central limit theorem. For $t \in \mathbb{C}$ we have $|\exp(ts_nX_n)| = \exp((\operatorname{Re} t)s_nX_n)$, so if $\mathbb{E}[\exp(\pm\varepsilon s_nX_n)] < \infty$, then $\mathbb{E}[\exp(ts_nX_n)] \in \mathbb{C}$ is well-defined for all $t \in \mathbb{C}$ with $|\operatorname{Re} t| \leq \varepsilon$. Moreover $t \mapsto \mathbb{E}[\exp(ts_nX_n)]$ is holomorphic in the open strip $|\operatorname{Re} t| < \varepsilon$. Now suppose that $\mathbb{E}[\exp(ts_nX_n)]$ is non-zero for all t in some open, connected set $U \subset \mathbb{C}$ containing 0. Then there exists a uniquely defined function $\varphi_n : U \to \mathbb{C}$ such that $\varphi_n(0) = 0$, φ_n is holomorphic in U, and

$$\mathbb{E}\Big[\mathrm{e}^{ts_nX_n}\Big] = \mathrm{e}^{s_n\varphi_n(t)} \qquad (t \in U \subset \mathbb{C}).$$

We write

$$\varphi_n(t) = \frac{1}{s_n}\log \mathbb{E}\Big[\mathrm{e}^{ts_nX_n}\Big]. \tag{4.3}$$

Theorem 4.3 (Bryc [4])**.** *Suppose that there exists some open neighborhood $U_\varepsilon = \{t \in \mathbb{C} \mid |t| < \varepsilon\}$ such that (i) $\mathbb{E}[\exp(ts_nX_n)] \neq 0$ for all $t \in U_\varepsilon$, and (ii) the pointwise limit*

$$\lim_{n\to\infty} \frac{1}{s_n}\log \mathbb{E}\Big[\mathrm{e}^{ts_nX_n}\Big] \in \mathbb{C}$$

exists for all $t \in U_\varepsilon$. Then $\varphi'(0)$ and $\sigma^2 := \varphi''(0) \geq 0$ exist, and the distribution of $\sqrt{s_n}(X_n - \mathbb{E}[X_n])$ converges weakly to the normal law $\mathcal{N}(0, \sigma^2)$.

Proof sketch. For $|t| < \varepsilon$, let $\varphi_n(t)$ be as in (4.3) and $\varphi(t) = \lim_{n\to\infty} \varphi_n(t)$. Fix $\delta \in (0, \varepsilon/2)$. As a first step, we show that

$$\sup_{n\in\mathbb{N}} \sup_{\substack{t\in\mathbb{C}:\\ |t|\leq\delta}} |\varphi_n(t)| < \infty. \tag{4.4}$$

By assumption, the sequences $(\varphi_n(\pm 2\delta))_{n\in\mathbb{N}}$ have finite limits and therefore they are bounded; let $M > 0$ be a common upper bound for $|\varphi_n(\pm 2\delta)|$, $n \in \mathbb{N}$. Consequently

$$\frac{1}{s_n}\log \mathbb{E}\Big[\mathrm{e}^{2\delta s_n|X_n|}\Big] \leq \frac{1}{s_n}\log \mathbb{E}\Big[\mathrm{e}^{2\delta s_nX_n} + \mathrm{e}^{-2\delta s_nX_n}\Big] \leq \frac{1}{s_n}\log 2 + M.$$

which is bounded. Remembering that $\operatorname{Re}\log f = \log|f|$ for all f[5], we have for $|t| \leq 2\delta$

$$\operatorname{Re}\varphi_n(t) = \frac{1}{s_n}\log\big|\mathbb{E}[\mathrm{e}^{ts_nX_n}]\big| \leq \frac{1}{s_n}\log \mathbb{E}[\mathrm{e}^{2\delta s_n|X_n|}]$$

[5] $f = \exp(\log f) = \exp(\operatorname{Re}\log f + \mathrm{i}\operatorname{Im}\log f)$ yields $|f| = \exp(\operatorname{Re}\log f)$.

hence $(\mathrm{Re}\,\varphi_n(t))_{n\in\mathbb{N}}$ is uniformly bounded from above in $B(0,2\delta)$. An extra argument is needed for a bound from below. The Borel-Carathéodory theorem from complex analysis allows us to pass from the real part to the function itself, in a smaller ball, and (4.4) follows.

In a second step, one notes that the functions $\varphi_n(t)$ are holomorphic in $B(0,2\delta)$, therefore by general theorems from complex analysis, pointwise convergence and the uniform bound (4.4) actually imply uniform convergence of φ_n and its derivatives in $B(0,\delta/2)$, moreover the limit function $\varphi(t)$ is holomorphic and limits and differentiation can be exchanged.

Third, we have for $|t|\leq \delta\sqrt{s_n}/2$

$$\begin{aligned}\mathbb{E}\Big[\mathrm{e}^{t\sqrt{s_n}(X_n-\mathbb{E}[X_n])}\Big] &= \exp\Big(s_n\big(\varphi_n(\tfrac{t}{\sqrt{s_n}})-\varphi_n'(0)\tfrac{t}{\sqrt{s_n}}\big)\Big)\\ &= \exp\Big(\tfrac12\varphi''(0)t^2+s_n\sup_{\substack{u\in B(0,\delta/2),\\ n\in\mathbb{N}}}|\varphi_n^{(3)}(u)|O\Big(s_n^{-3/2}\Big)\Big)\\ &\to \exp\Big(\tfrac12\varphi''(0)t^2\Big).\end{aligned}$$

It follows that

$$\mathbb{E}\Big[\mathrm{e}^{\mathrm{i}\tau\sqrt{s_n}(X_n-\mathbb{E}[X_n])}\Big]\to\exp\Big(-\tfrac12\sigma^2\tau^2\Big)$$

pointwise on $\mathbb{R}$. Lévy's continuity theorem then yields the required convergence in distribution. □

Theorem 4.3 provides an incentive for studying the pressure and the partition function at complex activities $z\in\mathbb{C}$; the zeros of the partition functions are related to condition (i) of the theorem.

4.2. Existence of the limit of the pressure.

For $\Lambda\in\mathcal{X}_\mathsf{b}$ and $h>0$, let

$$\partial_h\Lambda := \{x\in\mathbb{R}^d \mid \mathrm{dist}(x,\partial\Lambda)\leq h\}$$

Definition 4.4. *A sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of bounded Borel sets is a* van Hove sequence *if $|\Lambda_n|\to\infty$ and for all $h>0$,*

$$\lim_{n\to\infty}\frac{|\partial_h\Lambda_n|}{|\Lambda_n|}=0.$$

We specialize to translationally invariant pair interactions

$$V(\eta)=\begin{cases} v(x-y), & \eta=\delta_x+\delta_y,\\ 0, & N_{\mathbb{X}}(\eta)\neq 2.\end{cases}$$

The pair potential $v:\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ has a hard core r_{hc} if $v(x)=\infty$ for $|x|<r_{\mathrm{hc}}$, it is *lower regular* if $v(x)\geq -\psi(|x|)$ for some monotone decreasing function $\psi:\mathbb{R}\to\mathbb{R}_+$ with $\int_0^\infty r^{d-1}\psi(r)\mathrm{d}r<\infty$, and it is *upper regular* if $v(x)\leq\psi(|x|)$ whenever $|x|\geq b$, for some $b>0$ and $\psi(r)$ as before. The potential is *two-sided regular* if it is upper and lower regular.

Theorem 4.5. *Let $\mathbb{X}=\mathbb{R}^d$, $\lambda=\mathrm{Leb}$, and V a translationally invariant pair interaction. Assume that the pair potential has a hard core and is two-sided regular. Then the limit*

$$p(\beta,z)=\lim_{n\to\infty}p_{\Lambda_n}(\beta,z)\in\mathbb{R}$$

exists, for all $\beta,z>0$ and every van Hove sequence $(\Lambda_n)_{n\in\mathbb{N}}$. Moreover the limit does not depend on the van Hove sequence.

By Propositions 4.1 and 4.2, Theorem 4.5 implies limit theorems for the particle density $N_{\Lambda_n}/|\Lambda_n|$. Set

$$\rho(\beta,z) = z\frac{\partial}{\partial z}\beta p(\beta,z) = \frac{\partial}{\partial \mu} p(\beta,\mathrm{e}^{\beta\mu})\Big|_{\mu=\beta^{-1}\log z}.$$

if the partial derivative exists.

Corollary 4.6. *Let v be as in Theorem 4.5. Fix $\beta, z>0$. Suppose that the partial derivative of $p(\beta,z)$ with respect to z exists. Then for every van Hove sequence $(\Lambda_n)_{n\in\mathbb{N}}$, we have*

$$\lim_{n\to\infty} \mathsf{E}_{\Lambda_n}^{(\beta,z)}\left[\frac{N_{\Lambda_n}}{|\Lambda_n|}\right] = \rho(\beta,z)$$

and for all $\varepsilon>0$,

$$\lim_{n\to\infty} \mathsf{P}_{\Lambda_n}^{(\beta,z)}\left(\left|\frac{N_{\Lambda_n}}{|\Lambda_n|} - \rho(\beta,z)\right| \geq \varepsilon\right) = 0.$$

If the partial derivative $\partial_z p(\beta,z)$ does not exist, then in the limit $n\to\infty$ we only know that the density $N_{\Lambda_n}/|\Lambda_n|$ concentrates on an interval $[\rho_-(\beta,z),\rho_+(\beta,z)]$ where $\rho_\pm(\beta,z)$ are the right and left derivatives of $\bar{p}_\Lambda(\beta,\cdot)$ at $\mu=\beta^{-1}\log z$; we leave the precise formulation to the reader. In addition, there are analogous statements for the energy density $H_\Lambda/|\Lambda|$ related to the β-derivatives of $\beta\mapsto\beta p(\beta,z)$.

The key ingredient to the proof of the theorem is a form of subadditivity. Remember that if a real-valued sequence $(a_n)_{n\in\mathbb{N}}$ is sub-additive, i.e., $a_{m+n}\leq a_m+a_n$ for all $m,n\in\mathbb{N}$, then the limit $\lim_{n\to\infty} a_n/n$ exists in $\mathbb{R}\cup\{-\infty\}$ and is equal to $\inf_{\mathbb{N}}(a_n/n)$. Now, if the pair potential is non-negative and Λ_1, Λ_2 are two disjoint sets, it is not difficult to check that

$$\log \Xi_{\Lambda_1\cup\Lambda_2} \leq \log\Xi_{\Lambda_1} + \log\Xi_{\Lambda_2}. \tag{4.5}$$

We look first at the limit along cubes

$$Q_n = [0, 2^n L_0]^d \quad (n\in\mathbb{N}_0)$$

with fixed $L_0>0$. For non-negative interactions, the inequality (4.5) together with the translational invariance of the interaction leads to the monotonicity

$$\frac{1}{|Q_{n+1}|}\log\Xi_{Q_{n+1}} \leq \frac{2^d}{|Q_{n+1}|}\log\Xi_{Q_n} = \frac{1}{|Q_n|}\log\Xi_{Q_n}$$

and the existence of the limit along (Q_n) follows. For general pair potentials, we need to estimate error terms coming from the interaction between disjoint regions.

Let $r_{\mathrm{hc}}>0$ be the hard core of the pair potential, if it has one, and $r_{\mathrm{hc}}=0$ if v has no hard core. Set

$$\mathcal{N}^* := \{\eta = \sum_{j=1}^{\kappa}\delta_{x_j}\in\mathcal{N} \mid \forall i\neq j:\, |x_i-x_j|\geq r_{\mathrm{hc}}\}$$

and $\mathcal{N}^*_\Lambda := \mathcal{N}^*\cap\mathcal{N}_\Lambda$. Let

$$W\Big(\sum_{i=1}^n \delta_{x_i};\sum_{j=1}^{\kappa}\delta_{y_j}\Big) = \sum_{i=1}^n\sum_{j=1}^{\kappa} v(x_i-y_j)\in\mathbb{R}\cup\{\infty\} \tag{4.6}$$

be the interaction between a finite configuration $\eta=\sum_{j=1}^n\delta_{x_j}\in\mathcal{N}_{\mathsf{f}}$ and a possibly infinite configuration $\gamma=\sum_{j=1}^{\kappa}\delta_{y_j}\in\mathcal{N}^*$. Proceeding as in Example 3.2, one can check that for non-negative pair potentials or pair potentials with a hard core, we have

$$\sum_{j=1}^{\kappa} v_-(x_i;y_j) \le C \tag{4.7}$$

for some constant C that depends only on the pair potential v. As a consequence the sum (4.6) is well-defined even if $\kappa=\infty$, for all $\sum_{j=1}^{\kappa}\delta_{y_j}\in\mathcal{N}^*$.

Lemma 4.7. *Assume that v is non-negative or lower regular with a hard core. Then for every $\varepsilon>0$, there exists a constant $C_\varepsilon\ge 0$ such that for all $L\ge 1$, all $\eta\in\mathcal{N}^*_{[0,L]^d}$ and $\gamma\in\mathcal{N}^*$, we have*

$$W(\eta;\gamma_{\mathbb{X}\setminus[0,L]^d}) \ge -C_\varepsilon L^{d-1}-\varepsilon L^d.$$

Proof. If v is non-negative, then W is non-negative too and the inequality is trivial. If v is lower regular with a hard core, let $r_{\mathrm{hc}}>0$ and $\psi(r)$ be as in Example 3.2. Write $\eta=\sum_{i=1}^n\delta_{x_i}$ and $\gamma_{\mathbb{X}\setminus[0,L]^d}=\sum_{j=1}^{\kappa}\delta_{y_j}$. We have $|x_i-x_j|\ge r_{\mathrm{hc}}$ and $|y_i-y_j|\ge r_{\mathrm{hc}}$ for all $i\ne j$. Fix $\varepsilon>0$. For $M>0$, we split the interaction as

$$W(\eta;\gamma_{\mathbb{X}\setminus[0,L]^d})=\sum_{i=1}^n\sum_{j=1}^{\kappa}v(|x_i-y_j|)\mathbb{1}_{\{|x_i-y_j|\le M\}}+\sum_{i=1}^n\sum_{j=1}^{\kappa}v(|x_i-y_j|)\mathbb{1}_{\{|x_i-y_j|\ge M\}}. \tag{4.8}$$

The only points x_i contributing to the first sum are those that have distance smaller or equal to M to the boundary of the cube $[0,L]^d$; the number of such points is bounded by some constant c_1 times ML^{d-1}. Each such point contributes an interaction energy larger or equal to $-C$ with C as in (4.7), so altogether the first sum on the right-hand side of (4.8) is bounded from below by $-CMc_1L^{d-1}$. The second sum is bounded from below by a term of the order of $-c_2L^d\int_M^\infty r^{d-1}\psi(r)\mathrm{d}r$ with $c_2>0$ some constant. It can be made smaller than εL^d if we choose $M=M_\varepsilon$ large enough. We set $C_\varepsilon=-CM_\varepsilon c_1L^{d-1}$ and the proof is complete. □

Lemma 4.8. *Assume that v is non-negative or lower regular with a hard core. Then for every $\varepsilon>0$, there exists a constant $C_\varepsilon\ge0$ such that for all $n,k\in\mathbb{N}_0$, we have*

$$\Xi_{Q_{n+k}}\le\left(\Xi_{Q_k}\right)^{2^{nd}}\mathrm{e}^{\beta 2^{nd}[C_\varepsilon L_k^{d-1}+\varepsilon L_k^d]}.$$

Proof. Q_{n+k} is the union of 2^{kd} shifted copies $Q_k^{(j)}$ of Q_k. For $\eta\in\mathcal{N}_{Q_{n+k}}$ we decompose

$$H(\eta)=\sum_{j=1}^{2^{nd}}H(\eta_{Q_n^{(j)}})+\sum_{1\le i<j\le 2^{nd}}W(\eta_{Q_k^{(i)}};\eta_{Q_k^{(j)}}).$$

If v is non-negative, we deduce

$$\begin{aligned}\Xi_{Q_{n+k}}&\le\int_{N_{Q_{n+k}}}z^{N_{Q_{n+k}}(\eta)}\mathrm{e}^{-\beta\sum_{j=1}^{2^{nd}}H(\eta_{Q_k^{(j)}})}\mathrm{d}\tilde\lambda(\eta)\\&=\prod_{j=1}^{2^{nd}}\int_{N_{Q_k}}z^{N_{Q_k^{(j)}}}\mathrm{e}^{-\beta H}\mathrm{d}\tilde\lambda\\&=\prod_{j=1}^{2^{nd}}\Xi_{Q_k^{(j)}}=\left(\Xi_{Q_k}\right)^{2^{nd}}.\end{aligned} \tag{4.9}$$

If v is lower regular with a hard core, we note that the only relevant contributions to $\Xi_{Q_{n+k}}$ come from configurations $\eta \in \mathcal{N}^*_{Q_{n+k}}$. For such η we can estimate interactions between sub-cubes $Q_n^{(j)}$ in a way similar to Lemma 4.7, and we find

$$\sum_{1 \le i < j \le 2^{nd}} W(\eta_{Q_k^{(i)}}; \eta_{Q_k^{(j)}}) \le 2^{nd}(C_\varepsilon L_k^{d-1} + \varepsilon L_k)$$

for every $\varepsilon > 0$ and some constant $C_\varepsilon \ge 0$. From here the proof is completed by a chain of inequalities similar to (4.9). □

Lemma 4.9. *Assume that v is non-negative or lower regular with a hard core. The limit*

$$\beta p(\beta, z) = \lim_{n\to\infty} \frac{1}{|Q_n|} \log \Xi_{Q_n} \in [0, \infty)$$

exists.

Proof. Fix $\varepsilon > 0$ and let $C_\varepsilon \ge 0$ be as in Lemma 4.8. Keeping in mind that $L_{n+k} = 2^n L_k$ and $|Q_{n+k}| = 2^{nd}|Q_k|$, we have

$$\frac{1}{|Q_{n+k}|} \log \Xi_{Q_{n+k}} \le \frac{1}{|Q_k|} \log \Xi_{Q_k} + \beta\Big(\frac{C_\varepsilon}{L_k} + \varepsilon\Big).$$

We take the limit $m = n + k \to \infty$ and deduce

$$\limsup_{m\to\infty} \frac{1}{|Q_m|} \log \Xi_{Q_m} \le \frac{1}{|Q_k|} \log \Xi_{Q_k} + \beta\Big(\frac{C_\varepsilon}{L_k} + \varepsilon\Big).$$

Next we take the limit $k \to \infty$ and get

$$\limsup_{m\to\infty} \frac{1}{|Q_m|} \log \Xi_{Q_m} \le \liminf_{k\to\infty} \frac{1}{|Q_k|} \log \Xi_{Q_k} + \beta\varepsilon.$$

Finally the limit $\varepsilon \searrow 0$ shows that

$$\limsup_{m\to\infty} \frac{1}{|Q_m|} \log \Xi_{Q_m} \le \liminf_{k\to\infty} \frac{1}{|Q_k|} \log \Xi_{Q_k}$$

which shows that the limsup and the liminf have to be equal, and the limit $\beta p(\beta, z)$ exists. It is a non-negative finite number because of the bound $1 \le \Xi_\Lambda \le \exp(z e^{\beta B}|\Lambda|)$. □

Proof of Theorem 4.5. The main idea is to approximate domains Λ by unions of cubes. For $\boldsymbol{k} \in \mathbb{Z}^d$ and $a > 0$, let $Q(\boldsymbol{k}, a) = [k_1 a, (k_1+1)a) \times \cdots \times [k_d a, (k_d+1)a)$. For $\Lambda \in \mathcal{X}_\mathsf{b}$, let

$$N^-(\Lambda, a) := \#\{\boldsymbol{k} \in \mathbb{Z}^d \mid Q(\boldsymbol{k}, a) \subset \Lambda\}$$
$$N^+(\Lambda, a) := \#\{\boldsymbol{k} \in \mathbb{Z}^d \mid Q(\boldsymbol{k}, a) \cap \Lambda \ne \varnothing\}.$$

Note

$$\bigcup_{\substack{\boldsymbol{k}\in\mathbb{Z}^d:\\ Q(\boldsymbol{k},a)\subset\Lambda}} Q(\boldsymbol{k}, a) \subset \Lambda \subset \bigcup_{\substack{\boldsymbol{k}\in\mathbb{Z}^d:\\ Q(\boldsymbol{k},a)\cap\Lambda\ne\varnothing}} Q(\boldsymbol{k}, a)$$

Write $\Lambda^-(a)$ and $\Lambda^+(a)$ for the previous unions of cubes, so that $\Lambda^-(a) \subset \Lambda \subset \Lambda^+(a)$. If $x \in \Lambda^+(a) \setminus \Lambda^-(a)$, then $\operatorname{dist}(x, \partial\Lambda) \le \sqrt{d}a$. It follows that along every van Hove sequence,

$$\lim_{n\to\infty} \frac{|\Lambda_n^+(a) \setminus \Lambda_n^-(a)|}{|\Lambda_n|} \to 0.$$

Put differently, $|\Lambda_n^+(a)| - |\Lambda_n^-(a)| = o(|\Lambda_n|) = o(|\Lambda_n^+(a)|)$. Therefore for every fixed $a > 0$,

$$\lim_{n\to\infty} \frac{|\Lambda_n^-(a)|}{|\Lambda_n|} = \lim_{n\to\infty} \frac{|\Lambda_n^+(a)|}{|\Lambda_n|} = 1.$$

Next we note

$$\Xi_{\Lambda_n^-(a)} \le \Xi_{\Lambda_n} \le \Xi_{\Lambda_n^+(a)}.$$

Proceeding as in Lemma 4.8, we see that for every $\varepsilon > 0$, some $C_\varepsilon \ge 0$, and all $n \in \mathbb{N}$, $a > 0$, we have

$$\Xi_{\Lambda_n^+(a)} \le \exp\bigl(\beta N^+(\Lambda_n, a)(C_\varepsilon a^{d-1} + \varepsilon a^d)\bigr) \Xi_{[0,a]^d}^{N^+(\Lambda_n,a)}.$$

We can choose $a = L_m = 2^m L_0$. Taking first the limit $n \to \infty$, then the limit $m \to \infty$, and finally the limit $\varepsilon \searrow 0$, we get

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log \Xi_{\Lambda_n} \le \beta p(\beta, z).$$

For a lower bound we exploit the upper regularity of the pair potential. Fix $m \in \mathbb{N}$. Let us pick $a = L_m + b$ where $|v(r)| \le \psi(r)$ on $r \ge b$. For $\boldsymbol{k} \in \mathbb{Z}^d$, let

$$\begin{aligned} Q_b(\boldsymbol{k}, a) = \{x \in Q(\boldsymbol{k}, a) \mid \operatorname{dist}(x, \partial Q(\boldsymbol{k}, a) \ge b/2)\} \\ = \times_{j=1}^d [k_i a + b/2, (k_i + 1) - b/2). \end{aligned}$$

We have

$$\bigcup_{\substack{\boldsymbol{k}\in\mathbb{Z}^d: \\ Q(\boldsymbol{k},a)\subset\Lambda_n}} Q_b(\boldsymbol{k}, a) \subset \Lambda_n^-(a) \subset \Lambda_n. \tag{4.10}$$

We bound the partition function Ξ_{Λ_n} from below by the partition function for the union of cubes $Q_b(\boldsymbol{k}, a)$ as on the left-most side of (4.10). If x_i, x_j are in two distinct sub-cube interiors, then they have distance larger or equal to b and therefore $|v(x_i - x_j)| \le \psi(r)$. Arguments similar to Lemmas 4.7 and 4.8 show that

$$\frac{1}{|\Lambda_n|} \log \Xi_{\Lambda_n} \ge \frac{N^-(\Lambda_n, L_m + b) L_m^d}{|\Lambda_n|} \times \frac{1}{L_m^d} \log \Xi_{Q_m} - \beta(C_\varepsilon L_m^{-1} + \varepsilon).$$

Therefore

$$\liminf_{n\to\infty} \frac{1}{n} \log \Xi_{\Lambda_n} \ge \frac{L_m^d}{(L_m + b)^d} \times \frac{1}{L_m^d} \log \Xi_{Q_m} - \beta(C_\varepsilon L_m^{-1} + \varepsilon).$$

We take first the limit $m \to \infty$, then $\varepsilon \searrow 0$, and obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \Xi_{\Lambda_n} \ge \beta p(\beta, z).$$

Altogether we have shown that the limit along any van Hove sequence exists and is equal to the limit along the cubes Q_n. □

The result extends to multi-body interactions provided we can bound the interaction between distinct regions of space. Consider for example the energy of the

Widom-Rowlinson model. Let $\Lambda = [0,L]^d$, $\eta = \sum_{j=1}^n \delta_{x_j} \in \mathcal{N}_\Lambda$, and $\gamma \in \mathcal{N}_{\mathsf{f}}$ with $\gamma_{\mathbb{X}\setminus\Lambda} = \sum_{j=1}^\kappa \delta_{y_j}$. Then

$$\begin{aligned} H_\Lambda(\eta \mid \gamma) - H(\eta) &= H(\eta + \gamma_{\mathbb{X}\setminus\Lambda}) - H(\eta) - H(\gamma_{\mathbb{X}\setminus\Lambda}) \\ &= \left| \Big(\bigcup_{i=1}^n B(x_i,1) \Big) \cup \Big(\bigcup_{j=1}^k B(y_j,1) \Big) \right| - \left| \bigcup_{i=1}^n B(x_i,1) \right| - \left| \bigcup_{j=1}^k B(y_j,1) \right| \\ &= - \left| \Big(\bigcup_{i=1}^n B(x_i,1) \Big) \cap \Big(\bigcup_{j=1}^k B(y_j,1) \Big) \right| \geq -\big((L+1)^d - (L-1)^d \big). \end{aligned}$$

It follows that for some constant $C>0$ and all $L \geq 1$,

$$-CL^{d-1} \leq H_\Lambda(\eta \mid \gamma) - H(\eta) \leq 0.$$

The inequality extends to infinite $\gamma \in \mathcal{N}$ because the interaction potential of the Widom-Rowlinson model has finite range and $H_\Lambda(\eta \mid \gamma)$ depends only on $\gamma_{[-1,L+1]^d}$. The inequality replaces Lemma 4.7 and the upper regularity of the pair potential.

