GROUND STATES OF SEMI-LINEAR PDES

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Abstract. These are lecture notes from a course given at the summer school on 'Current topics in Mathematical Physics', held at Luminy in September 2013. We discuss ground state solutions for semi-linear PDEs in $\mathbb{R}^N$. In particular, we prove their existence, radial symmetry and uniqueness up to translations.

Contents

1. Introduction

The goal of these lectures is to prove the following two theorems. The first one concerns minimizers of a quotient that appears in certain Sobolev-type inequalities.

Theorem 1.1. *Let $2 < q < \infty$ if $N = 1, 2$ and $2 < q < 2N/(N-2)$ if $N \geq 3$. Then there is a positive, radial and decreasing function Q such that the infimum*

$$S_{N,q} = \inf_{0 \not\equiv u \in H^1(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^\theta \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{1-\theta}}{\left(\int_{\mathbb{R}^N} |u|^q \, dx\right)^{2/q}} \,, \qquad \theta = \frac{N}{2}\left(1 - \frac{2}{q}\right) \,, \tag{1.1}$$

is attained iff $u(x) = c\, Q(b(x-a))$ for some $a \in \mathbb{R}^N$, $b > 0$ and $c \in \mathbb{C} \setminus \{0\}$.

The second theorem concerns solutions of a semi-linear PDE.

Theorem 1.2. *Let $2 < q < \infty$ if $N = 1, 2$ and $2 < q < 2N/(N-2)$ if $N \geq 3$. Then there is a positive, radial and decreasing function Q such that any non-negative weak solution $0 \not\equiv u \in H^1(\mathbb{R}^N)$ of*

$$-\Delta u - u^{q-1} = -u \qquad \text{in } \mathbb{R}^N \tag{1.2}$$

is given by $u(x) = Q(x-a)$ for some $a \in \mathbb{R}^N$. Moreover, the linearization of (1.2) is non-degenerate in the sense that the kernel of $L = -\Delta - (q-1)u^{q-2} + 1$, considered as a self-adjoint operator in $L^2(\mathbb{R}^N)$, satisfies

$$\ker L = \operatorname{span}\{\partial_1 u, \ldots, \partial_N u\} \,.$$

The link between these two theorems is, of course, that (1.2) arises (possibly after rescaling and multiplying by a constant) as the Euler–Lagrange equation for the minimization problem (1.1).

Let us mention a few motivations for studying the minimization problem (1.1) and the equation (1.2).

(1) By a duality argument, the infimum (1.1) appears when asking 'how small can one make the lowest eigenvalue of a Schrödinger operator $-\Delta + V$ for a given L^p norm of V?' This isoperimetric problem for Schrödinger eigenvalues was introduced by Keller [Ke] and plays a key role in a famous conjecture of Lieb and Thirring [LiTh] about inequalities for sums of eigenvalues; see also [CaFrLi].

(2) The minimizer for (1.1) appears in the blow-up theory of the non-linear Schrödinger equation. Moreover, the methods to analyze minimal mass blow-up are closely related to the arguments we need to show that there is a solution to the minimization problem; see, e.g., the lecture notes [Ra] for more about this.

(3) By a scaling argument (write $\psi(x) = l^{-N/2} u(x/l)$), the minimization problem

$$\inf \left\{ \int_{\mathbb{R}^N} |\nabla \psi|^2 \, dx - \int_{\mathbb{R}^N} |\psi|^q \, dx : \int_{\mathbb{R}^N} |\psi|^2 \, dx = 1 \right\}$$

is equivalent to the problem discussed in Theorem 1.1, provided $q < 2(N+2)/N$. The latter minimization problem appears in many physical applications. For instance, with $q = 4$ and $N = 1$ in the analysis of the delta function Fermi gas [dLLi] or with $q = 5/2$ and $N = 3$ in the analysis of the two-component charged Bose gas [Dy, LiSo]. The case $q = 4$ and $N = 1, 2$ also appears in non-linear optics [ChGaTo, Kel, Ta, AkSuKh].

(4) From a purely mathematical point of view (1.1) and (1.2) serve as model cases for more complicated non-linearities. Before studying these more complicated (and possibly more realistic) non-linearities it is crucial to understand the model cases first in great detail.

The proof of Theorems 1.1 and 1.2 will take up the main part of these lectures, indeed, all sections except for the last one. In the last section we discuss the usefulness of the non-degeneracy statement in Theorem 1.2.

The case $N = 1$ in Theorems 1.1 and 1.2 is much simpler than its higher-dimensional analogue, since it is explicitly solvable. We discuss this further in Appendix A and focus otherwise mostly on $N \geq 2$.

In the remainder of this introduction we describe the main steps in the proof of Theorems 1.1 and 1.2 for $N \geq 2$.

Step 1. The Sobolev interpolation inequality. In Section 2 we prove that the infimum $S_{N,q}$, defined in (1.1), is strictly positive. In other words, we prove the Sobolev interpolation inequality

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\theta} \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{1-\theta} \geq S_{N,q} \left(\int_{\mathbb{R}^N} |u|^q \, dx\right)^{2/q}, \qquad \theta = \frac{N}{2}\left(1 - \frac{2}{q}\right),$$

for all $u \in H^1(\mathbb{R}^N)$. We follow classical arguments of Gagliardo [Ga] and Nirenberg [Ni].

Step 2. Existence of a minimizer. In Section 3 we shall show that the infimum in Theorem 1.1 is attained. It was observed by Weinstein [We1] that this follows from Lions' concentration compactness principle [Lio1, Lio2]. Here we present a different and older approach based on a compactness lemma by Lieb [Li2].

Before proceeding, let us note that any minimizer of (1.1) is a constant times a non-negative function. For the details of this argument we refer to Step 2 in the proof of Theorem A.1. We also note that, similarly as in Step 3 of that proof, an optimizer can be normalized so that it is a non-negative weak solution of the Euler–Lagrange equation (1.2). From the weak-strong maximum principle (Lemma 4.3) we infer that $u > 0$ in $\mathbb{R}^N$.

Step 3. Any minimizer is radial. In Section 4 we prove that any non-negative weak solution of (1.2) is radially symmetric with respect to some point and a decreasing function with respect to the distance from this point. This follows via the method of moving planes, which goes back to Alexandrov [Ale] and Serrin [Se] and was further developed and popularized by Gidas–Ni–Nirenberg [GiNiNi1, GiNiNi2]. Note that the result does *not* follow by Schwarz symmetrization, since there is no strict rearrangement inequality for $\int |\nabla u|^2\, dx$.

Step 4. Uniqueness of radial solutions. In Section 5 we prove uniqueness (up to translations) of positive finite energy solutions of (1.2). Having established radial symmetry, one has to prove uniqueness of positive finite energy solutions of the ODE

$$-u'' - (N-1)r^{-1}u' - u^{q-1} = -u \qquad \text{on } (0,\infty)\,.$$

This is a celebrated result of Kwong [Kw] which we state and prove in Theorem 5.1. (In fact, uniqueness of *energy-minimizing solutions* is an earlier result of Zhang [Zh]. We refer to Section 5 for further references.) We emphasize that for $N>1$ the above ODE is non-autonomous because of the first order term. This makes the uniqueness proof considerably harder than that in the one-dimensional case discussed in the appendix, where one deals with an autonomous equation.

2. Sobolev inequalities

In this section we prove that the infimum in (1.1) is strictly positive, that is, we prove Sobolev interpolation inequalities. Our arguments follow the proof due to Gagliardo [Ga] and Nirenberg [Ni] which goes via the quantity $\int |\nabla u|\, dx$ instead of the quantity $\int |\nabla u|^2\, dx$.

Theorem 2.1 (Isoperimetric inequality)**.** *Let $N\geq 2$. Then there is a constant $C_N>0$ such that for all $u\in \dot W^{1,1}(\mathbb{R}^N)$*

$$\int_{\mathbb{R}^N} |\nabla u|\, dx \geq C_N \left(\int_{\mathbb{R}^N} |u|^{N/(N-1)}\, dx \right)^{(N-1)/N} . \tag{2.1}$$

The constant C_N that our proof gives is explicit, but not sharp. The reason why we call (2.1) an *isoperimetric inequality* will be explained in Exercise 2.7, which also explains what the value of the sharp constant is.

The Gagliardo–Nirenberg argument of Theorem 2.1 relies on two lemmas. The first one is a *one-dimensional* analogue of the inequality we want to establish. The second one gives a method of how to pass from *smaller* to *higher* dimensions.

Lemma 2.2 (Easiest Sobolev inequality)**.** *If $u\in\dot W^{1,1}(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |u'|\, dx \geq 2 \sup_{x\in\mathbb{R}} |u(x)|\,.$$

Equality holds iff there is a constant $a\in\mathbb{R}$ such that u is non-decreasing on $(-\infty,a)$ and non-increasing on (a,∞).

Proof. Write $u(x) = \frac12 \left(\int_{-\infty}^x u'(y)\, dy - \int_x^\infty u'(y)\, dy \right)$ and take absolute values. □

In the statement of the next lemma, we shall use the following notation for $x\in\mathbb{R}^N$ and $1\leq j\leq N$,

$$\tilde x_j = (x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N) \in \mathbb{R}^{N-1}\,.$$

Then one has

Lemma 2.3 (Loomis–Whitney inequality). *Let $N \geq 2$ and let $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. Then the function $f(x) := f_1(\tilde{x}_1) \cdots f_N(\tilde{x}_N)$ belongs to $L^1(\mathbb{R}^N)$ and*

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{j=1}^{N} \|f_j\|_{L^{N-1}(\mathbb{R}^{N-1})} .$$

Proof. Note that this is an equality for $N = 2$. The main idea behind the proof is most clearly seen for $N = 3$, which we assume henceforth. We have

$$\int_{\mathbb{R}^3} |f(x)|\, dx = \iint_{\mathbb{R}\times\mathbb{R}} I(x_1, x_2)|f_3(x_1, x_2)|\, dx_1\, dx_2 ,$$

where

$$I(x_1, x_2) = \int_{\mathbb{R}} |f_1(x_2, x_3) f_2(x_1, x_3)|\, dx_3 .$$

By the Schwarz inequality, $I(x_1, x_2) \leq \sqrt{g_1(x_2)}\sqrt{g_2(x_1)}$, where

$$g_1(x_2) = \int_{\mathbb{R}} |f_1(x_2, x_3)|^2\, dx_3 \quad \text{and} \quad g_2(x_1) = \int_{\mathbb{R}} |f_2(x_1, x_3)|^2\, dx_3 .$$

Thus, again by Schwarz,

$$\begin{aligned}\int_{\mathbb{R}^3} |f(x)|\, dx &\leq \int_{\mathbb{R}} dx_1 \sqrt{g_2(x_1)} \left(\int_{\mathbb{R}} |f_3(x_1, x_2)|^2\, dx_2\right)^{1/2} \left(\int_{\mathbb{R}} g_1(x_2)\, dx_2\right)^{1/2} \\ &= \|f_1\|_2 \int_{\mathbb{R}} dx_1 \sqrt{g_2(x_1)}\sqrt{g_3(x_1)} ,\end{aligned}$$

where g_3 is defined similarly as g_1 and g_2. Applying Schwarz once again, we arrive at the claimed inequality for $N = 3$. The case $N \geq 4$ is proved similarly. □

Proof of Theorem 2.1. By an approximation argument we may assume that $u \in C_c^1(\mathbb{R}^N)$. Then, by Lemma 2.2, we have

$$|u(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |\partial_1 u(y_1, x_2, \ldots, x_N)|\, dy_1 = g_1(\tilde{x}_1) .$$

Defining g_2, …, g_N similarly with respect to the other coordinate directions and multiplying the corresponding inequalities we find that

$$|u(x)|^N \leq g_1(\tilde{x}_1) \cdots g_N(\tilde{x}_N) ,$$

which is the same as

$$|u(x)|^{N/(N-1)} \leq g_1(\tilde{x}_1)^{1/(N-1)} \cdots g_N(\tilde{x}_N)^{1/(N-1)} .$$

Applying Lemma 2.3 with $f_j = g_j^{1/(N-1)}$ we infer that

$$\int_{\mathbb{R}^N} |u(x)|^{N/(N-1)}\, dx \leq \prod_{j=1}^{N} \|g_j^{1/(N-1)}\|_{L^{N-1}(\mathbb{R}^{N-1})} = \prod_{j=1}^{N} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\partial_j u|\, dx\right)^{1/(N-1)} .$$

This is almost the inequality we wanted to prove. To arrive at the inequality stated in the theorem, we apply the arithmetic-geometric mean inequality,

$$\left(\prod_{j=1}^{N} a_j\right)^{1/N} \leq \frac{1}{N}\sum_{j=1}^{N} a_j$$

for $a_j \geq 0$, which yields

$$\left(\int_{\mathbb{R}^N} |u(x)|^{N/(N-1)}\, dx\right)^{(N-1)/N} \leq \frac{1}{2N}\sum_{j=1}^{N}\int_{\mathbb{R}^N} |\partial_j u|\, dx\,.$$

Finally, we bound the ℓ_1-norm of the gradient in $\mathbb{C}^N$ in terms of its ℓ_2-norm and complete the proof. □

Corollary 2.4 (Sobolev interpolation inequalities). *Let $2 \leq q \leq \infty$ if $N = 1$, let $2 \leq q < \infty$ if $N = 2$ and let $2 \leq q \leq 2N/(N-2)$ if $N \geq 3$. Then $u \in H^1(\mathbb{R}^N)$ implies $u \in L^q(\mathbb{R}^N)$ and there is a constant $S_{N,q} > 0$ such that for every $u \in H^1(\mathbb{R}^N)$ one has*

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{\theta}\left(\int_{\mathbb{R}^N} |u|^2\, dx\right)^{1-\theta} \geq S_{N,q}\left(\int_{\mathbb{R}^N} |u|^q\, dx\right)^{2/q}, \tag{2.2}$$

where $N/q = \theta(N-2)/2 + (1-\theta)N/2$.

Of course, the equation determining θ comes from scaling.

An important special case of (2.2) is, for $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq S_N\left(\int_{\mathbb{R}^N} |u|^{2N/(N-2)}\, dx\right)^{(N-2)/N}. \tag{2.3}$$

(Here and in the following we write $S_N = S_{N,2N/(N-2)}$ in the special case $q = 2N/(N-2)$.) This inequality, which was stated for functions in the inhomogeneous Sobolev space $H^1(\mathbb{R}^N)$, extends by continuity to the homogeneous space $\dot{H}^1(\mathbb{R}^N)$.

A simple consequence of (2.2) is the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq C_N|\{u \neq 0\}|^{-2/N}\int_{\mathbb{R}^N} |u|^2\, dx\,, \tag{2.4}$$

valid for functions $u \in H^1(\mathbb{R}^N)$ which vanish outside a set of finite measure. Indeed, in order to obtain (2.4) from (2.2) we only have to use Hölder's inequality

$$\int_{\mathbb{R}^N} |u|^2\, dx \leq |\{u \neq 0\}|^{(q-2)/2}\left(\int_{\mathbb{R}^N} |u|^q\, dx\right)^{2/q}.$$

Inequality (2.4) with the sharp constant is called *Faber–Krahn inequality*. Again the precise statement is that if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ with $\nabla u \in L^2(\mathbb{R}^N)$ and with $|\{u \neq 0\}| < \infty$, then $u \in L^2(\mathbb{R}^N)$ and the inequality holds.

Proof. We first note that by Hölder's inequality, if (2.2) holds for some q, it holds for all smaller values of $q \geq 2$ as well. Hence it suffices to derive (2.2) only for large values of q, and this is what we do in the following.

Case $N \geq 3$ and $q = 2N/(N-2)$. For the proof we assume, by an approximation argument, that $u \in C_c^1(\mathbb{R}^N)$. Then for $\alpha \geq 1$ we have $|\nabla |u|^\alpha| = \alpha |u|^{\alpha-1} |\nabla u|$ and hence, applying Theorem 2.1 with $|u|^\alpha$ in place of u, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |u|^{\alpha N/(N-1)}\,dx\right)^{(N-1)/N} &\leq \alpha C_N^{-1} \int_{\mathbb{R}^N} |u|^{\alpha-1} |\nabla u|\,dx \\ &\leq \alpha C_N^{-1} \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}^N} |u|^{2(\alpha-1)}\,dx\right)^{1/2}.\end{aligned}$$

Choosing $\alpha = (N-1)2/(N-2)$ yields (2.2) if we note that $\alpha N/(N-1) = 2N/(N-2) = 2(\alpha-1)$.

Case $N = 2$ and $q \geq 4$. We use the same argument as before, but this time we choose $\alpha = q/2$ depending on q and obtain

$$\left(\int_{\mathbb{R}^2} |u|^q\,dx\right)^{1/2} \leq \frac{q}{2\,C_2} \left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}^2} |u|^{q-2}\,dx\right)^{1/2}. \tag{2.5}$$

Since $q \geq 4$, the term $\int |u|^{q-2}\,dx$ can be estimated by Hölder in terms of $\int |u|^q\,dx$ and $\int |u|^2\,dx$. This yields the desired inequality.

Case $N = 1$ and $q = \infty$. We prefer to record this separately as Corollary 2.5, since in this case we actually obtain the sharp constant.

According to the remark made at the beginning of the proof that large q is enough, the proof of Corollary 2.4 is complete. □

Here is the announced sharp inequality in the one-dimensional case.

Corollary 2.5. *Let $N = 1$ and $q = \infty$. Then for every $u \in H^1(\mathbb{R})$*

$$\left(\int_{\mathbb{R}} |u'|^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}} |u|^2\,dx\right)^{1/2} \geq \sup_{x\in\mathbb{R}} |u(x)|^2, \tag{2.6}$$

with equality iff $u(x) = c\,h(b(x-a))$ for some $a \in \mathbb{R}$, $b > 0$ and $c \in \mathbb{C}$, where $h(x) = e^{-|x|}$.

Proof. The proof uses the same strategy as that of Corollary 2.4, except that we use Lemma 2.2 instead of Theorem 2.1. Namely, we apply the inequality of Lemma 2.2 to $|u|^2$ instead of u and use the Schwarz inequality for $\int |u||u'|\,dx$. We note that this Schwarz inequality is an equality iff $|u|$ and $|u'|$ are proportional. The only functions which satisfy this, together with the condition of Lemma 2.2 are the exponentials stated in Theorem 2.5. □

Remark 2.6 (Additive Sobolev inequality). In the literature, the Sobolev interpolation inequality (2.2) is often stated in the additive form

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx \geq S'_{N,q} \left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{2/q}, \tag{2.7}$$

the restrictions on q being the same as in Corollary 2.4. For us, the multiplicative form (2.2) will be more useful (besides making sense dimensionally, as we explain shortly). Let us nevertheless show that the two forms are completely equivalent and the respective optimal constants are related by

$$S_{N,q} = \theta^\theta (1-\theta)^{1-\theta} S'_{N,q}$$

with θ as in the corollary. Given (2.2) we arrive at (2.7) immediately by Young's inequality

$$ab \leq \theta a^{1/\theta} + (1-\theta) b^{1/(1-\theta)}\,, \tag{2.8}$$

valid for all $a, b \geq 0$ and $0 < \theta < 1$. The converse is only slightly more complicated. The key observation is that the two terms on the left side of (2.7) have a different behavior under scaling ('a different dimensionality'). To capitalize on this fact we replace $u(x)$ in (2.7) by $u(lx)$, where $l > 0$ is an arbitrary parameter. We obtain

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + l^{-2} |u|^2 \right) \, dx \geq S'_{N,q} l^{-2N\left(\frac{1}{q} - \frac{N-2}{2N}\right)} \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{2/q} . \tag{2.9}$$

Thus, for any fixed u, the term $\int |\nabla u|^2 \, dx$ is bounded from below by a function of l. Maximizing this function with respect to $l > 0$ we obtain (2.2).

As the reader might have already noticed, the same argument as in Corollaries 2.4 and 2.5 yields Sobolev inequalities in $W^{1,p}(\mathbb{R}^N)$ as well, that is, lower bounds for $\int |\nabla u|^p \, dx$ in terms of suitable L^q-norms. Since those will not play a role in what follows we do not state them explicitly.

Exercise 2.7. Here is the reason why we call (2.1) an isoperimetric inequality. The following argument is due to Federer and Fleming [FeFl] and Maz'ya [Ma1].

(1) Deduce from Theorem 2.1 that for any sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ one has
$$|\partial\Omega| \geq C_N |\Omega|^{(N-1)/N} \tag{2.10}$$
with the same constant C_N as in (2.1). Here $|\partial\Omega|$ stands for the $(N-1)$-dimensional surface measure of $\partial\Omega$, whereas $|\Omega|$ stands for the N-dimensional Lebesgue measure. For the proof, take u to be an approximation to the characteristic function of Ω.

(2) Denote by Ω^* the ball in $\mathbb{R}^N$, centered at the origin, with the same measure as Ω. Deduce that
$$|\partial\Omega| \geq C_N N^{-(N-1)/N} |\mathbb{S}^{N-1}|^{-1/N} |\partial\Omega^*| \,.$$
This is an isoperimetric inequality, which says that the boundary of any set cannot be much smaller (in a controlled way) than that of the ball with the same measure. Actually, it is known that balls have the smallest boundary among all sets of fixed measure, i.e., $|\partial\Omega| \geq |\partial\Omega^*|$.

(3) Prove the following elementary version of the *co-area formula*. For $u \in C_0^N(\mathbb{R}^N)$ real-valued and $g \in C_0^1(\mathbb{R}^N)$,
$$\int_{\mathbb{R}^N} g(x) |\nabla u(x)| \, dx = \int_0^\infty \int_{\{|u|=\tau\}} g(x) \, d\sigma(x) \, d\tau \,. \tag{2.11}$$

On the right side, $d\sigma$ is $(N-1)$-dimensional surface measure on $\{|u|=\tau\}$ which, according to Morse's theorem, is a smooth manifold for a.e. $\tau>0$. (This is where the smoothness assumption on u enters.) For the proof, show that

$$\int_{\mathbb{R}^N} w\nabla u\,dx = \int_0^\infty \int_{\{|u|=\tau\}} w\cdot\frac{\nabla u}{|\nabla u|}\,d\sigma(x)\,d\tau$$

for any $w\in C_0^1(\mathbb{R}^N,\mathbb{R}^N)$, pick $w=g(|\nabla u|^2+\varepsilon^2)^{-1/2}\nabla u$ and let $\varepsilon\to 0$. We remark that the co-area formula is valid in much greater generality. The proof (in particular, for Sobolev functions u) is rather involved; see, e.g., [Fe].

(4) Conversely, inequality (2.10) implies inequality (2.1). Prove this first for $u\in C_0^N(\mathbb{R}^N)$ real-valued by using the following consequence of (2.11),

$$\int_{\mathbb{R}^N}|\nabla u|\,dx = \int_0^\infty |\{|u|=\tau\}|\,d\tau\,.$$

In addition, one also needs the fact that for any non-increasing function f on $[0,\infty)$ with $\lim_{\tau\to\infty} f(\tau)=0$ and for any real $\nu>1$ one has

$$\int_0^\infty f(\tau)^{(\nu-1)/\nu}\,d\tau \geq \left(\frac{\nu}{\nu-1}\int_0^\infty f(\tau)\tau^{1/(\nu-1)}\,d\tau\right)^{(\nu-1)/\nu}\,. \tag{2.12}$$

For the proof of (2.12) write $f^{(\nu-1)/\nu}=\int_0^\infty \chi_{\{f^{(\nu-1)/\nu}>t\}}\,dt$ as a superposition of characteristic functions (*layer cake formula*). Since f is non-increasing, these are characteristic functions of intervals having zero as left endpoint. Now use Minkowski's inequality to reduce the proof to the case of a single characteristic function of the form described.

Exercise 2.8. The purpose of this exercise is to show that the Sobolev inequality (2.2) does *not* hold for $q=\infty$ if $N=2$. Prove this by examining the family of functions u_β with $u_\beta(x)=1$ if $|x|\leq 1$ and $u_\beta(x)=(1-\ln|x|/\ln\beta)_+$ if $|x|>1$.

Exercise 2.9. Show the following limiting form of the Sobolev inequality in the two-dimensional case. Namely, there are constants $\alpha, C>0$ such that for any $u\in H^1(\mathbb{R}^2)$ one has

$$\int_{\mathbb{R}^2}\left(\exp\left(\frac{\alpha|u|^2}{\|\nabla u\|^2}\right)-1\right)dx \leq C\frac{\|u\|^2}{\|\nabla u\|^2}\,. \tag{2.13}$$

This inequality from [Og] has its roots in a bound for functions with support in a given set of finite measure, which is independently due to Yudovič [Yu], Pohozaev [Po2] and Trudinger [Tr1]. For the proof of (2.13) proceed as follows:

(1) In arbitrary dimension N, use (2.7) to show that the optimal Sobolev constant in (2.2) can be expressed via the norm of a convolution operator,

$$\theta^\theta(1-\theta)^{1-\theta}S_{N,q}^{-1} = \sup\frac{\|k*f\|_q^2}{\|f\|^2}\,,$$

where $k(x)=(2\pi)^{-N}\int_{\mathbb{R}^N}(p^2+1)^{-1/2}e^{ip\cdot x}\,dp$.

(2) Specializing to $N=2$, prove that

$$k(x) = (2\pi)^{-1}|x|^{-1}e^{-|x|}\,.$$

(Hint: Since k is radially symmetric, one can assume that $x=(|x|,0)$. Now deform the p_1 integration to the 'cut' $i[\sqrt{p_2^2+1},\infty)$ and perform the p_2 integration.)

(3) Use (1), (2) and Young's inequality to get a lower bound on $S_{2,q}$ of the form

$$S_{2,q} \geq c\,(q+2)^{-(q+2)/q}\,.$$

for all $2\leq q<\infty$ and some $c>0$. Use this to prove that the series

$$\sum_{k=1}^\infty \frac{\alpha^k}{k!}\left(\frac{\|u\|_{2k}}{\|\nabla u\|}\right)^{2k} \leq \sum_{k=1}^\infty \frac{S_{2,2k}^{-k}\ \alpha^k}{k!}\ \frac{\|u\|^2}{\|\nabla u\|^2}$$

converges for $\alpha<c/(2e)$. (Note that the important point in the proof was a lower bound on $S_{2,q}$ which is almost as large as q^{-1} as $q\to\infty$. A straightforward application of (2.5) gives only a lower bound of the order of q^{-2}, which is not good enough to prove (2.13).

3. Existence of optimizers

Our goal in this section is to introduce some compactness methods to show that certain minimization problems in Sobolev spaces admit minimizers. In particular, we show that the infimum

$$S_{N,q} = \inf_{0\not\equiv u\in H^1(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^\theta \left(\int_{\mathbb{R}^N}|u|^2\,dx\right)^{1-\theta}}{\left(\int_{\mathbb{R}^N}|u|^q\,dx\right)^{2/q}}\,, \qquad \theta = \frac N2\left(1-\frac2q\right)\,,$$

is attained for $2<q<\infty$ if $N=1,2$ and $2<q<2N/(N-2)$ if $N\geq 3$.

If $(u_j)\subset H^1(\mathbb{R}^N)$ is a minimizing sequence for $S_{N,q}$ than by scaling and multiplication by a constant one can achieve that $\|\nabla u_j\|=1=\|u_j\|$ for all j. Thus, (u_j) is bounded in $H^1(\mathbb{R}^N)$ and, by the Banach–Alaoglu theorem, has a weakly convergent subsequence. This information, however, is not very useful in our context, as we shall explain now.

The difficulty that we have to overcome when proving the existence of an optimizer is that the above minimization problem is invariant under translations. These (noncompact) symmetry groups allow a minimizing sequence to converge weakly to zero. For example, if $u_j(x)=\varphi(x-a_j)$ for a fixed function φ and a sequence $|a_j|\to\infty$, then the quotient in the definition of $S_{N,q}$ remains invariant, but u_j converges weakly to zero in $H^1(\mathbb{R}^N)$. Roughly speaking, the results in this section say that translations are the *only* ways in which compactness in $H^1(\mathbb{R}^N)$ can fail.

Results of this kind go back to Lieb [Li2] (see also [BrLi2] for an application to minimization problems) and were later termed 'concentration compactness principle' by Lions [Lio1, Lio2]. Earlier methods to prove the existence of a minimizer of the similar variational problem are based on Schwarz symmetrization, see [Tal, Str, Li1].

The concentration compactness method has an important advantage over the symmetrization method: it shows that any minimizing sequence is relatively compact in $H^1(\mathbb{R}^N)$ (up to translations). We will see an application of this fact in Section 6.

3.1. **Two results from measure theory.** In this preliminary subsection we collect two measure theoretic results that will play a key role in our proofs of existence of optimizers.

In order to motivate the first result, we recall that Fatou's lemma states that the pointwise limit f of a sequence of functions $(f_j) \subset L^p(X)$ satisfies

$$\liminf_{j\to\infty} \int_X |f_j|^p\, dx \geq \int_X |f|^p\, dx \tag{3.1}$$

for any $p > 0$. Here, in general, one cannot expect equality. The next lemma [BrLi], however, provides the 'missing term' in Fatou's lemma.

Lemma 3.1 (Brézis–Lieb lemma)**.** *Let (X, dx) be a measure space and let (f_j) be a bounded sequence in $L^p(X)$, $0 < p < \infty$, which converges pointwise a.e. to a function f. Then*

$$\lim_{j\to\infty} \int_X \left||f_j|^p - |f_j - f|^p - |f|^p\right|\, dx = 0\,.$$

Proof. We write $f_j = f + g_j$ with $g_j \to 0$ pointwise a.e. and estimate for any $\varepsilon > 0$

$$\int \left||f_j|^p - |f_j - f|^p - |f|^p\right|\, dx \leq \varepsilon \int |g_j|^p\, dx + \int G_j^{(\varepsilon)}\, dx$$

with

$$G_j^{(\varepsilon)} = \left(\left||f + g_j|^p - |g_j|^p - |f|^p\right| - \varepsilon |g_j|^p\right)_+\,.$$

We shall prove that (g_j) is bounded in L^p and that $\int G_j^{(\varepsilon)}\, dx \to 0$ as $j \to \infty$ for every fixed $\varepsilon > 0$. Since $\varepsilon > 0$ can be taken arbitrarily small, this will imply the assertion.

Since $\|f_j\|_p \leq C$ we have $\|f\|_p \leq C$ by (3.1), and therefore

$$\int |g_j|^p\, dx = \int |f_j - f|^p\, dx \leq 2^p \int (|f_j|^p + |f|^p)\, dx \leq 2^{p+1} C\,,$$

as claimed. In order to prove the claim about $G_j^{(\varepsilon)}$ we first note that for any $\varepsilon > 0$ (and $p > 0$) there is a C_ε such that for all numbers $a, b \in \mathbb{C}$

$$\left||a + b|^p - |b|^p\right| \leq \varepsilon |b|^p + C_\varepsilon |a|^p\,.$$

This implies that

$$\left||f + g_j|^p - |g_j|^p - |f|^p\right| \leq \left||f + g_j|^p - |g_j|^p\right| + |f|^p \leq \varepsilon |g_j|^p + (1 + C_\varepsilon)|f|^p\,,$$

and hence $G_j^{(\varepsilon)} \leq (1 + C_\varepsilon)|f|^p$. Since $g_j \to 0$ a.e. one has $G_j^{(\varepsilon)} \to 0$ a.e. as well, and since $|f|^p$ is integrable, dominated convergence implies that $\int G_j^{(\varepsilon)}\, dx \to 0$. This completes the proof. $\square$

Our second measure theoretic result in this section is taken from [FrLiLo]. It gives a useful condition for preventing a sequence of functions from converging to zero in measure.

Lemma 3.2 (pqr theorem)**.** *For any $0 < p < q < r \leq \infty$ and any $C_p, c_q, C_r > 0$ there are $\varepsilon > 0$ and $\delta > 0$ such that for any function f on a measure space X the inequalities $\|f\|_p \leq C_p$, $\|f\|_q \geq c_q$ and $\|f\|_r \leq C_r$ imply*

$$|\{|f| > \varepsilon\}| \geq \delta\,.$$

In other words, if the L^p and L^r norms are controlled from above and the L^q norm is controlled from below for a sequence of f's, then this sequence cannot converge to zero in measure.

Proof. By means of the layer cake representation we find that

$$\|f\|_q^q = q \int_0^\infty |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda\,.$$

For any $0 < \varepsilon \leq M < \infty$ we obtain from the q-norm bound

$$q^{-1} c_q^q \leq \int_0^\varepsilon |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda + \int_\varepsilon^M |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda + \int_M^\infty |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda\,.$$

We show that the first and the last term on the right side can be made arbitrarily small by choosing ε small and M large. Namely, we bound

$$\int_0^\varepsilon |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda \leq \varepsilon^{q-p} \int_0^\varepsilon |\{|f| > \lambda\}| \lambda^{p-1}\, d\lambda \leq \varepsilon^{q-p} p^{-1} \|f\|_p^p \leq \varepsilon^{q-p} p^{-1} C_p^p$$

and similarly

$$\int_M^\infty |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda \leq M^{q-r} r^{-1} C_r^r\,.$$

Now we can choose ε so small and M so large that

$$\varepsilon^{q-p} \frac{q}{p} C_p^p + M^{q-r} \frac{q}{r} C_r^r < c_q^q\,.$$

Then

$$|\{|f| > \varepsilon\}| \geq \frac{q}{M^q - \varepsilon^q} \int_\varepsilon^M |\{|f| > \lambda\}| \lambda^{q-1}\, d\lambda \geq \frac{c_q^q - \varepsilon^{q-p} \frac{q}{p} C_p^p - M^{q-r} \frac{q}{r} C_r^r}{M^q - \varepsilon^q}\,,$$

which is a positive number. This proves the claim. □

3.2. **A concentration compactness lemma.** The following theorem describes bounded sequences in the Sobolev space $H^1(\mathbb{R}^N)$.

Theorem 3.3 (First concentration compactness lemma)**.** *Let $N \geq 1$ and let (u_j) be a bounded sequence in $H^1(\mathbb{R}^N)$. Then one of the following alternatives occurs.*

(1) *(u_j) converges to zero in $L^q(\mathbb{R}^N)$ for every $2 < q < \infty$ if $N = 1, 2$ and for every $2 < q < 2N/(N-2)$ if $N \geq 3$.*

(2) *There is a subsequence* (u_{j_m}) *and a sequence* $(a_m) \subset \mathbb{R}^N$ *such that*

$$v_m(x) := u_{j_m}(x + a_m)$$

converges weakly in $H^1(\mathbb{R}^N)$ *to a function* $v \not\equiv 0$*. Moreover,* (v_m) *converges to* v *a.e. and in* $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ *for every* $q < \infty$ *if* $N = 1, 2$ *and for every* $q < 2N/(N-2)$ *if* $N \geq 3$.

The key in the proof is the following result due to Lieb [Li2].

Lemma 3.4 (Lieb's translation lemma)**.** *Let* (u_j) *be a bounded sequence of functions in* $H^1(\mathbb{R}^N)$ *and suppose that there are* $\varepsilon, \delta > 0$ *such that*

$$|\{|u_j| > \varepsilon\}| \geq \delta \tag{3.2}$$

for all j*. Then there is a sequence* $(a_j) \subset \mathbb{R}^N$ *such that the translated sequence* $u_j(x + a_j)$ *has a subsequence that converges weakly in* $H^1(\mathbb{R}^N)$ *to a function that is* not identically zero.

Before turning to the proof of Lemma 3.4, let us show that it implies the concentration compactness lemma.

Proof of Theorem 3.3. Assume that (1) does not hold, that is, there is a $q \in (2, \infty)$ if $N = 1, 2$ or a $q \in (2, 2N/(N-2))$ if $N \geq 3$ such that $\limsup_{j\to\infty} \|u_j\|_q > 0$. Our goal is to apply the *pqr* theorem (Lemma 3.2). Since (1) is not valid we have a lower bound on the q-norm. We still need two upper bounds on norms. By assumption the $p = 2$-norm of the u_j's is bounded from above. Moreover, by Sobolev inequalities (2.2) there is an $r > q$ such that the r-norm of the u_j's is bounded from above as well. Thus, the *pqr* theorem (Lemma 3.2) tells us that the non-vanishing assumption (3.2) in Lemma 3.4 is satisfied.

Thus, Lemma 3.4 provides us with vectors a_m and a subsequence such that the translated sequence v_m has a weak limit v in $H^1(\mathbb{R}^N)$ that is not identically zero. The fact that v_m converges to v strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for every $q < \infty$ if $N = 1, 2$ and for every $q < 2N/(N-2)$ if $N \geq 3$ follows from Rellich's compactness lemma (see, e.g., [LiLo, Thm. 8.9]). Thus, on every compact set there is a subsequence that converges pointwise a.e. on that set. Taking an increasing sequence of compact sets (e.g., balls with radii tending to infinity) and employing a diagonal argument gives us a subsequence that converges pointwise a.e. on all of $\mathbb{R}^N$. This concludes the proof. □

We now turn to the proof of Lieb's compactness lemma, which is a slight modification of that in [Li2].