Theorem 4.10. *Let $\mathbb{X} = \mathbb{R}^d$, $\lambda = \mathrm{Leb}$, and H the energy of the Widom-Rowlinson model. Then the limit*

$$p(\beta,z) = \lim_{n\to\infty} p_{\Lambda_n}(\beta,z) \in \mathbb{R}$$

exists, for all $\beta, z > 0$ and every van Hove sequence $(\Lambda_n)_{n\in\mathbb{N}}$. Moreover the limit does not depend on the van Hove sequence.

The proof is similar to the proof of Theorem 4.5 and therefore omitted.

4.3. **A first look at cluster expansions.** For the ideal gas, the pressure can be computed explicitly and is given by $\beta p(\beta,z) = z$. For general pair potentials, there is in general no closed-form expression, however we may hope for a Taylor expansion for small z. Indeed the pressure in finite volume

$$\beta p_\Lambda(\beta,z) = \frac{1}{|\Lambda|} \log\Big(1 + z|\Lambda| + \frac{z^2}{2}\int_{\Lambda^2} \mathrm{e}^{-\beta v(x_1-x_2)} \mathrm{d}x_1 \mathrm{d}x_2 + \cdots \Big)$$

is the logarithm of a power series, so it should have an expansion itself. In this section we show that such an expansion is indeed possible, and the expansion can be organized in a "graphical" way. In a later chapter we discuss expansions of the correlation functions as well, which is a way of estimating differences between the Gibbs measure and the Poisson point process with intensity measure $z\,\mathrm{Leb}$ (ideal gas at activity z).

The key trick is to expand the Boltzmann weight $\exp(-\beta H)$ in terms of *Mayer's f-function*

$$f(x,y) = \mathrm{e}^{-\beta v(x-y)} - 1 \qquad (x,y \in \mathbb{R}^d)$$

(to lighten notation we suppress the β-dependence of f). We have

$$\mathrm{e}^{-\beta \sum_{1\leq i<j\leq n} v(x_i-x_j)} = \prod_{1\leq i<j\leq n} \big(1 + f(x_i,x_j)\big) = \sum_E \prod_{\{i,j\}\in E} f(x_i,x_j)$$

where sum runs over collections $E \subset \{\{i,j\} \mid i,j \in [n],\, i \neq j\}$ and the product over the empty set $E = \varnothing$ is 1. The sum is interpreted as a sum over graphs.

Definition 4.11.

(a) *A* graph *(undirected, no self-edges, no multiple edges) is a pair* $G = (V, E)$ *consisting of a set* V *and a subset* $E \subset \{\{v, w\} \mid v, w \in V, v \neq w\}$*. Elements of* V *are called* vertices*, elements of* E *are* edges*.*
(b) *A graph* $G = (V, W)$ *is* connected *if for all* $v, w \in V$ *with* $v \neq w$ *there exist* $v_1, \ldots, v_n \in V$ *such that* $\{v, v_1\}$*,* $\{v_1, v_2\}, \ldots, \{v_n, w\}$ *are in* E*.*

We write $\mathcal{G}(V)$ and $\mathcal{C}(V)$ for the collections of graphs and connected graphs with vertex set V. For $V = [n]$ we abbreviate $\mathcal{G}_n = \mathcal{G}([n])$ and $\mathcal{C}_n = \mathcal{C}([n])$. For V a finite set, $G = (V, E)$ a graph with vertex set V, and $\boldsymbol{x} \in \mathbb{X}^V$, we define

$$w(G; (x_i)_{i \in V}) = \prod_{\{i,j\} \in E} f(x_i, x_j)$$

and

$$w_\Lambda(G) = \int_{\Lambda^n} w(G; x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Then

$$\prod_{1 \leq i < j \leq n} (1 + f(x_i, x_j)) = \sum_{G \in \mathcal{G}_n} w(G; x_1, \ldots, x_n)$$

and

$$\Xi_\Lambda(\beta, z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{G \in \mathcal{G}_n} w_\Lambda(G). \tag{4.11}$$

The partition function, as a function of z, is the *exponential generating function* for the family of weighted graphs. Taking the logarithm of $\Xi_\Lambda(\beta, z)$ eliminates graphs that are not connected.

Theorem 4.12. *The power series* $\sum_n \frac{z^n}{n!} |\sum_{G \in \mathcal{C}_n} w_\Lambda(G)|$ *has a strictly positive radius of convergence* $R_\Lambda(\beta)$*, and we have, for all* $z \in \mathbb{C}$ *with* $|z| < R_\Lambda(\beta)$*,*

$$\log \Xi_\Lambda(\beta, z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{G \in \mathcal{C}_n} w_\Lambda(G).$$

Remark. Suppose that the pair potential vanishes when $|x| \geq R$. Then $f(x_i, x_j) = 0$ for $|x_i - x_j| \geq R$ and for every connected graph $G \in \mathcal{C}_n$, if $w(G; x_1, \ldots, x_n) \neq 0$, then $x_1, \ldots, x_n$ is R-connected in the sense that for all $i \neq j$, either $|x_i - x_j| \leq R$ or we can find a sequence $i_1, \ldots, i_k$ such that for all $\ell \in 1, \ldots, k$, $|x_{i_\ell} - x_{i_{\ell-1}}| \leq R$, where $i_0 = i$ and $i_{k+1} = j$. So the only relevant contribution to $w_\Lambda(G)$ comes from connected configurations or "clusters". The expansion of $\log \Xi_\Lambda(\beta, z)$ is a *cluster expansion.*

The proof of Theorem 4.12 starts from the observation that every graph $G \in \mathcal{G}(V)$ splits into connected components $G_1, \ldots, G_r$. Their vertex sets $V_1, \ldots, V_r$ form a partition of V, and the weights satisfy

$$w(G; \boldsymbol{x}_V) = \prod_{j=1}^{r} w(G_j; \boldsymbol{x}_{V_j}), \quad w_\Lambda(G) = \prod_{j=1}^{r} w_\Lambda(G_j).$$

Let $\mathcal{P}_n$ be the collection of set partitions of $[n]$, i.e.,

$$\mathcal{P}_n = \Big\{ \{V_1, \ldots, V_r\} \mid r \in \mathbb{N}, V_1, \ldots, V_r \subset [n] \text{ non-empty and disjoint}, \cup_{i=1}^r V_i = [n] \Big\}.$$

Set

$$\Phi(V) = \sum_{G\in\mathcal{C}(V)} w_\Lambda(G).$$

Then

$$\sum_{G\in\mathcal{G}_n} w_\Lambda(G) = \sum_{r=1}^{n} \sum_{\{V_1,\dots,V_r\}\in\mathcal{P}_n} \Phi(V_1)\cdots\Phi(V_r). \tag{4.12}$$

Lemma 4.13. *The weight $\Phi(V)$ depends on $\#V$ alone.*

Proof. Let V and V' be finite sets with $\#V = \#V'$ and σ a bijection from V onto V'. The bijection σ induces a bijection from $\mathcal{C}(V)$ onto $\mathcal{C}(V')$ via

$$G = (V,E) \mapsto G_\sigma = \Big(V', \Big\{\{\sigma(v),\sigma(w)\} \mid \{v,w\}\in E\Big\}\Big).$$

The graph G_σ is obtained from G by relabelling the vertices of G via $v\mapsto\sigma(v)$. To each $\boldsymbol{x}\in\mathbb{X}^V$ we assign $\boldsymbol{x}_\sigma \in V'$ by defining $(x_\sigma)_j = x_{\sigma^{-1}(j)}$. Then

$$w(G_\sigma;\boldsymbol{x}_\sigma) = w(G;\boldsymbol{x}).$$

It follows that $w_\Lambda(G_\sigma) = w_\Lambda(G)$ and then $\Phi(V) = \Phi(V')$. □

Proof of Theorem 4.12. We write $\log\Xi_\Lambda = \log[1+(\Xi_\Lambda - 1)]$. Let

$$|z| < (\log 2)\mathrm{e}^{-\beta B}/|\Lambda|. \tag{4.13}$$

We estimate

$$\big|\Xi_\Lambda(\beta,z) - 1\big| \le \sum_{n=1}^{\infty} \frac{|z|^n}{n!}(\mathrm{e}^{\beta B}|\Lambda|)^n = \exp(|z|\mathrm{e}^{\beta B}|\Lambda|) - 1 < 1.$$

For $u\in(-1,1)$, the series $\log(1+u) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}u^n$ is absolutely convergent. The statement extends to complex $u\in\mathbb{C}$ with $|u|<1$, then $\log(1+u) = \mathrm{Log}(1+u)$ is the principal branch of the logarithm $\mathrm{Log}\, z = \log|z| + \mathrm{i}\mathrm{Arg}\, z$ with $\mathrm{Arg}\, z\in(-\pi,\pi]$. As a consequence

$$\log\Xi_\Lambda(\beta,z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\sum_{k=1}^{\infty}\frac{z^k}{k!}\int_{\Lambda^k}\mathrm{e}^{-\beta H_k}\mathrm{d}\boldsymbol{x}\right)^n$$

with

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=1}^{\infty}\frac{|z|^k}{k!}\int_{\Lambda^k}\mathrm{e}^{-\beta H_k}\mathrm{d}\boldsymbol{x}\right)^n < \infty.$$

Because of the absolute convergence, we can change the order of summation and get

$$\log\Xi_\Lambda(\beta,z) = \sum_{n=1}^{\infty}\frac{z^n}{n!}A_n$$

with

$$A_n = \sum_{r=1}^{n}\sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r\\ k_1+\dots+k_r=n}} (-1)^{r-1}\frac{n!}{k_1!\cdots k_r!}\prod_{j=1}^{r}\left(\int_{\Lambda^{k_j}}\mathrm{e}^{-\beta H_{k_j}(\boldsymbol{x})}\mathrm{d}\boldsymbol{x}\right). \tag{4.14}$$

In order to show that $A_n = \Phi([n])$, we show that it satisfies a set of equations similar to (4.12). We have

$$\begin{aligned}\exp\Big(\sum_{n=1}^\infty \frac{z^n}{n!}A_n\Big) &= 1+\sum_{r=1}^\infty \frac{1}{r!}\Big(\sum_{k=1}^\infty \frac{z^k}{k!}A_k\Big)^r \\ &= 1+\sum_{r=1}^\infty \frac{1}{r!}\sum_{k_1,\dots,k_r\in\mathbb{N}}\prod_{j=1}^r\Big(\frac{z^{k_j}}{k_j!}A_{k_j}\Big) \\ &= 1+\sum_{n=1}^\infty \frac{z^n}{n!}\sum_{\substack{r,k_1,\dots,k_r\in\mathbb{N}:\\ k_1+\cdots+k_r=n}}\frac{n!}{r!k_1!\cdots k_r!}A_{k_1}\cdots A_{k_r}.\end{aligned} \tag{4.15}$$

The inner sum is a sum over set partitions, since

$$\begin{aligned}\sum_{r\in\mathbb{N},\{V_1,\dots,V_r\}\in\mathcal{P}_n}\prod_{j=1}^r A_{\#V_j} &= \sum_{r=1}^n\frac{1}{r!}\sum_{\substack{(V_1,\dots,V_r):\\ \{V_1,\dots,V_r\}\in\mathcal{P}_n}}\prod_{j=1}^r A_{\#V_j} \\ &= \sum_{r=1}^n\frac{1}{r!}\sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r:\\ k_1+\cdots+k_r=n}}\sum_{\substack{(V_1,\dots,V_r):\\ \{V_1,\dots,V_r\}\in\mathcal{P}_n}}\mathbb{1}_{\{\forall j:\,\#V_j=k_j\}}\prod_{j=1}^r A_{k_j} \\ &= \sum_{r=1}^n\frac{1}{r!}\sum_{\substack{(k_1,\dots,k_r)\in\mathbb{N}^r:\\ k_1+\cdots+k_r=n}}\binom{n}{k_1,\cdots,k_r}\prod_{j=1}^r A_{k_j}.\end{aligned}$$

Comparing with (4.15) we get

$$\Xi_\Lambda(\beta,z) = \exp\Big(\sum_{n=1}^\infty\frac{z^n}{n!}A_n\Big) = \sum_{n=1}^\infty\frac{z^n}{n!}\sum_{\substack{\{V_1,\dots,V_r\}\in\mathcal{P}_n\\ r\in\mathbb{N}}}\prod_{j=1}^r A_{\#V_j},$$

moreover $\sum_{n=1}^\infty |z|^n|A_n|/n! < \infty$ for z as in (4.13). Expansion coefficients are unique, so Eq. (4.11) yields

$$\sum_{G\in\mathcal{G}_n} w_\Lambda(G) = \sum_{r=1}^n\sum_{\{V_1,\dots,V_r\}\in\mathcal{P}_n}\prod_{j=1}^r A_{\#V_j} = A_n + \sum_{r=2}^n\sum_{\{V_1,\dots,V_r\}\in\mathcal{P}_n}\prod_{j=1}^r A_{\#V_j}$$

for all $n\in\mathbb{N}$. A simple induction over n shows that given the $w_\Lambda(G)$'s, the set of equations has a unique solution. By Eq. (4.12) and Lemma 4.13, the numbers $\Phi([k])$ solve the system of equations as well, therefore $A_n = \Phi([n])$ for all n. □

Remark (Möbius inversion). Let $p = \{W_1,\dots,W_m\}$ and $q = \{V_1,\dots,V_r\}$ be two partitions of some finite set $V\subset\mathbb{N}$. We say $p\preceq q$ if p is a refinement of q, i.e., if we can label the sets in p as $W'_{k,j}$, $k=1,\dots,r$, $j=1,\dots,n_k$, in such a way that for each k, $\{W'_{k,1},\dots,W'_{k,n_k}\}$ is a partition of V_k. Suppose we are given a family of weights $a(V)$ on finite subsets $V\subset\mathbb{N}$. We can extend it to partitions as $A(p)=\prod_{V\in p}a(V)$. Let us define

$$B(q) = \sum_{q\preceq p} A(q)$$

and $b(V) = B(\{V\})$, then

$$b([n]) = \sum_{r=1}^{n} \sum_{\{V_1,\dots,V_r\}\in\mathcal{P}_n} a(V_1)\cdots a(V_r)$$

which is precisely the type of relation from (4.12). Eq. (4.14) is related to a Möbius inversion.

In Theorem 4.12 one would like to pass to the limit $|\Lambda| \to \infty$ along van Hove sequences $(\Lambda_n)_{n\in\mathbb{N}}$. In a later chapter we show that for small $|z|$, this is indeed possible, and that

$$\liminf_{n\to\infty} R_{\Lambda_n}(\beta) > 0.$$

In Theorem 4.12 we had only proven the crude bound $R_{\Lambda_n} \geq \mathrm{const}/|\Lambda_n|$, which is clearly not enough. Here we content ourselves with the observation that the expansion coefficients converge in the infinite-volume limit.

Lemma 4.14. *Suppose that* $\int_{\mathbb{R}^d} |\exp(-\beta v(x)) - 1|\mathrm{d}x < \infty$*. Then for all* $n \in \mathbb{N}$ *and* $G \in \mathcal{C}_n$*,*

$$\int_{(\mathbb{R}^d)^{n-1}} |w(G;0,x_2,\dots,x_n)|\mathrm{d}x_2\cdots\mathrm{d}x_n < \infty.$$

Furthermore for all $n \in \mathbb{N}$ *and every van Hove sequence* $(\Lambda_k)_{k\in\mathbb{N}}$*, we have*

$$\lim_{k\to\infty} \frac{1}{|\Lambda_k|} \sum_{G\in\mathcal{C}_n} w_{\Lambda_k}(G) = \sum_{G\in\mathcal{C}_n} \int_{(\mathbb{R}^d)^{n-1}} w(G;0,x_2,\dots,x_n)\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

Lemma 4.14 suggests the formula

$$\beta p(\beta,z) = z + \sum_{n=2}^{\infty} \frac{z^n}{n!} \sum_{G\in\mathcal{C}_n} \int_{(\mathbb{R}^d)^{n-1}} w(G;0,x_2,\dots,x_n)\mathrm{d}x_2\cdots\mathrm{d}x_n, \tag{4.16}$$

at least for small z, but for now we don't even know that the series (4.16) has a positive radius of convergence.

Proof. Given $G = ([n],E)$, by removing edges if needed, we can find a subgraph T that is a tree, i.e., $T = ([n],E')$ with $E' \subset E$ and T has no cycles. Stable pair potentials satisfy $v(x-y) \geq -2B$, so we can estimate

$$|w(G;0,x_2,\dots,x_n)| \leq (\mathrm{e}^{2\beta B}-1)^{n^2} w(T;0,x_2,\dots,x_n).$$

The n^2 is a rough upper bound for the number of edges that have to be removed from the graph to obtain a tree. It is a general result from graph theory that the number of edges in a tree on n vertices has $n-1$ edges. We get

$$\int_{(\mathbb{R}^d)^{n-1}} |w(G;0,x_2,\dots,x_n)|\mathrm{d}x_2\cdots\mathrm{d}x_n \leq (\mathrm{e}^{2\beta B}-1)^{n^2} \left(\int_{\mathbb{R}^d} |\mathrm{e}^{-\beta v(x)}-1|\mathrm{d}x\right)^{n-1} < \infty.$$

Next we note that for each van Hove sequence (Λ_k) and each fixed n, G

$$\begin{aligned}\frac{1}{|\Lambda_k|}\int_{\Lambda_k^n} w(G;x_1,\dots,x_n)\mathrm{d}\boldsymbol{x} &= \frac{1}{|\Lambda_k|}\int_{\Lambda_k^n} w(G;0,x_2-x_1,\dots,x_n-x_1)\mathrm{d}\boldsymbol{x}\\ &= \int_{(\mathbb{R}^d)^{n-1}} \alpha_k(\boldsymbol{x}')w(G;0,x_2',\dots,x_n')\mathrm{d}\boldsymbol{x}'\end{aligned}$$

with

$$\begin{aligned}\alpha_k(\boldsymbol{x}') &= \frac{1}{|\Lambda_k|}\int_{\Lambda_k} \mathbb{1}_{\Lambda_k^n}(x_1, x_1 + x_2', \dots, x_1 + x_n')\mathrm{d}x_1 \\ &= \frac{1}{|\Lambda_k|}\,|\Lambda_k \cap (\Lambda_k - x_2') \cap \cdots \cap (\Lambda_k - x_n')| \\ &\to 1 \qquad (k \to \infty).\end{aligned}$$

Clearly $0 \leq \alpha_k(\boldsymbol{x}') \leq 1$ for all $\boldsymbol{x}' \in (\mathbb{R}^d)^{n-1}$. Dominated convergence yields

$$\lim_{k\to\infty} \frac{1}{|\Lambda_k|} w_{\Lambda_k}(G) = \int_{(\mathbb{R}^d)^{n-1}} w(G; 0, \boldsymbol{x}')\mathrm{d}\boldsymbol{x}'. \qquad \square$$

4.4. **Summary.**

- We take infinite-volume limits along van Hove sequences, for which the boundary is negligible compared to the bulk.
- Under suitable assumptions on the interaction, the infinite-volume limit of the pressure along van Hove sequences exists and the value of the limit is independent of the precise choice of sequences—it doesn't matter whether we take limits along disks or cubes.
- The existence of the limit of the pressure implies the existence of the limit of rescaled cumulant generating functions for the particle density with respect to the Gibbs measure. By general results on real-valued random variables, we get limit laws for the particle density: if $\partial_z p$ exists at (β, z), then the particle density converges in probability, otherwise we only know that it concentrates on some interval defined in terms of left and right partial derivatives.
- In general, there is no explicit formula for the pressure. However for small z, pair potentials, in finite volume, we can give an expansion in powers of z. The expansion coefficients are expressed as sums over connected graphs, the expansion is an example of a cluster expansion.

4.5. **Exercises.**

Exercise 4.1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{R}$. Show that $\lim_{n\to\infty} a_n/n = \inf_{n\in\mathbb{N}} a_n/n$.
Hint: for $p \in \mathbb{N}$ and $k = mp + q$, compare a_k/k and a_p/p.

Exercise 4.2. Let I be a non-empty open interval and $f : I \to \mathbb{R}$ a convex function, i.e.,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$. Show that:

(a) For all $a, b, c \in I$ with $a < b < c$,

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}.$$

(b) The limits $f'(x+) = \lim_{h\searrow 0}[f(x+h) - f(x)]/h$, $f'(x-) = \lim_{h\searrow 0}[f(x-h) - f(x)]/(-h)$ exist, for each $x \in I$.

(c) For all $a, b \in I$ with $a < b$, we have $f'(a-) \leq f'(a+) \leq f'(b-) \leq f'(b+)$.

(d) For all $x_0, x \in I$, we have

$$f(x) \geq f(x_0) + f'(x_0+)(x - x_0), \quad f(x) \geq f(x_0) + f'(x_0-)(x - x_0).$$

Exercise 4.3. A sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of bounded Borel sets in $\mathbb{R}^d$ is a *Fisher sequence* if there exist a non-negative function $s(\alpha)$ with $\lim_{\alpha\to 0} s(\alpha) = 0$ and some $\alpha_0 > 0$ such that for all sufficiently small α and all sufficiently large n,

$$\frac{|\partial_{\alpha\,\mathrm{diam}(\Lambda_n)}\Lambda_n|}{|\Lambda_n|} \le s(\alpha).$$

($s(\alpha)$ is called *shape function.*) Let $d = 2$ and $\Lambda_n := [0, n^2] \times [0, n]$. Show that $(\Lambda_n)_{n\in\mathbb{N}}$ is a van Hove sequence but not a Fisher sequence.

Exercise 4.4. Fix $a > 0$. For $n \in \mathbb{N}$ and $L > 0$, define

$$Z_n(L) := \frac{1}{n!}\int_{[0,L]^n} \prod_{1\le i<j\le n} \mathbb{1}_{\{|x_i - x_j| > a\}} \mathrm{d}\boldsymbol{x}.$$

(a) Show that $Z_n(L) = \frac{1}{n!}(L - (n-1)a)^n$ for all $n \in \mathbb{N}$ and $L > (n-1)a$.
(b) Let $\rho \in (0, 1/a)$ and $(L_n)_{n\in\mathbb{N}}$ a sequence with $n/L_n \to \rho$. Compute

$$f(\rho) := -\lim_{n\to\infty} \frac{1}{L_n} \log Z_n(L).$$

(c) Set $a = 1$. Let $p(z) := \sup_{\rho>0}(\rho \log z - f(\rho))$. Show that (i) $\rho \mapsto (\rho \log z - f(\rho))$ has a unique maximizer $\rho(z)$, (ii) $zp'(z) = \rho(z)$, and (iii)

$$p(z) = \frac{\rho(z)}{1 - \rho(z)}.$$

Exercise 4.5. Let $Z_n(L)$ and $f(\rho)$ be as in Exercise 4.4, with $a = 1$. Show that

$$\lim_{L\to\infty} \frac{1}{L} \log\left(1 + \sum_{n=1}^{\infty} z^n Z_n(L)\right) = \sup_{\rho>0}(\rho \log z - f(\rho)).$$

Hint: with the help of the inequality $n! \ge (n/\mathrm{e})^n$ and Exercise 4.4, check that

$$Z_n(L) \le \mathrm{e}^{-(L+1) f(\frac{n}{L+1})},$$

then show that contributions from those n for which n/L is far from $\rho(z)$ are negligible.

Exercise 4.6. Let $\mathbb{X} = \mathbb{R}^d$, $\lambda = \mathrm{Leb}$, and $v : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ a pair potential that has a hard core and is two-sided regular. Let $p(\beta, z)$ be the pressure from Theorem 4.5; remember that it was defined starting from empty boundary conditions. Show that for all $\gamma \in \mathcal{N}^*$ and every van Hove sequence $(\Lambda_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \log \Xi_{\Lambda_n|\gamma}(\beta, z) = \beta p(\beta, z).$$

Hint: Compare $\Xi_{\Lambda_n|\gamma}$ and $\Xi_{\Lambda_n'|0} = \Xi_{\Lambda_n'|0}$, where Λ_n' is equal to Λ_n or slightly different, using a suitable adaptation of Lemma 4.7.

Exercise 4.7.

(a) Let $\mathfrak{S}_n$ be the set of permutations on $\{1, \dots, n\}$ and

$$S(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \#\mathfrak{S}_n, \quad C(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \#\{\sigma \in \mathfrak{S}_n \mid \sigma \text{ is a cycle}\}$$

the exponential generating functions of permutations and cycles. Show that $S(z) = \exp(C(z))$.

(b) Let X be a real-valued random variable such that $\mathbb{E}[\mathrm{e}^{tX}] < \infty$ for all $t \in (-\varepsilon, \varepsilon)$ and some $\varepsilon > 0$. The j-th cumulants is given as

$$\kappa_j = \frac{\mathrm{d}^j}{\mathrm{d}t^j} \log \mathbb{E}[\mathrm{e}^{tX}]\Big|_{t=0}.$$

Show that for all $n \in \mathbb{N}$, we have

$$\mathbb{E}[X^n] = \sum_{r=1}^{n} \sum_{\{V_1,\dots,V_r\}\in\mathcal{P}_n} \kappa_{\#V_1} \cdots \kappa_{\#V_r}$$

where $\mathcal{P}_n$ is the collection of set partitions of $[n] = \{1, \dots, n\}$.

5. Gibbs measures in infinite volume

If $\lambda(\mathbb{X}) = \infty$, then the partition function for $\Lambda = \mathbb{X}$ is infinite and we can no longer define Gibbs measure as in finite volume. Instead, Gibbs measures are defined by structural equations.

5.1. A structural property of finite-volume Gibbs measures.

Fix a non-empty set $\Lambda \in \mathcal{X}_{\mathsf{b}}$ and a boundary condition $\gamma \in \mathcal{N}$ with $\Xi_{\Lambda|\gamma} < \infty$. Let $\Delta \in \mathcal{X}_{\mathsf{b}}$ with $\Delta \subset \Lambda$. We observe

$$H_\Lambda(\eta \mid \gamma_{\mathbb{X}\setminus\Lambda}) = H_\Delta(\eta_\Delta \mid \eta_{\Lambda\setminus\Delta} + \gamma_{\mathbb{X}\setminus\Lambda}) + H_\Lambda(\eta_{\Lambda\setminus\Delta} \mid \gamma_{\mathbb{X}\setminus\Lambda}) \tag{5.1}$$

for all $\eta \in \mathcal{N}_\Lambda$ and

$$\int_{\mathcal{N}_\Lambda} f(\eta)\mathrm{d}\tilde{\lambda}(\eta) = \int_{\mathcal{N}_{\Lambda\setminus\Delta}} \left(\int_{\mathcal{N}_\Delta} f(\eta' + \eta'')\mathrm{d}\tilde{\lambda}(\eta') \right) \mathrm{d}\tilde{\lambda}(\eta'') \tag{5.2}$$

for all non-negative measurable f. Eqs. (5.1) and (5.2) imply that for every non-negative, measurable f, we have

$$\begin{aligned}&\int_{\mathcal{N}_\Lambda} f(\eta) z^{N_\Lambda(\eta)} \mathrm{e}^{-\beta H_\Lambda(\eta|\gamma)} \mathrm{d}\tilde{\lambda}(\eta) \\ &= \int_{\mathcal{N}_{\Lambda\setminus\Delta}} \left(\int_{\mathcal{N}_\Delta} f(\eta' + \eta'') z^{N_\Delta(\eta')} \mathrm{e}^{-\beta H_\Delta(\eta'|\eta'' + \gamma_{\mathbb{X}\setminus\Lambda})} \mathrm{d}\tilde{\lambda}(\eta') \right) z^{N_\Lambda(\eta'')} \mathrm{e}^{-\beta H_\Lambda(\eta''|\gamma)} \mathrm{d}\tilde{\lambda}(\eta'').\end{aligned}$$

Dividing by $\Xi_{\Lambda|\gamma}$, we obtain

$$\int_{\mathcal{N}_\Lambda} f \mathrm{d}\mathsf{P}_{\Lambda|\gamma} = \int_{\mathcal{N}_{\Lambda\setminus\Delta}} \left(\int_{\mathcal{N}_\Delta} f(\eta' + \eta'') z^{N_\Delta(\eta')} \mathrm{e}^{-\beta H_\Delta(\eta'|\eta'' + \gamma_{\mathbb{X}\setminus\Lambda})} \mathrm{d}\tilde{\lambda}(\eta') \right) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\eta''). \tag{5.3}$$

Applying the identity to $f = 1$ we get $1 = \int_{\mathcal{N}_{\Lambda\setminus\Delta}} \Xi_{\Delta|\eta''+\gamma_{\mathbb{X}\setminus\Lambda}} \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\eta'')$, hence

$$\Xi_{\Delta|\eta''+\gamma_{\mathbb{X}\setminus\Lambda}} < \infty \text{ for } \mathsf{P}_{\Lambda|\gamma}\text{-almost all } \eta'' \in \mathcal{N}_{\Lambda\setminus\Delta}.$$

This allows us to express the inner integral on the right-hand side in (5.3) in terms of

$$\bar{f}(\zeta) = \int_{\mathcal{N}_\Delta} f(\eta' + \zeta_{\Lambda\setminus\Delta}) \mathrm{d}\mathsf{P}_{\Delta|\zeta}(\eta')$$

as

$$\begin{aligned}\int_{\mathcal{N}_\Lambda} f \mathrm{d}\mathsf{P}_{\Lambda|\gamma} &= \int_{\mathcal{N}_{\Lambda\setminus\Delta}} \Xi_{\Delta|\eta''+\gamma_{\mathbb{X}\setminus\Lambda}} \bar{f}(\eta'') \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\eta'') \\ &= \int_{\mathcal{N}_{\Lambda\setminus\Delta}} \left(\int_{\mathcal{N}_\Delta} \bar{f}(\eta'') z^{N_\Delta(\eta')} \mathrm{e}^{-\beta H_\Delta(\eta'|\eta'' + \gamma_{\mathbb{X}\setminus\Lambda})} \mathrm{d}\tilde{\lambda}(\eta') \right) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\eta'') \\ &= \int_{\mathcal{N}_\Lambda} \bar{f}(\zeta) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\zeta).\end{aligned}$$

In the last line we have applied Eq. (5.3) to $\bar{f}$. Thus we have shown

$$\int_{\mathcal{N}_\Lambda} f \mathrm{d}\mathsf{P}_{\Lambda|\gamma} = \int_{\mathcal{N}_\Lambda} \left(\int_{\mathcal{N}_\Delta} f(\eta + \zeta_{\Lambda\setminus\Delta}) \mathrm{d}\mathsf{P}_{\Delta|\zeta+\gamma_{\mathbb{X}\setminus\Lambda}}(\eta) \right) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\zeta) \tag{5.4}$$

for all non-negative measurable f.

5.2. **DLR equations.** In infinite volume we define Gibbs measures by relations similar to (5.4). The equations are named after Dobrushin, Lanford and Ruelle. Let

$$\mathcal{R}_\Delta = \{\gamma \in \mathcal{N} \mid \Xi_{\Lambda|\gamma} < \infty\}.$$

Definition 5.1. *Fix $\beta, z > 0$. A probability measure* P *on* $(\mathcal{N}, \mathfrak{N})$ satisfies the DLR conditions *if for all non-empty $\Delta \in \mathcal{X}_\mathsf{b}$, we have $\mathsf{P}(\mathcal{R}_\Delta) = 1$ and*

$$\int_{\mathcal{N}} f \mathrm{d}\mathsf{P} = \int_{\mathcal{N}} \Big(\int_{\mathcal{N}_\Delta} f(\eta + \gamma_{\Delta^c}) \mathrm{d}\mathsf{P}_{\Delta|\gamma}(\eta) \Big) \mathrm{d}\mathsf{P}(\gamma) \qquad \text{(DLR)}$$

for all non-negative observables f. A measure that satisfies the DLR conditions is called a (grand-canonical) Gibbs measure. The set of Gibbs measures is denoted $\mathscr{G}(\beta, z)$.

The DLR conditions can be reformulated with the help of conditional probabilities. It is convenient to set $\mathsf{P}_{\Delta|\gamma} = 0$ if $\gamma \in \mathcal{N} \setminus \mathcal{R}_\Delta$. For $\Lambda \in \mathcal{X}$, let $\mathfrak{N}_\Lambda = \sigma(N_B : B \subset \Lambda, B \in \mathcal{X})$. Equivalently, $\mathfrak{N}_\Lambda$ is the σ-algebra generated by the $(\mathcal{N}, \mathfrak{N})$-valued map $\gamma \mapsto \gamma_\Lambda$ (see Exercise (2.2)).

Proposition 5.2. *Fix $\beta, z > 0$. A probability measure P on $(\mathcal{N}, \mathfrak{N})$ satisfies the DLR conditions if and only if for all non-empty $\Delta \in \mathcal{X}_\mathsf{b}$ and all $B \in \mathfrak{N}$, the map $\gamma \mapsto \mathsf{P}_{\Delta|\gamma}(B)$ is a version of the conditional expectation $\mathsf{E}[\mathbb{1}_B(\eta_\Delta) \mid \mathfrak{N}_{\mathbb{X}\setminus\Delta}]$.*

We write

$$\mathsf{P}(\eta_\Delta \in B \mid \eta_{\mathbb{X}\setminus\Delta} = \gamma_{\mathbb{X}\setminus\Delta}) = \mathsf{P}_{\Delta|\gamma}(B). \qquad (5.5)$$

Proof. "$\Rightarrow$" Suppose that P satisfies (DLR). Let $\Delta \in \mathcal{X}_\mathsf{b}$ be a non-empty set and $B \in \mathfrak{N}$. The map $\gamma \mapsto \mathsf{P}_{\Delta|\gamma}(B)$ is measurable, moreover $\mathsf{P}_{\Delta|\gamma}(B) = \mathsf{P}_{\Delta|\gamma_{\Delta^c}}(B)$. Therefore $\gamma \mapsto \mathsf{P}_{\Delta|\gamma}(B)$ is measurable with respect to $\mathfrak{N}_{\Delta^c}$ (see Exercise (2.2)). Let $g : \mathcal{N} \to \mathbb{R}_+$ be $\mathfrak{N}_{\Delta^c}$-measurable. By Exercise 2.2, $g(\eta) = g(\eta_{\Delta^c})$ for all $\eta \in \mathcal{N}$. (DLR) applied to $f(\eta) = g(\eta_{\Delta^c}) \mathbb{1}_B(\eta)$ yields

$$\mathsf{E}[g(\eta)\mathbb{1}_B(\eta_\Delta)] = \int_{\mathcal{N}} g(\gamma) \mathsf{P}_{\Delta|\gamma}(B) \mathrm{d}\mathsf{P}(\gamma) = \mathsf{E}\big[g(\cdot)\mathsf{P}_{\Delta|\cdot}(B)\big]. \qquad (5.6)$$

This holds true for all non-negative $\mathfrak{N}_{\Delta^c}$-measurable g, thus $\gamma \mapsto \mathsf{P}_{\Delta|\gamma}(B)$ is a version of the conditional expectation $\mathsf{E}[\mathbb{1}_B(\eta_\Delta) \mid \mathfrak{N}_{\mathbb{X}\setminus\Delta}]$.

"$\Leftarrow$" Suppose that P satisfies (5.5). Fix $\Delta \in \mathcal{X}_\mathsf{b}$. By (5.6), Eq. (DLR) holds true for all non-negative observables f of the form $f(\eta) = g(\eta_{\Delta^c})\mathbb{1}_B(\eta)$ with $g : \mathcal{N} \to \mathbb{R}_+$ measurable and $B \in \mathfrak{N}$, $B \subset \mathcal{N}_\Delta$. A Dynkin system argument shows that (DLR) holds true for all indicator functions of sets in $\mathfrak{N}$. Taking monotone limits we find that (DLR) holds true for all non-negative measurable f.

Eq. (5.6) applied to $B = \mathcal{N}$ and the constant function $g = \mathbf{1}$ shows $1 = \int_{\mathcal{N}} \mathsf{P}_{\Delta|\gamma}(\mathbb{X}) \mathrm{d}\mathsf{P}(\gamma)$. Since $\mathsf{P}_{\Delta|\gamma}(\mathbb{X}) \in \{0, 1\}$ for all $\gamma \in \mathcal{N}$, it follows that $\mathsf{P}_{\Delta|\eta_{\Delta^c}}(\mathbb{X}) = 1$ for P-almost all γ, hence $\mathsf{P}(\mathcal{R}_\Delta) = 1$. $\square$

Another reformulation of the DLR conditions highlights some analogies with invariant measures of Markov chains. It is expressed with a family of kernels

$$\pi_\Delta : \mathcal{N} \times \mathfrak{N} \to [0, \infty) \qquad (\Delta \in \mathcal{X}_\mathsf{b},\ \Delta \neq \varnothing) \qquad (5.7)$$

given by

$$\pi_\Delta(\gamma, A) := \mathsf{P}_{\Delta|\gamma}\Big(\{\eta \in \mathcal{N}_\Delta \mid \eta + \gamma_{\Delta^c} \in A\}\Big). \qquad (5.8)$$

Notice that $\mathsf{P}_{\Delta|\gamma}$ is a measure on $\mathcal{N}_\Lambda$ but $\pi_\Delta(\gamma,\cdot)$ is a measure on $\mathcal{N}$. In abstract terms, $\pi_\Delta(\gamma,\cdot)$ is the image of $\mathsf{P}_{\Delta|\gamma}$ under the map

$$\mathcal{N}_\Delta \to \mathcal{N}, \quad \eta \mapsto \eta + \gamma_{\Delta^c} \tag{5.9}$$

which transforms a configuration η in Δ into a configuration on the whole space $\mathbb{X}$ by adding the particles from the boundary condition γ_{Δ^c}. The kernels act from the left on functions and from the right on measures:

$$(\pi_\Delta f)(\gamma) := \int_{\mathcal{N}} \pi_\Delta(\gamma, \mathrm{d}\eta) f(\eta) = \int_{\mathcal{N}_\Delta} f(\eta + \gamma_{\Delta^c}) \mathrm{d}\mathsf{P}_{\Delta|\gamma}(\eta), \tag{5.10}$$

$$\mathsf{P}\pi_\Delta(B) = \int_{\mathcal{N}} \mathrm{d}\mathsf{P}(\gamma)\pi_\Delta(\gamma, B), \tag{5.11}$$

and there is a notion of composition or product given by

$$\big(\pi_\Lambda \pi_\Delta\big)(\gamma, B) = \int_{\mathcal{N}} \pi_\Lambda(\gamma, \mathrm{d}\eta)\pi_\Delta(\eta, B). \tag{5.12}$$

Notice that $\pi_\Lambda(\gamma,\mathbb{X}) \in \{0,1\}$ for all $\gamma \in \mathcal{N}$ (π_Λ is a *quasi-probability kernel*).

Proposition 5.3. P *satisfies the DLR-conditions if and only if for all non-empty* $\Delta \in \mathcal{X}_{\mathsf{b}}$*, we have*

$$\mathsf{P}\pi_\Delta = \mathsf{P}.$$

Proof. The equation $\mathsf{P}\pi_\Delta = \mathsf{P}$ holds true if and only if for every measurable $f : \mathcal{N} \to \mathbb{R}_+$, we have

$$\int_{\mathcal{N}} (\pi_\Delta f) \mathrm{d}\mathsf{P} = \int_{\mathcal{N}} f \mathrm{d}\mathsf{P}.$$

The left-hand side is equal to $\int_{\mathcal{N}} (\int_{\mathcal{N}_\Delta} f(\eta + \gamma_{\Delta^c}) \mathrm{d}\mathsf{P}_{\Delta|\gamma}(\eta)) \mathrm{d}\mathsf{P}(\gamma)$. The proposition now follows from Proposition 5.1. □

The kernels π_Δ have some interesting properties. Let $\mathcal{X}_{\mathsf{b}}^*$ be the non-empty bounded Borel sets.

Proposition 5.4. *The family of kernels* $(\pi_\Delta)_{\Delta \in \mathcal{X}_{\mathsf{b}}^*}$ *form a* specification *with respect to* $(\mathcal{R}_\Delta)_{\Delta \in \mathcal{X}_{\mathsf{b}}^*}$*, which means:*

(a) $\pi_\Delta(\gamma,\cdot)$ *is a probability measure on* $\mathcal{N}$*, for every* $\Delta \in \mathcal{X}_{\mathsf{b}}^*$ *and* $\gamma \in \mathcal{R}_\Delta$.
(b) $\pi_\Delta(\gamma, A) = 0$ *for every* $\Delta \in \mathcal{X}_{\mathsf{b}}^*$, $\gamma \in \mathcal{N} \setminus R_\Delta$ *and all* $A \in \mathfrak{N}$.
(c) The map $\gamma \mapsto \pi_\Delta(\gamma, A)$ *is* $\mathfrak{N}_{\mathbb{X}\setminus\Delta}$*-measurable, for every* $\Delta \in \mathcal{X}_{\mathsf{b}}^*$ *and* $A \in \mathfrak{N}$.
(d) $\pi_\Delta(\cdot, A) = \mathbb{1}_{A \cap \mathcal{R}_\Delta}(\cdot)$ *if* $\Delta \in \mathcal{X}_{\mathsf{b}}^*$, $A \in \mathfrak{N}_{\mathbb{X}\setminus\Delta}$.
(e) $\pi_\Lambda \pi_\Delta = \pi_\Lambda$ *whenever* $\Delta \subset \Lambda$.

The proof of parts (a) to (d) is left as an exercise, we only prove (e). Property (d) leads to the following: if $f : \mathcal{N} \to \mathbb{R}_+$ satisfies $f(\eta) = f(\eta_{\mathbb{X}\setminus\Delta})$ for all $\eta \in \mathcal{N}$, then $\pi_\Delta f = \mathbb{1}_{\mathcal{R}_\Delta} f$, which is quite natural: intuitively, $\pi_\Delta f$ is just the function f after averaging out what happens inside Δ. If f does not depend on what happens inside Δ, then the averaging should not change anything, except for the indicator $\mathbb{1}_{\mathcal{R}_\Delta}$ that is inherited from the definition of π_Δ.

Proof of Proposition 5.4(e). If $\gamma \in \mathcal{N} \setminus \mathcal{R}_\Lambda$, then $\pi_\Lambda \pi_\Delta(\gamma, A) = \pi_\Lambda(\gamma, A) = 0$ for all $A \in \mathfrak{N}$. If $\gamma \in \mathcal{R}_\Lambda$ and f is a non-negative observable, we simply rewrite (5.4) as

$$\begin{aligned}\pi_\Lambda f(\gamma) &= \int_{\mathcal{N}_\Lambda} f(\eta + \gamma_{\Lambda^c}) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\eta) \\ &= \int_{\mathcal{N}_\Lambda} \Big(\int_{\mathcal{N}_\Delta} f(\eta + \zeta_{\Lambda \setminus \Delta} + \gamma_{\mathbb{X} \setminus \Lambda}) \mathrm{d}\mathsf{P}_{\Delta|\zeta + \gamma_{\mathbb{X}\setminus\Lambda}}(\eta) \Big) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\zeta) \\ &= \int_{\mathcal{N}_\Lambda} (\pi_\Delta f)(\zeta + \gamma_{\mathbb{X}\setminus\Lambda}) \mathrm{d}\mathsf{P}_{\Lambda|\gamma}(\zeta) = (\pi_\Lambda \pi_\Delta f)(\gamma).\end{aligned}$$

Choosing $f = \mathbb{1}_A$ we find $\pi_\Lambda(\gamma, A) = (\pi_\Lambda \pi_\Delta)(\gamma, A)$. □

So the property $\pi_\Lambda \pi_\Delta = \pi_\Lambda$ is essentially a reformulation of the structural property (5.4) of finite-volume Gibbs measures noted in Section 5.1. To conclude, we also formulate the analogue of Eq. (5.3).

Definition 5.5. P *satisfies* Ruelle's equation *if for all measurable* $F : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ *and all* $\Delta \in \mathcal{X}_\mathsf{b}$*, we have*

$$\int_{\mathcal{N}} F \mathrm{d}\mathsf{P} = \int_{\mathcal{N}_{\mathbb{X}\setminus\Delta}} \Big(\int_{\mathcal{N}_\Delta} F(\eta + \gamma_{\Delta^c}) z^{N_\Delta(\eta)} \mathrm{e}^{-\beta H_\Delta(\eta|\gamma)} \mathrm{d}\tilde{\lambda}(\eta) \Big) \mathrm{d}\mathsf{P}(\gamma). \tag{R}$$

In Theorem 5.10 below we show that the DLR conditions are equivalent to Ruelle's equation.

5.3. **Existence.** A priori it is not clear that the set $\mathscr{G}(\beta, z)$ of Gibbs measures is non-empty. The probability question at the heart of the existence problem is: given a family of kernels $(\pi_\Delta)_{\Delta \in \mathcal{X}_\mathsf{b}^*}$, is it possible to find a probability measure P such that $\mathsf{P}(\eta_\Delta \in A \mid \eta_{\Delta^c} = \gamma_{\Delta^c}) = \pi_\Delta(\gamma, A)$? In general, the answer is negative, see Exercise 5.2, but additional conditions related to decay at infinity of interactions and quasi-locality of π_Δ in γ ensure a positive answer.