Proof of Lemma 3.4. Let us begin by showing the following inequality, valid for any function $u \in H^1(\mathbb{R}^N)$, $N \geq 1$, and any $r > 0$,

$$\int_{\mathbb{R}^N} |u|^2\,dx \leq C'_N \sup_{a\in\mathbb{R}^N} |\{B_r(a) \cap \{u \neq 0\}|^{2/N} \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx + c_N r^{-2} \int_{\mathbb{R}^N} |u|^2\,dx \right). \tag{3.3}$$

Here $B_r(a)$ is the ball of radius r centered at $a\in\mathbb{R}^N$.

Let χ be a real-valued Lipschitz function with $\|\chi\|=1$ and support in the unit ball $B_1(0)$. Then by the so-called *IMS localization formula*,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(|\nabla(\chi_{a,r}u)|^2 - r^{-2}\|\nabla\chi\|^2|\chi_{a,r}u|^2\right)dx\,da\,,$$

where $\chi_{a,r}(x) = r^{-N/2}\chi((x-a)/r)$. According to the (non-sharp) Faber–Krahn inequality (2.4) we have

$$\int_{\mathbb{R}^N} |\nabla(\chi_{a,r}u)|^2\,dx \geq C_N |B_r(a)\cap\{u\neq 0\}|^{-2/N}\int_{\mathbb{R}^N} |\chi_{a,r}u|^2\,dx$$

and, therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &\geq \int_{\mathbb{R}^N}\left(C_N|B_r(a)\cap\{u\neq 0\}|^{-2/N} - r^{-2}\|\nabla\chi\|^2\right)\int_{\mathbb{R}^N}|\chi_{a,r}u|^2\,dx\,da\\ &\geq \left(C_N\inf_{a\in\mathbb{R}^N}|B_r(a)\cap\{u\neq 0\}|^{-2/N} - r^{-2}\|\nabla\chi\|^2\right)\int_{\mathbb{R}^N}|u|^2\,dx\,.\end{aligned}$$

This is the claimed inequality.

We now show that (3.3) implies the statement of the lemma. Indeed, let (u_j) be a bounded sequence in $H^1(\mathbb{R}^N)$ satisfying (3.2). Then $v_j := (|u_j|-\varepsilon/2)_+$ belongs to $H^1(\mathbb{R}^N)$ (see, e.g., [LiLo, Thm. 6.18]) and satisfies

$$\int_{\mathbb{R}^N}|\nabla v_j|^2\,dx \leq \int_{\mathbb{R}^N}|\nabla u_j|^2\,dx \leq \sup_k\int_{\mathbb{R}^N}|\nabla u_k|^2\,dx =: C<\infty$$

and, by the Markov inequality,

$$\int_{\mathbb{R}^N}|v_j|^2\,dx \geq (\varepsilon/2)^2|\{v_j>\varepsilon/2\}| = (\varepsilon/2)^2|\{|u_j|>\varepsilon\}| \geq (\varepsilon/2)^2\delta\,.$$

Thus, from (3.3), with $u=v_j$ and $r=1$,

$$1\leq C_N'\sup_{a\in\mathbb{R}^N}|\{B_1(a)\cap\{v_j>0\}|^{2/N}\left(\frac{C}{(\varepsilon/2)^2\delta}+c_N\right).$$

This proves that for every j there is an $a_j\in\mathbb{R}^N$ such that

$$\frac{1}{2}\leq C_N'|\{B_1(a_j)\cap\{v_j>0\}|^{2/N}\left(\frac{C}{(\varepsilon/2)^2\delta}+c_N\right).$$

Thus,

$$\begin{aligned}\int_{B_1(a_j)}|u_j|\,dx &\geq (\varepsilon/2)\,|B_1(a_j)\cap\{|u_j|>\varepsilon/2\}|\\ &= (\varepsilon/2)\,|B_1(a_j)\cap\{v_j>0\}|\\ &\geq (\varepsilon/2)(2C_N')^{-N/2}\left(\frac{C}{(\varepsilon/2)^2\delta}+c_N\right)^{-N/2}.\end{aligned}$$

Now consider the translated sequence $\tilde{u}_j(x) = u_j(x + a_j)$, which satisfies

$$\int_{B_1(0)} |\tilde{u}_j|\, dx = \int_{B_1(a_j)} |u_j|\, dx \geq (\varepsilon/2)(2C'_N)^{-N/2} \left(\frac{C}{(\varepsilon/2)^2 \delta} + c_N \right)^{-N/2} . \tag{3.4}$$

Since $(\tilde{u}_j)$ is bounded in H^1, it has a weakly convergent subsequence. By Rellich's compactness lemma (see, e.g., [LiLo, Thm. 8.9]) this subsequence converges strongly in $L^1(B_1(0))$. Now (3.4) shows that the limit is not identically zero. This proves the lemma. □

3.3. **Existence of minimizers in the subcritial case.** Our goal in this subsection is to prove the following theorem.

Theorem 3.5 (Existence of minimizers)**.** *Let* $2 < q < \infty$ *if* $N = 1, 2$ *and* $2 < q < 2N/(N-2)$ *if* $N \geq 3$*. Then the infimum*

$$\inf_{0 \not\equiv u \in H^1(\mathbb{R}^N)} \frac{\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^\theta \left(\int_{\mathbb{R}^N} |u|^2\, dx\right)^{1-\theta}}{\left(\int_{\mathbb{R}^N} |u|^q\, dx\right)^{2/q}} , \qquad \theta = \frac{N}{2}\left(1 - \frac{2}{q}\right) ,$$

is attained. Moreover, any minimizing sequence is relatively compact in $H^1(\mathbb{R}^N)$ *up to translations, dilations and multiplication by constants.*

We know already from the Sobolev inequality (2.2) that the infimum is strictly positive.

With the concentration compactness lemma (Theorem 3.3) at hand, the proof of Theorem 3.5 is rather short. This lemma takes care of the loss of compactness due to translations and produces a non-trivial weak limit u. What still needs to be proved is that the L^q-norm does not decrease in the limit. To exclude this, the idea is to consider simultaneously a weak limit point u of a minimizing sequence (u_j) and the remainder $u_j - u$. The general proof strategy goes back to [Li3]. Brézis and Lieb [BrLi2] combined this technique with Lieb's translation lemma (Lemma 3.4) to solve minimization problems posed on $\mathbb{R}^N$.

Proof. Let (u_j) be a minimizing sequence for the infimum in the theorem, which we denote by $S_{N,q}$. After multiplying u_j by a constant and after rescaling we may assume that

$$\int_{\mathbb{R}^N} |\nabla u_j|^2\, dx = \int_{\mathbb{R}^d} |u_j|^2\, dx = 1 \tag{3.5}$$

for all j. Thus, (u_j) is a bounded sequence in $H^1(\mathbb{R}^N)$. Since $S_{N,q} > 0$ and

$$\int_{\mathbb{R}^N} |u_j|^q\, dx = S_{N,q}^{-q/2} + o(1) ,$$

the first alternative in Theorem 3.3 does not occur. Therefore, we infer from that theorem that, after a translation if necessary, (u_j) converges weakly in $H^1(\mathbb{R}^N)$ and a.e. to a function $u \not\equiv 0$. We introduce the 'remainder' $r_j = u_j - u$. The weak convergence in $H^1(\mathbb{R}^N)$ implies that

$$1 = \int_{\mathbb{R}^N} |\nabla u_j|^2\,dx = \int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} |\nabla r_j|^2\,dx + o(1) \tag{3.6}$$

and

$$1 = \int_{\mathbb{R}^N} |u_j|^2\,dx = \int_{\mathbb{R}^N} |u|^2\,dx + \int_{\mathbb{R}^N} |r_j|^2\,dx + o(1)\,. \tag{3.7}$$

Moreover, the almost everywhere convergence together with the Brézis–Lieb lemma (Lemma 3.1) implies that

$$S_{N,q}^{-q/2} + o(1) = \int_{\mathbb{R}^N} |u_j|^q\,dx = \int_{\mathbb{R}^N} |u|^q\,dx + \int_{\mathbb{R}^N} |r_j|^q\,dx + o(1)\,. \tag{3.8}$$

We now use the elementary inequality

$$(t_1+t_2+t_3)^{\theta q/2}(m_1+m_2+m_3)^{(1-\theta)q/2} \geq t_1^{\theta q/2} m_1^{(1-\theta)q/2} + t_2^{\theta q/2} m_2^{(1-\theta)q/2} + t_3^{\theta q/2} m_3^{(1-\theta)q/2}$$

for all $t_1, t_2, t_3, m_1, m_2, m_3 \geq 0$. Indeed, by Hölder's inequality in $\mathbb{R}^3$ applied to the vectors $(t_1^\theta, t_2^\theta, t_3^\theta)$ and $(m_1^{1-\theta}, m_2^{1-\theta}, m_3^{1-\theta})$, the left side is bounded from below by

$$\left(t_1^\theta m_1^{1-\theta} + t_2^\theta m_2^{1-\theta} + t_3^\theta m_3^{1-\theta}\right)^{q/2},$$

and this in turn can be bounded from below by the claimed expression if we use (twice) the elementary inequality

$$(a+b)^{q/2} \geq a^{q/2} + b^{q/2} \tag{3.9}$$

for $a, b \geq 0$. (Here we use $q \geq 2$.)

From this inequality and (3.6), (3.7) and (3.8) we deduce that

$$\begin{aligned}
1 &= \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} |\nabla r_j|^2\,dx + o(1)\right)^{\theta q/2} \left(\int_{\mathbb{R}^N} |u|^2\,dx + \int_{\mathbb{R}^N} |r_j|^2\,dx + o(1)\right)^{(1-\theta)q/2} \\
&\geq \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\theta q/2} \left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{(1-\theta)q/2} + \left(\int_{\mathbb{R}^N} |\nabla r_j|^2\,dx\right)^{\theta q/2} \left(\int_{\mathbb{R}^N} |r_j|^2\,dx\right)^{(1-\theta)q/2} + o(1) \\
&\geq \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\theta q/2} \left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{(1-\theta)q/2} + S_{N,q}^{q/2} \int_{\mathbb{R}^N} |r_j|^q\,dx + o(1) \\
&= \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\theta q/2} \left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{(1-\theta)q/2} + S_{N,q}^{q/2} \left(S_{N,q}^{-q/2} - \int_{\mathbb{R}^N} |u|^q\,dx\right) + o(1)\,.
\end{aligned}$$

Subtracting 1 from both sides, this becomes

$$0 \geq \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\theta q/2} \left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^{(1-\theta)q/2} - S_{N,q}^{q/2} \int_{\mathbb{R}^N} |u|^q\,dx + o(1)\,,$$

which (recall that $u \not\equiv 0$) means that u is a minimizer.

To prove relative compactness of the minimizing sequence, we need to prove that u_j converges to u *strongly* in $H^1(\mathbb{R}^N)$, which follows if we can prove that $\int |\nabla u|^2\,dx = \int |u|^2\,dx = 1$. Our proof will rely on the fact that the inequality

$$(t_1+t_2)^{\theta q/2}(m_1+m_2)^{(1-\theta)q/2} \geq t_1^{\theta q/2} m_1^{(1-\theta)q/2} + t_2^{\theta q/2} m_2^{(1-\theta)q/2}$$

is *strict*, unless $t_1 = m_1 = 0$ or $t_2 = m_2 = 0$. (This follows from conditions of equality in Hölder's inequality and in inequality (3.9).) Now let $T_1 = \int |\nabla u|^2\,dx$ and $M_1 = \int |u|^2\,dx$ and note that $0 < T_1 \leq 1$ and $0 < M_1 \leq 1$. It follows from (3.6) and (3.7) that

$$T_2 = 1 - T_1 = \lim_{j\to\infty} \int |\nabla r_j|^2\,dx\,, \qquad M_2 = 1 - M_1 = \lim_{j\to\infty} \int |r_j|^2\,dx\,.$$

Thus, if we had $T_1 < 1$ or $M_1 < 1$, then we would have the strict inequality

$$1 = (T_1+T_2)^{\theta q/2}(M_1+M_2)^{(1-\theta)q/2} > T_1^{\theta q/2} M_1^{(1-\theta)q/2} + T_2^{\theta q/2} M_2^{(1-\theta)q/2}\,,$$

which, because of the bound

$$T_2^{\theta q/2} M_2^{(1-\theta)q/2} \geq S_{N,q}^{q/2} \lim_{j\to\infty} \int_{\mathbb{R}^N} |r_j|^q\,dx = 1 - S_{N,q}^{q/2} \int_{\mathbb{R}^N} |u|^q\,dx\,,$$

would lead to a contradiction. Thus, $T_1 = M_1 = 1$, as we set out to prove. □

4. The method of moving planes

4.1. **Symmetry of positive solutions of semi-linear equations.** We now introduce a method to prove that positive solutions of semi-linear PDEs are necessarily radial and decreasing. Thus, while Schwarz symmetrization tells one that if certain functionals have a minimizer, they also have a radial decreasing minimizer, a consequence of the theorem in this section is that *any* minimizer is radial decreasing. The argument uses the so-called *method of moving planes.*

Theorem 4.1 (Symmetry of positive solutions)**.** *Let $N \geq 2$ and let f be a function on $[0,\infty)$ satisfying*

$$\frac{f(b)-f(a)}{b-a} \leq -\tau^2 + Cb^\alpha \qquad \text{for all } 0 \leq a < b \tag{4.1}$$

for some $C \geq 0$, $\tau > 0$ and $\alpha > 0$. Assume that $u \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > \alpha N/2$ is non-negative and solves

$$-\Delta u = f(u)\,. \tag{4.2}$$

Then u is radial with respect to some point and (unless $u \equiv 0$) strictly decreasing with respect to the distance from this point. The same conclusion holds for $\tau = 0$, provided $N \geq 3$ and $u \in \dot H^1(\mathbb{R}^N) \cap L^{\alpha N/2}(\mathbb{R}^N)$.

Note that we do not assume f to be continuous. We only require a one-sided condition.

Example 4.2. The function $f(u) = u^{q-1} - u$ with $q > 2$ satisfies (4.1). To see this, we write for $a < b$,

$$\frac{f(b) - f(a)}{b - a} = -1 + (q-1) \int_0^1 ((b-a)t + a)^{q-2} \, dt \,.$$

Since $(b-a)t + a \leq b$ for $0 \leq t \leq 1$, we obtain (4.1) with $\tau = 1$, $C = q - 1$ and $\alpha = q - 2$.

Thus, the non-linearity of our equation (1.2) satisfies assumption (4.1). We also observe that for q as in Theorem 1.2 the integrability condition in Theorem 4.1 is automatically satisfied for H^1 functions due to the Sobolev interpolation inequality (2.2). Thus, Theorem 4.1 implies that u in Theorem 1.2 is radial with respect to some point and a decreasing function with respect to the radius.

The method of moving planes goes back to Alexandrov [Ale] and was popularized by Serrin [Se] and, in particular, by Gidas–Ni–Nirenberg [GiNiNi1, GiNiNi2]. It has been extended in various directions and our theorem gives only a glimpse of the power of the method. In particular, we assume for simplicity that $u \in L^q(\mathbb{R}^N)$ which, in some sense, means that u vanishes at infinity. This assumption allows us to follow an argument by Terracini [Te] which is mostly based on Sobolev inequalities and uses only weak versions of the Hopf lemma and the maximum principle. We discuss those in the following subsection, before turning to the main proof of Theorem 4.1.

4.2. **Maximum principles.** Our proof of Theorem 4.1 is based on two versions of the maximum principle. To motivate the first one, assume that $-w'' + Vw = 0$ on an interval (a, b) and $w \geq 0$ on that interval. If there is a point $c \in (a, b)$ with $w(c) = 0$, then (at least if w is smooth) $w'(c) = 0$ as well and therefore, by uniqueness for second order ODEs, $w \equiv 0$. The following lemma is a PDE version of this result. It is a weak version of the strong maximum principle due to Trudinger [Tr2].

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^N$ be an open and connected set and let $V \in L^1_{\mathrm{loc}}(\Omega)$. Assume that $w \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, |V| \, dx)$ satisfies*

$$\begin{aligned} -\Delta w + Vw &\geq 0 \qquad \text{in } \Omega \,, \\ w &\geq 0 \qquad \text{in } \Omega \,. \end{aligned} \tag{4.3}$$

Then either $w \equiv 0$ or else $|\{w = 0\} \cap \Omega| = 0$.

The first equation in (4.3) means, by definition, that

$$\int_\Omega (\nabla \psi \cdot \nabla w + V \psi w) \, dx \geq 0 \tag{4.4}$$

for all $0 \leq \psi \in H^1(\Omega) \cap L^2(\Omega, |V| \, dx)$ with compact support in Ω.

For readers familiar with the notion of capacity we note that our proof of Lemma 4.3 shows that either $w \equiv 0$ or $\{w = 0\}$ has zero capacity.

Proof. Let B be a ball with $\overline{B} \subset \Omega$. We are going to show that if $|\{w = 0\} \cap B| > 0$, then $w \equiv 0$ on B. Since Ω is connected, this implies the result.