Theorem 5.6. *Fix* $\beta, z > 0$ *and let* H *be a locally stable energy function with finite range. Then* $\mathscr{G}(\beta, z) \neq \varnothing$*.*

Proof. Let $\Lambda_n = B(0, n)$ and $\mathsf{P}_n(\cdot) = \pi_{\Lambda_n}(0, \cdot)$ be the finite volume Gibbs measure in Λ_n with empty boundary conditions. The local stability of the energy and Proposition 3.15 ensure that the sequence $(\mathsf{P}_n)_{n \in \mathbb{N}}$ satisfies the Ruelle bound $(\mathscr{R}_\xi)$ with n-independent ξ. By Theorem 2.53, $(\mathsf{P}_n)_{n\in\mathbb{N}}$ admits a locally convergent subsequence $\mathsf{P}_{n_j} \xrightarrow{\text{loc}} \mathsf{P}$. Fix a non-empty set $\Delta \in \mathcal{X}_\mathsf{b}$. Taking n large enough, we may assume $\Delta \subset \Lambda_n$. By Proposition 5.4(e), we have $\mathsf{P}_n \pi_\Delta = \mathsf{P}_n$, hence

$$\int_{\mathcal{N}} (\pi_\Delta f) \mathrm{d}\mathsf{P}_n = \int_{\mathcal{N}} f \mathrm{d}\mathsf{P}_n \tag{5.13}$$

for all non-negative measurable f. Suppose that f is a bounded local observable, i.e., $f \in \mathcal{A}_B$ for some $B \in \mathcal{X}_\mathsf{b}$. Then $\pi_\Delta f$ is local as well. Indeed, as the energy H has finite range R, we have

$$\mathsf{P}_{\Delta|\gamma} = \mathsf{P}_{\Delta|\gamma_{\Delta^+}} = \mathsf{P}_{\Delta|(\gamma_{B\cup\Delta^+})_{\Delta^+}} = \mathsf{P}_{\Delta|\gamma_{B\cup\Delta^+}}$$

with $\Delta^+ = \{x \in \mathbb{X} \mid \mathrm{dist}(x, \Delta) \leq R\}$. We note

$$f(\eta + \gamma_{\Delta^c}) = f(\eta_B + \gamma_{\Delta^c \cap B}) = f\big(\eta + (\gamma_{B\cup\Delta^+})_{\Delta^c}\big)$$

and therefore

$$(\pi_\Delta f)(\gamma) = (\pi_\Delta f)(\gamma_{B\cup\Delta^+}).$$

It follows that $\pi_\Delta f$ is a bounded local observable, we can pass to the limit along the subsequence (P_{n_j}) in (5.13), and (DLR) holds true for P and bounded local f. A monotone class argument then shows that this holds true for all $f \in \mathscr{L}^\infty(\mathcal{N}, \mathfrak{N})$ and it follows that $\mathsf{P} \in \mathscr{G}(\beta, z)$. In particular, $\mathscr{G}(\beta, z) \neq \varnothing$. □

In general sequences of finite volume Gibbs measures need not converge, however every accumulation point is in $\mathscr{G}(\beta, z)$.

Theorem 5.7. *Assume that the energy is locally stable and has finite range. Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{X}_{\mathsf{b}}$ such that every bounded set Δ is eventually contained in some Λ_n. Let $\gamma \in \mathcal{N}$ be such that $H(\zeta) < \infty$ for all $\zeta \in \mathcal{N}_{\mathsf{f}}$ with $\zeta \leq \gamma$. Then the sequence $(\mathsf{P}_{\Lambda_n|\gamma})_{n\in\mathbb{N}}$ has a locally convergent subsequence, and every accumulation point is in $\mathscr{G}(\beta, z)$.*

The proof is similar to the proof of Theorem 5.6 and therefore omitted.

The proof of Theorem 5.6 is easily adapted to the more general case where (i) all finite-volume Gibbs measures P_Λ with empty boundary conditions satisfy Ruelle's bound $(\mathscr{R}_\xi)$ for some Λ-independent ξ, and (ii) π_Δ maps local functions F to quasi-local functions $\pi_\Delta F$. Specifications $(\pi_\Delta)_{\Delta\in\mathcal{X}_{\mathsf{b}}^*}$ that satisfy condition (ii) are called *quasi-local specifications*. Unfortunately, for particles in $\mathbb{X} = \mathbb{R}^d$, the specifications at hand are usually not quasi-local, and existence proofs become more involved.

We restrict to pair interactions

$$V(\eta) = \begin{cases} v(x,y), & \eta = \delta_x + \delta_y, \\ 0, & N_{\mathbb{X}}(\eta) \neq 2 \end{cases}$$

where $v : \mathbb{X} \times \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ is *stable*, i.e.,

$$\sum_{1\leq i<j\leq n} v(x_i, x_j) \geq -Bn \tag{S}$$

for some $B \geq 0$ and all $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{X}$. We also assume the integrability condition

$$\int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x,y)} - 1| \mathrm{d}\lambda(y) < \infty \tag{I}$$

for all $\beta > 0$ and for all $x \in \mathbb{X}$.

Theorem 5.8. *Let v be a pair potential that satisfies* (S) *and* (I). *Suppose that the finite-volume Gibbs measures P_Λ with empty boundary conditions satisfy Ruelle's bound $(\mathscr{R}_\xi)$ for some Λ-independent ξ. Then $\mathscr{G}(\beta, z) \neq \varnothing$.*

The theorem goes back to Ruelle [49], our presentation follows closely Kuna [31]. It is proven in Section 5.7.

5.4. **GNZ equation.** Georgii [18] and Nguyen and Zessin [38] devised a characterization of Gibbs measures equivalent to the DLR conditions. The starting point is another decomposition of the energy, this time singling out a particle x rather than a domain Δ: Let $W(x;\eta) = H(\delta_x \mid \eta)$. Then for all $\eta \in \mathcal{N}_{\mathsf{f}}$ and $x \in \mathbb{X}$,

$$H(\eta + \delta_x) = W(x;\eta) + H(\eta) \tag{5.14}$$

and

$$z^{N_{\mathbb{X}}(\eta+\delta_x)}\mathrm{e}^{-\beta H(\eta+\delta_x)} = z\mathrm{e}^{-\beta W(x;\eta)} \times z^{N_{\mathbb{X}}(\eta)}\mathrm{e}^{-\beta H(\eta)}. \tag{5.15}$$

Definition 5.9. P *satisfies the GNZ-equation if for all measurable* $F : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$*, we have*

$$\int_{\mathcal{N}} \left(\sum_{x \in S_\eta} n_x(\eta) F(x,\eta) \right) \mathrm{d}\mathsf{P}(\eta) = \int_{\mathbb{X}} \left(\int_{\mathcal{N}} F(x, \eta+\delta_x) z\mathrm{e}^{-\beta W(x;\eta)} \mathrm{d}\mathsf{P}(\eta) \right) \mathrm{d}\lambda(x). \tag{GNZ}$$

(GNZ) can also be written as

$$\mathsf{E}\left[\int_{\mathbb{X}} F(x,\eta)\mathrm{d}\eta(x)\right] = \int_{\mathbb{X}} \mathsf{E}\left[z\mathrm{e}^{-\beta W(x;\eta)} F(x,\eta+\delta_x)\right] \mathrm{d}\lambda(x).$$

For example, if $F(x,\eta) = \frac{1}{\lambda(\Delta)} \mathbb{1}_\Delta(x) N_{B(x,r)}(\eta - \delta_x)$ for some $r > 0$, then $\int_{\mathbb{X}} F(x,\eta)\mathrm{d}\eta(x)$ represents the average number of r-neighbors of points $x \in S_\eta \cap \Delta$.

Theorem 5.10. *Fix* $\beta, z > 0$*, and* P *a probability measure on* $\mathcal{N}$*. The following conditions are equivalent:*

(a) P *satisfies the DLR equations.*
(b) P *satisfies the Ruelle equation.*
(c) P *satisfies the GNZ equation.*

Proof. (DLR) implies (GNZ): Let $F : \mathbb{X} \times \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ be a measurable map. Suppose that for some $\Delta \in \mathcal{X}_\mathsf{b}$ and some $f, g : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$, we have

$$F(x,\eta) = \mathbb{1}_\Delta(x) f(x, \eta_\Delta) g(\eta_{\Delta^c}) \qquad (x \in \mathbb{X}, \eta \in \mathcal{N}). \tag{5.16}$$

Set $f_n(x; x_1, \ldots, x_n) = f(x; \delta_{x_1} + \cdots + \delta_{x_n})$ and $H_{\Delta,n}(x_1, \ldots, x_n \mid \gamma) = H_\Delta(\delta_{x_1} + \cdots + \delta_{x_n} \mid \gamma)$. Then for all $\gamma \in \mathcal{R}_\Delta$, we have

$$\begin{aligned}
&\int_{\mathcal{N}_\Delta} \Big(\sum_{x \in S_\eta} n_x(\eta) f(x,\eta) \Big) \mathrm{d}\mathsf{P}_{\Delta|\gamma}(\eta) \\
&= \frac{1}{\Xi_{\Delta|\gamma}} \sum_{n=1}^\infty \frac{z^n}{n!} \int_{\Delta^n} \sum_{j=1}^n f_n(x_j; x_1, \ldots, x_n) \mathrm{e}^{-\beta H_{\Delta,n}(x_1,\ldots,x_n|\gamma)} \mathrm{d}\lambda^n(\boldsymbol{x}) \\
&= \frac{1}{\Xi_{\Delta|\gamma}} \sum_{n=1}^\infty \frac{z^n}{n!} \int_{\Delta^n} n f_n(x_1; x_1, \ldots, x_n) \mathrm{e}^{-\beta[W(x_1; \sum_{j=2}^n \delta_{x_j} + \gamma_{\Delta^c}) + H_{\Delta,n-1}(x_2,\ldots,x_n|\gamma)]} \mathrm{d}\lambda^n(\boldsymbol{x}) \\
&= \frac{1}{\Xi_{\Delta|\gamma}} \int_\Delta \left\{ \sum_{m=0}^\infty \frac{z^m}{m!} \int_{\Delta^m} f_{m+1}(x; y_1, \ldots, y_m) z\mathrm{e}^{-\beta[W(x; \sum_{j=1}^m \delta_{y_j} + \gamma_{\Delta^c}) + H_{\Delta,m}(\boldsymbol{y}|\gamma)]} \mathrm{d}\lambda^m(\boldsymbol{y}) \right\} \mathrm{d}\lambda(x) \\
&= \frac{1}{\Xi_{\Delta|\gamma}} \int_\Delta \left\{ \int_{\mathcal{N}_\Delta} f(x;\eta) z\mathrm{e}^{-\beta W(x;\eta+\gamma_{\Delta^c})} \mathrm{d}\mathsf{P}_{\Delta|\gamma}(\eta) \right\} \mathrm{d}\lambda(x).
\end{aligned}$$

We multiply with $g(\gamma_{\Delta^c})$, note

$$F(x, \eta + \gamma_{\Delta^c}) = \mathbb{1}_\Delta(x) f(x,\eta) g(\gamma_{\Delta^c}) \qquad (x \in \mathbb{X},\ \eta \in \mathcal{N}_\Delta, \gamma \in \mathcal{N}_{\Delta^c}),$$

integrate over γ with respect to P on both sides, use the DLR equations and find that (GNZ) holds true for F.

The identity applies, in particular, to $F(x,\eta) = \mathbb{1}_\Delta(x)\mathbb{1}_{\{N_\Delta \le n\}}(\eta)\mathbb{1}_{B\cap C}(\eta)$ with $\Delta \in \mathcal{X}_\mathsf{b}$, $n \in \mathbb{N}$, $B \in \mathfrak{N}_\Delta$ and $C \in \mathfrak{N}_{\mathbb{X}\setminus\Delta}$. Moreover for such F the expression in (GNZ) is bounded by

$$\mathsf{E}\left[\sum_{x\in S_\eta} n_x(\eta)\mathbb{1}_\Delta(x)\mathbb{1}_{\{N_\Delta\le n\}}\right] = \mathsf{E}\left[N_\Delta \mathbb{1}_{\{N_\Delta \le n\}}\right] \le n < \infty.$$

The sets $B\cap C$ with $B \in \mathfrak{N}_\Delta$ and $C \in \mathfrak{N}_{\mathbb{X}\setminus\Delta}$ form a π-system generating $\mathfrak{N}$. A Dynkin system argument then shows that (GNZ) holds true for functions $F(x,\eta) = \mathbb{1}_\Delta(x)\mathbb{1}_{\{N_\Delta\le n\}}(\eta)\mathbb{1}_A(\eta)$ with $A \in \mathfrak{N}$. Taking monotone limits, we find that Eq. (GNZ) holds true for all indicator functions of Cartesian products $\Delta \times A \in \mathcal{X}\otimes\mathfrak{N}$.

Now we may view the left and right sides of (GNZ), when applied to indicator functions, as the definition of measures μ and ν on $\mathcal{X}\otimes\mathfrak{N}$. We have just checked that those measures coincide on the generating π-system of Cartesian products, furthermore we have seen that $\mu(\Delta\times\{N_\Delta\le n\}) = \nu(\Delta\times\{N_\Delta \le n\}) \le n < \infty$ for all $\Delta\in\mathcal{X}_\mathsf{b}$, $n\in\mathbb{N}$. In particular, μ and ν are σ-finite. It follows that $\mu=\nu$, i.e., Eq. (GNZ) holds true for the indicator functions of all sets in $\mathcal{X}\otimes\mathfrak{N}$. Consequently it holds true for all measurable non-negative F.

(GNZ) implies (R): Let $F:\mathcal{N}\to\mathbb{R}_+\cup\{\infty\}$ be a non-negative observable and $\Delta\in\mathcal{X}_\mathsf{b}$. The GNZ equation applied to $G(x,\eta) = \mathbb{1}_\Delta(x)\mathbb{1}_{\{N_\Delta=m\}}F(\eta)$ with $m\in\mathbb{N}$ yields

$$\int_\mathcal{N} m\mathbb{1}_{\{N_\Delta=m\}}F\mathrm{d}\mathsf{P} = \int_\Delta\left(\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=m\}}(\eta+\delta_x)F(\eta+\delta_x)\mathrm{d}\mathsf{P}(\eta)\right)\mathrm{d}\lambda(x)$$

hence

$$\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=m\}}F\mathrm{d}\mathsf{P} = \frac{1}{m}\int_\Delta\left(\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=m-1\}}(\eta)F(\eta+\delta_x)z\mathrm{e}^{-\beta W(x_1;\eta)}\mathrm{d}\mathsf{P}(\eta)\right)\mathrm{d}\lambda(x_1).$$

If $m\ge 2$, then in the inner integral, we keep $x=x_1$ fixed and apply (GNZ) to $G(x,\eta)=\mathbb{1}_\Delta(x)\mathbb{1}_{\{N_\Delta=m-1\}}F(\eta+\delta_{x_1})\exp(-\beta W(x_1;\eta))$. We find

$$\begin{aligned}&\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=m\}}F\mathrm{d}\mathsf{P}\\ &=\frac{1}{m(m-1)}\int_{\Delta^2}\left(\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=m-2\}}(\eta)F(\eta+\delta_{x_1}+\delta_{x_2})z^2\mathrm{e}^{-\beta[W(x_1;\eta+\delta_{x_2})+W(x_2;\eta)]}\mathrm{d}\mathsf{P}(\eta)\right)\mathrm{d}\lambda^2(\boldsymbol{x}).\end{aligned}$$

We iterate the procedure. Exploiting

$$W(x_1;\eta+\delta_{x_2}+\cdots+\delta_{x_m})+W(x_2;\eta+\delta_{x_3}+\cdots+\delta_{x_m})+\cdots+W(x_m;\eta) = H(\delta_{x_1}+\cdots+\delta_{x_m}\mid\eta),$$

we find

$$\begin{aligned}&\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=m\}}F\mathrm{d}\mathsf{P}\\ &\quad=\frac{1}{m!}\int_{\Delta^m}\left(\int_\mathcal{N}\mathbb{1}_{\{N_\Delta=0\}}(\eta)F\big(\eta+\delta_{x_1}+\cdots+\delta_{x_m}\big)z^m\mathrm{e}^{-\beta H(\delta_{x_1}+\cdots+\delta_{x_m}\mid\eta)}\mathrm{d}\mathsf{P}(\eta)\right)\mathrm{d}\lambda^m(\boldsymbol{x})\\ &\quad=\int_{\mathcal{N}_{\mathbb{X}\setminus\Delta}}\left(\frac{z^m}{m!}\int_{\Delta^m}F(\eta+\delta_{x_1}+\cdots+\delta_{x_m})\mathrm{e}^{-\beta H(\delta_{x_1}+\cdots+\delta_{x_m}\mid\eta)}\mathrm{d}\lambda^m(\boldsymbol{x})\right)\mathrm{d}\mathsf{P}(\eta).\end{aligned}$$

For $m=0$ we have

$$\int_{\mathcal{N}} \mathbb{1}_{\{N_\Delta=0\}} F \mathrm{d}\mathsf{P} = \int_{\mathcal{N}_{\mathbb{X}\setminus\Delta}} F(0+\eta)\mathrm{d}\mathsf{P}(\eta).$$

Summing over $m\in\mathbb{N}_0$, we find that (R) holds true for F.

(R) implies (DLR): Let F be a non-negative observable. We have

$$\begin{aligned}\int_{\mathcal{N}} F\mathrm{d}\mathsf{P} &= \int_{\mathcal{N}_{\mathbb{X}\setminus\Delta}} \Bigl(\int_{\mathcal{N}_\Delta} F(\eta+\gamma) z^{N_\Delta(\eta)} \mathrm{e}^{-\beta H_\Delta(\eta|\gamma)} \mathrm{d}\tilde\lambda(\eta)\Bigr)\mathrm{d}\mathsf{P}(\gamma)\\ &= \int_{\mathcal{N}_{\mathbb{X}\setminus\Delta}} (\pi_\Delta F)(\gamma)\Xi_{\Delta|\gamma}\mathrm{d}\mathsf{P}(\gamma)\\ &= \int_{\mathcal{N}_{\mathbb{X}\setminus\Delta}} \Bigl(\int_{\mathcal{N}_\Delta} (\pi_\Delta F)(\gamma) z^{N_\Delta(\eta)} \mathrm{e}^{-\beta H_\Delta(\eta|\gamma)} \mathrm{d}\tilde\lambda(\eta)\Bigr)\mathrm{d}\mathsf{P}(\gamma)\\ &= \int_{\mathcal{N}} (\pi_\Delta F)\mathrm{d}\mathsf{P}\end{aligned}$$

hence $\mathsf{P}\pi_\Delta = \mathsf{P}$. For $F=\mathbf{1}$ we get $1=\int_{\mathcal{N}} \Xi_{\Delta|\gamma}\mathrm{d}\mathsf{P}(\gamma)$, hence $\Xi_{\Delta|\gamma}<\infty$ P-a.s. and $\mathsf{P}(\mathcal{R}_\Delta)=1$. □

The GNZ equation can be iterated, leading to a multivariate form that is helpful in evaluating correlation functions.

Proposition 5.11 (Multivariate GNZ equation)**.** *Let $\mathsf{P}\in\mathscr{G}(\beta,z)$. Then we have, for all $m\in\mathbb{N}$ and all measurable $F:\mathbb{X}^m\times\mathcal{N}\to\mathbb{R}_+\cup\{\infty\}$,*

$$\begin{aligned}&\mathsf{E}\Bigl[\int_{\mathbb{X}^m} F(\boldsymbol{x};\eta)\mathrm{d}\eta^{(m)}(\boldsymbol{x})\Bigr]\\ &\qquad= \int_{\mathbb{X}^m} \mathsf{E}\Bigl[F(\boldsymbol{x};\eta+\delta_{x_1}+\cdots+\delta_{x_m}) z^m \mathrm{e}^{-\beta H(\delta_{x_1}+\cdots+\delta_{x_m}|\eta)}\Bigr]\mathrm{d}\lambda^m(\boldsymbol{x}).\end{aligned}\tag{MGNZ}$$

Proof. The proof is by induction over m. For $m=1$, Eq. (MGNZ) reduces to the GNZ equation and there is nothing to prove. Now suppose that Eq. (MGNZ) holds true for $m-1$ and all measurable non-negative test functions. Let $F:\mathbb{X}^{m+1}\times\mathcal{N}\to\mathbb{R}_+\cup\{\infty\}$. We have, for every $\kappa\in\mathbb{N}_0\cup\{\infty\}$, $y_1,y_2,\ldots\in\mathbb{X}$, and $\eta=\sum_{j=1}^\kappa \delta_{y_j}$,

$$\begin{aligned}\int_{\mathbb{X}^m} F(\boldsymbol{x};\eta)\mathrm{d}\eta^{(m)}(\boldsymbol{x}) &= \sum_{(j_1,\ldots,j_m)}^{\neq} F(y_{j_1},\ldots,y_{j_m};\sum_{i=1}^\kappa \delta_{y_i})\\ &= \sum_{j_1=1}^\kappa \sum_{(j_2,\ldots,j_m)}^{\neq} \mathbb{1}_{\{\forall \ell\neq 1:\, j_\ell\neq j_1\}} F(y_{j_1},\ldots,y_{j_m};\sum_{i=1}^\kappa \delta_{y_i})\\ &= \int_{\mathbb{X}} G(x_1;\eta)\mathrm{d}\eta(x_1)\end{aligned}$$

where

$$G(x_1;\eta) = \int_{\mathbb{X}^{m-1}} F(x_1,x_2,\ldots,x_m;\eta)\mathrm{d}\eta_{x_1}^{(m-1)}(x_2,\ldots,x_m),\quad \eta_{x_1}:=\eta-\delta_{x_1}.$$

The GNZ equation shows

$$\mathsf{E}\Bigl[\int_{\mathbb{X}^m} F(\boldsymbol{x};\eta)\mathrm{d}\eta^{(m)}(\boldsymbol{x})\Bigr] = \int_{\mathbb{X}} \mathsf{E}\Bigl[G(x_1;\eta+\delta_{x_1}) z\mathrm{e}^{-\beta W(x_1;\eta)}\Bigr]\mathrm{d}\lambda(x_1)\tag{5.17}$$

Notice

$$G(x_1;\eta+\delta_{x_1}) = \int_{\mathbb{X}^{m-1}} F(x_1,x_2,\dots,x_m;\eta+\delta_{x_1})\mathrm{d}\eta^{(m-1)}(x_2,\dots,x_m).$$

Thus

$$G(x_1;\eta+\delta_{x_1})z\mathrm{e}^{-\beta W(x_1;\eta)} = \int_{\mathbb{X}^{m-1}} H_{x_1}(x_2,\dots,x_m;\eta)\mathrm{d}\eta^{(m-1)}(x_2,\dots,x_m)$$

with $H_{x_1}(x_2,\dots,x_m;\eta) = F(x_1,x_2,\dots,x_m;\eta)z\mathrm{e}^{-\beta W(x_1;\eta)}$. The induction hypothesis yields

$$\begin{aligned}&\mathsf{E}\Big[G(x_1;\eta+\delta_{x_1})z\mathrm{e}^{-\beta W(x_1;\eta)}\Big]\\ &= \int_{\mathbb{X}^{m-1}} \mathsf{E}\Big[F(x_1,x_2,\dots,x_m;\eta)z^m\mathrm{e}^{-\beta[W(x_1;\eta+\delta_{x_2}+\dots+\delta_{x_m})+H(\delta_{x_2}+\dots+\delta_{x_m}\mid\eta)]}\Big]\\ &\qquad\qquad \mathrm{d}\lambda(x_2)\cdots\mathrm{d}\lambda(x_m).\end{aligned}$$

We insert this identity into Eq. (5.17), exploit

$$H(\delta_{x_1}+\dots+\delta_{x_m}\mid\eta) = W(x_1;\eta+\delta_{x_2}+\dots+\delta_{x_m}) + H(\delta_{x_2}+\dots+\delta_{x_m}\mid\eta),$$

and obtain (MGNZ). □

5.5. **Correlation functions and Mayer-Montroll equation.** From the multivariate GNZ equation, we obtain an infinite-volume version of Lemma 3.12 as a simple consequence.

Proposition 5.12. *Let* $\mathsf{P}\in\mathscr{G}(\beta,z)$. *The correlation functions of* P *exist and satisfy*

$$\rho_n(x_1,\dots,x_n) = \int_{\mathcal{N}} z^n\mathrm{e}^{-\beta H(\delta_{x_1}+\dots+\delta_{x_n}\mid\eta)}\mathrm{d}\mathsf{P}(\eta) \tag{5.18}$$

for all $n\in\mathbb{N}$ *and* λ^n*-almost all* $(x_1,\dots,x_n)\in\mathbb{X}^n$.

In particular, the one-particle density is

$$\rho_1(x) = \int_{\mathcal{N}} z\mathrm{e}^{-\beta W(x;\eta)}\mathrm{d}\mathsf{P}(\eta).$$

Proof. The proposition is an immediate consequence of the multivariate GNZ equation: let $m\in\mathbb{N}$ and $f:\mathbb{X}^m\to\mathbb{R}_+\cup\{\infty\}$ measurable, then (MGNZ) applied to $F(\boldsymbol{x};\eta) = f(\boldsymbol{x})$ yields

$$\mathsf{E}\Big[\int_{\mathbb{X}^m} f(\boldsymbol{x})\mathrm{d}\eta^{(m)}\Big] = \int_{\mathbb{X}^m} f(\boldsymbol{x})\mathsf{E}\Big[z^m\mathrm{e}^{-\beta H(\sum_{j=1}^m\delta_{x_j}\mid\eta)}\Big]\mathrm{d}\lambda^m(\boldsymbol{x}).$$

As the identity holds true for all non-negative measurable f, and the left-hand side is equal to $\int_{\mathbb{X}^m} f\mathrm{d}\alpha_m$ by definition of the factorial moment measure α_m, it follows that the factorial moment measure is absolutely continuous with respect to λ^m with Radon-Nikodým derivative

$$\rho_m(\boldsymbol{x}) = \frac{\mathrm{d}\alpha_m}{\mathrm{d}\lambda^m}(\boldsymbol{x}) = \mathsf{E}\Big[z^m\mathrm{e}^{-\beta H(\sum_{j=1}^m\delta_{x_j}\mid\eta)}\Big] \qquad \lambda^m\text{-a.e.}$$

This proves the claim. □

For pair interactions $v(x,y)$, Proposition 5.12 can be combined with the Theorem 2.40 on Laplace functionals.

Lemma 5.13. *Suppose that the pair potential $v(x,y)$ is stable* (S) *and satisfies the integrability condition* (I). *Let $W(x_1,\dots,x_n;y) := \sum_{j=1}^n v(x_j-y)$. Then for all $n\in\mathbb{N}$ and $x_1,\dots,x_n\in\mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} |\mathrm{e}^{-\beta W(x_1,\dots,x_n;y)}-1|\mathrm{d}y \le \mathrm{e}^{2\beta n B}\sum_{j=1}^n \int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x_j,y)}-1|\mathrm{d}\lambda(y) < \infty.$$

Proof. Fix $y\in\mathbb{R}^d$ and set $\varphi_j := \exp(-\beta v(x_j-y))-1$. We have

$$\prod_{j=1}^n (1+\varphi_j) - 1 = \varphi_n \prod_{j=1}^{n-1}(1+\varphi_j) + \varphi_{n-1}\prod_{j=1}^{n-2}(1+\varphi_j) + \cdots + \varphi_2(1+\varphi_1)+\varphi_1$$

and

$$\prod_{j=1}^k (1+\varphi_j) = \mathrm{e}^{-\beta\sum_{j=1}^k v(x_j-y)} \le \mathrm{e}^{\beta 2Bk} \le \mathrm{e}^{\beta 2Bn}$$

because $v(x,y) = H_2(x,y)\ge -2B$ for all $x,y\in\mathbb{X}$. Hence

$$\left|\prod_{j=1}^n (1+\varphi_j) - 1\right| \le \mathrm{e}^{2\beta Bn}\sum_{j=1}^n |\varphi_j|.$$

We integrate on both sides with respect to y and obtain the required inequality. □

Theorem 5.14. *Let $\mathsf{P}\in\mathscr{G}(\beta,z)$. Assume that the interaction is a pair potential that satisfies* (S) *and* (I), *and that P satisfies Ruelle's bound ($\mathscr{R}_\xi$). Then the correlation functions ρ_n satisfy the* Mayer-Montroll equations*: for all $n\in\mathbb{N}$ and λ^n-almost all $(x_1,\dots,x_n)\in\mathbb{X}^n$, we have*

$$\rho_n(x_1,\dots,x_n) = z^n \mathrm{e}^{-\beta H(x_1,\dots,x_n)} \times\left(1+\sum_{k=1}^\infty \frac{1}{k!}\int_{(\mathbb{X})^n}\prod_{j=1}^k\big(\mathrm{e}^{-\beta W(x_1,\dots,x_n;y_j)}-1\big)\rho_k(y_1,\dots,y_k)\mathrm{d}\lambda^k(\boldsymbol{y})\right). \tag{MM}$$

Proof. By Proposition (5.18),

$$\rho_n(x_1,\dots,x_n) = z^n\mathrm{e}^{-\beta H(x_1,\dots,x_n)}\mathsf{E}\big[\mathrm{e}^{-\beta\sum_{y\in S_\eta} W(x_1,\dots,x_n;y)n_y(\eta)}\big].$$

Because of Lemma 5.13, we can apply Theorem 2.40 to $f(y) = W(x_1,\dots,x_n;y)$ and (MM) follows. □

5.6. **Kirkwood-Salsburg equation.** Another set of integral equations expresses the correlation function $\rho_{n+1}(x_0,\dots,x_n)$ with a point x_0 singled out in terms of the interaction of x_0 with points from η and the whole set of correlation functions $(\rho_k)_{k\in\mathbb{N}}$. It is the correlation function sibling of the GNZ equation. Historically, though, integral equations such as Mayer-Montroll and Kirkwood-Salsburg appeared long before the DLR or GNZ equation.

To lighten notation, we drop the Dirac symbols when there is no risk of confusion; so we write $H(x_1,\dots,x_n\mid\eta)$ for $H(\delta_{x_1}+\cdots+\delta_{x_n}\mid\eta)$, $W(x_0;x_1,\dots,x_n)$ for $W(x_0;\delta_{x_1}+\cdots+\delta_{x_n})$, etc.

Theorem 5.15. *Let* P *be a probability measure that satisfies Ruelle's bound* $(\mathscr{R}_\xi)$. *Assume that the pair interaction satisfies* (S) *and* (I). *Then* $\mathsf{P} \in \mathscr{G}(\beta, z)$ *if and only if the correlation functions satisfy the* Kirkwood-Salsburg equations*:*

$$\rho_{n+1}(x_0, x_1, \ldots, x_n) = z\mathrm{e}^{-\beta W(x_0; x_1, \ldots, x_n)}$$
$$\times \left(\rho_n(x_1, \ldots, x_n) + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{(\mathbb{X})^k} \prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) \rho_{n+k}(x_1, \ldots, x_n, \boldsymbol{y}) \mathrm{d}\lambda^k(\boldsymbol{y}) \right) \tag{KS}$$

for all $n \in \mathbb{N}_0$ *and* λ^n*-almost all* $(x_0, x_1, \ldots, x_n) \in \mathbb{X}^{n+1}$, *with the convention* $\rho_0 = 1$.

Proof of $P \in \mathscr{G}(\beta, z) \Rightarrow$ (KS). We decompose

$$H(x_0 \cdots x_n \mid \eta) = W(x_0; x_1, \ldots, x_n) + W(x_0 \mid \eta) + H(x_1, \ldots, x_n \mid \eta)$$

and obtain from Proposition 5.12 that

$$\rho_{n+1}(x_0, x_1, \ldots, x_n) = z\mathrm{e}^{-\beta W(x_0; x_1, \ldots, x_n)} \mathsf{E}\Big[z^n \mathrm{e}^{-\beta [W(x_0; \eta) + H(x_1, \ldots, x_n \mid \eta)]} \Big]$$

We can further expand

$$\mathrm{e}^{-\beta W(x_0; \eta)} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) \mathrm{d}\eta^{(k)}(\boldsymbol{y}), \tag{5.19}$$

which is absolutely convergent for P-almost all $\eta \in \mathcal{N}$ because of

$$\mathsf{E}\left[1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{X}^k} \left| \prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) \right| \mathrm{d}\eta^{(k)}(\boldsymbol{y}) \right]$$
$$\leq \exp\left(\xi \int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x_0, y)} - 1| \mathrm{d}\lambda(y) \right) < \infty, \tag{5.20}$$

see the proof of Theorem 2.40. Applying (MGNZ) we get

$$\begin{aligned}
&\mathsf{E}\Big[z^n \int_{\mathbb{X}^k} \prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) \mathrm{e}^{-\beta H(x_1, \ldots, x_n \mid \eta)} \mathrm{d}\eta^{(k)}(\boldsymbol{y}) \Big] \\
&\quad = \int_{\mathbb{X}^k} \mathsf{E}\Big[\prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) z^{n+k} \mathrm{e}^{-\beta [H(x_1, \ldots, x_n \mid \delta_{y_1} + \cdots + \delta_{y_k} + \eta) + H(y_1, \ldots, y_k \mid \eta)]} \Big] \mathrm{d}\lambda^k(\boldsymbol{y}) \\
&\quad = \int_{\mathbb{X}^k} \prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) \mathsf{E}\Big[z^{n+k} \mathrm{e}^{-\beta H(x_1, \ldots, x_n, y_1, \ldots, y_k \mid \eta)} \Big] \mathrm{d}\lambda^k(\boldsymbol{y}) \\
&\quad = \int_{\mathbb{X}^k} \prod_{j=1}^{k} (\mathrm{e}^{-\beta v(x_0, y_j)} - 1) \rho_{n+k}(x_1, \ldots, x_n, \boldsymbol{y}) \mathrm{d}\lambda^k(\boldsymbol{y}).
\end{aligned} \tag{5.21}$$

Similarly,

$$\mathsf{E}\Big[z^n \int_{\mathbb{X}^k} \prod_{j=1}^k |\mathrm{e}^{-\beta v(x_0,y_j)} - 1| \mathrm{e}^{-\beta H(x_1,\dots,x_n|\eta)} \mathrm{d}\eta^{(k)}(\boldsymbol{y})\Big]$$
$$= \int_{\mathbb{X}^k} \prod_{j=1}^k |\mathrm{e}^{-\beta v(x_0,y_j)} - 1| \rho_{n+k}(x_1,\dots,x_n,\boldsymbol{y}) \mathrm{d}\lambda^k(\boldsymbol{y}).$$

Combining with Ruelle's moment bound ($\mathscr{R}_\xi$), we get

$$\sum_{k=1}^\infty \frac{1}{k!} \mathsf{E}\Big[z^n \int_{\mathbb{X}^k} \prod_{j=1}^k \Big|\mathrm{e}^{-\beta v(x_0,y_j)} - 1\Big| \mathrm{e}^{-\beta H(x_1,\dots,x_n|\eta)} \mathrm{d}\eta^{(k)}(\boldsymbol{y})\Big]$$
$$\leq \xi^n \exp\Big(\xi \int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x_0,y)} - 1| \mathrm{d}\lambda(y)\Big) < \infty. \quad (5.22)$$

This allows us to sum up the expressions (5.21) (after multiplication with $1/k!$) and to exchange summation and expectations and the claim follows. □

For the other direction, we show that (KS) implies the GNZ-equation in the form

$$\mathsf{E}\Big[\sum_{x \in S_\eta} n_x(\eta) F(x, \eta - \delta_x)\Big] = \int_{\mathbb{X}} \mathsf{E}\Big[F(x_0,\eta) z \mathrm{e}^{-\beta W(x_0;\eta)}\Big] \mathrm{d}\lambda(x_0) \quad (5.23)$$

The equivalence with (GNZ) is readily recognized upon setting $\tilde{F}(x;\eta) := F(x;\eta - \delta_x)$. Theorem 5.10 then shows $\mathsf{P} \in \mathscr{G}(\beta, z)$. We consider first the case

$$F(x_0,\eta) = \mathbb{1}_\Delta(x_0) \int_{\mathbb{X}^n} g \mathrm{d}\eta^{(n)} \quad (5.24)$$

with $\Delta \in \mathcal{X}_\mathsf{b}$ and $g : \mathbb{X}^n \to \mathbb{R}$ a bounded measurable map with bounded support.

Lemma 5.16. *Let P be a probability measure that satisfies Ruelle's bound ($\mathscr{R}_\xi$), v a stable, tempered pair potential, and H the associated energy. Let $F : \mathbb{X} \times \mathcal{N} \to \mathbb{R}$ be as above. Then*

$$\int_{\mathbb{X}} \mathsf{E}\Big[F(x_0,\eta) z \mathrm{e}^{-\beta W(x_0;\eta)}\Big] \mathrm{d}\lambda(x_0)$$
$$= \int_{\mathbb{X}^{n+1}} \mathbb{1}_\Delta(x_0) g(x_1,\dots,x_n) z \mathrm{e}^{-\beta W(x_0;x_1,\dots,x_n)}$$
$$\times \left(\sum_{k=0}^\infty \frac{1}{k!} \int_{\mathbb{X}^k} \prod_{j=1}^k (\mathrm{e}^{-\beta v(x_0,y_j)} - 1) \rho_{n+k}(\boldsymbol{x},\boldsymbol{y}) \mathrm{d}\lambda^k(\boldsymbol{y})\right) \mathrm{d}\lambda^{n+1}(\boldsymbol{x}). \quad (5.25)$$

We postpone the proof of the lemma and complete first the proof of Theorem 5.15.

Proof of (KS) $\Rightarrow \mathsf{P} \in \mathscr{G}(\beta, z)$. Let F be as in (5.24). The right-hand side of Eq. (5.23) is given by Lemma 5.16, the left-hand side is

$$\mathsf{E}\Big[\sum_{x \in S_\eta} F(x, \eta - \delta_x)\Big] = \mathsf{E}\Big[\int_{\mathbb{X}^{n+1}} \mathbb{1}_\Delta(x_0) g(x_1,\dots,x_n) \mathrm{d}\eta^{(n)}(x_1,\dots,x_n)\Big]$$
$$= \int_{\mathbb{X}^{n+1}} \mathbb{1}_\Delta(x_0) g(x_1,\dots,x_n) \rho_{n+1}(x_0,x_1,\dots,x_n) \mathrm{d}\lambda^{n+1}(\boldsymbol{x}). \quad (5.26)$$

The Kirkwood-Salsburg equations show that the right-hand sides of (5.25) and (5.26) are equal, hence (5.23) holds true for F given by (5.24). The usual arguments then show that the statement extends first to all maps of the form $F(x_0,\eta) = \mathbb{1}_\Delta(x_0)G(\eta)$ with $\Delta \in \mathcal{X}_\mathsf{b}$ and $G \in \mathcal{A}^{\mathrm{loc}}$, and then to all non-negative observables. Theorem 5.10 then guarantees $\mathsf{P} \in \mathscr{G}(\beta, z)$. □

Proof of Lemma 5.16. In order to evaluate the right side of (5.23), we note that for $\eta = \sum_{j=1}^{\kappa} \delta_{y_j}$

$$F(x_0,\eta)\mathrm{e}^{-\beta W(x_0;\eta)} = n!\mathbb{1}_\Delta(x_0) \sum_{\substack{J\subset[\kappa]\\ \#J=n}} \Big(g\big(\boldsymbol{y}_J\big)\mathrm{e}^{-\beta W(x_0;\boldsymbol{y}_J)}\Big)\mathrm{e}^{-\beta W(x_0;\boldsymbol{y}_{[\kappa]\setminus J})}$$

where $[\kappa] = \{1,\dots,\kappa\}$ if $\kappa \in \mathbb{N}$ and $[\kappa] = \mathbb{N}$ if $\kappa = \infty$, and $\boldsymbol{y}_I = (y_i)_{i\in I}$. Let us abbreviate

$$G(\boldsymbol{y}_J) = g\big(\boldsymbol{y}_J\big)\mathrm{e}^{-\beta W(x_0;\boldsymbol{y}_J)}, \quad h(\boldsymbol{y}_I) := \prod_{i\in I}\big(\mathrm{e}^{-\beta v(x_0,y_i)} - 1\big).$$

Then

$$\begin{aligned} F(x_0,\eta)\mathrm{e}^{-\beta W(x_0;\eta)} &= n!\mathbb{1}_\Delta(x_0) \sum_{\substack{J\subset[\kappa]\\ \#J=n}} G(\boldsymbol{y}_J) \sum_{\substack{I\subset[\kappa]\setminus J\\ \#I<\infty}} h(\boldsymbol{y}_I) \\ &= n!\mathbb{1}_\Delta(x_0) \sum_{\substack{L\subset[\kappa]\\ n\le\#L<\infty}} \sum_{\substack{I\subset L\\ \#(L\setminus I)=n}} h(\boldsymbol{y}_I)G(\boldsymbol{y}_{L\setminus I}) \end{aligned}$$

Therefore

$$\begin{aligned} &\mathsf{E}\big[F(x_0,\eta)\mathrm{e}^{-\beta W(x_0;\eta)}\big] \\ &= n!\mathbb{1}_\Delta(x_0)\sum_{m=n}^{\infty}\frac{1}{m!}\int_{\mathbb{X}^m}\left(\sum_{\substack{I\subset[m]\\ \#I=m-n}} h(\boldsymbol{y}_I)G(\boldsymbol{y}_{[m]\setminus I})\right)\rho_m(\boldsymbol{y})\mathrm{d}\lambda^m(\boldsymbol{y}) \\ &= n!\mathbb{1}_\Delta(x_0)\sum_{m=n}^{\infty}\frac{1}{m!}\int_{\mathbb{X}^m}\binom{m}{n}G(y_1,\dots,y_n)h(y_{n+1},\dots,y_m)\rho_m(\boldsymbol{y})\mathrm{d}\lambda^m(\boldsymbol{y}) \\ &= \mathbb{1}_\Delta(x_0)\sum_{k=0}^{\infty}\frac{1}{k!}\int_{\mathbb{X}^{n+k}} G(x_1,\dots,x_n)h(y_1,\dots,y_k)\rho_{n+k}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\lambda^n(\boldsymbol{x})\mathrm{d}\lambda^k(\boldsymbol{y}) \end{aligned}$$

and the proof is readily concluded. □

5.7. **Proof of Theorem 5.8.** Let $(\Lambda_\ell)_{\ell\in\mathbb{N}}$ be a sequence in $\mathcal{X}_\mathsf{b}$ such that every bounded set $B \in \mathcal{X}_\mathsf{b}$ is eventually contained in one of the Λ_ℓ's. For example, $\Lambda_\ell = B(0,\ell)$. Let $\rho_k^{(\ell)}$ be the k-point correlation function of $\mathsf{P}_{\Lambda_\ell}$. By assumption, $\rho_k^{(\ell)} \le \xi^k$ on Λ_ℓ^k. Passing to a subsequence if need be, we may assume that $\mathsf{P}_{\Lambda_\ell}$ converges locally to some measure P with correlation functions ρ_n. The k-point correlation functions satisfy a finite volume version of the Kirkwood-Salsburg equations, namely,

$$\rho^{(\ell)}_{n+1}(x_0,x_1,\dots,x_n) = z\mathrm{e}^{-\beta W(x_0;x_1,\dots,x_n)} \times \left(\rho^{(\ell)}_n(x_1,\dots,x_n) + \sum_{k=1}^\infty \frac{1}{k!}\int_{\Lambda_\ell^k}\prod_{k=1}^n (\mathrm{e}^{-\beta v(x_0,y_j)}-1)\rho^{(\ell)}_{n+k}(x_1,\dots,x_n,\boldsymbol{y})\mathrm{d}\lambda^k(\boldsymbol{y})\right) \tag{5.27}$$

on Λ_ℓ^{n+1}, with the convention $\rho_0^{(\ell)} = 1$. Choosing the version of the correlation functions that satisfies the formula from Lemma 3.12 pointwise, we may assume without loss of generality that (5.27) holds true for all $x_0, x_1, \dots, x_n \in \Lambda_\ell$ and all $n \in \mathbb{N}$. Ruelle's bound, the integrability of $y \mapsto \exp(-\beta v(x_0, y)) - 1$, and the weak convergence of the correlation functions imply that the sum on the right-hand side of (5.27) converges to the same expression with $\rho^{(\ell)}_{n+k}$ replaced with ρ_{n+k}. This holds true for all $n \in \mathbb{N}_0$ and all $x_0 \in \mathbb{X}$. For $n = 0$, we obtain

$$\lim_{\ell\to\infty} \rho_1^{(\ell)}(x_0) = z\left(1 + \prod_{k=1}^n (\mathrm{e}^{-\beta v(x_0,y_j)} - 1)\rho_k(\boldsymbol{y})\mathrm{d}\lambda^k(\boldsymbol{y})\right). \tag{5.28}$$

and the one-point correlation function converges pointwise. But pointwise convergence implies weak* convergence in $L^\infty(\mathbb{X}, \mathcal{X}, \lambda)$, so the pointwise limit equals the weak* limit, which is $\rho_1(x_0)$. It follows that the correlation functions of P satisfy (KS) for $n = 0$. An induction over n shows that $\rho_n^{(\ell)}$ converges pointwise to ρ_n, for all $n \in \mathbb{N}_0$, and that the correlation functions of P satisfy (KS) for all $n \in \mathbb{N}_0$. Theorem 5.15 then guarantees $\mathsf{P} \in \mathscr{G}(\beta, z)$. In particular, $\mathscr{G}(\beta, z) \neq \varnothing$. □

5.8. **Uniqueness for small z.** In this section we assume that the pair potential is locally stable,

$$\sum_{j=1}^n v(x, y_j) \geq -2B \tag{5.29}$$

for some $B \geq 0$ and all $n \in \mathbb{N}$, $x, y_1, \dots, y_n \in \mathbb{X}$ with $H(x, y_1, \dots, x_n) <, \infty$, and we ask that the integrability condition (I) holds true with some uniformity in x,

$$C(\beta) = \sup_{x\in\mathbb{X}} \int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x,y)} - 1| \mathrm{d}\lambda(y) < \infty. \tag{5.30}$$

A sufficient condition for (5.29) and (5.30) to hold true, in $\mathbb{X} = \mathbb{R}^d$, is that $v(x, y) = v(0, y - x)$ is translationally invariant, has a hard core, and is two-sided regular as in Theorem 4.5 .

Theorem 5.17. *Assume that the pair potential satisfies* (5.29) *and* (5.30)*. Then for all $\beta, z > 0$ with*

$$z < \mathrm{e}^{-2\beta B - 1} C(\beta)^{-1}, \tag{5.31}$$

the Gibbs measure is unique, $\#\mathscr{G}(\beta, z) = 1$.

Example 5.18 (Hard spheres). Let $\mathbb{X} = \mathbb{R}^d$, $\lambda = \mathrm{Leb}$ and $v(x, y) = \infty \mathbb{1}_{\{|x-y|\leq r\}}$ (with the convention $0 \cdot \infty = 0$). The pair potential is locally stable with $B = 0$ and $C(\beta) = |B(0, r)|$. The theorem shows that the condition

$$z|B(0, r)| < \frac{1}{\mathrm{e}}$$

is sufficient for the Gibbs measure to be unique.