Thus, let B be a ball and let B' be a larger open ball with $\overline{B} \subset B' \subset \overline{B'} \subset \Omega$. We choose a real-valued function $\zeta \in C_0^\infty(B')$ with $\zeta \equiv 1$ on B. Then $(w+\varepsilon)^{-1}\zeta^2$ is non-negative, has compact support in Ω and belongs to $H^1(\Omega) \cap L^2(\Omega, |V|\,dx)$ for any $\varepsilon > 0$. Assumption (4.4) with $\psi = (w+\varepsilon)^{-1}\zeta^2$ implies that

$$\int_\Omega \zeta^2 (w+\varepsilon)^{-2} |\nabla w|^2 \,dx \leq \int_\Omega \left(2\zeta (w+\varepsilon)^{-1} \nabla\zeta \cdot \nabla w + \zeta^2 V (w+\varepsilon)^{-1} w \right) dx \,.$$

The right side is bounded from above by

$$\begin{aligned} &2 \left(\int_\Omega \zeta^2 (w+\varepsilon)^{-2} |\nabla w|^2 \,dx \right)^{1/2} \left(\int_\Omega |\nabla \zeta|^2 \,dx \right)^{1/2} + \int_\Omega \zeta^2 V_+ \,dx \\ &\leq \frac{1}{2} \int_\Omega \zeta^2 (w+\varepsilon)^{-2} |\nabla w|^2 \,dx + \int_\Omega \left(2|\nabla\zeta|^2 + \zeta^2 V_+ \right) dx \,, \end{aligned}$$

and therefore we have shown that

$$\int_\Omega \zeta^2 (w+\varepsilon)^{-2} |\nabla w|^2 \,dx \leq 2 \int_\Omega \left(2|\nabla\zeta|^2 + \zeta^2 V_+ \right) dx =: C$$

with a constant C independent of ε. In terms of $u_\varepsilon = \ln(1 + w/\varepsilon)$, the previous inequality can be written as

$$\int_B |\nabla u_\varepsilon|^2 \,dx \leq \int_\Omega \zeta^2 |\nabla u_\varepsilon|^2 \,dx \leq C \,.$$

On the other hand, let $K = \{w = 0\} \cap B$ and assume that $|K| > 0$. Since u_ε vanishes on this set, a version of the Poincaré inequality (proved via contradiction in the same way as the 'usual' Poincaré inequality with a mean value zero condition; see, e.g., [LiLo, Thm. 8.11]) implies that

$$\int_B |\nabla u_\varepsilon|^2 \,dx \geq C_P \int_B |u_\varepsilon|^2 \,dx \,,$$

where $C_P > 0$ depends on $|B|$, $|K|$ and N, but not on u_ε. Combining the last two inequalities we learn that

$$\int_B |u_\varepsilon|^2 \,dx \leq C_P^{-1} C$$

for all $\varepsilon > 0$. Since $u_\varepsilon \to \infty$ pointwise on $B \setminus K$ as $\varepsilon \to 0$, we infer from the monotone convergence theorem that $|B \setminus K| = 0$. That is, $w \equiv 0$ a.e. on B, as claimed. $\square$

To motivate our next lemma and its proof we assume that $w \in H^1(\mathbb{R}^N)$ satisfies $-\Delta w + Vw \geq 0$ for some *non-negative* V. We want to conclude that $w \geq 0$. Indeed, since $w_- \in H^1(\mathbb{R}^N)$, we can use this function in the weak form of this equation and obtain $-\int_{w(x)<0} \left(|\nabla w|^2 + V|w|^2 \right) dx \geq 0$. Since the integrand is non-negative, this implies that w is non-negative, as claimed. In the following we show that the same conclusion remains valid in $N \geq 3$ if V is 'not too negative'. We also find a similar result for $N = 2$.

Lemma 4.4. *Let $\gamma > 0$ if $N = 2$ and $\gamma \geq 0$ if $N \geq 3$, let $\Omega \subset \mathbb{R}^N$ be an open set and let $V_- \in L^{\gamma+N/2}(\Omega)$ and $V_+ \in L^1_{\mathrm{loc}}(\Omega)$. Let $\tau > 0$ if $\gamma > 0$ and $\tau \geq 0$ if $\gamma = 0$ and assume that $w \in H^1_0(\Omega) \cap L^2(\Omega, V_+\, dx)$ satisfies in the weak sense*

$$-\Delta w + Vw \geq -\tau^2 w \qquad \text{in } \Omega\,. \tag{4.5}$$

If

$$\int_{\{x\in\Omega:\, w(x)<0\}} V_-^{\gamma+N/2}\, dx < \tau^{2\gamma} \left(L^{(1)}_{\gamma,N}\right)^{-1},$$

then $w \geq 0$ in Ω. Here $L^{(1)}_{\gamma,N} = \gamma^\gamma\, (N/2)^{N/2}\, (\gamma + N/2)^{-(\gamma+N/2)} S_{N,2(\gamma+N/2)/(\gamma+N/2-1)}^{-(\gamma+N/2)}$.

If $\tau = \gamma = 0$ we interpret $\tau^{2\gamma} = 1$.

Proof. As in the discussion before the proposition, $w_- \in H^1_0(\Omega) \cap L^2(\Omega, V_+\, dx)$ and therefore the weak definition of (4.5) implies that

$$\int_\Omega \left(|\nabla w_-|^2 + V w_-^2\right) dx \leq -\tau^2 \int_\Omega w_-^2\, dx$$

Let us assume $\gamma > 0$. We bound the left side from below using Hölder's inequality with $1/(\gamma+N/2)+2/q = 1$ and the Sobolev inequalities (2.2) with $\theta = (N/2)(1-2/q)$,

$$\begin{aligned}
\int_\Omega \left(|\nabla w_-|^2 + V w_-^2\right) dx &\geq \int_\Omega |\nabla w_-|^2\, dx - \|V_-\chi_{\{w<0\}}\|_{\gamma+N/2} \|w_-\|_q^2 \\
&\geq \int_\Omega |\nabla w_-|^2\, dx - S_{N,q}^{-1} \|V_-\chi_{\{w<0\}}\|_{\gamma+N/2} \|\nabla w_-\|^{2\theta} \|w_-\|^{2(1-\theta)} \\
&\geq -\theta^{\theta/(1-\theta)} (1-\theta) S_{N,q}^{-1-N/(2\gamma))} \|V_-\chi_{\{w<0\}}\|_{\gamma+N/2}^{1+N/(2\gamma)} \|w_-\|^2\,.
\end{aligned}$$

For the last inequality we minimized over $\|\nabla w_-\|$. Thus, we have shown that if $w_- \not\equiv 0$ then

$$\theta^{\theta/(1-\theta)} (1-\theta) S_{N,q}^{-1-N/(2\gamma))} \|V_-\chi_{\{w<0\}}\|_{\gamma+d/2}^{1+N/(2\gamma)} \geq \tau^2\,,$$

that is,

$$\|V_-\chi_{\{w<0\}}\|_{\gamma+d/2}^{\gamma+N/2} \geq \theta^{-\gamma\theta/(1-\theta)} (1-\theta)^{-\gamma}\, S_{N,q}^{\gamma+N/2}\, \tau^{2\gamma} = \frac{(\gamma+N/2)^{\gamma+N/2}}{(N/2)^{N/2}\,\gamma^\gamma} S_{N,q}^{\gamma+N/2}\, \tau^{2\gamma}\,.$$

This implies the statement of the proposition for $\gamma > 0$. The case $\gamma = 0$ if $N \geq 3$ is handled similarly using the Sobolev inequality (2.3). □

4.3. **Proof of Theorem 4.1.** Before beginnning with the proof of Theorem 4.1 we observe the following strategy of how to prove that a function is radially symmetric functions.

Lemma 4.5. *Let u be a non-negative, measurable function on $\mathbb{R}^N$ such that $|\{u > \tau\}| < \infty$ for any $\tau > 0$. Assume that for any $e \in \mathbb{S}^{N-1}$ there is an $a \in \mathbb{R}$ such that*

$$u(x) = u(x - 2(x\cdot e - a)e) \quad \text{a.e. } x \in \mathbb{R}^N\,. \tag{4.6}$$

Then there is an $x_0 \in \mathbb{R}^N$ and a function $\tilde u$ on $[0,\infty)$ such that $u(x) = \tilde u(|x - x_0|)$ for a.e. $x \in \mathbb{R}^N$.

In other words, if for every direction there is a hyperplane normal to this direction such that u is reflection symmetric with respect to this hyperplane (this is (4.6)), then u is radial with respect to some origin.

Proof. We shall prove the following

Claim. Let $E \subset \mathbb{R}^N$ be a set of finite measure and let $e \in \mathbb{S}^{N-1}$ and $a \in \mathbb{R}$. If E is invariant (up to a set of measure zero) both under the map $x \mapsto -x$ and under the map $x \mapsto x - 2(x \cdot e - a)e$, then $a = 0$, unless $|E| = 0$.

Before proving this claim, let us show that it implies the statement of the lemma. We fix an orthonormal basis $e_1, \ldots, e_N$ of $\mathbb{R}^N$. Then, after translating u if necessary, we may assume that $u(\ldots, x_j, \ldots) = u(\ldots, -x_j, \ldots)$ for every $1 \leq j \leq N$ and a.e. $x \in \mathbb{R}^N$. (Note that (4.6) remains valid, possibly with a different a.) In particular, $u(x) = u(-x)$ for a.e. $x \in \mathbb{R}^N$.

Now we fix $\tau > 0$ and let $E = \{u > \tau\}$. Then (4.6) and the property $u(x) = u(-x)$ for a.e. $x \in \mathbb{R}^N$ imply that E satisfies the assumptions of the claim for every $e \in \mathbb{S}^{N-1}$ and some $a \in \mathbb{R}$ (depending on e). We infer that E is reflection symmetric with respect to any hyperplane passing through the origin. Thus, E is radial with respect to the origin and, since τ is arbitrary, u is radial with respect to the origin, as required.

Thus, it remains to prove the claim. Replacing e by $-e$ if necessary, we may assume that $a \geq 0$. By the invariance under that map $x \mapsto x - 2(x \cdot e - a)e$ we have

$$|\{x \in E : \ x \cdot e > a\}| = |\{x \in E : \ x \cdot e < a\}| \, .$$

On the other hand, by the invariance under $x \mapsto -x$ we have

$$|\{x \in E : \ x \cdot e > 0\}| = |\{x \in E : \ x \cdot e < 0\}| \, .$$

We conclude from this that $|\{x \in E : 0 < x \cdot e < a\}| = 0$.

Now assume that $a > 0$. By iterating reflections and inversions we shall deduce that $|E| = 0$. Indeed, since E is invariant under $x \mapsto x - 2(x \cdot e - a)e$, the fact that $|\{x \in E : 0 < x \cdot e < a\}| = 0$ implies that $|\{x \in E : \ 0 < x \cdot e < 2a\}| = 0$ and then, since E is invariant under $x \mapsto -x$, $|\{x \in E : \ -2a < x \cdot e < 2a\}| = 0$. Using again the reflection invariance we find $|\{x \in E : \ -2a < x \cdot e < 4a\}| = 0$ and then by inversion invariance $|\{x \in E : \ -4a < x \cdot e < 4a\}| = 0$. Continuing similarly, we obtain $|\{x \in E : \ -2na < x \cdot e < 2na\}| = 0$ and, since $|\{x \in E : \ -2na < x \cdot e < 2na\}| \to |E|$ for $a > 0$ by monotone convergence, we conclude that $|E| = 0$. This proves the claim. $\square$

We now turn to the proof of Theorem 4.1. We denote coordinates in $\mathbb{R}^N$ by $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$. Our goal is to prove that there is a $\lambda_* \in \mathbb{R}$ such that $u(2\lambda_* - x_1, x') = u(x_1, x')$ for all $x \in \mathbb{R}^N$ and that for all $x' \in \mathbb{R}^{N-1}$ the function $x_1 \mapsto u(x_1, x')$ is strictly decreasing for $x_1 > \lambda_*$. Since the choice of the x_1-axis is arbitrary, this, together with Lemma 4.5 will imply the radial symmetry part of the theorem. We will comment on the monotonicity after the proof.

We first introduce some notation. For $\lambda \in \mathbb{R}$ and a point $x = (x_1, x')$ we denote by $x^\lambda := (2\lambda - x_1, x')$ its reflection on the hyperplane $\{x_1 = \lambda\}$. Moreover, the reflection of a function v on $\mathbb{R}^N$ is defined by $v_\lambda(x) := v(x^\lambda)$. Assume now that u solves (4.2) and put

$$\Lambda := \{\mu \in \mathbb{R} : \text{ for all } \lambda > \mu \text{ and for all } x \text{ with } x_1 < \lambda \text{ one has } u(x) \geq u_\lambda(x)\} .$$

Moreover, let $w_\lambda := u - u_\lambda$.

Step 1. Λ is non-empty and bounded below. Since $-\Delta$ commutes with reflections, the function w_λ satisfies the equation

$$-\Delta w_\lambda + V w_\lambda = -\tau^2 w_\lambda \qquad \text{with } V := -\frac{g(u) - g(u_\lambda)}{u - u_\lambda} ,$$

where $g(a) = f(a) + \tau^2 a$. Since $w_\lambda \in H^1_0(\{x_1 < \lambda\})$ we can apply Lemma 4.4. To do so, we write $q = \alpha(\gamma + N/2)$ for some $\gamma > 0$. If $N \geq 3$ and $\tau = 0$, we take $\gamma = 0$. Then, using (4.1), we have on the set $\{w_\lambda < 0\}$ the bound $V \geq -C u_\lambda^\alpha$ and, therefore,

$$\int_{\{x_1 < \lambda, w_\lambda < 0\}} V_-^{\gamma + N/2} \, dx \leq C^{\gamma + N/2} \int_{\{x_1 < \lambda\}} u_\lambda^q \, dx = C^{\gamma + N/2} \int_{\{x_1 > \lambda\}} u^q \, dx .$$

Since $u \in L^q(\mathbb{R}^N)$ there is a $\mu \in \mathbb{R}$ such that

$$C^{\gamma + N/2} \int_{\{x_1 > \lambda\}} u^q \, dx < \tau^{2\gamma} \left(L^{(1)}_{\gamma, N} \right)^{-1}$$

for all $\lambda > \mu$. Hence Lemma 4.4 implies that $w_\lambda \geq 0$ on $\{x_1 < \lambda\}$ for all $\lambda > \mu$, i.e., $\mu \in \Lambda$.

It is easy to see that Λ is bounded from below. Indeed, otherwise $u(x)$ were non-increasing with respect to x_1 for any x', which would contradict $u \in L^q(\mathbb{R}^N)$ – unless $u \equiv 0$, of course.

Step 2. Proof that $u_{\lambda_*} = u$, where

$$\lambda_* := \inf \Lambda .$$

One has $w_\lambda \geq 0$ in $\{x_1 < \lambda\}$ for all $\lambda > \lambda_*$ and hence, since $w_\lambda \to w_{\lambda_*}$ in L^q, also $w_{\lambda_*} \geq 0$ in $\{x_1 < \lambda_*\}$. According to the maximum principle (Lemma 4.3) the claim will follow if we can prove that w_{λ_*} is not positive almost everywhere on $\{x_1 < \lambda_*\}$. We argue by contradiction, assuming that $w_{\lambda_*} > 0$ a.e. We define

$$I(\lambda) := \int \chi_{\{x_1 < \lambda\}} \chi_{\{w_\lambda < 0\}} u_\lambda^q \, dx = \int \chi_{\{x_1 > \lambda\}} \chi_{\{w_\lambda > 0\}} u^q \, dx$$

and claim that there is a $\delta > 0$ such that $I(\lambda) < C^{-\gamma - N/2} \tau^{2\gamma} \left(L^{(1)}_{\gamma, N} \right)^{-1}$ for all $\lambda \in (\lambda_* - \delta, \lambda_*)$. Assuming this for the moment, we deduce as in Step 1 that $w_\lambda \geq 0$ on $\{x_1 < \lambda\}$ for all $\lambda \in (\lambda_* - \delta, \lambda_*)$, in contradiction to the definition of λ_*.

In order to prove the above claim we shall show that any sequence (λ_n) with $\lambda_n \to \lambda_*$ has a subsequence (λ_{n_j}) with $I(\lambda_{n_j}) \to I(\lambda_*) = 0$. Indeed, since translations are continuous in L^q, one has $u_{\lambda_n} \to u_{\lambda_*}$ in L^q and hence $u_{\lambda_{n_j}} \to u_{\lambda_*}$ a.e. for a subsequence.

Thus $w_{\lambda_{n_j}} \to w_{\lambda_*}$ a.e. Since $w_{\lambda_*} > 0$ a.e. on $\{x_1 < \lambda_*\}$ and therefore $w_{\lambda_*} < 0$ a.e. on $\{x_1 > \lambda_*\}$, this implies also that $\chi_{\{w_{\lambda_{n_j}}<0\}} \to \chi_{\{w_{\lambda_*}<0\}}$ a.e. (Note that the strict negativity of w_{λ_*} is crucial at this point.) Since $\chi_{\{x_1>\lambda_n\}} \to \chi_{\{x_1>\lambda_*\}}$ pointwise and $u^q \in L^1$, the assertion follows by dominated convergence. This completes the proof of the Step 2 and, therefore as explained before, by means of Lemma 4.5, the proof of the radial symmetry part of Theorem 4.1.

Note that we also have shown that $x_1 \mapsto u(x_1, x')$ is non-increasing on (λ_*, ∞). Since u is radial with respect to some point, this means that u is a non-increasing function of the radius. In particular, if we write $\partial_j u = u'(r)x_j/r$ then $u' \leq 0$ on $(0,\infty)$. Since u' solves the second-order equation

$$\left(-\partial_r^2 - (N-1)r^{-1}\partial_r + (N-1)r^{-2} - f'(u(r))\right) u' = 0$$

and is not identically zero, we conclude by ODE uniqueness that u' is strictly negative. Thus, u is strictly decreasing. This completes the proof of Theorem 4.1.

5. Uniqueness of ground states

5.1. **Kwong's uniqueness theorem.** In this section we prove uniqueness (up to translation) of positive H^1-solutions of the equation $-\Delta u - u^{q-1} = -u$ in $\mathbb{R}^N$, $N \geq 2$. According to our discussion following Example 4.2 we know that any positive solution is radial with respect to some point and therefore it suffices to prove uniqueness for positive, finite-energy solutions of the ODE

$$-\partial_r^2 u - \frac{N-1}{r}\partial_r u - u^{q-1} = -u \qquad \text{in } (0,\infty) \tag{5.1}$$

with boundary condition $u'(0) = 0$. We note that N enters this equation only as a parameter and, in fact, we will never use the fact that it is integer. Thus, from now on in this section we replace N by an arbitrary real number $\nu > 1$.

It turns out that uniqueness of positive solutions of (5.1) with finite energy is a remarkably deep result, which is due to Kwong [Kw]. Earlier works include [Co, McLSe], as well as [Zh], where uniqueness was proved for a certain notion of 'ground state solutions'. Kwong's proof was later simplified by McLeod [McL] and we follow mostly his arguments and their exposition in [Tao]. For more general uniqueness results obtained by a different method we refer for instance to [SeTa].

Here is Kwong's theorem.

Theorem 5.1. *Let $\nu > 1$ and $q > 2$. Then the problem*

$$\begin{cases} -\partial_r^2 Q - \frac{\nu-1}{r}\partial_r Q - Q^{q-1} = -Q & \text{in } (0,\infty)\,, \\ Q'(0) = 0\,, \qquad \lim_{r\to\infty} Q(r) = 0\,, \\ Q > 0 & \text{in } (0,\infty)\,, \end{cases} \tag{5.2}$$

has at most one solution.

We emphasize that this theorem is a conditional result which does *not* assume the existence of a solution of (5.2). It is not difficult, however, to show that a solution exists for every $q > 2$ if $1 < \nu \leq 2$ and for every $2 < q < 2\nu/(\nu-2)$ if $\nu > 2$. (We know this already for integer values of ν.) Moreover, using Pohozaev's identity one can show that (5.2) does not have a solution if $\nu > 2$ and $q \geq 2\nu/(\nu-2)$.

Remark 5.2. We note that the boundary conditions in (5.2) are pointwise conditions and not 'energy' conditions. We leave it as a simple exercise to show that if Q satisfies (5.2) then $Q \in H^1((0,\infty), r^{\nu-1}dr)$, that is,

$$\int_0^\infty (|Q'|^2 + |Q|^2) r^{\nu-1}\, dr < \infty \,.$$

Conversely, if $0 < Q \in H^1((0,\infty), r^{\nu-1}dr)$ satisfies

$$\int_0^\infty \left(\psi' Q' - \psi Q^{q-1} + \psi Q\right) r^{\nu-1}\, dr = 0 \qquad \text{for all } \psi \in H^1((0,\infty), r^{\nu-1}dr)\,,$$

then Q is smooth and satisfies $\partial_r Q(0) = 0$ and $\lim_{r\to\infty} Q(r) = 0$.

Throughout this subsection the parameters ν and q are fixed and satisfy the assumptions of this theorem.

We are going to prove Theorem 5.1 using the *shooting method*. The use of this method in connection with uniqueness proofs originates in the works of Kolodner [Ko] and Coffman [Co]. It consists in considering for any $a \in (0,\infty)$ the solution Q_a of

$$\begin{cases} -\partial_r^2 Q_a - \frac{\nu-1}{r}\partial_r Q_a - |Q_a|^{q-2} Q_a = -Q_a \\ Q_a(0) = a\,, \qquad Q_a'(0) = 0\,. \end{cases} \tag{5.3}$$

By standard results about initial value problems, there is a unique solution Q_a for each $a > 0$ and this solution is defined on all of $[0,\infty)$. Moreover, Q_a depends smoothly on a. (We leave it as a simple exercise to show that the singular term $(\nu-1)r^{-1}$ does not destroy the smoothness of Q at the origin.)

We are going to classify the shooting parameter a as follows,

$$\begin{aligned} S_+ &= \{a > 0 :\ \inf Q_a > 0\}\,, \\ S_0 &= \{a > 0 :\ \inf Q_a = 0\}\,, \\ S_- &= \{a > 0 :\ \inf Q_a < 0\}\,. \end{aligned}$$

This is a decomposition of the interval $(0,\infty)$ into three disjoint sets. Since Theorem 5.1 is only interesting if a solution to the problem in the theorem exists, we will in the following assume that

$$S_0 \neq \emptyset\,, \tag{5.4}$$

Theorem 5.1 follows immediately from

Proposition 5.3 (Classification of initial conditions)**.** *Assume* (5.4)*. Then there is an $a_0 > 0$ such that*

$$S_+ = (0, a_0)\,, \quad S_0 = \{a_0\}\,, \quad S_- = (a_0, \infty)\,.$$

To appreciate the statement of this proposition we will consider in the next subsection the (much simpler) case $\nu = 1$. We shall prove by elementary means the analogue of Proposition 5.3.

5.2. **A warm-up problem.** In the case $\nu = 1$ we obtain the autonomous equation

$$\ddot{Q} = -V'(Q) \qquad \text{in } \mathbb{R}\,, \tag{5.5}$$

where we now denote the independent variable by t and differentiation with respect to t by a dot. The case we have in mind is

$$V(Q) = \frac{1}{q}|Q|^q - \frac{1}{2}|Q|^2\,, \tag{5.6}$$

but our discussion here applies to general real C^1 functions V on $\mathbb{R}$.

The key observation, on which everything in this subsection hinges, is that the *energy*

$$E(t) = \frac{1}{2}|\dot{Q}(t)|^2 + V(Q(t))$$

is *independent* of t. This can be easily verified by differentiation.

One consequence of energy conservation is that the solution remains for all times in the *potential well* $\{Q \in \mathbb{R} :\ V(Q) \leq E\}$, where $E = E(t_0)$ for some arbitrary time t_0.

Another consequence is that if I is an interval such that $\dot{Q}$ does not vanish in the interior of I (which is the same thing as $V(Q) - E$ does not vanish), then Q solves in I

$$\text{either } \dot{Q} = \sqrt{2(E - V(Q))} \qquad \text{or } \dot{Q} = -\sqrt{2(E - V(Q))}\,, \tag{5.7}$$

depending on the (constant) sign of $\dot{Q}$ in I. Equations (5.7) is a first-order ODE with separated variables, and therefore Q is given implicitly by the equation

$$t = t_0 - \int_{Q_0}^{Q(t)} \frac{dq}{\sqrt{2(E - V(q))}}\,, \tag{5.8}$$

where $t_0 \in I$, $Q(t_0) = Q_0$ and, to be specific, $\dot{Q}(t_0) < 0$.

For simplicity, in the following we only discuss the case $\dot{Q}(t_0) = 0$. (We will have $t_0 \in \partial I$ and $\dot{Q} < 0$ in I, so (5.8) is still valid.)

Proposition 5.4. *Let V be a real C^1 function on $\mathbb{R}$, let $E > \inf V$ and assume that there are $A < B$ with*

$$V(A) = V(B) = E\,, \qquad V(q) < E \textit{ if } q \in (A, B)$$

and

$$V'(B) > 0\,. \tag{5.9}$$

Let $t_0 \in \mathbb{R}$. *Then there is a unique solution* Q *of* (5.5) *satisfying* $Q(t_0) = B$ *and* $\dot Q(t_0) = 0$. *Let*

$$T = 2\int_A^B \frac{dq}{\sqrt{2(E - V(q))}}\,.$$

If $T < \infty$, *then* Q *is periodic with minimal period* T. *If* $T = \infty$, *then* Q *is strictly decreasing with* $\lim_{t\to\infty} Q(t) = A$.

Note the assumption (5.9). Of course, if $V'(B) = 0$, then $Q \equiv B$ is a constant solution.

It is easy to see that because of (5.9) the integral defining T converges at $q = B$ and any possible divergence comes from the behavior of V at A. In particular, if $V'(A) < 0$, then $T < \infty$. Conversely, if $V'(A) = 0$ and V is twice differentiable at A, then $T = \infty$.

Proof. Since $(E - V(q))^{-1/2}$ is integrable near $q = B$ by (5.9), it is easy to see that (5.8) with $Q_0 = B$ defines a function Q in a neighborhood of $t = t_0$ with $Q(t_0) = B$. This function is differentiable and satisfies the first and the second equation in (5.7) on the left and on the right of t_0, respectively, and $\dot Q(t_0) = 0$. Equations (5.7) together with the C^1 regularity of V imply that Q solves (5.5) in a neighborhood of t_0.

Conversely, as explained before, any solution of (5.5) satisfies (5.7) and therefore (5.8). This proves existence and uniqueness of Q near $t = t_0$.

Let us introduce $t_1 := \sup\{t \geq t_0 :\ \dot Q < 0 \text{ on } (t_0, t)\}$ and $C := \lim_{t\to t_1} Q(t)$, which exists by monotonicity, although its finiteness is not clear a-priori. But, indeed, $C \geq A$ for, if $Q(t) = A$ for some $t \in (t_0, t_1)$, then $\dot Q(t) = 0$ by energy conservation. Thus, in particular, $C > -\infty$ and, therefore, $\lim_{t\to t_1} \dot Q(t) = 0$. (This is clear if $t_1 = \infty$ and, if $t_1 < \infty$, we use the fact that if $\dot Q(t_1) < 0$, then the solution could be extended beyond t_1 with negative derivative.) Now energy conservation implies that $C = A$.

Finally, we pass to the limit $t \to t_1$ in (5.8). If $t_1 = \infty$, we find that $T = \infty$. We extend the solution by reflection at $t = t_0$ to all of $\mathbb{R}$. Since $\dot Q(t_0) = 0$, the extended function is C^2 on $\mathbb{R}$ and solves the required equation (5.5). This completes the proof for $t_1 = \infty$.

In the case $t_1 < \infty$, (5.8) yields

$$t_1 - t_0 = \int_A^B \frac{dq}{\sqrt{2(E - V(q))}} = \frac{T}{2}\,.$$

We extend Q by reflection to the interval $[t_1, 2t_1 - t_0]$ and then by translation to all of $\mathbb{R}$. By construction this is a periodic function of period T. Using $\dot Q(t_0) = \dot Q(t_1) = 0$ we verify that we have found a solution of (5.5). This proves the claim. □

Finally, let us return to the special case where V is of the form (5.6) with some $q > 2$. Similarly to the previous subsection, for $a > 0$ we denote by Q_a the solution of (5.5) satisfying $Q_a(0) = a$ and $\dot Q_a(0) = 0$. Then clearly $Q_1 \equiv 1$. Moreover, Proposition 5.4 together with the remark following the proposition implies that Q_a is periodic for $a \in (0,1) \cup (1,a_0) \cup (a_0,\infty)$, where $a_0 = (q/2)^{1/(q-2)}$. (This is the point where $V(a_0) = V(0) = 0$ and $V'(0) = 0$.) Moreover, Q_{a_0} is a strictly decreasing function tending to zero at infinity. This proves that

$$S_+ = (0, (q/2)^{1/(q-2)})\,, \quad S_0 = \{(q/2)^{1/(q-2)}\}\,, \quad S_- = ((q/2)^{1/(q-2)}, \infty)\,.$$

Let us compare this with Kwong's theorem. Proposition 5.3 says that the situation does not change qualitatively if the non-autonomous term $(\nu-1)t^{-1}\dot{Q}$ is added to the left side of (5.5). The situation does change quantitatively, however, as below we shall see that $a_0 > (q/2)^{1/(q-2)}$ for $\nu > 1$.

5.3. **Strategy of the proof of Theorem 5.1.** We begin by observing that

$$S_- \text{ is open}\,. \tag{5.10}$$

Indeed, this is a simple consequence of the continuous dependence of Q_a on a. The fact that S_+ is open as well (and contains a neighborhood of zero) is more involved. We state it as a lemma which we shall prove in the following subsection.

Lemma 5.5. *The set S_+ is open. Moreover, $(0,(q/2)^{1/(q-2)}] \subset S_+$.*

According to this lemma, (5.10) and (5.4) there is an $a_0 > (q/2)^{1/(q-2)}$ such that

$$(0, a_0) \subset S_+ \quad \text{and} \quad a_0 \in S_0\,. \tag{5.11}$$

Thus, to prove Proposition 5.3 we need to show that any $a > a_0$ belongs to S_-. We shall do this by analyzing the first zero of Q_a in $(0,\infty)$ for $a \in S_-$. (This zero exists by definition of S_-.) We denote this zero by R_a, so that

$$Q_a(r) > 0 \text{ in } [0, R_a) \quad \text{and} \quad Q_a(R_a) = 0\,. \tag{5.12}$$

We set $R_a = +\infty$ for $a \in S_0$. We observe that

$$\text{the function } a \mapsto R_a \text{ is continuous on } S_0 \cup S_-\,. \tag{5.13}$$

By continuity at a point $a \in S_0$ we mean that if a sequence $(a_n) \subset S_-$ converges to $a \in S_0$, then $R_{a_n} \to R_a = \infty$. The proof of (5.13) relies on the unique solvability of the Cauchy problem, which excludes double zeroes, and the smooth dependence of Q_a on a. We leave it as an exercise.

We are now ready to state the key observation in the proof of Proposition 5.3.

Proposition 5.6 (Monotonicity of zeroes)**.** *Let a_0 be defined by* (5.11)*. Then $(a_0, a_0+\varepsilon) \subset S_-$ for some $\varepsilon > 0$. Moreover, define*

$$a_- := \sup\{a > a_0 : \ (a_0, a) \subset S_-\} \in (a_0, +\infty]\,.$$

Then the function $a \mapsto R_a$ is strictly decreasing on (a_0, a_-).

Before discussing the proof of this proposition, let us see how it implies Proposition 5.3.

Proof of Proposition 5.3 given Proposition 5.6. By (5.10) and Lemma 5.5 either $a_- = \infty$ or $a_- \in S_0$ and, in order to prove Proposition 5.3, we have to exclude the second possibility. By the second statement of Proposition 5.6, $a \mapsto R_a$ is a strictly decreasing function on (a_0, a_-) and therefore $\limsup_{a\to a_-} R_a < \infty$. By (5.13) this is incompatible with $a_- \in S_0$ (which would mean $R_{a_-} = \infty$), and therefore proves Proposition 5.3. □

Thus, everything is reduced to studying the monotonicity of the first zero R_a of Q_a. The following lemma gives a convenient condition for this in terms of the function

$$\delta_a := \frac{\partial Q}{\partial a}\,.$$

Note that this is the (unique) solution of

$$\begin{cases} -\partial_r^2\delta_a - \frac{\nu-1}{r}\partial_r\delta_a - (q-1)|Q_a|^{q-2}\delta_a = -\delta_a \\ \delta_a(0) = 1\,, \qquad \delta_a'(0) = 0\,. \end{cases} \tag{5.14}$$

Lemma 5.7. *If $a \in S_-$ and $\delta_a(R_a) < 0$, then $\frac{dR}{da}(a) < 0$. Moreover, if $a \in S_0$ and $\lim_{r\to\infty}\delta_a(r) = -\infty$, then $(a, a+\varepsilon) \subset S_-$ for some $\varepsilon > 0$.*

Proof. We differentiate the equation $Q_a(R_a) = 0$ with respect to a at a point in S_- and obtain

$$\frac{dR}{da} = -\frac{\delta_a(R_a)}{\partial_r Q_a(R_a)} \tag{5.15}$$

Because Cauchy uniqueness for equation (5.3), we have $\partial_r Q_a(R_a) < 0$ and therefore the monotonicity stated in Proposition 5.6 is equivalent to $\delta_a(R_a) < 0$, proving the first part of the lemma.

The proof for $a \in S_0$ is similar (treating $R_a = +\infty$ in the same way as $R_a < \infty$ for $a \in S_-$) and is left as an exercise; see also [McL, Lemma 3]. □

We prove Proposition 5.6 via Lemma 5.7, that is, we prove that $\lim_{r\to\infty}\delta_{a_0}(r) = -\infty$ and that $\delta_a(R_a) < 0$ for every $a \in (a_0, a_-)$. To do this we employ a *continuity argument*. That is, we verify a certain property of a at $a = a_0$ and show that it persists as a is increased continuously above a_0 and stays in S_-. The property in question is that δ_a has exactly one zero in $(0, R_a)$ and that $\delta_a(R_a) < 0$. (If $a \in S_0$ the last condition should be interpreted as $\lim_{r\to\infty}\delta_a(r) = -\infty$.)

The continuity method relies on two lemmas, which we state now and prove later in Subsection 5.5.

Lemma 5.8. *For every $a \in S_0 \cup S_-$, δ_a has at least one zero in $(0, R_a)$. Moreover, for a_0 from (5.11), δ_{a_0} has exactly one zero in $(0, \infty)$.*

The second lemma is the technical main result which lies at the heart of the proof.

Lemma 5.9. *Let $a \in S_0 \cup S_-$ and assume that δ_a has exactly one zero in $(0, R_a)$. Then $\delta_a(R_a) < 0$ if $a \in S_-$ and $\lim_{r\to\infty}\delta_a(r) = -\infty$ if $a \in S_0$.*

We now reduce the proof of Proposition 5.6 to the proof of these two lemmas.

Proof of Proposition 5.6 given Lemmas 5.8 and 5.9. Let a_0 be defined by (5.11). According to Lemma 5.8, δ_{a_0} has a unique zero in $(0,\infty)$ and therefore, by Lemma 5.9, $\lim_{r\to\infty}\delta_{a_0}(r) = -\infty$. By the second statement in Lemma 5.7 this implies that $(a_0, a_0+\varepsilon) \subset S_-$ for some $\varepsilon > 0$. This proves the first statement of Proposition 5.6.

Now we introduce the set

$$A = \{a > a_0 :\ a \in S_- ,\ \delta_a \text{ has exactly one zero in } (0, R_a)\}$$

and define

$$a_1 = \sup\{a > a_0 :\ (a_0, a) \subset A\} \in [a_0, +\infty] .$$

Then, by Lemma 5.9, $\delta_a(R_a) < 0$ for $a \in A$ and therefore, by the first statement in Lemma 5.7, $a \mapsto R_a$ is strictly decreasing on (a_0, a_1). Clearly, $a_1 \leq a_-$ and to conclude the proof we need to show that

$$a_1 \geq a_- . \tag{5.16}$$

The proof of this fact will be based on several properties of the set A. First note that since δ_{a_0} has a unique zero in $(0,\infty)$ and satisfies $\lim_{r\to\infty}\delta_{a_0}(r) = -\infty$, the continuous dependence on a shows that $(a_0, a_0+\varepsilon) \subset A$ for some $\varepsilon > 0$. Thus, $A \neq \emptyset$ and $a_1 > a_0$. A similar argument, based again on the continuous dependence on a and on the fact that $\delta_a(R_a) < 0$ for $a \in A$ (Lemma 5.9), shows that

$$\text{if } a \in A ,\ \text{then } [a, a+\varepsilon) \subset A \text{ for some } \varepsilon > 0 . \tag{5.17}$$

We now prove that

$$\text{if } a \in S_- \text{ and } (a-\varepsilon, a) \subset A \text{ for some } \varepsilon > 0 ,\ \text{then } a \in A . \tag{5.18}$$

Indeed, by assumption δ_b has exactly one zero in $(0, R_b)$ for all $b \in (a-\varepsilon, a)$ and by Lemma 5.8 δ_a has at least one zero in $(0, R_a)$. Moreover, $\delta_b(0) = 1$ for all b. Finally, δ_a cannot have a double zero because it solves a second-order equation. All these facts imply that δ_a has exactly one zero in $(0, R_a)$, which proves (5.18).

With (5.17) and (5.18) at hand it is easy to prove (5.16). We argue by contradiction and assume that $a_1 < a_-$. Then $a_1 \in S_-$ and $(a_0, a_1) \subset A$, so that by (5.18), $a_1 \in A$. Thus, by (5.17), $[a_1, a_1+\varepsilon) \subset A$, which contradicts the definition of a_1. This completes the proof of Proposition 5.6. □

It remains to prove the lemmas. This is the content of the following subsections.

5.4. **Analysis of S_+.** The analysis in this subsection bears some similarities with the arguments in the autonomous case $\nu = 1$. It is based on the function

$$E_a(r) = \frac{1}{2}|\partial_r Q_a(r)|^2 + V(Q_a(r))$$

with V from (5.6) and the *monotonicity formula*

$$\partial_r E_a(r) = -\frac{\nu-1}{r}|\partial_r Q_a(r)|^2 , \tag{5.19}$$

which implies that E_a is non-increasing.

As a first consequence of this formula and the fact that

$$E_a(r) \geq \inf V > -\infty\,,$$

we infer that $E_a(\infty) = \lim_{r\to\infty} E_a(r)$ exists and is finite. We also infer that any Q_a is a bounded function.