For the proof we show that the solution to the Kirkwood-Salsburg equations is unique. First we rewrite the equations as a linear equation in a suitable Banach space. For $\xi > 0$, let E_ξ be the space of sequences $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ of real-valued functions $\rho_n \in L^\infty(\mathbb{X}^n, \mathcal{X}^{\otimes n}, \lambda^n)$ with

$$|\rho_n(x_1,\dots,x_n)| \le C_{\boldsymbol{\rho}}\, \xi^n$$

for some $C_{\boldsymbol{\rho}} \ge 0$ and all $n \in \mathbb{N}$. Let $||\boldsymbol{\rho}||_\xi$ be the smallest constant $C_{\boldsymbol{\rho}}$. The space $(E_\xi, ||\cdot||_\xi)$ is a Banach space. For $\boldsymbol{\rho} \in E_\xi$, we define a new sequence $(\mathbf{K}\boldsymbol{\rho})_{n\in\mathbb{N}}$ by

$$(\mathbf{K}\boldsymbol{\rho})_1(x_0) = \sum_{k=1}^\infty \frac{1}{k!}\int_{(\mathbb{X})^k} \prod_{j=1}^k \big(\mathrm{e}^{-\beta v(x_0,y_j)} - 1\big)\rho_k(\boldsymbol{y})\mathrm{d}\lambda^k(\boldsymbol{y})$$

and for $n \in \mathbb{N}$,

$$(\mathbf{K}\rho)_{n+1}(x_0,x_1,\dots,x_n) = z\mathrm{e}^{-\beta W(x_0;x_1,\dots,x_n)} \times \Big(\rho_n(x_1,\dots,x_n) + \sum_{k=1}^\infty \frac{1}{k!}\int_{(\mathbb{X})^k}\prod_{j=1}^k\big(\mathrm{e}^{-\beta v(x_0,y_j)}-1\big)\rho_{n+k}(x_1,\dots,x_n,\boldsymbol{y})\mathrm{d}\lambda^k(\boldsymbol{y})\Big).$$

Let $\boldsymbol{e} = (e_n)_{n\in\mathbb{N}}$ be the sequence defined by $e_1(x_0) = 1$ and $e_n = 0$ for $n \ge 2$. The Kirkwood-Salsburg equations become

$$\boldsymbol{\rho} = z\boldsymbol{e} + z\boldsymbol{K}\boldsymbol{\rho}. \tag{5.32}$$

Lemma 5.19. *Assume that the pair potential satisfies* (5.29) *and* (5.30). *Let* $\xi > 0$. *Then* $\mathbf{K}\boldsymbol{\rho} \in E_\xi$ *for all* $\boldsymbol{\rho} \in E_\xi$, *and*

$$||\mathbf{K}\boldsymbol{\rho}||_\xi \le \frac{1}{\xi}\mathrm{e}^{2\beta B}\mathrm{e}^{\xi C(\beta)}||\boldsymbol{\rho}||_\xi$$

Thus $\mathbf{K} : E_\xi \to E_\xi$ is a bounded linear operator with operator norm $||\mathbf{K}||_\xi \le \frac{1}{\xi}\mathrm{e}^{2\beta B}\mathrm{e}^{\xi C(\beta)}$.

Proof. We have

$$|(\mathbf{K}\boldsymbol{\rho})_1(x_0)| \le ||\boldsymbol{\rho}||\sum_{k=1}^\infty \frac{1}{k!}\int_{(\mathbb{X})^k}\prod_{j=1}^k\big|\mathrm{e}^{-\beta v(x_0,y_j)} - 1\big|\xi^k\mathrm{d}\lambda^k(\boldsymbol{y}) = (\mathrm{e}^{\xi C(\beta)} - 1)||\boldsymbol{\rho}||$$

and for $n \in \mathbb{N}$,

$$|(\mathbf{K}\boldsymbol{\rho})_{n+1}(x_0,\dots,x_n)| \le ||\boldsymbol{\rho}||\,\xi^n\mathrm{e}^{2\beta B}\mathrm{e}^{\xi C(\beta)}.$$

Therefore for all $n \in \mathbb{N}_0$,

$$|(\mathbf{K}\boldsymbol{\rho})_{n+1}| \le \xi^{n+1} \times \frac{1}{\xi}\mathrm{e}^{2\beta B}\mathrm{e}^{\xi C(\beta)}||\boldsymbol{\rho}||$$

and the lemma follows. □

Lemma 5.20. *Assume that the pair potential satisfies* (5.29) *and* (5.30). *Suppose that* (β, z) *satisfies* (5.31). *Then we can choose* $\xi > 0$ *such that*

$$z\,||\mathbf{K}||_\xi \le z\xi^{-1}\mathrm{e}^{2\beta B}\mathrm{e}^{\xi C(\beta)} < 1$$

and the equation $\boldsymbol{\rho} = z\boldsymbol{e} + z\mathbf{K}\boldsymbol{\rho}$, $\boldsymbol{\rho} \in E_\xi$, *has a unique solution.*

Proof. We have

$$z\,||\mathbf{K}||_\xi \le zC(\beta)\mathrm{e}^{2\beta B}\frac{\exp(\xi C(\beta))}{\xi C(\beta)}.$$

The right-hand side is strictly smaller than 1 if and only if

$$zC(\beta)\mathrm{e}^{2\beta B} < \xi C(\beta)\mathrm{e}^{-\xi C(\beta)}. \tag{5.33}$$

The function $x \mapsto x\exp(-x)$ attains its maximum $1/\mathrm{e}$ at $x = 1$, so if (5.33) holds true, then necessarily

$$zC(\beta)\exp(2\beta B) < \frac{1}{\mathrm{e}} = \sup_{\xi>0} \xi C(\beta)\mathrm{e}^{-\xi C(\beta)}.$$

i.e., (5.33) implies (5.31). Conversely, if $zC(\beta)\exp(2\beta B) < 1/\mathrm{e}$, then we can find $\xi > 1$ such that (5.33) holds true and $z||\mathbf{K}||_\xi < 1$. We have $z\boldsymbol{e} \in E_\xi$ because of

$$z \le z\mathrm{e}^{2\beta B} \le \xi\mathrm{e}^{-\xi C(\beta)} \le \xi.$$

The operator $(\mathrm{id} - z\mathbf{K})$ in E_ξ has a bounded inverse operator given by a Neumann series. It follows that the equation $\boldsymbol{\rho} = z\boldsymbol{e} + z\mathbf{K}\boldsymbol{\rho}$ in E_ξ has a unique solution, given by

$$\boldsymbol{\rho} = (\mathrm{id} - z\mathbf{K})^{-1}z\boldsymbol{e} = z\boldsymbol{e} + \sum_{\ell=1}^{\infty} z^{\ell+1}\mathbf{K}^\ell\boldsymbol{e}. \tag{5.34}$$

The lemma is proven. □

Proof of Theorem 5.17. We already know from Theorem 5.8 that $\#\mathscr{G}(\beta, z) \ge 1$. Pick $\mathsf{P} \in \mathscr{G}(\beta, z)$. Moreover by Proposition 5.12 and the local stability (5.29) ensure that $\rho_n \le z^n\mathrm{e}^{2\beta Bn}$. Let $\xi > 0$ be as in Lemma 5.20. Then $z\mathrm{e}^{2\beta Bn} \le \xi$, hence $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in E_\xi$. By Theorem 5.15, the correlation functions solve the Kirkwood-Salsburg equation $\boldsymbol{\rho} = z\boldsymbol{e}+z\mathbf{K}\boldsymbol{\rho}$. This determines $\boldsymbol{\rho}$ uniquely by Lemma 5.20. Since P satisfies Ruelle's moment bound, Theorem 2.42 in turn shows that P is uniquely determined by the correlation functions, so altogether $\mathsf{P} \in \mathscr{G}(\beta, z)$ is uniquely determined. □

5.9. **Summary.**

- In infinite volume, Gibbs measures are defined by structural property called DLR conditions after Dobrushin, Lanford and Ruelle: for each bounded observation window Δ, the behavior of the point process inside Δ conditioned on the outside $\eta_{\mathbb{X}\setminus\Delta} = \gamma_{\mathbb{X}\setminus\Delta}$ is governed by the finite volume Gibbs measure $\mathsf{P}_{\Delta|\gamma}$ with boundary condition γ.
- The DLR conditions are equivalent to the GNZ equation named after Georgii, Nguyen and Zessin. The GNZ equation singles out points x of the configuration rather than regions of space Δ.
- For well-behaved interactions, the set $\mathscr{G}(\beta, z)$ of infinite-volume Gibbs measures is non-empty, and every accumulation point of sequences of finite-volume Gibbs measures lies in $\mathscr{G}(\beta, z)$. Infinite-volume Gibbs measures are not necessarily unique.
- The correlation functions ρ_n of a Gibbs measure exist and are given by z^n times the expected value of the exponential of an interaction term.
- For well-behaved pair potentials, under Ruelle's bound $(\mathscr{R}_\xi)$, a probability measure P is a Gibbs measure if and only if its correlation functions satisfy a set of integral equations, the Kirkwood-Salsburg equations. They allow

us to map the uniqueness problem for Gibbs measures to a linear fixed point problem $\boldsymbol{\rho} = z\boldsymbol{e} + z\mathbf{K}\boldsymbol{\rho}$ in a suitable Banach space. For small z, the fixed point equation involves a contraction and the Gibbs measure is unique.

5.10. **Exercises.**

Exercise 5.1. Check properties (a) to (d) in Proposition 5.4.

Exercise 5.2. Let $\mathbb{X} = \mathbb{Z}$, $\lambda(B) = \#B$. For a finite non-empty set Δ, define

$$\pi_\Delta(\gamma, A) = \begin{cases} \frac{1}{\#\Delta}\sum_{x\in\Delta} \mathbb{1}_A(\delta_x), & N_{\mathbb{X}\setminus\Delta}(\gamma) = 0,\\ \mathbb{1}_A(\gamma_{\mathbb{X}\setminus\Delta}), & \text{else.}\end{cases}$$

(a) Show that $\pi_\Lambda\pi_\Delta = \pi_\Lambda$ for all $\Delta, \Lambda \in \mathcal{X}_{\mathsf{b}}$ with $\Delta \subset \Lambda$.
(b) Let $\mathscr{G}(\pi)$ be the set of probability measures P for which $\mathsf{P}\pi_\Delta = \mathsf{P}$ for all non-empty $\Delta \in \mathcal{X}_{\mathsf{b}}$. Show that for every $\mathsf{P} \in \mathscr{G}(\pi)$, $\mathsf{P}(N_{\mathbb{X}} = 1) = 1$.
(c) Show that $\mathscr{G}(\pi) = \varnothing$.

(See Friedli and Velenik [16, Exercise 6.15] for a variant with spin systems.)

Exercise 5.3. Let $v : \mathbb{X}\times\mathbb{X} \to \mathbb{R}\cup\{\infty\}$ be a stable pair potential, so that in particular, $v(x,y) \geq -2B > -\infty$ for all $x, y \in \mathbb{X}$. Fix $x \in \mathbb{X}$. Consider the following statements:

(a) $\int_{\mathbb{X}} |\exp(-\beta v(x,y)) - 1| \mathrm{d}\lambda(y) < \infty$ for some $\beta > 0$.
(b) $\int_{\mathbb{X}} |\exp(-\beta v(x,y)) - 1| \mathrm{d}\lambda(y) < \infty$ for all $\beta > 0$.
(c) There exists a measurable set $A = A_x \subset \mathbb{X}$ with

$$\lambda(A) < \infty, \qquad \int_{\mathbb{X}\setminus A} |v(x,y)| \mathrm{d}\lambda(y) < \infty.$$

Show that (a)$\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Exercise 5.4 (Convolution I). Let $\mathcal{P}_{\mathsf{f}}(\mathbb{N})$ be the collection of finite subsets of $\mathbb{N}$ (including the empty set $\varnothing \in \mathcal{P}_{\mathsf{f}}(\mathbb{N})$). We call a function $f : \mathcal{P}_{\mathsf{f}}(\mathbb{N}) \to \mathbb{R}$ *exchangeable* if $f(I)$ depends on the cardinality of $I \subset \mathbb{N}$ alone. The generating function of f is the formal power series

$$G_f(z) = f(\varnothing) + \sum_{n=1}^\infty \frac{z^n}{n!} f\bigl([n]\bigr).$$

For two exchangeable functions f, g, we define a new function $f * g$ by

$$\bigl(f * g\bigr)(I) = \sum_{J\subset I} f(J) g(I\setminus J).$$

Show that $G_{f*g}(z) = G_f(z) G_g(z)$.

Exercise 5.5 (Convolution II). For $f, g, : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$ we define a new function $f * g : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$ by

$$(f*g)\Bigl(\sum_{i=1}^n \delta_{x_i}\Bigr) = \sum_{I\subset[n]} f\Bigl(\sum_{i\in I}\delta_{x_i}\Bigr) g\Bigl(\sum_{i\in[n]\setminus I}\delta_{x_i}\Bigr)$$

with the usual convention that the sum over the empty set is 0; so in particular, $f*g(0) = f(0)g(0)$. Show that:

(a) $*$ is commutative and associative, i.e., $f*g = g*f$ and $f*(g*h) = (f*g)*h$ for all $f, g, h : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$.
(b) $f * \mathbb{1}_{\{N_{\mathbb{X}}=0\}} = f$ for all $f : \mathcal{N}_{\mathsf{f}} \to \mathbb{R}$.

(c) $\int_{\mathcal{N}_\mathsf{f}} (f*g)\mathrm{d}\tilde\lambda = \int_{\mathcal{N}_\mathsf{f}} f\mathrm{d}\tilde\lambda \int_{\mathcal{N}_\mathsf{f}} g\mathrm{d}\tilde\lambda$ for all f,g that are either both non-negative or both integrable with respect to $\tilde\lambda$.

(d) Show that Kf, Kg, and $K(f*g)$, as functions from $\mathcal{N}_\mathsf{f}$ to $\mathbb{R}$, satisfy $Kf = f*\mathbf{1}$ and $K(f*g) = f*(Kg)$.

Exercise 5.6. Let $\boldsymbol{K}$ be the Kirkwood-Salsburg operator and $\boldsymbol{e}$ the seqeunce from Section 5.6. Compute $z\boldsymbol{e} + z^2\boldsymbol{K}\boldsymbol{e} + z^3\boldsymbol{K}^2\boldsymbol{e}$. Express the result with the help of the graph weights $w(G;x_1,\dots,x_n)$ from the cluster expansions.

Exercise 5.7 (Convolution III, relation with Kirkwood-Salsburg). Let P be a probability measure on $(\mathcal{N},\mathfrak{N})$ with correlation functions ρ_n. For $f:\mathcal{N}_\mathsf{f}\to\mathbb{R}_+$ and $g:\mathcal{N}\to\mathbb{R}_+$, define $f*g:\mathcal{N}\to\mathbb{R}_+\cup\{\infty\}$ by

$$(f*g)(\eta) = f(0)g(\eta) + \sum_{k=1}^\infty \frac{1}{k!}\int_{\mathbb{X}^k} f\big(\delta_{x_1}+\cdots+\delta_{x_k}\big) g\big(\eta - \delta_{x_1} - \cdots - \delta_{x_k}\big)\mathrm{d}\eta^{(k)}(\boldsymbol{x}).$$

Show that:

(a) $K(f*g) = f*(Kg)$ on $\mathcal{N}$.

(b) The expected value with respect to P satisfies

$$\begin{aligned}\mathsf{E}\Big[f*(Kg)\Big] &= \sum_{\ell=0}^\infty \frac{1}{\ell!}\int_{\mathbb{X}^\ell} f(\delta_{x_1}+\cdots+\delta_{x_\ell})\\ &\qquad\times\Big(\sum_{m=0}^\infty\frac{1}{m!}\int_{\mathbb{X}^m} g(\delta_{y_1}+\cdots+\delta_{y_m})\rho_{\ell+m}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\lambda^m(\boldsymbol{y})\Big)\mathrm{d}\lambda^\ell(\boldsymbol{x}).\end{aligned}$$

(c) Fix $x_0\in\mathbb{X}$ and $g:\mathcal{N}_f\to\mathbb{R}_+$. Set $h(\delta_{y_1}+\cdots+\delta_{y_n}) = \prod_{j=1}^n(\mathrm{e}^{-\beta v(x_0,y_j)}-1)$ and $\tilde g(\eta) = g(\eta)\exp(-\beta W(x_0;\eta))$. Use (a) and (b) to explain, formally (don't worry about convergence of sums and integrals), the identity

$$\begin{aligned}\mathsf{E}\Big[(Kg)(\eta)\mathrm{e}^{-\beta W(x_0;\eta)}\Big] &= \mathsf{E}\Big[\tilde g * Kh\Big]\\ &= \sum_{\ell=0}^\infty\frac{1}{\ell!}\int_{\mathbb{X}^\ell} g(\delta_{x_1}+\cdots+\delta_{x_\ell})\mathrm{e}^{-\beta W(x_0;x_1,\dots,x_\ell)}\\ &\quad\times\Big(\sum_{m=0}^\infty\frac{1}{m!}\int_{\mathbb{X}^m}\prod_{i=1}^m(\mathrm{e}^{-\beta v(x_0,y_i)}-1)\rho_{\ell+m}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\lambda^m(\boldsymbol{y})\Big)\mathrm{d}\lambda^m(\boldsymbol{x}).\end{aligned}$$

6. Phase transition for the Widom-Rowlinson model

We have already encountered the Widom-Rowlinson model with energy function $|\bigcup_{i=1}^n B(x_i,1)| - n|B(0,1)|$. It was introduced by Widom and Rowlinson [55] as a model for liquid-vapor transitions and is one of the few continuum models for which a phase transition is rigorously known [50, 6]. Our goal is to show that there exists some $\beta_c < \infty$ such that at

$$\beta > \beta_c, \quad z = z_c(\beta) := \beta \mathrm{e}^{-\beta|B(0,1)|} \tag{6.1}$$

there is more than one Gibbs measure; moreover the partial derivative $\partial_z p(\beta, z)$ of the pressure has a jump discontinuity. The interpretation is that of a *phase transition* and *phase coexistence*: as the line $z = z_c(\beta)$ in the (β, z)-plane is crossed from small to larger z, the system switches from a low-density phase (vapor) to a high-density phase (liquid); on the line, both phases coexist, much in the same way as vapor and liquid water coexist at the boiling point.

The proof builds on a "two-color" variant of the model. Throughout this chapter, $\mathbb{X} = \mathbb{R}^2$, λ is the Lebesgue measure. The proof presented here follows closely Ruelle [50] and is a variant of the Peierls argument for the Ising model, see e.g. [16, Chapter 3.7.2]. A completely different proof that uses notions from stochastic geometry and percolation was given by J. Chayes, L. Chayes and Kotecký [6], see also Chapter 10 in the survey by Georgii, Häggström and Maes [22].

6.1. **The two-color Widom-Rowlinson model. Color symmetry.**

Define $H : \mathcal{N}_{\mathsf{f}} \times \mathcal{N}_{\mathsf{f}} \to \mathbb{R}_+$ by

$$H(\delta_{x_1} + \cdots + \delta_{x_m}, \delta_{y_1} + \cdots + \delta_{y_n}) = \begin{cases} 0, & |x_i - y_j| \geq 1 \ \forall i \in [m], j \in [n], \\ \infty, & \text{else.} \end{cases}$$

Equivalently,

$$H(\delta_{x_1} + \cdots + \delta_{x_m}, \delta_{y_1} + \cdots + \delta_{y_n}) = \sum_{i=1}^m \sum_{j=1}^n v(x_i - y_j)$$

with $v(x-y) = \infty \mathbb{1}_{\{|x-y| \geq 1\}}$. We view $H(\eta, \eta')$ as the energy for a system with two types of particles, say blue and red. Same-color particles don't interact, particles of different color have a hard core interaction. Notice that the interaction has finite range. Conditional energies $H(\eta, \eta | \gamma, \gamma')$ and energies with boundary conditions $H_\Lambda(\eta, \eta' \mid \gamma, \gamma')$ are defined in a way analogous to single-color cases.

The connection of the two-color with the one-color model is understood with an explicit computation: For $z_1, z_2 > 0$ and $\gamma_1, \gamma_2 \in \mathcal{N}$, we define

$$\Xi_{\Lambda|\gamma_1,\gamma_2}(z_1, z_2) := \sum_{m,n=0}^\infty \frac{z_1^m}{m!} \frac{z_2^n}{n!} \int_{\Lambda^m} \int_{\Lambda^n} \mathrm{e}^{-H_\Lambda(\sum_i \delta_{x_i}, \sum_j \delta_{y_j} | \gamma_1, \gamma_2)} \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y}.$$

The summand for $m = n = 0$ is to be read as 1, summands with $m \geq 1$ and $n = 0$ as $\frac{1}{m!} z^m \int_{\Lambda^m} \exp(-H_\Lambda(\sum_i \delta_{x_i}, 0 | \gamma_1, \gamma_2)) \mathrm{d}\boldsymbol{x}$, similarly for $m = 0$ and $n \geq 1$. It is instructive to integrate out one color. For empty boundary conditions $\gamma_1 = \gamma_2 = 0$, we have

$$\begin{aligned}\Xi_\Lambda(z_1,z_2) &= \sum_{m=0}^\infty \frac{z_1^m}{m!}\int_{\Lambda^m}\Big(\sum_{n=0}^\infty \frac{z_2^n}{n!}\int_{\mathbb{X}^n} \mathbb{1}_{\{\forall i,j:\, |x_i-y_j|\geq 1\}}\mathrm{d}\boldsymbol{y}\Big)\mathrm{d}\boldsymbol{x}\\ &= \sum_{m=0}^\infty \frac{z_1^m}{m!}\int_{\Lambda^m} \mathrm{e}^{z_1|\Lambda\setminus\cup_{i=1}^n B(x_i,1)|}\mathrm{d}\boldsymbol{x}\\ &= \mathrm{e}^{z_1|\Lambda|}\sum_{m=0}^\infty \frac{z^m}{m!}\int_{\Lambda^m}\mathrm{e}^{-\beta[|\Lambda\cap(\cup_{i=1}^m B(x_i,1))|-m|B(0,1)|]}\mathrm{d}\boldsymbol{x} \end{aligned}\tag{6.2}$$

with

$$\beta = z_2,\quad z = z_1 \mathrm{e}^{-z_2|B(0,1)|}.\tag{6.3}$$

In the exponent in (6.2) we recognize essentially the energy of $\delta_{x_1}+\cdots+\delta_{x_n}$ for the one-color Widom-Rowlinson model. Thus integrating out a color in the two-color model we end up with the one-color model. Notice

$$z_1 = z_2 \;\Leftrightarrow\; z = \beta \mathrm{e}^{-\beta|B(0,1)|},\tag{6.4}$$

so the curve $z = z_c(\beta)$ from (6.1) corresponds, in two-color model, to the two colors having the same activity $z_1 = z_2$.

Pressure and Gibbs measure. The notions of pressure and Gibbs measure generalize as follows to the two-color model: we define

$$p(z_1,z_2) = \lim \frac{1}{|\Lambda_n|}\log \Xi_{\Lambda_n}(z_1,z_2)\tag{6.5}$$

with the limit taken along van-Hove sequences $(\Lambda_n)_{n\in\mathbb{N}}$; the limit exists and is independent of the precise choice of the sequence. A measure P on $\mathcal{N}\times\mathcal{N}$ is a Gibbs measure at (z_1,z_2) if

$$\int_{\mathcal{N}^2} F\mathrm{d}\mathsf{P} = \int_{\mathcal{N}^2_{\Lambda^c}}\Big(\int_{\mathcal{N}^2_\Lambda} F\big(\eta+\gamma_{\Lambda^c},\eta'+\gamma'_{\Lambda^c}\big)\mathrm{e}^{-H_\Lambda(\eta,\eta'|\gamma,\gamma')}\mathrm{d}\tilde\lambda(\eta)\mathrm{d}\tilde\lambda(\eta')\Big)\mathrm{d}\mathsf{P}(\gamma,\gamma')\tag{6.6}$$

for all non-negative measurable $F:\mathcal{N}^2\to\mathbb{R}_+$, with

$$H_\Lambda(\eta,\eta'|\gamma,\gamma') = \begin{cases} 0, & \text{if } \min\big[\mathrm{dist}(S_\eta, S_{\eta'+\gamma'_{\Lambda^c}}),\mathrm{dist}(S_{\eta'},S_{\eta+\gamma_{\Lambda^c}})\big]\geq 1,\\ \infty, & \text{else}\end{cases}$$

(remember that $S_\eta\subset\mathbb{R}^d$ is just the support of η, i.e., the set of particle locations). Eq. (6.6) is a two-color version of Ruelle's equation, but we could just as well have written down the analogue of the DLR conditions or a variant of the GNZ equation. The set of Gibbs measures is denoted $\mathscr{G}(z_1,z_2)$; again it is non-empty.