Proof of Lemma 5.5. First part. We prove that $(0,(q/2)^{1/(q-2)}] \subset S_+$. First, let $a \in (0,(q/2)^{1/(q-2)})$. Then, by (5.19),

$$V(Q_a(r)) \leq E_a(r) \leq E_a(0) = V(a) = \frac{1}{q}a^q - \frac{1}{2}a^2 < 0\,.$$

Since $V(a) < 0$ we have

$$\{Q \in \mathbb{R} :\ V(Q) \leq V(a)\} = [-Q_>, -Q_<] \cup [Q_<, Q_>]$$

for some $0 < Q_< \leq Q_> < \infty$ (depending on a). Since $Q_a(0) = a > 0$, we conclude by continuity that

$$Q_< \leq Q_q(r) \leq Q_> \qquad \text{for all } r \geq 0\,.$$

Thus, $\inf Q_a \geq Q_< > 0$ and therefore $a \in S_+$.

Now let $a = (q/2)^{1/(q-2)}$. Then, by continuity, $\inf_{[0,\varepsilon]} Q_a > 0$ for some $\varepsilon > 0$. Moreover, by (5.19) and the fact that Q_a is non-constant, $E_a(\varepsilon) < 0$. Now the same argument as before shows that $\inf_{[\varepsilon,\infty)} Q_a > 0$ and we are done. □

Lemma 5.10 (Monotonicity of solutions). (1) *If $a \in S_+ \cap (0,1)$, then there is an $\tilde R_a > 0$ such that $\partial_r Q_a(r) > 0$ for all $r \in (0,\tilde R_a)$ and $\partial_r Q_a(\tilde R_a) = 0$. Moreover, $Q_a(\tilde R_a) > 1$.*

(2) *If $a = 1$, then $Q_a \equiv 1$.*

(3) *If $a \in S_+ \cap (1,\infty)$, then there is an $\tilde R_a > 0$ such that $\partial_r Q_a(r) < 0$ for all $r \in (0,\tilde R_a)$ and $\partial_r Q_a(\tilde R_a) = 0$. Moreover, $Q_a(\tilde R_a) < 1$.*

(4) *If $a \in S_0$, then $\partial_r Q_a(r) < 0$ for all $r \in (0,\infty)$.*

(5) *If $a \in S_-$, then $\partial_r Q_a(r) < 0$ for all $r \in (0,R_a]$.*

Proof. Let us assume $a > 1$. (The case $0 < a < 1$ is similar and the case $a = 1$ is clear.) Let $r_* \geq 0$ such that $\partial_r Q_a(r_*) = 0$. (Such a point always exists, e.g., $r_* = 0$.) From (5.3) we learn that

$$\partial_r^2 Q_a(r_*) = -Q_a(r_*)\left(|Q_a(r_*)|^{q-2} - 1\right)\,,$$

By unique solvability of (5.3) we have $Q_a(r_*) \neq 0,1$ (since $a \neq 0,1$). Thus, Q_a has a local maximum (minimum, respectively) at r_* if either $Q_a(r_*) > 1$ or $-1 < Q_a(r_*) < 0$ (either $Q_a(r_*) < -1$ or $0 < Q_a(r_*) < 1$, respectively). In particular, taking $r_* = 0$, we conclude that Q_a has a local maximum at zero.

Thus, only the following cases can occur

(i) Q_a decreases up to some $r_{**} > 0$, where Q_a has a local minimum and where $0 < Q_a(r_{**}) < 1$;

(ii) Q_a decreases up to some $r_{**} > 0$, where Q_a has a local minimum and where $Q_a(r_{**}) < -1$;

(iii) Q_a decreases for all $r \geq 0$.

We claim that for $a \in S_0 \cup S_-$ case (i) cannot occur. Indeed, for $a \in S_-$ one has $E_a(R_a) > 0$. In case (i) one would have $E_a(r_{**}) < 0$ and $R_a > r_{**}$, which contradicts (5.19). The argument for $a \in S_0$ is similar with $E_a(\infty) = 0$.

Thus, if $a \in S_-$, then either (ii) or (iii) occurs (with $\lim_{r\to\infty} Q(r) < 0$ in case (iii)), which implies the assertion in this case. Similarly, if $a \in S_0$, then (iii) occurs (with $\lim_{r\to\infty} Q(r) = 0$), which again implies the assertion in this case.

Thus, it remains to discuss the case $a \in S_+$. Clearly, in this case (ii) cannot occur, and it remains to prove the (iii) does not occur. We argue by contradition and assume that Q_a decreases for all $r \geq 0$. Let $\alpha = \lim_{r\to\infty} Q_a(r)$. Since $a \in S_+$, we have $\alpha > 0$. Moreover, it follows from (5.3) that $\lim_{r\to\infty} \partial_r^2 Q_a(r)$ exists and equals $-\alpha(\alpha^{q-2} - 1)$. But the limit of $\partial_r^2 Q_a$, if it exists, is necessarily equal to zero, and therefore $\alpha = 1$.

The following part of the argument is taken from [BeLiPe]; see [Kw] for a different argument. Let $v(r) = r^{(\nu-1)/2}(Q_a(r) - 1)$. Then $v \geq 0$ and, by a simple computation,

$$-\partial_r^2 v + \left(\frac{(\nu-1)(\nu-3)}{4r^2} - \frac{Q_a(r)^{q-1} - Q_a(r)}{Q_a(r) - 1} \right) v = 0\,.$$

As $r \to \infty$, $(Q_a(r)^{q-1} - Q_a(r))/(Q_a(r) - 1) \to q - 1$ and therefore there is an $R \geq 0$ such that

$$\frac{(\nu-1)(\nu-3)}{4r^2} - \frac{Q_a(r)^{q-1} - Q_a(r)}{Q_a(r) - 1} \leq -\frac{q-1}{2} \qquad \text{for all } r \geq R\,.$$

In particular, $\partial_r^2 v < 0$ in $[R, \infty)$ and, therefore, $\partial_r v$ is decreasing towards a limit $L \in \mathbb{R} \cup \{-\infty\}$. If $L < 0$, then $v(r) \to -\infty$ as $r \to \infty$, which contradicts $v \geq 0$. On the other hand, if $L \geq 0$, then $\partial_r v \geq 0$ on $[R, \infty)$ and, in particular, $v \geq v(R)$ on $[R, \infty)$. Thus, from the equation, $-\partial_r^2 v \geq (q-1)v(R)/2 > 0$. This implies $\partial_r v(r) \to -\infty$ as $r \to \infty$, contradicting $L \geq 0$. Thus we have shown that Q_a cannot be decreasing on $[0, \infty)$, which concludes the proof of the lemma for $a > 1$. $\square$

Remark 5.11. By a similar argument one can show that for $a \in S_+ \cap (1, \infty)$, Q_a has infinitely many local maxima and minima and if $0 = r_0 < r_2 < r_4 < \ldots$ and $0 < r_1 < r_3 < \ldots$ denote the points where Q_a attains its local maxima and minima, respectively, then

$$Q_a(r_0) > Q_a(r_2) > \ldots > 1 > \ldots > Q_a(r_3) > Q_a(r_1)\,.$$

A similar statement is valid for $a \in S_+ \cap (0, 1)$.

Proof of Lemma 5.5. Second part. According to the first part that we have already shown, it suffices to prove that $S_+ \cap (1, \infty)$ is open. Let $a \in S_+ \cap (1, \infty)$. Then we know from Lemma 5.10 that there is an $\tilde{R}_a$ such that $\partial_r Q_a(\tilde{R}_a) = 0$ and $0 < Q_a(\tilde{R}_a) < 1$. Thus, $E_a(\tilde{R}_a) < 0$. Since $Q_a(r)$ and $\partial_r Q_a(r)$ depend continuously on r, we have $E_b(\tilde{R}_a) < 0$ for all b in a neighborhood of a. By the same argument as in the first part of the proof, this implies $\inf_{[\tilde R_a,\infty)} Q_b > 0$ for all b in this neighborhood. On the other hand, since $\inf_{[0,\tilde R_a]} Q_a > 0$ (because of $a \in S_+$), we have in view of the continuous dependence also $\inf_{[0,\tilde R_a]} Q_b > 0$ for all b in a (possibly smaller) neighborhood of a. Therefore a neighborhood of a belongs to S_+, which concludes the proof. $\square$

The idea of the following proof is from [KwZh].

Proof of Lemma 5.8. First part. We prove that δ_{a_0} has at most one zero in $(0,\infty)$.

Since Q_{a_0} is decreasing by Lemma 5.10, the function $Q_1 - Q_{a_0} = 1 - Q_{a_0}$ changes sign exactly once. We now consider the sign changes of the function $Q_a - Q_{a_0}$ as a is continuously increased from $a = 1$ towards $a = a_0$. Our goal is to show that there is exactly one sign change for any $a \in [1, a_0)$.

The idea is that, since Q_{a_0} and Q_a solve the same differential equation, $Q_a - Q_{a_0}$ cannot have a double zero. Thus, in order to change the number of sign changes, a zero must enter the interval $[0,\infty)$ either through the origin or through infinity. We shall show that both possibilities do not occur.

The function $Q_a - Q_{a_0}$ is equal to $a - a_0 < 0$ at the origin and, since $a \in S_+$, positive and bounded away from zero at infinity, locally uniformly in $a \in [1, a_0)$. Indeed, according to Lemma 5.10 there is an $\tilde R_a$ such that $E_a(\tilde R_a) < 0$. Since $E_a(r)$ depends continuously on a we learn that $E_b(\tilde R_a) \le -\varepsilon < 0$ for all b in a neighborhood of a fixed $a \in [1, a_0)$. By the argument in the proof of Lemma 5.5 this implies that $Q_b(r) \ge \delta > 0$ for all $r \ge \tilde R_a$ and all b in this neighborhood. If $R \ge \tilde R_a$ is such that $Q_{a_0} \le \delta/2$ in $[R,\infty)$, we learn that $Q_b - Q_{a_0} \ge \delta/2$ in $[R,\infty)$ and all b in a neighborhood of a fixed $a \in [1, a_0)$. This bound, together with the fact that $Q_a - Q_{a_0}$ cannot have a double zero, shows that $Q_a - Q_{a_0}$ has exactly one sign change for any $a \in [1, a_0)$.

To complete the proof, it suffices to note that if $\delta_{a_0} = \lim_{a\to a_0}(Q_a - Q_{a_0})/(a - a_0)$ had more than one sign change, then $Q_a - Q_{a_0}$ would have more than one sign change for all a sufficiently close to a_0. But we have just seen that this is impossible. This proves the second assertion in Lemma 5.8. $\square$

5.5. **Analysis of $S_0 \cup S_-$.** While initial positions $a \in S_+$ can be discussed using the energy E_a similarly as in the autonomous case, one needs a different argument to treat $a \in S_0 \cup S_-$. The proofs of Lemmas 5.8 and 5.9 in [McL] (similarly to [Kw]) are based on Sturm oscillation theory. Following [Tao], we avoid this general theory and derive the needed facts directly using the Wronskian

$$r^{\nu-1}(f\partial_r g - g\partial_r f)\,.$$

The key computation is

$$\partial_r\left(r^{\nu-1}(f\partial_r g - g\partial_r f)\right) = r^{\nu-1}\left(f\left(\partial_r^2 g + \frac{\nu-1}{r}\partial_r g\right) - g\left(\partial_r^2 f + \frac{\nu-1}{r}\partial_r f\right)\right)\,. \tag{5.20}$$

Proof of Lemma 5.8. Second part. We begin by proving that for every $a \in S_-$, δ_a has at least one zero in $(0, R_a)$.

Using equations (5.3) and (5.14) for Q_a and δ_a, as well as (5.20), we compute

$$\partial_r \left(r^{\nu-1} \left(Q_a \partial_r \delta_a - \delta_a \partial_r Q_a \right) \right) = -(q-2) r^{\nu-1} |Q_a|^{q-2} Q_a \delta_a \,,$$

and obtain after integration

$$r^{\nu-1} \left(Q_a \partial_r \delta_a - \delta_a \partial_r Q_a \right) = -(q-2) \int_0^r s^{\nu-1} |Q_a(s)|^{q-2} Q_a(s) \delta_a(s) \, ds \,. \tag{5.21}$$

We now argue by contradiction and assume that the $\delta_a \geq 0$ in $(0, R_a)$. Then, by (5.21), in $(0, R_a)$,

$$\begin{aligned} \partial_r \frac{\delta_a}{Q_a} &= \frac{Q_a \partial_r \delta_a - \delta_a \partial_r Q_a}{Q_a^2} \\ &= -(q-2) r^{-\nu+1} Q_a(r)^{-2} \int_0^r s^{\nu-1} Q_a(s)^{q-1} \delta_a(s) \, ds \leq 0 \,, \end{aligned}$$

Since $\delta_a(0) = 1$ and $Q_a(0) = a$, integration shows that

$$\frac{\delta_a(r)}{Q_a(r)} \leq \frac{1}{a} \qquad \text{for } r \in [0, R_a) \,.$$

Thus, $\delta_a(r) \leq Q_a(r)/a$ for $r \in [0, R_a)$ and by continuity also at $r = R_a$. This shows that $\delta_a(R_a) \leq 0$ and, since $\delta_a \geq 0$ on $(0, R_a)$, that $\delta_a(R_a) = 0$.

We now evaluate (5.21) at $r = R_a$. Since the left side vanishes and since $Q_a > 0$ in $[0, R_a)$, we conclude that $\delta_a \equiv 0$ in $[0, R_a)$, which contradicts $\delta_a(0) = 1$. This proves the first assertion of Lemma 5.8 for $a \in S_-$.

It remains to prove that also for $a \in S_0$, δ_a has at least one zero in $(0, \infty)$. The proof is similar to the case $a \in S_-$, but the property that a solution 'vanish'/'does not vanish' at R_a has to be replaced by the property that a solution is 'exponentially decreasing'/'exponentially increasing' at ∞; see also [Tao, Lemma B.12]. □

The proof of Lemma 5.9 is based on similar arguments.

Proof of Lemma 5.9. Similarly to the proof of Lemma 5.8 we give the proof only in the case $a \in S_-$ and leave the modifications for $a \in S_0$ to the reader; see again [Tao, Appendix B] for that case.

Thus, let $a \in S_-$ and let r_a be the (unique) zero of δ_a in $(0, R_a)$. We introduce the function

$$P_a(r) = Q_a(r) + c_a r \partial_r Q_a(r) \,, \qquad c_a = -\frac{Q_a(r_a)}{r_a \partial_r Q_a(r_a)} \,.$$

Note that the constant c_a is chosen in such a way that P_a vanishes at r_a. A short computation shows that

$$-\partial_r^2 P_a - \frac{\nu-1}{r} \partial_r P_a - (q-1) |Q_a|^{q-2} P_a + P_a = -2 c_a Q_a - (q-2-2c_a) |Q_a|^{q-2} Q_a \,,$$

which, together with (5.14) and (5.20), implies that

$$\partial_r \left(r^{\nu-1} \left(P_a \partial_r \delta_a - \delta_a \partial_r P_a \right) \right) = -r^{\nu-1} \left(2c_a + (q-2-2c_a)|Q_a|^{q-2} \right) Q_a \delta_a \,. \tag{5.22}$$

Since $\delta_a(r_a) = P_a(r_a) = 0$, the left side integrates to zero over $[0, r_a]$ and therefore there must be an $r' \in (0, r_a)$ such that

$$2c_a + (q-2-2c_a) Q_a(r')^{q-2} \geq 0 \,.$$

We claim that this implies that

$$2c_a + (q-2-2c_a) Q_a^{q-2} \geq 0 \qquad \text{in } [r', R_a] \,. \tag{5.23}$$

Indeed, first note that $c_a \geq 0$, which follows from the fact that Q_a is decreasing in $[0, R_a]$ (Lemma 5.10). Thus, if $c_a \leq (q-2)/2$, then (5.23) is trivially true, and if $c_a > (q-2)/2$, then what we have shown is $Q_a(r')^{q-2} \leq 2c_a/(2c_a - q + 2)$, and (5.23) follows again from the monotonicity of Q_a.

According to (5.23),

$$\left(2c_a + (q-2-2c_a) Q_a^{q-2} \right) Q_a \delta_a \leq 0 \qquad \text{in } [r_a, R_a] \,.$$

With this information at hand we integrate (5.22) over $[r_a, R_a]$ and infer that

$$P_a(R_a) \partial_r \delta_a(R_a) - \delta_a(R_a) \partial_r P_a(R_a) > 0 \,.$$

(The strictness comes from the fact that the derivative does not vanish identically.)

We claim that this implies $\delta_a(R_a) < 0$. Indeed, since $\delta_a(0) = 1$ and δ_a has a unique zero in $(0, R_a)$, we know that $\delta_a(R_a) \leq 0$. To complete the proof we argue by contradiction and assume that $\delta_a(R_a) = 0$ (which implies that $\partial_r \delta_a(R_a) \geq 0$). Then the previous inequality reads $P_a(R_a) \partial_r \delta_a(R_a) > 0$, which is the same as

$$c_a \partial_r Q_a(R_a) \partial_r \delta_a(R_a) > 0$$

This is a contradiction, because $c_a \geq 0$, $\partial_r Q_a(R_a) \leq 0$ and $\partial_r \delta_a(R_a) \geq 0$. This completes the proof. □

5.6. **Non-degeneracy.** In this subsection we prove the non-degeneracy statement in Theorem 1.2. Assume that $N \geq 2$ and that q satisfies the assumptions of that theorem. Let u be the unique positive, finite energy solution of (1.2) which is radial with respect to a given point, which we take to be the origin. We are interested in the linearization

$$L = -\Delta - (q-1) u^{q-2} + 1$$

considered as a self-adjoint operator in $L^2(\mathbb{R}^N)$. Clearly, by differentiating equation (1.2) with respect to the j-th coordinate, we find that $\partial_j u \in \ker L$. The main result of this subsection is that these N function span $\ker L$. This is essentially a consequence of the proof of Kwong's theorem. It was first noted by Weinstein [We2].

To prove this, we first observe that L is a Schrödinger operator with *radial* potential. From the theory of spherical harmonics (or, equivalently, the representation theory of $SO(N)$) it follows that L is unitarily equivalent to an operator

$$\bigoplus_{\ell=0}^{\infty} L_\ell \otimes 1_{\mathbb{C}^{\nu_{\ell,N}}}$$

in the Hilbert space

$$\bigoplus_{\ell=0}^{\infty} L^2((0,\infty), r^{N-1}\,dr) \otimes \mathbb{C}^{\nu_{\ell,N}}$$

for some integers $\nu_{\ell,N}$. These integers are explicit, but we only need to know that $\nu_{0,N} = 1$ and $\nu_{1,N} = N$ for any $N \geq 2$. The operators L_ℓ are given by

$$L_\ell = -\partial_r^2 - (N-1)r^{-1}\partial_r + \ell(\ell + N - 2)r^{-2} - (q-1)u(r)^{q-2} + 1\,.$$

Together with Dirichlet boundary conditions these are self-adjoint operators in the Hilbert space $L^2((0,\infty), r^{N-1}\,dr)$. (More precisely, by 'Dirichlet boundary conditions' we mean that the Friedrichs extension of the corresponding operators defined on $C_0^\infty(0,\infty)$. Here we use the fact that $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ is dense in $H^1(\mathbb{R}^N)$.) Our assertion about the operator L will follow if we can show that

$$\ker L_\ell = \{0\} \quad \text{if } \ell \neq 1\,, \qquad \dim\ker L_1 = 1\,.$$

Note that the operator L_0 is the same operator that appeared in the proof of Kwong's theorem and that we proved there that $\ker L_0 = \{0\}$. (Technically speaking, L_0 is considered with a Neumann boundary condition. Thus, by ODE uniqueness, any solution in $\ker L_0$ would be a multiple of δ_{a_0} in (5.14), but in Lemma 5.9 we have shown that δ_{a_0} is not square-integrable at infinity. Thus, $\ker L_0 = \{0\}$.)

Let us now discuss $\ell = 1$. If we identify u with a radial function, we have $\partial_j u = u' x_j / r$ and a short computation yields that $L_1 u' = 0$. Moreover, one can show that u is C^∞ and decays exponentially together with all its derivatives. Thus, $u' \in H^1((0,\infty), r^{N-1}\,dr)$ and therefore $u' \in \ker L_1$. Since u is a decreasing function, u' is non-positive. By the Perron–Frobenius theorem u' is the unique ground state of L_1. Thus, $\dim\ker L_1 = 1$, as claimed.

Finally, for $\ell \geq 2$ we note that, due to the $\ell(\ell + N - 2)r^{-2}$-term, $L_\ell > L_{\ell-1}$ in the sense that $\operatorname{dom} L_\ell \subset \operatorname{dom} L_{\ell-1}$ and $(\psi, L_\ell\psi) > (\psi, L_{\ell-1}\psi)$ for all $0 \not\equiv \psi \in \operatorname{dom} L_\ell$. Since the essential spectrum of all operators L_ℓ starts at 1, we deduce that either $\inf\operatorname{spec} L_\ell = \inf\operatorname{spec} L_{\ell-1} = 1$ or $\inf\operatorname{spec} L_\ell > \inf\operatorname{spec} L_{\ell-1}$. In any case we have $\inf\operatorname{spec} L_\ell > \inf\operatorname{spec} L_1 = 0$. This concludes the proof of the non-degeneracy statement in Theorem 1.2.

The proof of both Theorems 1.1 and 1.2 is now complete.

6. A remainder estimate in a Sobolev inequality

6.1. **Setting of the problem and main result.** In this section we consider the minimization problem

$$E_{N,q} = \inf \left\{ \mathcal{E}[\psi] : \ \psi \in H^1(\mathbb{R}^N) \,, \ \int_{\mathbb{R}^N} \psi^2 \, dx = 1 \right\} \tag{6.1}$$

for the functional

$$\mathcal{E}[\psi] = \int_{\mathbb{R}^N} |\nabla \psi|^2 \, dx - \left(\int_{\mathbb{R}^N} |\psi|^q \, dx \right)^{2/q} . \tag{6.2}$$

Here, and everywhere in this section, we only consider *real-valued* functions ψ. Moreover, we assume throughout that $2 < q < \infty$ if $N = 1, 2$ and $2 < q < 2N/(N-2)$ if $N \geq 3$.

Under these assumption we shall see shortly that $E_{N,q}$ is finite. Let

$$\mathcal{G} = \left\{ \psi \in H^1(\mathbb{R}^N) : \ \mathcal{E}[\psi] = E_{N,q} \,, \ \int_{\mathbb{R}^N} \psi^2 \, dx = 1 \right\} .$$

We shall also see shortly that $\mathcal{G} \neq \emptyset$. The main result of this section is the following theorem, which addresses the stability of the minimization problem (6.1). It quantifies that, if the energy $\mathcal{E}[\psi]$ for a certain normalized ψ is close to the minimal energy $E_{N,q}$, then ψ is close to a minimizer.

Theorem 6.1. *Let $2 < q < \infty$ if $N = 1, 2$ and $2 < q < 2N/(N-2)$ if $N \geq 3$. Then there is a constant $c_{q,d} > 0$ such that*

$$\int_{\mathbb{R}^N} |\nabla \psi|^2 \, dx - \left(\int_{\mathbb{R}^N} |\psi|^q \, dx \right)^{2/q} \geq E_{N,q} + c_{q,d} \inf_{\varphi \in \mathcal{G}} \|\psi - \varphi\|_{H^1}^2$$

for all $\psi \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \psi^2 \, dx = 1$.

Remark 6.2. The constant $c_{q,d}$ will be provided by a compactness argument and hence its value cannot be computed or even estimated numerically, unlike $E_{N,q}$.

Theorem 6.1 is from [CaFrLi]. It uses a technique that was introduced by Bianchi and Egnell [BiEg] to prove stability for the homogeneous Sobolev inequality (2.3).

As stated Theorem 6.1 does not really use Kwong's theorem (Theorem 5.1). In Lemma 6.4 we use Kwong's theorem to identify the set $\mathcal{G}$. This, however, is not used further in the proof of Theorem 6.1. The second point where Kwong's theorem seems to enter is Lemma 6.6, where we show the non-degeneracy of the linearization around a minimizer. Since we are only concerned with *minimizers*, not with general solutions of the corresponding Euler–Lagrange equation, there is an independent non-degeneracy proof, however, which we briefly sketch in Remark 6.7.

6.2. **Some preliminaries.** We begin by showing that the infimum (6.1) is finite. This variational problem may be put in a more familiar form by replacing $\psi(x)$ with $\lambda^{N/2}\psi(\lambda x)$ and optimizing over λ. This leads to

$$E_{N,q} = -\theta^{1/(1-\theta)}(1-\theta)S_{N,q}^{-1/(1-\theta)}\,, \qquad \theta = \frac{N}{2}\left(1-\frac{2}{q}\right)\,. \tag{6.3}$$

Thus, the determination of $E_{N,q}$ comes down to the determination of the best constant $S_{N,q}$ in the Sobolev interpolation inequality (2.2), which we have shown to be strictly positive in Corollary 2.4. This shows that $E_{N,q}$ is finite.

Next, we prove existence of a minimizer and show that any minimizing sequence approaches the set of minimizers in the H^1 norm.

Lemma 6.3. *The infimum* (6.1) *is attained. Moreover, if* $(\psi_n) \subset H^1(\mathbb{R}^N)$ *is a minimizing sequence for* $E_{N,q}$*, then*

$$\lim_{n\to\infty} \inf_{\varphi\in\mathcal{G}} \|\psi_n - \varphi\|_{H^1} = 0\,.$$

This can be deduced either directly from Theorem 3.5 via scaling or by repeating (and slightly modifying) the proof of that theorem.

We next use Theorem 5.1 to identify the set $\mathcal{G}$ of minimizers.

Lemma 6.4. *There is a radial, strictly decreasing function* Q *such that*

$$\mathcal{G} = \left\{\sigma Q(\cdot - a) :\ a\in\mathbb{R}^N\,,\ \sigma = \pm 1\right\}\,.$$

The function Q *satisfies* $\int Q^2\,dx = 1$ *and*

$$-\Delta Q - \|Q\|_q^{2-q} Q^{q-1} = EQ\,, \tag{6.4}$$

where $E = E_{N,q}$*.*

The key step in the proof of Theorem 6.1 is the following 'local version' of Theorem 6.1, which establishes the desired inequality under the additional assumption that the distance from the set of optimizers is small. The precise statement reads as follows.

Lemma 6.5. *There are constants* $\varepsilon > 0$ *and* $c_{q,d} > 0$ *such that*

$$\int_{\mathbb{R}^N} |\nabla\psi|^2\,dx - \left(\int_{\mathbb{R}^N} |\psi|^q\,dx\right)^{2/q} \geq E_{N,q} + c_{q,d}\inf_{\varphi\in\mathcal{G}} \|\psi-\varphi\|_{H^1}^2$$

for all $\psi\in H^1(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^N}\psi^2\,dx = 1$ *and* $\inf_{\varphi\in\mathcal{G}}\|\psi-\varphi\|_{H^1}\leq\varepsilon$*.*

Assuming Lemma 6.5 for the moment we complete the

Proof of Theorem 6.1. As a preliminary remark, let us show that if $(\psi_n)\subset H^1(\mathbb{R}^N)$ is a sequence with $\int_{\mathbb{R}^N}\psi_n^2\,dx = 1$ and

$$\int_{\mathbb{R}^N} |\nabla\psi_n|^2\,dx - \left(\int_{\mathbb{R}^N} |\psi_n|^q\,dx\right)^{2/q} \leq E_{N,q} + (1-\rho)\inf_{\varphi\in\mathcal{G}} \|\psi_n-\varphi\|_{H^1}^2 \tag{6.5}$$

for some $\rho > 0$, then the sequence $(\inf_{\varphi\in\mathcal{G}}\|\psi_n-\varphi\|_{H^1})$ is bounded.

Indeed, it suffices to show that both $(\|\nabla\psi_n\|)$ and $(\|\nabla\varphi\|)_{\varphi\in\mathcal{G}}$ are bounded. It follows easily from (2.2) that $M = \sup_{\varphi\in\mathcal{G}}\|\nabla\varphi\| < \infty$. (Instead of (2.2) one can also use a virial type argument to compute that $\|\nabla\varphi\|^2 = N(q-2)/(2N+2q-Nq)|E_{N,q}|$ for any $\varphi\in\mathcal{G}$.) Thus, we can bound $\inf_{\varphi\in\mathcal{G}}\|\psi_n-\varphi\|_{H^1}^2 \leq (1+\sigma)\|\nabla\psi_n\|^2+(1+\sigma^{-1})M^2+4$ for any $\sigma>0$. Therefore from (6.5) with the choice $\sigma = \rho/(2(1-\rho))$,

$$\frac{\rho}{2}\int_{\mathbb{R}^N}|\nabla\psi_n|^2\,dx - \left(\int_{\mathbb{R}^N}|\psi_n|^q\,dx\right)^{2/q} \leq E_{N,q} + (1-\rho)\left(\rho^{-1}(2-\rho)M^2+4\right)\,.$$

Again by (2.2) this implies that $(\|\nabla\psi_n\|)$ is bounded, which proves the preliminary remark.

Let us now turn to the proof of Theorem 6.1. We argue by contradiction and assume there is a sequence $(\psi_n)\subset H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\psi_n^2\,dx = 1$ and

$$\int_{\mathbb{R}^N}|\nabla\psi_n|^2\,dx - \left(\int_{\mathbb{R}^N}|\psi_n|^q\,dx\right)^{2/q} \leq E_{N,q} + \delta_n\inf_{\varphi\in\mathcal{G}}\|\psi_n-\varphi\|_{H^1}^2 \tag{6.6}$$

where $\delta_n\to 0$. Thus, (6.5) holds with any fixed $\rho>0$ for all sufficiently large n. The preliminary remark then implies that $(\inf_{\varphi\in\mathcal{G}}\|\psi_n-\varphi\|_{H^1})$ is bounded. By (6.6) this means that (ψ_n) is a minimizing sequence and then, by Lemma 6.3, that $\inf_{\varphi\in\mathcal{G}}\|\psi_n-\varphi\|_{H^1}\to 0$. Thus, for all sufficiently large n, $\inf_{\varphi\in\mathcal{G}}\|\psi_n-\varphi\|_{H^1}\leq\varepsilon$ with $\varepsilon>0$ from Lemma 6.5. But the inequality from Lemma 6.5 contradicts (6.6) with $\delta_n\to 0$. This completes the proof. □

Thus, we have reduced the proof of Theorem 6.1 to the proof of Lemma 6.5, which we will prove in the remaining two subsections.

6.3. **Non-degeneracy of the linearization.** The main ingredient in our proof of Lemma 6.5 is the following theorem which can be deduced from Kwong's results for $N\geq 2$ (and well-known results for $N=1$); see, however, also Remark 6.7 for an alternative argument.

Theorem 6.6. *Let $2<q<\infty$ if $N=1,2$ and $2<q<2N/(N-2)$ if $N\geq 3$. Let Q and E be the function and the number from Theorem 5.1 and consider the self-adjoint operator*

$$H = -\Delta - (q-1)\|Q\|_q^{2-q}Q^{q-2} - E + (q-2)\|Q\|_q^{2-2q}|Q^{q-1}\rangle\langle Q^{q-1}|$$

in $L^2(\mathbb{R}^N)$. Then $H\geq 0$ and

$$\ker H = \operatorname{span}\{Q,\partial_1 Q,\ldots,\partial_N Q\}\,.$$

Proof of Theorem 5.6. First, note that H is the Hessian of the minimization problem for $E_{N,q}$, in the sense that

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0}\left(\left\|\nabla\frac{Q+\varepsilon\varphi}{\|Q+\varepsilon\varphi\|}\right\|^2 - \left\|\frac{Q+\varepsilon\varphi}{\|Q+\varepsilon\varphi\|}\right\|_q^2\right) = (\varphi,H\varphi)$$

for every $\varphi \in H^1(\mathbb{R}^N)$. This is shown by the same arguments as in the proof of Lemma 6.5 below. Thus, since Q is a minimizer, $H \geq 0$, proving the first claim.

The inclusion $\operatorname{span}\{Q, \partial_1 Q, \ldots, \partial_d Q\} \subset \ker H$ is easy. Indeed, equation (6.4) implies that $Q \in \ker H$ and, differentiating (6.4), we also infer that $\partial_j Q \in \ker H$ for any $j = 1, \ldots, N$. (Note that $\partial_j Q \in H^1(\mathbb{R}^N)$ by standard regularity and decay results for Q.)

Let us prove the opposite inclusion. We argue similarly as in Subsection 5.6. Since Q is a radial function, the operator H commutes with rotation and, therefore, can be analysed separated in each angular momentum channel. (In dimension $d = 1$, this means separately on even and odd functions.) On functions orthogonal to radial functions the rank one operator $|Q^{q-1}\rangle\langle Q^{q-1}|$ vanishes and therefore on this subspace the operator H coincides with the operator $-\Delta - (q-1)\|Q\|_q^{2-q} Q^{q-2} - E$, which is just a rescaling of the operator in Theorem 1.2. That theorem therefore implies that the kernel of H on the subspace of functions orthogonal to radial functions is given by the span of the $\partial_j Q$'s.

Thus, it remains to prove that the operator H when restricted to angular momentum $l = 0$ (that is, to radial functions) has only a one-dimensional kernel spanned by Q. To prove this, let η be a radial function in $\ker H$ with $(Q, \eta) = 0$. Then

$$L\eta = \alpha Q^{q-1}\,, \tag{6.7}$$

where $\alpha = -(q-2)\|Q\|_q^{2-2q}(Q^{q-1}, \eta)$ and

$$L = -\Delta - (q-1)\|Q\|_q^{2-q} Q^{q-2} - E\,.$$

Because of the Euler–Lagrange equation (6.4) satisfied by Q we have

$$L(\eta - \beta Q) = 0$$

for $\beta = -(q-2)^{-1}\|Q\|_q^{q-2}\alpha$, that is, $\eta - \beta Q \in \ker L$. Now the non-degeneracy part of Theorem 1.2 (and an explicit computation in $N = 1$) implies that $\eta - \beta Q \equiv 0$. Since $(Q, \eta) = 0$ this implies $\eta \equiv 0$, as claimed. □

Remark 6.7. In the previous proof we used Kwong's work to show that for *any* positive solution Q of (6.4) the linearization L, when restricted to radial functions, does not have a zero eigenvalue. (L is defined in the previous proof.) Here we briefly present an independent proof of this fact for *minimizing* functions Q, which is the situation at hand. The argument is due to [ChGuNaTs].

We argue by contradiction and assume that there is a radial function $\psi \in \ker L$. The key observation is that ψ, when considered as a function on $(0, \infty)$ of the radius, has a unique zero r_0. Indeed, since Q is a minimizer for $E_{N,q}$ we have, as in the previous proof, $H \geq 0$. Since H differs from L only by a rank one operator, we infer that L has at most one negative eigenvalue. On the other hand, since $(Q, LQ) = -(q-2)\|Q\|_q^2 < 0$ by the equation for Q, the operator L, when restricted to radial functions, has at least one negative eigenvalue. Thus, 0 is the second eigenvalue of the radial operator

$-\partial_r^2-(N-1)r^{-1}\partial_r-(q-1)\|Q\|_q^{2-q}Q^{q-2}$ in $L^2((0,\infty),r^{N-1}dr)$. Sturm–Liouville theory then implies that ψ has a unique zero.

Let us now show that

$$\int_{\mathbb{R}^N} Q^{q-1}\psi\,dx = 0 = \int_{\mathbb{R}^N} Q\psi\,dx\,. \tag{6.8}$$

The first equation comes from

$$0=\int_{\mathbb{R}^d}\left(\nabla Q\cdot\nabla\psi-\frac{q-1}{\|Q\|_q^{q-2}}Q^{q-1}\psi+EQ\psi\right)dx=-\frac{q-2}{\|Q\|_q^{q-2}}\int_{\mathbb{R}^d}Q^{q-1}\psi\,dx\,,$$

where we used the equations for ψ and for Q. To prove the second equation we first note that $Q_\lambda(x)=Q(\lambda x)$ satisfies

$$-\Delta Q_\lambda-\frac{\lambda^2}{\|Q\|_q^{q-2}}Q_\lambda^{q-1}=\lambda^2 EQ_\lambda\,.$$

Differentiating with respect to λ at $\lambda=1$ we obtain

$$L\dot{Q}_1=2EQ+\frac{2}{\|Q\|_q^{q-2}}Q^{q-1}\,,$$

where $\dot{Q}_1=x\cdot\nabla Q=r\partial_r Q$. Regularity and decay theory for Q imply that $\dot{Q}_1\in H^1(\mathbb{R}^N)$. Thus,

$$0=\int_{\mathbb{R}^d}\left(\nabla\dot{Q}_1\cdot\nabla\psi-\frac{q-1}{\|Q\|_q^{q-2}}Q^{q-2}\dot{Q}_1\psi+E\dot{Q}_1\psi\right)dx=2\int_{\mathbb{R}^d}\left(EQ\psi+\frac{1}{\|Q\|_q^{q-2}}Q^{q-1}\psi\right)dx\,,$$

where we used the equations for ψ and for $\dot{Q}_1$. Since $E\neq 0$, this gives the second equality in (6.8).