Color symmetry. A key observation is the following: for all $z_1,z_2>0$,

- $H(\eta,\eta') = H(\eta',\eta)$—the energy is invariant with respect to the change-of-color map $(\eta,\eta')\mapsto(\eta',\eta)$.
- $p(z_1,z_2) = p(z_2,z_1)$.
- If $\mathsf{P}\in\mathscr{G}(z_1,z_2)$, then the image Q of P under the change-of-color map $(\eta,\eta')\mapsto(\eta',\eta)$ is in $\mathscr{G}(z_2,z_1)$.
- If $z_1=z_2=z$ and $\mathsf{P}\in\mathscr{G}(z,z)$, then the image Q of P under the change-of-color map is in $\mathscr{G}(z,z)$ as well.

As a consequence, if there exists a Gibbs measure $\mathsf{P} \in \mathscr{G}(z,z)$ that is not invariant under the change-of-color map, then $\#\mathscr{G}(z,z) \geq 2$. In that case we speak of *spontaneous symmetry breaking*: the energy and the activity are invariant under change of color but there is a Gibbs measure that isn't.

In finite volume, the only way to break the symmetry is via boundary conditions: the finite volume Gibbs measure P_Λ with empty boundary conditions is color-symmetric, but with boundary conditions that are not color-blind, this is no longer true. Therefore we are going to look at sequences of finite-volume measures with different boundary conditions and show that for large $z_1 = z_2$, the color preference induced by the boundary condition survives in the infinite volume limit.

6.2. **Contours. Peierls argument.** From now on $z_1 = z_2 = z$ and the dimension is $d = 2$. We want to construct an infinite-volume measure $\mathsf{P} \in \mathscr{G}(z,z)$ that breaks the color symmetry. The intuitive picture is the following: Let us think of the system as a mixture of red and blue disks of radius $1/2$ with the constraint that disks of opposite color are not allowed to overlap. If a Gibbs measure is color-symmetric, it should have red and blue particles in equal proportion. Because of the hard-core interaction between distinct colors, regions occuped by red and blue particles are separated by corridors that have no particles at all. But at large z, we may expect that the system has a high density and it is highly unlikely that a given region of space stays empty. Therefore the system should want to avoid changes of color.

To make this picture precise, we discretize space into little cells and introduce a notion of contours. Set $a := \frac{1}{3\sqrt{2}}$. For $k = (k_1, k_2) \in \mathbb{Z}^2$, let $Q_k = [k_1 a, (k_1+1)a) \times [k_2 a, (k_2+1)a)$. Two distinct cells Q_k and Q_ℓ are neighbors if they share an edge or a corner, i.e., if $\overline{Q_k} \cap \overline{Q_\ell} \neq \varnothing$. The side length is chosen in such a way that the diagonal of a cube of sidelength $3a$, comprising 3×3 little cells, has length 1.

We only look at rectangular domains Λ that are unions of finitely many cells, and impose blue boundary conditions: let $\chi = (0, \chi^{\mathsf{B}})$ with $N_{Q_k}(\chi^{\mathsf{B}}) \geq 1$ for all $k \in \mathbb{Z}^2$.

Let $(\eta^{\mathsf{R}}, \eta^{\mathsf{B}}) \in \mathcal{N}_\Lambda^2$ be a configuration such that $(\eta^{\mathsf{R}}, \eta^{\mathsf{B}} + \chi^{\mathsf{B}}_{\Lambda^c})$ is admissible, i.e., opposite-color particles have mutual distance ≥ 1: we have $|x-y| \geq 1$ for all $x \in S_{\eta^{\mathsf{R}}}$ and $y \in S_{\eta^{\mathsf{B}} + \chi^{\mathsf{B}}_{\Lambda^c}}$. Given $(\eta^{\mathsf{R}}, \eta^{\mathsf{B}}) \in \mathcal{N}_\Lambda^2$ we mark the cells Q_k as follows:

- A cell is blue if it contains a blue particle from η or χ_{Λ^c}, and red if it contains a red particle. (Note that it cannot contain particles of differing colors.) A cell is white if it contains no particles.
- In addition to the coloring, we shade every red cell and all its neighboring eight cells. A shaded cell and a blue cell can never be neighbors.

We may view $(\eta^{\mathsf{R}}, \eta^{\mathsf{B}})$ as a random variable with distribution $\mathsf{P}_{\Lambda|\chi}$, then the coloring and shading are random as well. The boundary

$$\Gamma = \Gamma(\eta^{\mathsf{R}}, \eta^{\mathsf{B}} + \chi^{\mathsf{B}}_{\Lambda^c}) := \partial\Big(\bigcup_{\substack{k \in \mathbb{Z}^2,\ Q_k \subset \Lambda: \\ Q_k \text{ shaded}}} Q_k \Big)$$

of the union of shaded regions is a collection of finite-length polygonal lines $\ell_1, \dots, \ell_m$ contained in Λ. The boundary Γ is the union of connected groups of lines, precisely: A group of lines $\ell_1, \dots, \ell_k$ is 1-connected if for all $i, j \in \{1, \dots, k\}$, there is a finite sequence $i_0, \dots, i_q$ in $\{1, \dots, k\}$ such that $i_0 = i$, $i_q = j$, and $\operatorname{dist}(\ell_{i_r}, \ell_{i_{r+1}}) \leq 1$ for all $r \in \{1, \dots, q\}$. We may write $\Gamma = \gamma_1 \cup \dots \cup \gamma_m$ where each γ_r is a 1-connected group of lines, and m is an integer depending on Γ. We call each 1-connected group γ_i of lines a contour.

For a given contour γ, a point $x \in \mathbb{R}^2 \setminus \gamma$ is *interior* to γ if any continuous path in Λ connecting x to $\partial\Lambda$ crosses γ an odd number of times. The contour is called an *outer contour* for $(\eta^{\mathsf{R}}, \eta^{\mathsf{B}} + \chi^{\mathsf{B}}_{\Lambda^c})$ if it can be reached from $\partial\Lambda$ by a continuous path in Λ that does not cross Γ. Let $\mathscr{C}_\Lambda(\eta^{\mathsf{R}}, \eta^{\mathsf{B}})$ be the collection of outer contours.

The *length* of a contour is the sum of the lengths of the contained lines.

Lemma 6.1. *Let Λ and χ be as above, $z_1 = z_2 = z$, and γ a group of 1-connected lines (contour) of length ℓa with $\ell \in \mathbb{N}$. Then*

$$\mathsf{P}_{\Lambda|\chi}\Big(\gamma \in \mathscr{C}_\Lambda(\eta^{\mathsf{R}}, \eta^{\mathsf{B}})\Big) \leq \mathrm{e}^{-\ell a^2 z/2}.$$

Proof. Let

$$\Omega_\Lambda := \big\{(\eta^{\mathsf{R}}, \eta^{\mathsf{B}}) \in \mathcal{N}_\Lambda^2 \mid \mathrm{dist}(S_{\eta^{\mathsf{R}}}, S_{\eta^{\mathsf{B}} + \chi^{\mathsf{B}}_{\Lambda^c}}) \geq 1\big\}$$

be the set of configurations without color conflicts. Note $\mathsf{P}_{\Lambda|\chi}(\Omega_\Lambda) = 1$. Set

$$A := \Omega_\Lambda \cap \{\eta \in \mathcal{N}_\Lambda^2 \mid \gamma \in \mathscr{C}_\Lambda(\eta^{\mathsf{R}}, \eta^{\mathsf{B}})\}.$$

We want to show that A is unlikely. Roughly, the idea of the proof is to map each configuration $\omega \in A$ to a configuration ω'' that is much more likely. We do this by first flipping colors in the interior of γ and then filling up the empty corridor located along γ with blue particles; the previous color flip ensures that filling up the corridor does not create color conflicts, and the filling up removes the empty corridor, which should result in a much more likely configuration.

More rigorously, we define a new event B in two steps. First, given $\omega = (\eta^{\mathsf{R}}, \eta^{\mathsf{B}}) \in A$, define a new configuration ω' by flipping all colors in the interior $\mathrm{Int}(\gamma)$ of γ, i.e.,

$$\omega' = \big(\eta^{\mathsf{R}}_{\Lambda \setminus \mathrm{Int}(\gamma)} + \eta^{\mathsf{B}}_{\mathrm{Int}(\gamma)}, \eta^{\mathsf{B}}_{\Lambda \setminus \mathrm{Int}(\gamma)} + \eta^{\mathsf{R}}_{\mathrm{Int}(\gamma)}\big).$$

Then $\omega' \in \Omega_\Lambda$, i.e., no color conflicts have been produced. Indeed, let $x, y \in S_{\eta^{\mathsf{R}} + \eta^{\mathsf{B}}} \subset \Lambda$ be two points of the configuration ω with $|x - y| < 1$. Because of $\omega \in \Omega_\Lambda$, we know that x and y have the same color, i.e., either they are both in $S_{\eta^{\mathsf{R}}}$ or they are both in $S_{\eta^{\mathsf{B}}}$. We distinguish cases:

- If x and y are both in $\Lambda \setminus \mathrm{Int}(\gamma)$, their colors have not changed and so they have the same color after the color flip.
- If x and y are both in $\mathrm{Int}(\gamma)$, then they both change colors and stay compatible—either they were both blue before the flip and are both red after the flip, or vice-versa.
- If $x \in \mathrm{Int}(\gamma)$ and $y \in \Lambda \setminus \mathrm{Int}(\gamma)$, we might be worried about a color conflict.

let $A' \subset \mathcal{N}_\Lambda^2$ be the collection of □

Lemma 6.2. *Let $n(\ell)$ be the number of of contours of length ℓa enclosing Q_0. Then*

$$n(\ell) \leq \ell 3^{2\ell}.$$

Lemma 6.3. *Let Λ and χ be as above. Assume $Q_0 \subset \Lambda$ and $z_1 = z_2 = z$ with z large enough so that $9\exp(-a^2 z/2) < 1$. Then*

$$\mathsf{P}_{\Lambda|\chi}\Big(\exists \gamma \in \mathscr{C}_\Lambda(\eta^{\mathsf{R}}, \eta^{\mathsf{B}}) : \gamma \text{ encloses } Q_0\Big) \leq \frac{9\exp(-a^2 z/2)}{(1 - 9\exp(-a^2 z/2))^2} =: g(z).$$

So when z is large, the probability is small, and the bound is uniform in Λ.

6.3. **Phase transition for the two-color Widom-Rowlinson model.**

Proposition 6.4. *Let Λ be a union of little cells Q_k, with $Q_0 \subset \Lambda$, and χ a blue boundary condition as above. Then as $z_1 = z_2 = z \to \infty$,*

$$\mathsf{P}_{\Lambda|\chi}\big(Q_0 \text{ is red}\big) = O\big(\mathrm{e}^{-a^2 z/2}\big), \quad \mathsf{P}_{\Lambda|\chi}\big(Q_0 \text{ is blue}\big) = 1 + O\big(\mathrm{e}^{-a^2 z/2}\big),$$

uniformly in Λ.

Theorem 6.5. *Consider the two-color Widom-Rowlinson model in $\mathbb{R}^2$. There exists $z_c < \infty$ such that for all $z_1 = z_2 = z > z_c$, we have $\#\mathscr{G}(z, z) \geq 2$.*

Remark (Phase transition). In the mathematical literature, non-uniqueness of Gibbs measures is sometimes adopted as a definition of phase transition, see e.g. [20, Def. 2.11]. Another definition is that a phase transition occurs at (z_1^0, z_2^0) if the pressure $p(z_1, z_2)$ is non-analytic at (z_1^0, z_2^0), see [27, Chapter 7]. For general models, these two definitions need not coincide! For the two-color Widom-Rowlinson model, however, points (z, z) with z large should correspond to a phase transition in both senses: the partial derivatives $\rho_1 = z_1 \partial_1 p(z_1, z_2)$, which represents an average density of red particles, should have a jump discontinuity at the line (z, z) with $z > z_c$.

6.4. **Phase transition for the one-color Widom-Rowlinson model.**

Theorem 6.6. *Consider the one-color Widom-Rowlinson model in $\mathbb{R}^2$. For $\beta > 0$, let $z_c(\beta) := \beta \exp(-\beta |B(0,1)|)$. There exists $\beta_c < \infty$ such that for all $\beta > \beta_c$, $\#\mathscr{G}(\beta, z_c(\beta)) \geq 2$.*

6.5. **Summary.**

- The Widom-Rowlinson model comes in two variants, a two-color and a one-color model. Roughly, the one-color model is obtained from the two-color model by looking at marginals. The parameter correspondence is such that the color-symmetry line $z_1 = z_2$ in the two-color model orresponds to an activity-temperature curve $z = z_c(\beta) = \beta \exp(-\beta |B(0,1)|)$ in the one-color model.
- Both models display a phase transition in the sense of non-uniqueness of Gibbs measures.
- The two-color model has a color symmetry. In order to prove phase transition in the sense of non-uniqueness of Gibbs measures, we show that when $z_1 = z_2 = z$ is large, the color symmetry is spontaneously broken.
- Intuitively, if $z_1 = z_2 = z$ is large, then borders between regions of different colors are penalized because they are a no man's land (the hardcore exclusion between opposite colors imposes that they are separted by empty corridors), but empty space should be unlikely at high z. Therefore color preferences induced by boundary conditions should survive even in the thermodynamic limit.
- The intuition is formalized by discretizing space and working with blue boundary conditions. Admissible configurations are associated with colorings and shadings of discretized space, leading to a picture of red islands in a blue sea—islands may have lakes and islands within lakes—and a notion of contours.
- For the Peierls-type argument one checks (1) the probability of finding a given long contour is exponentially small in the contour length, and (2)

the number of contours of a given length that enclose the origin is at most exponentially large in ℓ. For large z, (1) wins over (2).

7. Cluster expansions

Let us come back to pair potentials $v(x,y)$ that are stable and satisfy $C(\beta) := \sup_{x\in\mathbb{X}} \int_{\mathbb{X}} |f(x,y)| \mathrm{d}\lambda(y) < \infty$, with $f(x,y) = f_\beta(x,y) = \exp(-\beta v(x,y)) - 1$. Remember from Section 4.3 that

$$\log \Xi_\Lambda(\beta, z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\Lambda^n} \sum_{G\in\mathcal{C}_n} w(G; x_1, \dots, x_n) \mathrm{d}\lambda^n(\boldsymbol{x})$$

with $w(G; x_1, \dots, x_n) = \prod_{\{i,j\}\in E(G)} f(x_i, x_j)$. Section 4.3 dealt with translationally invariant pair potentials in $\mathbb{R}^d$, but the arguments generalize to the present setup in a straightforward way. The goal of this chapter is to provide a bound for the radius of convergence that is uniform in Λ. The key idea is to map graphs to trees and group graphs that project to the same tree, systematizing and refining the crude argument from the proof of Lemma 4.14.

7.1. **Tree-graph inequality for non-negative interactions.**

One fairly natural way of mapping a connected graph to a tree is as follows. Fix $n \in \mathbb{N}$ and $G \in \mathcal{C}_n$. Construct a tree $T \in \mathcal{T}_n$ by successively adding edges as follows:

- Add all edges $\{1, i\}$ from G that are incident to 1. The vertices $i_1, \dots, i_m$ such that $\{1, i\} \in E(G)$ form the first generation of the tree, the vertex 1 is treated as the root.
- Label the first-generation vertices in increasing order, i.e., $i_1 < \dots < i_m$.
  - Go through the edges $\{i_1, j_1\}, \dots, \{i_1, j_r\}$ (in increasing order $j_1 < \dots < j_r$) in $E(G)$ that emanate from i_1; for each edge, add it if it doesn't create a loop and skip it if does create a loop.
  - Repeat the previous step for the other first generation vertices, in increasing order (first i_2, then i_3, etc.)

  The vertices j_ℓ appearing in this step form the second generation of the tree.
- Repeat the previous step for the second generation vertices, then third generation, etc., until there are no more edges to add.

The procedure ends with a tree $T \in \mathcal{T}_n$. Let $\pi : \mathcal{C}_n \to \mathcal{T}_n$ be the map $G \mapsto T$ thus constructed.

It is instructive to investigate which edges $\{i, j\}$ from G have been discarded. For $i \in [n]$, let $d(i)$ be the graph distance to the root 1; equivalently, the number of the generation to which i belongs. Suppose that $\{i, j\} \in E(G) \setminus E(T)$ and assume without loss of generality that $d(i) < d(j)$. Then $d(j)$ cannot be larger than $d(i) + 2$; if it was, then it would connect to some same-generation cousin of i and in the iterative construction of $E(T)$ we would have added it when going through the generation of i. Thus we are left with two cases:

- either $d(i) = d(j)$, i.e., i and j belong to the same generation,
- or $d(j) = d(i) + 1$. In that case let i' be the parent of j in T (i.e. $\{i', j\} \in E(T)$, $d(i') = d(j) - 1$). Then we must have $i' < i$, since otherwise we would have added $\{i, j\}$ before examining $\{i', j\}$.

Thus $E(G) \setminus E(T) \subset E'(T)$ where $E'(T)$ consists of all edges $\{i, j\}$ that satisfy

- $d(i) = d(j)$ (edge within a generation), or
- $d(j) = d(i) + 1$ and $i' < i$, with i' the parent of j (edges toward a younger uncle).

Any graph G that is mapped to a given tree T, i.e., $\pi(G) = T$, satisfies $E(T) \subset E(G) \subset E(T) \cup E'(T)$.

Conversely, given $T \in \mathcal{T}_n$, define $E'(T)$ as above and let $R(T)$ be the graph with vertex set $E(T) \cup E'(T)$. Then if $G \in \mathcal{C}_n$ satisfies $E(T) \subset E(G) \subset E(R(T))$, we have $\pi(G) = T$. Thus $\pi(G) = T$ if and only if the graph is in the "interval" of subgraphs of $R(T)$ that also contain T.

Definition 7.1. *A map $\pi : \mathcal{C}_n \to \mathcal{T}_n$ is a* tree partition scheme *if for every $T \in \mathcal{T}_n$, there is a graph $R(T) \in \mathcal{C}_n$ such that*

$$\pi^{-1}(\{T\}) = [T, R(T)] := \{G \in \mathcal{C}_n \mid E(T) \subset E(G) \subset E(R(T))\}.$$

Equivalently, a tree partition scheme is a set partition of the collection of connected graphs into intervals $[T, R(T)]$ indexed by trees T.

Example 7.2 (Penrose partition scheme). The map π described above is a tree partition scheme, sometimes called *Penrose partition scheme*.

Proposition 7.3. *Suppose that $v \geq 0$ on $\mathbb{X}^2$. Then for every $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in \mathbb{X}^n$, we have*

$$\Big|\sum_{G \in \mathcal{C}_n} w(G; x_1, \ldots, x_n)\Big| \leq \sum_{T \in \mathcal{T}_n} |w(T; x_1, \ldots, x_n)|.$$

Proof. Let $\pi : \mathcal{C}_n \to \mathcal{T}_n$ be a tree partition scheme, for example, the Penrose partition scheme, and $R(T)$ as in Definition 7.1. Then

$$\begin{aligned}\sum_{G \in \mathcal{C}_n} w(G; \boldsymbol{x}) &= \sum_{T \in \mathcal{T}_n} \sum_{\substack{G \in \mathcal{C}_n: \\ \pi(G) = T}} w(G; \boldsymbol{x}) \\ &= \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} f(x_i, x_j) \sum_{E' \subset E(R(T)) \setminus E(T)} \prod_{\{i,j\} \in E'} f(x_i, x_j) \\ &= \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} f(x_i, x_j) \prod_{\{i,j\} \in E(R(T)) \setminus E(T)} (1 + f(x_i, x_j)). \qquad (7.1)\end{aligned}$$

We take absolute values, use the triangle inequality and $1 + f(x_i, x_j) = \mathrm{e}^{-\beta v(x_i, x_j)} \in [0, 1]$, and obtain the required inequality. □

Remark (Alternating sign property). Since every tree has $n-1$ edges and $f(x_i, x_j) \leq 0$ and $1 + f(x_i, x_j) \geq 0$, Eq. (7.1) shows that the weight has the sign of $(-1)^{n-1}$, i.e.,

$$\sum_{G \in \mathcal{C}_n} w(G; \boldsymbol{x}) = (-1)^{n-1} \Big|\sum_{G \in \mathcal{C}_n} w(G; \boldsymbol{x})\Big|.$$

Remark (Tree-graph identity for hard-core interactions). Suppose that $v(x, y) \in \{0, \infty\}$ for all $x, y \in \mathbb{X}^2$, e.g., $v(x, y) = \infty$ if $\mathrm{dist}(x, y) < 1$ and 0 otherwise. Then $f(x, y) \in \{-1, 0\}$ for all x, y. Let us write $x \,\iota\, y$ ("x intersects y") if $f(x, y) = -1$, $v(x, y) = \infty$. Let $G(\boldsymbol{x}) \in \mathcal{G}_n$ be the graph with edge set $\{\{i, j\} \mid x_i \,\iota\, x_j\}$. Then

$$\begin{aligned}&\sum_{G \in \mathcal{C}_n} w(G; \boldsymbol{x}) \\ &= (-1)^{n-1} \#\{T \in \mathcal{T}_n \mid E(T) \subset E(G(\boldsymbol{x})),\ \big(E(R(T)) \setminus E(T)\big) \cap E(G(\boldsymbol{x})) = \varnothing\}. \qquad (7.2)\end{aligned}$$

Example 7.4 (Singletons $\mathbb{X} = \{x\}$ and expansion of the logarithm). Consider the Penrose partition scheme and suppose that $\mathbb{X} = \{x\}$ is a singleton, λ is the counting measure $\lambda(\{x\}) = 1$, and $v(x,x) = \infty$. Then $G(\boldsymbol{x})$ is the complete graph (i.e., it contains all edges $\{i,j\}$) and the identity of the previous remark simplifies to

$$\sum_{G\in\mathcal{C}_n} w(G;\boldsymbol{x}) = (-1)^{n-1}\#\{T\in\mathcal{T}_n \mid T = R(T)\}.$$

A tree satisfies $R(T) = T$ if and only if 1 has outdegree 1 and the tree is linear, i.e., if there exists an enumeration $1 = i_1,\dots,i_n$ of the vertices $1,\dots,n$ such that $E(T) = \{\{1,i_2\},\{i_2,i_3\},\dots,\{i_{n-1},i_n\}\}$. The number of such trees is $(n-1)!$. Then by (7.2),

$$\sum_{G\in\mathcal{C}_n} w(G;\boldsymbol{x}) = (-1)^{n-1}(n-1)!$$

and for $|z|<1$,

$$\sum_{n=1}^{\infty}\frac{z^n}{n!}(-1)^{n-1}(n-1)! = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n = \log(1+z),$$

in agreement with

$$\begin{aligned}\Xi_{\mathbb{X}}(z) &= 1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{x_1,\dots,x_n\in\mathbb{X}} \mathrm{e}^{-\beta\sum_{1\le i<j\le n} v(x_i,x_j)}\\ &= 1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\mathrm{e}^{-\beta\frac{n(n-1)}{2}v(x,x)} = 1+z.\end{aligned}$$

7.2. **Tree-graph inequality for stable interactions.** If the pair potential $v(x,y)$ takes negative values, we can no longer estimate $1+f\le 1$ and need to be more careful. Arecent approach [45, 54] rests on a specific choice of a tree partition scheme.

Suppose that the set of edges $\{\{i,j\}\mid 1\le i<j\le n\}$ is equipped with a total order $\prec$. Given a graph $G\in\mathcal{C}_n$, let $e_1\prec e_2\cdots\prec e_m$ be an enumeration of the edges of G. We define a tree T by adding edges successively as follows:

- First, add the smallest edge e_1 from G. Add e_2 as well.
- Next, add e_3 unless it creates a loop.
- Repeat: go through the edges $e_4, e_5,\dots$ in that order and for each edge, either keep it if it doesn't create a loop, or discard it if it does. Stop when all edges of G have been examined.

This results in a graph T that has no loops and satisfies $E(T)\subset E(G)$. If $\{i,j\}\in E(G)\setminus E(T)$, adding $\{i,j\}$ would have created a loop, which means that i and j are connected by a path in G consisting of edges $e\prec\{i,j\}$ that have been added in the iterative procedure described above before reaching $\{i,j\}$. In particular, i and j are connected by a path in T. Thus T is loop-free and connected, i.e., a tree.