It now follows from (6.8) that

$$\int_{\mathbb{R}^N}\psi Q\left(Q^{q-2}-Q(r_0)^{q-2}\right)dx=0\,.$$

On the other hand, since Q is positive and non-increasing, the function $\psi Q\left(Q^{q-2}-Q(r_0)^{q-2}\right)$ has a constant sign and is not identically zero. This is the desired contradiction.

The proof strategy in the previous remark is quite robust and has been generalized, for instance, to fractional powers of the Laplacian [FraLe, FraLeSy].

6.4. **Proof of Lemma 6.5.** With Theorem 6.6 at hand we can finally give the

Proof of Lemma 6.5. Let $\psi\in H^1(\mathbb{R}^N)$ with $\int\psi^2\,dx=1$. After a translation and a change of sign, if necessary, we may assume that

$$\inf_{\varphi\in\mathcal{G}}\|\psi-\varphi\|_{H^1}=\|\psi-Q\|_{H^1}\,.$$

This implies that

$$(\psi-Q,\partial_j Q)_{H^1}=0\qquad\text{for all } j=1,\ldots,d\,. \tag{6.9}$$

Let us introduce $j = \psi - Q$ and note that, since $\int \psi^2\, dx = 1 = \int Q^2\, dx$,

$$2\int_{\mathbb{R}^N} Qj\, dx + \int_{\mathbb{R}^N} j^2\, dx = 0\,. \tag{6.10}$$

We now make use of the fact that

$$\left| |a+b|^q - a^q - qa^{q-1}b - \frac{q(q-1)}{2}a^{q-2}b^2 \right| \leq C\left(a^{q-2-\theta}|b|^{2+\theta} + |b|^q\right)$$

for all $a > 0$, $b \in \mathbb{R}$ and some C (depending only on $q > 2$) and $\theta = \min\{q-2, 1\}$. Thus, by Hölder's inequality,

$$\|\psi\|_q^q = \|Q\|_q^q + q\int_{\mathbb{R}^N} Q^{q-1}j\, dx + \frac{q(q-1)}{2}\int_{\mathbb{R}^N} Q^{q-2}j^2\, dx + O(\|j\|_q^{2+\theta} + \|j\|_q^q) \tag{6.11}$$

with an implied constant depending only on q and d (through $\|Q\|_q$). Moreover, by Hölder and Sobolev inequalities

$$\begin{aligned}&\left| q\int_{\mathbb{R}^N} Q^{q-1}j\, dx + \frac{q(q-1)}{2}\int_{\mathbb{R}^N} Q^{q-2}j^2\, dx + O(\|j\|_q^{2+\theta} + \|j\|_q^q) \right| \\ &\qquad \leq \text{const }\left(\|j\|_q + \|j\|_q^q\right) \leq \text{const }\left(\|j\|_{H^1} + \|j\|_{H^1}^q\right)\end{aligned}$$

with constants depending only on q and N. We conclude that there is an $\varepsilon > 0$ (depending only on q and N) such that

$$\left|\|\psi\|_q^q - \|Q\|_q^q\right| \leq \frac{1}{2}\|Q\|_q^q \qquad \text{provided } \|j\|_{H^1} \leq \varepsilon\,.$$

For such j we can take the $2/q$-th power of (6.11) and obtain

$$\begin{aligned}\|\psi\|_q^2 =& \|Q\|_q^2 + 2\|Q\|_q^{2-q}\int_{\mathbb{R}^N} Q^{q-1}j\, dx \\ &+ (q-1)\|Q\|_q^{2-q}\int_{\mathbb{R}^N} Q^{q-2}j^2\, dx - (q-2)\|Q\|_q^{2-2q}\left(\int_{\mathbb{R}^N} Q^{q-1}j\, dx\right)^2 \\ &+ O(\|j\|_q^{2+\theta})\end{aligned}$$

with an implied constant depending only on q and N. Recalling equation (6.4) for Q and condition (6.10) for j we obtain

$$\begin{aligned}\|\nabla\psi\|^2 - \|\psi\|_q^2 &= E + 2\int_{\mathbb{R}^N} \nabla Q\cdot\nabla j\, dx - 2\|Q\|_q^{2-q}\int_{\mathbb{R}^N} Q^{q-1}j\, dx \\ &\quad + \|\nabla j\|^2 - (q-1)\|Q\|_q^{2-q}\int_{\mathbb{R}^N} Q^{q-2}j^2\, dx \\ &\quad + (q-2)\|Q\|_q^{2-2q}\left(\int_{\mathbb{R}^N} Q^{q-1}j\, dx\right)^2 + O(\|j\|_q^{2+\theta}) \\ &= E + (j, Hj) + O(\|j\|_q^{2+\theta})\,.\end{aligned} \tag{6.12}$$

We now define

$$k = j - (Q,j)Q - \sum_{i=1}^{N}\frac{(\partial_i Q, j)}{\|\partial_i Q\|^2}\partial_i Q$$

and note that, according to Theorem 5.1, k is L^2-orthogonal to the kernel of H. Since the essential spectrum of H starts at $-E>0$ there is a constant $g>0$ such that

$$(j,Hj)=(k,Hk)\geq g\|k\|_2^2\,.$$

On the other hand, it is easy to see that there is a constant $C>0$ such that

$$H\geq -\Delta - C\,.$$

(Indeed, one can take $C=\big\|(q-1)\|Q\|_q^{2-q}Q^{q-2}+E\big\|_\infty=(q-1)\|Q\|_q^{2-q}Q(0)^{q-2}+E$.) Thus, for every $0<\rho<1$,

$$(j,Hj)=(k,Hk)\geq g(1-\rho)\|k\|_2^2+\rho\|\nabla k\|_2^2-\rho C\|k\|_2^2$$

and, upon choosing $\rho=g/(g+C+1)$,

$$(j,Hj)=(k,Hk)\geq \frac{g}{g+C+1}\|k\|_{H^1}^2\,.$$

Recalling the orthogonality conditions (6.9) and (6.10) we compute

$$\begin{aligned}\|k\|_{H^1}^2&=\|j\|_{H^1}^2+|(Q,j)|^2\|Q\|_{H^1}^2+\sum_{i=1}^N\frac{|(\partial_iQ,j)|^2}{\|\partial_iQ\|^4}\|\partial_iQ\|_{H^1}^2-2(Q,j)(j,Q)_{H^1}\\&=\|j\|_{H^1}^2+|(Q,j)|^2\|Q\|_{H^1}^2+\sum_{i=1}^N\frac{|(\partial_iQ,j)|^2}{\|\partial_iQ\|^4}\|\partial_iQ\|_{H^1}^2+\|j\|_2^2(j,Q)_{H^1}\,.\end{aligned}$$

Here we used the fact that the ∂_iQ's are H^1 orthogonal among each other and to Q. This simply follows from the fact that ∂_iQ is a radial function times the spherical harmonic $x_i/|x|$ of degree one. Thus,

$$(j,Hj)\geq\frac{g}{g+C+1}\|j\|_{H^1}^2+O(\|j\|_{H^1}^3)\,.$$

We insert this bound into (6.12) and obtain, after decreasing ε if necessary,

$$\|\nabla\psi\|^2-\|\psi\|_q^2\geq E+\frac{g}{2(g+C+1)}\|j\|_{H^1}^2\,.$$

This completes the proof of Lemma 6.5. ☐

Appendix A. The one-dimensional case

It turns out that in dimension $N=1$ the value of the optimal constant $S_{1,q}$ in (1.1) can be computed explicitly.

Theorem A.1. *Let $N=1$ and $2<q<\infty$. Then*

$$\left(\int_{\mathbb{R}}|u'|^2\,dx\right)^\theta\left(\int_{\mathbb{R}}|u|^2\,dx\right)^{1-\theta}\geq S_{1,q}\left(\int_{\mathbb{R}}|u|^q\,dx\right)^{2/q},\quad \theta=\frac12\left(1-\frac2q\right),$$

where

$$S_{1,q}=\frac{(q+2)^{(q+2)/2q}}{2^{(q+2)/q}(q-2)^{(q-2)/2q}}\left(\frac{\sqrt{\pi}\,\Gamma(\frac{q}{q-2})}{\Gamma(\frac{q}{q-2}+\frac12)}\right)^{(q-2)/q}$$

with equality iff $u(x) = cQ(b(x-a))$ *for some* $a \in \mathbb{R}$, $b > 0$ *and* $c \in \mathbb{C}$, *where*

$$Q(x) = \cosh^{-2/(q-2)} x\,.$$

This theorem is due to Sz.-Nagy [Na], who has a short and clever proof. Our arguments are closer to those of Keller [Ke].

Proof. We consider the minimization problem

$$S_{1,q} := \inf_{0 \not\equiv u \in H^1(\mathbb{R})} \frac{\left(\int_{\mathbb{R}} |u'|^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} |u|^2\,dx\right)^{1-\theta}}{\left(\int_{\mathbb{R}} |u|^q\,dx\right)^{2/q}}\,.$$

According to Corollary 2.4 this defines a strictly positive number.

Step 1. The infimum $S_{1,q}$ is attained. This can be shown by a compactness argument, see Theorem 3.5.

Step 2. We claim that any minimizer is a complex multiple of a non-negative function.

We recall that $u \in H^1(\mathbb{R})$ implies $|u| \in H^1(\mathbb{R})$ and $||u|'| \leq |u'|$ a.e.; see, e.g., [LiLo, Thm. 6.17]. This implies that there is a non-negative minimizer, but what we claim is that *any* minimizer is non-negative up to multiplication with a constant.

If we write $u = v + iw$ with v and w real functions, then

$$\begin{aligned}
\left(\int_{\mathbb{R}} |u'|^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} |u|^2\,dx\right)^{1-\theta} &= \left(\int_{\mathbb{R}} \left((v')^2 + (w')^2\right)\,dx\right)^{\theta} \left(\int_{\mathbb{R}} \left(v^2 + w^2\right)\,dx\right)^{1-\theta} \\
&\geq \left(\int_{\mathbb{R}} (v')^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} v^2\,dx\right)^{1-\theta} \\
&\quad + \left(\int_{\mathbb{R}} (w')^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} w^2\,dx\right)^{1-\theta}
\end{aligned}$$

by Hölder's inequality in $\mathbb{R}^2$. We also have, by the triangle inequality

$$\left(\int_{\mathbb{R}} |u|^q\,dx\right)^{2/q} = \|v^2 + w^2\|_{q/2} \leq \|v^2\|_{q/2} + \|w^2\|_{q/2} = \left(\int_{\mathbb{R}} |v|^q\,dx\right)^{2/q} + \left(\int_{\mathbb{R}} |w|^q\,dx\right)^{2/q}$$

This inequality is strict unless $v \equiv 0$ or $w^2 = \lambda^2 v^2$ for some $\lambda \geq 0$. Therefore, if $U = V + iW$ is a minimizer for $S_{1,q}$, then either one of V and W is identically equal to zero or else both V and W are optimizers and $|W| = \lambda|V|$ for some $\lambda > 0$. For any real $u \in H^1(\mathbb{R})$ its positive and negative parts $u_\pm$ belong to $H^1(\mathbb{R})$ and satisfy $u'_\pm = \pm\chi_{\{\pm u > 0\}} u'$ in the sense of distributions. (This can be proved similarly to [LiLo, Thm. 6.17].) Thus, for real u, similarly as before,

$$\begin{aligned}
\left(\int_{\mathbb{R}} (u')^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} u^2\,dx\right)^{1-\theta} &\geq \left(\int_{\mathbb{R}} |u'_+|^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} u_+^2\,dx\right)^{1-\theta} \\
&\quad + \left(\int_{\mathbb{R}} |u'_-|^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}} u_-^2\,dx\right)^{1-\theta}
\end{aligned}$$

and

$$\left(\int_{\mathbb{R}} |u|^q \, dx\right)^{2/q} \leq \left(\int_{\mathbb{R}} u_+^q \, dx\right)^{2/q} + \left(\int_{\mathbb{R}} u_-^q \, dx\right)^{2/q} .$$

Since the function $x \mapsto x^{2/q}$ is strictly concave, the latter inequality is strict unless u has a definite sign. Therefore, if $U = V + iW$ is a minimizer for $S_{1,q}$, then both V and W have a definite sign. We conclude that any minimizer is a complex multiple of a non-negative function.

Step 3. Any (non-negative, without loss of generality) minimizer u satisfies the Euler–Lagrange equation

$$-u'' - \lambda u^{q-1} = -\mu u \tag{A.1}$$

with Lagrange multipliers λ and μ. Let us show that λ and μ are both positive. Indeed, by the optimality of u we have for every real $\varphi \in H^1(\mathbb{R})$,

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} \ln \frac{\left(\int_{\mathbb{R}} (u' + \varepsilon\varphi')^2 \, dx\right)^{\theta} \left(\int_{\mathbb{R}} (u + \varepsilon\varphi)^2 \, dx\right)^{1-\theta}}{\left(\int_{\mathbb{R}} |u + \varepsilon\varphi|^q \, dx\right)^{2/q}} .$$

Working out the derivative, we obtain

$$0 = 2\theta \frac{\int_{\mathbb{R}} \varphi' u' \, dx}{\int_{\mathbb{R}} (u')^2 \, dx} + 2(1-\theta) \frac{\int_{\mathbb{R}} \varphi u \, dx}{\int_{\mathbb{R}} (u)^2 \, dx} - 2 \frac{\int_{\mathbb{R}} \varphi u^{q-1} \, dx}{\int_{\mathbb{R}} |u|^q \, dx} .$$

Since φ is an arbitrary H^1 function, this means that u is a weak solution of

$$0 = \theta \frac{-u''}{\int_{\mathbb{R}} (u')^2 \, dx} + (1-\theta) \frac{u}{\int_{\mathbb{R}} (u)^2 \, dx} - \frac{u^{q-1}}{\int_{\mathbb{R}} |u|^q \, dx} .$$

In particular, λ and μ are both positive, as claimed.

Hence, after a scaling and after multiplication by a positive constant, we may assume that $\lambda = \mu = 1$.

Step 4. Any H^1 solution of (A.1) is, in fact, C^2. To prove this simple regularity result, we first note that $|\int u'v' \, dx| = |\int (u^{q-1} - u) v \, dx| \leq \text{const } \|v\|_2$, since $u \in L^2 \cap L^{2(q-1)}$ by Corollary 2.4. Since this bound holds for any $v \in H^1$, we conclude that $u \in H^2$. We now recall that any H^1 function in one dimension is continuous and therefore any H^2 function is C^1. Thus, $u \in C^1$. Moreover, because of the equation, u'', the weak derivative of u', coincides in L^2-sense with the continuous function $u - u^{q-1}$, and hence $u \in C^2$, as claimed.

Step 5. This is the main part of the proof! We show that the only non-negative and non-zero solution in $H^1(\mathbb{R}) \cap C^2(\mathbb{R})$ of (A.1) with $\mu = \nu = 1$ is given by

$$u(x) = \left(\frac{q}{2}\right)^{1/(q-2)} \cosh^{-2/(q-2)} \left(\frac{q-2}{2}(x-a)\right) \tag{A.2}$$

for some $a \in \mathbb{R}$. Once this is proved, the value of the constant follows by a straightforward (but tedious) computation, using the fact that

$$\int_{\mathbb{R}} \cosh^{-\alpha} x \, dx = \sqrt{\pi} \frac{\Gamma(\alpha/2)}{\Gamma((\alpha+1)/2)} .$$

To prove (A.2), we multiply the equation by u' and find the first integral

$$-\frac{1}{2}(u')^2 - \frac{1}{q}u^q = -\frac{1}{2}u^2 + C$$

for some constant C. Since $u \in H^1(\mathbb{R})$, we have $\lim_{|x|\to\infty} u(x) = 0$ and we deduce from the previous formula that $\lim_{|x|\to\infty}(u'(x))^2$ exists and is given by $-C$. From this we conclude that $C = 0$ and consequently

$$u' = \pm\sqrt{u^2 - \frac{2}{q}u^q}\,.$$

(When solving the quadratic equation for u' a sign ambiguity arises. This ambiguity will disappear later in the proof. It is important, however, that the sign can only change at points where the square root vanishes, that is, at points where $u = (q/2)^{q-2}$. Here we used the regularity of u and also the fact that u is strictly positive, which follows uniqueness of ODE initial value problems.)

We have derived an equation with separate variables for u which, in principle, can be solved in terms of an anti-derivative of $(u^2 - \frac{2}{q}u^q)^{-1/2}$. We proceed somewhat differently and introduce, following, e.g., [Fab], the function $v(u) = \sqrt{1 - \frac{2}{q}u^{q-2}}$. The equation becomes $u' = \pm uv$. On the other hand, we compute

$$\frac{dv}{du} = -\frac{q-2}{q}\frac{u^{q-3}}{v} = -\frac{q-2}{2}\frac{1-v^2}{uv}$$

and obtain

$$\frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} = \mp\frac{q-2}{2}(1-v^2)\,.$$

Recalling that $(1-v^2)^{-1}$ has anti-derivative arctanh we can integrate this equation and find that

$$x - a = \mp\frac{2}{q-2}\operatorname{arctanh} v\,,$$

that is,

$$\tanh\left(\mp\frac{q-2}{2}(x-a)\right) = v = \sqrt{1 - \frac{2}{q}u^{q-2}}\,.$$

(We emphasize that, in principle, a could depend on the $\pm$ sign. It does not, however, by the regularity of u.) Since tanh is odd, we conclude that

$$1 - \cosh^{-2}\left(\frac{q-2}{2}(x-a)\right) = \tanh^2\left(\frac{q-2}{2}(x-a)\right) = 1 - \frac{2}{q}u^{q-2}\,,$$

which is what we claimed in (A.2). □

Finally we discuss the linearization of (A.1) (with $\lambda = \mu = 1$) around its solution (A.2). We obtain the operator

$$L = -\frac{d^2}{dx^2} - (q-1)u^{q-2} + 1 = -\frac{d^2}{dx^2} - \frac{q(q-1)}{2}\cosh^{-2}\beta x + 1\,, \quad \beta = \frac{q-2}{2}\,.$$

By the same argument as in arbitrary dimension we have $u' \in \ker L$. The fact that this function already spans $\ker L$ simply follows from the fact that the eigenvalue multiplicity of any one-dimensional Schrödinger operator (at least with potential in $L^1(\mathbb{R})$) is one. Indeed, under this assumption on the potential a simple iteration argument shows that the equation $-\psi'' + V\psi = E\psi$ with $E < 0$ has two linearly independent solutions $\psi_\pm$ behaving as $e^{\mp\sqrt{-E}x}\psi_\pm \to 1$ as $x \to \infty$. Thus, at most one solution is in L^2.

Finally, we mention that the operator L can be explicitly diagonalized. This can be done using a single creation and annihilation operator (mathematically, a Darboux transform) much in the same way as for the hydrogen atom. We leave this as an amusing exercise to the reader.

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