Conversely, fix a tree $T\in\mathcal{T}_n$. Let $E'(T)$ be the collection of edges $\{i,j\}$ that are not in T and satisfy the following: every edge e in the path $\gamma_{i,j}$ connecting i to j in T satisfies $e\prec\{i,j\}$. Let $R(T)\in\mathcal{C}_n$ be the graph with edge set $E(T)\cup E'(T)$. Then for every $G\in[T,R(T)]$, we have $\pi(G) = T$. It follows that $\pi^{-1}(\{T\}) = [T,R(T)]$. Thus we have checked:

Lemma 7.5. *For every total order $\prec$ on $\{\{i,j\}\mid 1\le i<j\le n\}$, the map $\pi = \pi_{\prec}:\mathcal{C}_n\to\mathcal{T}_n$ described above is a tree partition scheme.*

The key idea for the proof of a more general tree-graph identity is to pick a total order $\prec$ on edges such that $\{i,j\} \mapsto v(x_i,x_j)$ is increasing, i.e.,

$$\{i,j\} \prec \{k,\ell\} \ \Rightarrow v(x_i,x_j) \leq v(x_k,x_\ell). \tag{7.3}$$

The order depends on $x_1,\dots,x_n$ but this is okay since we only need to prove a tree-graph inequality pointwise, i.e., at fixed $\boldsymbol{x}$. Another proof ingredient is a clever decomposition of Mayer's f-function.

Lemma 7.6. *For all $\beta > 0$ and $x,y \in \mathbb{X}$, we have*

$$\bigl|\mathrm{e}^{-\beta v(x_i,x_j)} - 1\bigr| = \mathrm{e}^{(\beta v(x_i,x_j))_-}\bigl(1 - \mathrm{e}^{-\beta |v(x_i,x_j)|}\bigr).$$

Proof. Let $u := \beta v(x,y)$. If $u \geq 0$, then $u_- = 0$ and

$$|\mathrm{e}^{-u} - 1| = 1 - \mathrm{e}^{-u} = \mathrm{e}^{u_-}\bigl(1 - \mathrm{e}^{-|u|}\bigr).$$

If $u < 0$, then $|u| = -u = -u_-$ and

$$|\mathrm{e}^{-u} - 1| = \mathrm{e}^{-u} - 1 = \mathrm{e}^{-u}\bigl(1 - \mathrm{e}^{u}\bigr) = \mathrm{e}^{u_-}\bigl(1 - \mathrm{e}^{-|u|}\bigr)$$

and the lemma follows. $\square$

Theorem 7.7. *Let $v : \mathbb{X}^2 \to \mathbb{R} \cup \{\infty\}$ be a stable pair potential with stability constant B. Then for all $n \in \mathbb{N}$ and $x_1,\dots,x_n \in \mathbb{X}^n$, we have*

$$\begin{aligned}\Bigl|\sum_{G \in \mathcal{C}_n} w(G;x_1,\dots,x_n)\Bigr| &\leq \mathrm{e}^{\beta B n} \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} \bigl(1 - \mathrm{e}^{-\beta |u(x_i,x_j)|}\bigr) \\ &\leq \mathrm{e}^{\beta B n} \sum_{T \in \mathcal{T}_n} \bigl|w(T;x_1,\dots,x_n)\bigr|.\end{aligned}$$

Proof. Fix $n \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{X}^n$. Pick a total order $\prec$ on $\{\{i,j\} \mid 1 \leq i < j \leq n\}$ such that (7.3) holds true and consider the associated tree partition scheme $\pi = \pi_\prec$ with $R(T)$ and $E'(T)$ as defined above. Then

$$\begin{aligned}\sum_{G \in \mathcal{C}_n} w(G;x_1,\dots,x_n) &= \sum_{T \in \mathcal{T}_n} \sum_{\substack{G \in \mathcal{C}_n: \\ \pi(G) = T}} w(G;x_1,\dots,x_n) \\ &= \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} f(x_i,x_j) \prod_{\{i,j\} \in E'(T)} \bigl(1 + f(x_i,x_j)\bigr)\end{aligned}$$

and

$$\Bigl|\sum_{G \in \mathcal{C}_n} w(G;x_1,\dots,x_n)\Bigr| \leq \sum_{T \in \mathcal{T}_n} \prod_{\{i,j\} \in E(T)} \bigl|\mathrm{e}^{-\beta v(x_i,x_j)} - 1\bigr| \prod_{\{i,j\} \in E'(T)} \mathrm{e}^{-\beta v(x_i,x_j)}.$$

Given $T \in \mathcal{T}_n$, let T_- be the graph with edge set $\{\{i,j\} \in E(T) \mid v(x_i,x_j) < 0\}$. Lemma 7.6 yields

$$\begin{aligned}&\Bigl|\sum_{G \in \mathcal{C}_n} w(G;x_1,\dots,x_n)\Bigr| \\ &\leq \sum_{T \in \mathcal{T}_n} \Bigl(\prod_{\{i,j\} \in E(T)} \bigl(1 - \mathrm{e}^{-\beta |v(x_i,x_j)|}\bigr)\Bigr) \mathrm{e}^{-\beta \sum_{\{i,j\} \in E(T_-)} v(x_i,x_j) - \beta \sum_{\{i,j\} \in E'(T)} v(x_i,x_j)}.\end{aligned} \tag{7.4}$$

Now, T_- is a forest, i.e., a collection of subtrees $T_1,\dots,T_r$ whose vertex sets $V_1,\dots,V_k$ form a partition of some subset of $\{1,\dots,n\}$. Any edge $\{i,j\}$ from T or $E'(T)$ connecting two distinct subtrees satisfies $v(x_i,x_j) \geq 0$: for T this is true by definition of T_- and $T_1, \ldots, T_r$, for $E'(T)$ this follows because $v(x_i, x_j) \geq v(x_k, x_\ell) \geq 0$ for any edge $\{k, \ell\}$ in the path connecting i to j in T. Discarding edges from $E'(T)$ that connect different subtrees, we find

$$\sum_{\{i,j\}\in E(T_-)} v(x_i, x_j) + \sum_{\{i,j\}\in E'(T)} v(x_i, x_j) \geq \sum_{m=1}^{r} \Big(\sum_{\substack{\{i,j\}\in E(T_-):\\ i,j\in V_m}} v(x_i, x_j) + \sum_{\substack{\{i,j\}\in E'(T):\\ i,j\in V_m}} v(x_i, x_j) \Big). \quad (7.5)$$

Two distinct edges $i, j \in V_m$ are connected by a path of edges $\{k, \ell\}$ in T_-, i.e., $v(x_k, x_\ell) < 0$. Consequently if $i, j \in V_m$ and $v(x_i, x_j) \geq 0$, then $\{i, j\} \in E'(T)$. Therefore if $i, j \in V_m$ with $i \neq j$ and $\{i, j\} \notin E(T_-) \cup E'(T)$, then $v(x_i, x_j) < 0$, hence

$$\sum_{\substack{\{i,j\}\in E(T_-)\cup E'(T):\\ i,j\in V_m}} v(x_i, x_j) \geq \sum_{\substack{i,j\in V_m:\\ i<j}} v(x_i, x_j) \geq -B\,\#V_m. \quad (7.6)$$

The inequalities (7.5) and (7.6) yield

$$\sum_{\{i,j\}\in E(T_-)\cup E'(T)} v(x_i, x_j) \geq -B \sum_{m=1}^{r} \#V_m \geq -Bn.$$

We plug this estimate into (7.4) and obtain the first inequality of the theorem. The second inequality follows from $(1 - \mathrm{e}^{-|u|}) \leq |1 - \mathrm{e}^{-u}|$. $\square$

7.3. Activity expansion of the pressure.

Theorem 7.8. *Let $v : \mathbb{X}^2 \to \mathbb{R} \cup \{\infty\}$ be a stable pair potential with stability constant $B \geq 0$. Suppose that $C(\beta) := \sup_{x\in\mathbb{X}} \int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x,y)} - 1| \mathrm{d}\lambda(y) < \infty$ and*

$$z\mathrm{e}^{\beta B} C(\beta) \leq \frac{1}{\mathrm{e}}.$$

Then for every $x_1 \in \mathbb{X}$, we have

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\mathbb{X}^{n-1}} \Big| \sum_{G\in\mathcal{C}_n} w(G; x_1, \ldots, x_n) \Big| \mathrm{d}\lambda(x_2) \cdots \mathrm{d}\lambda(x_n) \leq C(\beta)^{-1} \sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} \mathrm{e}^{-n} < \infty.$$

Proof. By Theorem 7.7, we have

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\mathbb{X}^{n-1}} \Big| \sum_{G\in\mathcal{C}_n} w(G; x_1, \ldots, x_n) \Big| \mathrm{d}\lambda(x_2) \cdots \mathrm{d}\lambda(x_n) \leq \sum_{n=1}^{\infty} \frac{z^n}{n!} \mathrm{e}^{\beta B n} \sum_{T\in\mathcal{T}_n} \int_{\mathbb{X}^{n-1}} \big| w(T; x_1, \ldots, x_n) \big| \mathrm{d}\lambda(x_2) \cdots \mathrm{d}\lambda(x_n).$$

Each tree has $n - 1$ edges and we can estimate

$$\int_{\mathbb{X}^{n-1}} \big| w(T; x_1, \ldots, x_n) \big| \mathrm{d}\lambda(x_2) \cdots \mathrm{d}\lambda(x_n) \leq C(\beta)^{n-1},$$

which gives

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\mathbb{X}^{n}} \Big| \sum_{G\in\mathcal{C}_n} w(G; x_1, \ldots, x_n) \Big| \mathrm{d}\lambda(x_2) \cdots \mathrm{d}\lambda(x_n) \leq \sum_{n=1}^{\infty} \frac{z^n}{n!} C(\beta)^{n-1} \mathrm{e}^{\beta B n}\, \#\mathcal{T}_n.$$

By Cayley's theorem, the number of non-rooted, labelled trees with vertex set $\{1,\dots,n\}$ is $\#\mathcal{T}_n = n^{n-2}$ and the first inequality of the theorem follows. The second follows from Stirling's formula: we have

$$\frac{n^{n-2}}{n!}\mathrm{e}^{-n} \sim \frac{1}{n^2}\frac{(n/\mathrm{e})^n}{\sqrt{2\pi n}(n/\mathrm{e})^n} = \frac{1}{n^2\sqrt{2\pi n}}.$$

Since $\sum_n \big(n^2\sqrt{2\pi n}\big)^{-1} < \infty$, the series $\sum_n \frac{n^{n-2}}{n!}\mathrm{e}^{-n}$ is convergent. □

Corollary 7.9. *Under the assumptions of Theorem 7.8, we have, for all non-empty* $\Lambda \in \mathcal{X}_\mathsf{b}$,

$$\frac{1}{|\Lambda|}\log \Xi_\Lambda(\beta,z) = \sum_{n=1}^\infty \frac{z^n}{n!}\frac{1}{|\Lambda|}\int_{\Lambda^n}\Big(\sum_{G\in\mathcal{C}_n} w(G;x_1,\dots,x_n)\Big)\mathrm{d}\lambda^n(\boldsymbol{x})$$

with

$$\sum_{n=1}^\infty \frac{z^n}{n!}\frac{1}{|\Lambda|}\int_{\Lambda^n}\Big|\sum_{G\in\mathcal{C}_n} w(G;x_1,\dots,x_n)\Big|\mathrm{d}\lambda^n(\boldsymbol{x}) \le \sum_{n=1}^\infty \frac{n^{n-2}}{n!}\mathrm{e}^{-n} < \infty.$$

In particular, the radius of convergence $R_\Lambda(\beta)$ of the expansion of the pressure in finite volume satisfies

$$R_\Lambda(\beta) \ge \frac{\mathrm{e}^{-\beta B}}{\mathrm{e}\, C(\beta)}. \tag{7.7}$$

The lower bound is uniform in the volume Λ.

Theorem 7.10. *Suppose* $\mathbb{X} = \mathbb{R}^d$, $\lambda = \mathrm{Leb}$. *Suppose that v is a translationally invariant pair potential $v(x,y) = v(0,y-x)$ that is stable with stability constant B and satisfies* $C(\beta) = \int_{\mathbb{R}^d}|\mathrm{e}^{-\beta v(0,y)}-1|\mathrm{d}y < \infty$. *Assume* $z\mathrm{e}^{\beta B}C(\beta) < 1/\mathrm{e}$. *Then the pressure is given by*

$$\beta p(\beta,z) = \sum_{n=1}^\infty \frac{z^n}{n!}\int_{(\mathbb{R}^d)^{n-1}}\Big(\sum_{G\in\mathcal{C}_n} w(G;0,x_2,\dots,x_n)\Big)\mathrm{d}x_2\cdots\mathrm{d}x_n$$

with absolutely convergent series.

Proof. The absolute convergence of the series follows from Theorem 7.8. By Theorem 4.12, we have, for every $\Lambda\in\mathcal{X}_\mathsf{b}$,

$$\frac{1}{|\Lambda|}\log \Xi_\Lambda(\beta,z) = \sum_{n=1}^\infty \frac{z^n}{n!}\frac{1}{|\Lambda|}\int_{\Lambda^n}\Big(\sum_{G\in\mathcal{C}_n} w(G;x_1,\dots,x_n)\Big)\mathrm{d}\boldsymbol{x} \tag{7.8}$$

whenever $0 \le z < R_\Lambda(\beta)$, with $R_\Lambda(\beta)$ the radius of convergence of the power series, which by (7.7) covers the case $z\mathrm{e}^{\beta B}C(\beta) \le 1/\mathrm{e}$. Let $(\Lambda_k)_{k\in\mathbb{N}}$ be a van Hove sequence. By Lemma 4.14, we have for each fixed n

$$\lim_{k\to\infty}\frac{z^n}{n!}\frac{1}{|\Lambda_k|}\int_{\Lambda_k^n}\Big(\sum_{G\in\mathcal{C}_n} w(G;x_1,\dots,x_n)\Big)\mathrm{d}\boldsymbol{x} = \frac{z^n}{n!}\int_{(\mathbb{R}^d)^{n-1}}\Big(\sum_{G\in\mathcal{C}_n} w(G;0,x_2,\dots,x_n)\Big)\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

Just as in the proof of Theorem 7.8, we see that

$$\Big|\frac{z^n}{n!}\int_{(\mathbb{R}^d)^{n-1}}\Big(\sum_{G\in\mathcal{C}_n} w(G;0,x_2,\dots,x_n)\Big)\mathrm{d}x_2\cdots\mathrm{d}x_n\Big| \le \frac{1}{C(\beta)}\frac{n^{n-2}}{n!}z^n$$

for all n, k. Monotone convergence shows that we may exchange summation and limits in (7.8) and the theorem follows. $\square$

The condition $z\mathrm{e}^{\beta B} C(\beta) \leq 1/\mathrm{e}$ is quite similar to the condition $z\mathrm{e}^{2\beta B} C(\beta) < 1/\mathrm{e}$ encountered in Section 5.8 on uniqueness of Gibbs measures via the Kirkwood-Salsburg equation. In fact for non-negative interactions we may pick $B = 0$ and the two conditions coincide except that one allows for equality and the other one does not. Combining the results, we see that for small z, there is absence of phase transition in two senses: (1) $\#\mathscr{G}(\beta, z) = 1$, (2) $z \mapsto \beta p(\beta, z)$ is analytic.

7.4. **Summary.**

- For stable pair potentials with $C(\beta) = \sup_{x \in \mathbb{X}} \int_{\mathbb{X}} |\mathrm{e}^{-\beta v(x,y)} - 1| \mathrm{d}\lambda(y) < \infty$, a sufficient condition for the expansion of the pressure in powers of the activity to converge is that $z\mathrm{e}^{\beta B} C(\beta) \leq 1/\mathrm{e}$.
- The estimates are uniform in the volume. For translationally invariant pair potentials, they allow us to exchange summation and the infinite-volume limit, leading to a formula for $\beta p(\beta, z)$ at small z as a power series in z.
- The proof builds on tree-graph inequalities and tree partition schemes. For non-negative interactions, it doesn't matter which tree partition scheme we work with and we may choose the Penrose partition scheme. For general stable interactions, we have worked with a partition scheme defined with a total order on edges such that $\{i, j\} \mapsto v(x_i, x_j)$ is increasing.

Appendix A. Monotone class theorems

Let Ω be a set. A π-system is a class $\mathcal{C} \subset \mathcal{P}(\Omega)$ that is closed under finite intersections ($A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$). A *Dynkin system*, also called λ-*system*, is a class $\mathcal{D} \subset \mathcal{P}(\Omega)$ that contains Ω, is closed with respect to proper differences ($A, B \in \mathcal{D}$, $A \subset B \Rightarrow B \setminus A \in \mathcal{D}$) and under countable unions of disjoint sets.

Theorem A.1 (Dynkin system theorem)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of subsets of Ω with $\mathcal{C} \subset \mathcal{D}$. Suppose that $\mathcal{C}$ is a π-system and that $\mathcal{D}$ is a Dynkin system. Then $\sigma(\mathcal{C}) \subset \mathcal{D}$.*

See [24, Theorem 1.1]. Every Dynkin system is in particular a *monotone class*: it is closed under countable increasing unions and countable decreasing intersections, i.e., $A_n \in \mathcal{D}$, $A_n \subset A_{n+1}$ implies $\cup_{n\in\mathbb{N}} A_n \in \mathcal{D}$, and $A_n \in \mathcal{M}$, $A_n \supset A_{n+1}$ implies $\cap_{n\in\mathbb{N}} A_n \in \mathcal{D}$. Therefore Theorem A.1 is also called a *monotone class theorem*. Variants of the theorem ask *less* of $\mathcal{D}$ (e.g. that it be a monotone class), but *more* of $\mathcal{C}$ (e.g. that it be an *algebra*), see [33, Theorem A.2].

There are also monotone class theorems for functions, called *functional monotone class theorems*. For example, the following holds true.

Theorem A.2 (Functional monotone class theorem)**.** *Let Ω be a set, $\mathcal{M}$ a set of bounded maps $f : \Omega \to \mathbb{R}$, and $\mathcal{K} \subset \mathcal{M}$. Suppose that:*

- *$\mathcal{K}$ is closed with respect to multiplication, i.e., $f, g \in \mathcal{K} \Rightarrow fg \in \mathcal{K}$.*
- *The constant function $\mathbf{1}$ is in $\mathcal{M}$.*
- *$\mathcal{M}$ is a linear vector space.*
- *$\mathcal{M}$ is closed with respect to pointwise monotone limits of uniformly bounded, non-negative sequences: if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ with $f_n \geq 0$ and $f_n \nearrow f$ for some bounded function $f : \Omega \to \mathbb{R}$, then $f \in \mathcal{M}$.*

Then $\mathcal{M}$ contains all $\sigma(\mathcal{K})$-measurable bounded functions.

The theorem is often proven under the additional assumption that $\mathcal{M}$ is closed with respect to uniform convergence, see e.g. [2, Theorem 2.12.9]. As observed by Sharpe [51, p. 365], however, every monotone vector space is closed with respect to uniform convergence, so the extra assumption is not needed. The following proof is taken from the website planetmath.[6]

Proof of Theorem A.2. *Step 1: $\mathcal{M}$ is closed with respect to uniform convergence.* Let $f : \Omega \to \mathbb{R}$ be a bounded function and $(f_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{M}$ with $||f - f_n||_\infty = \sup_{x\in\Omega} |f(x) - f_n(x)| \to 0$. Passing to a subsequence if necessary, we may assume that $||f_m - f_n||_\infty \leq 2^{-n}$ for all $m, n \in \mathbb{N}$ with $m \geq n$. Set $g_n := f_n - 2^{1-n} + 2 + ||f||_\infty$. One checks that $g_n \in \mathcal{M}$, $g_n \geq 0$, $g_n \leq g_{n+1}$, and $g_n \nearrow f + 2 + ||f||_\infty$, and concludes that $f \in \mathcal{M}$.

For the remaining steps, let $\mathcal{H}_1$ consist of the linear combinations of functions in $\mathcal{K}$ and the constant functions, and $\mathcal{H} = \overline{\mathcal{H}_1}$ the closure of $\mathcal{H}_1$ with respect to uniform convergence. Then $\mathcal{H} \subset \mathcal{M}$ by Step 1, and one can check that $\mathcal{H}_1$ and $\mathcal{H}$ are closed with respect to multiplication.

[6] `http://planetmath.org/sites/default/files/texpdf/41387.pdf`, downloaded 30 Oct. 2017.

Step 2: *If $f, g \in \mathcal{H}$, then* $\max(f,g) \in \mathcal{H}$ *and* $\min(f,g) \in \mathcal{H}$. Since $\mathcal{H}$ is a vector space and

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2},$$

it is enough to show that $f \in \mathcal{H}$, f bounded, implies $|f| \in \mathcal{H}$. Since $\mathcal{H}$ is a vector space and closed under pairwise products, we have $p(f) \in \mathcal{H}$ for every polynomial $p : \mathbb{R} \to \mathbb{R}$ and $f \in \mathcal{H}$. By the Weierstrass approximation theorem, there exists a sequence of polynomials (p_n) such that $p_n(y) \to |y|$ uniformly on the compact interval $[-||f||_\infty, ||f||_\infty]$. Then $p_n(f(x)) \to |f(x)|$ uniformly on Ω. It follows that $|f| \in \mathcal{H}$.

Step 3: *$\mathcal{M}$ contains all indicator functions of the form $\mathbb{1}_{\{f>a\}}$, $a \in \mathbb{R}$, $f \in \mathcal{K}$.* For $f \in \mathcal{K}$ and $n \in \mathbb{N}$, define $f_n := \min(1, n(f-a)_+)$ where $(f-a)_+ = \max(f-a, 0)$. Then $f_n \in \mathcal{H} \subset \mathcal{M}$ by Step 2. Clearly $f_n \geq 0$. One checks that $f_n \nearrow \mathbb{1}_{\{f>a\}}$ and concludes that the indicator is in $\mathcal{M}$.

Step 4: *There is a π-system $\mathcal{C}$ with $\sigma(\mathcal{C}) = \sigma(\mathcal{K})$ such that $\mathcal{M}$ contains all indicator functions $\mathbb{1}_C$, $C \in \mathcal{C}$.* Let $\mathcal{C}$ be the collection of subsets $A \subset \Omega$ such that $f_n \nearrow \mathbb{1}_A$ for some sequence $(f_n)_{n\in\mathbb{N}}$ of non-negative functions in $\mathcal{H}$. By Step 3, $\mathcal{C}$ contains all sets of the form $\{f > a\}$ with $f \in \mathcal{K}$ and $a \in \mathbb{R}$. These sets generate $\sigma(\mathcal{K})$, so we also have $\sigma(\mathcal{K}) = \sigma(\mathcal{C})$. Moreover $\mathcal{C}$ is a π-system because $f_n \nearrow \mathbb{1}_A$, $g_n \nearrow \mathbb{1}_B$ implies $f_n g_n \nearrow \mathbb{1}_{A\cap B}$, and because $\mathcal{H}$ is closed under multiplication.

Step 5: *$\mathcal{M}$ contains all indicator functions of the form $\mathbb{1}_A$, $A \in \sigma(\mathcal{K})$*: Let $\mathcal{D}$ be the collection of sets $A \subset \mathbb{R}$ for which $\mathbb{1}_A \in \mathcal{M}$ and $\mathcal{C}$ a π-system as in Step 4. Then $\mathcal{C} \subset \mathcal{D}$, $\mathcal{D}$ is a Dynkin system, hence $\sigma(\mathcal{K}) = \sigma(\mathcal{C}) \subset \mathcal{D}$ by Theorem A.1. Consequently $\mathbb{1}_A \in \mathcal{M}$ for all $A \in \sigma(\mathcal{K})$.

It follows that $\mathcal{M}$ contains all linear combinations of indicator functions $\mathbb{1}_A$ with $A \in \sigma(\mathcal{K})$, and all bounded functions that are pointwise monotone limits of such linear combinations. This concludes the proof. □

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