EFFICIENT ALLOCATIONS UNDER LAW-INVARIANCE: A UNIFYING APPROACH

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Abstract. We study the problem of optimising the aggregated utility within a system of agents under the assumption that individual utility assessments are *law-invariant*: they rank Savage acts merely in terms of their distribution under a fixed reference probability measure. We present a unifying framework in which optimisers can be found which are *comonotone allocations* of an aggregated quantity. Our approach can be localised to arbitrary rearrangement invariant commodity spaces containing at least all bounded wealths. The aggregation procedure is a substantial degree of freedom in our study. Depending on the choice of aggregation, the optimisers of the optimisation problems are allocations of a wealth with desirable economic efficiency properties, such as (weakly, biased weakly, and individually rationally) Pareto efficient allocations, core allocations, and systemically fair allocations.

Keywords: efficient allocations, law-invariant utilities, comonotone improvement, (weak) Pareto efficiency, fair allocations

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1. Introduction

A substantial driver in the development of mathematical economics has been the theory of general equilibrium, surveyed, among many others, by Debreu [16] and Mas-Colell & Zame [29]. It mainly analyses whether and how a *pure exchange economy* populated by a finite number of agents can share a commodity in an efficient way. Efficiency is always to be understood against the backdrop of potentially varying *individual preferences* the agents have concerning the shares of the commodity they receive.

We shall refer to such such sharing schemes as *allocations*. The most prominent notion of their efficiency is *Pareto efficiency*, a systemic notion of stability and efficiency of an economy which means that no agent can improve her share without worsening the share of another agent. Formally, suppose the agents are represented by the set $\{1,...,n\}$, share a common good X, and entertain preferences $\preceq_i$, $i \in \{1,...,n\}$, concerning the share they are to receive. Then a sharing $\mathbf{X} = (X_1,...,X_n)$ is Pareto efficient if any other sharing scheme $\mathbf{Y} = (Y_1,...,Y_n)$ which satisfies $X_i \prec Y_i$ for some agent i necessarily satisfies $Y_j \prec X_j$ for another agent $j \neq i$.

Pareto efficient allocations can be analysed particularly well if the individual preferences $\preceq_i$, $i \in \{1,...,n\}$, admit a numerical representation: if $\mathcal{X}$ denotes the set of commodities agent i accepts as her share, a function $\mathfrak{U}_i : \mathcal{X} \to [-\infty,\infty)$ is a numerical representation of the preference relation $\preceq_i$ if i weakly prefers Y to X if, and only if, $\mathfrak{U}_i(X) \leq \mathfrak{U}_i(Y)$. We will refer to $\mathfrak{U}_i$ as a *utility function.*[1]

Let us assume now that the commodity space involved in such a problem is a vector space $\mathcal{X}$. Given a good $X \in \mathcal{X}$ which is to be shared, an allocation of X is any vector $\mathbf{X} = (X_1,...,X_n)$ with the property $X_1 + ... + X_n = X$. This assumption of perfect substitution means that in principle, any sharing of X is hypothetically feasible for the agents. Suppose furthermore that each individual preference relation $\preceq_i$ can be numerically represented by a utility function $\mathfrak{U}_i : \mathcal{X} \to [-\infty,\infty)$, $i \in \{1,...,n\}$.

A key observation which will be the guiding thread of our investigations, initially due to Negishi, is the following: suppose that suitable positive weights $w_1,...,w_n > 0$ can be found such that the allocation $\mathbf{X}^* = (X_1^*,...,X_n^*)$ satisfies

$$\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) \leq \sum_{i=1}^n w_i \mathfrak{U}_i(X_i^*) \in \mathbb{R},$$

where $\mathbf{X} = (X_1,...,X_n) \in \mathcal{X}^n$ is an arbitrary allocation of X. Then $\mathbf{X}^*$ is indeed a Pareto efficient allocation of X.

Let us abstract this this example which we shall get back to at a later stage of the paper. The allocation $\mathbf{X}^*$ is a maximiser for the optimisation problem

$$\Lambda(\mathfrak{U}(\mathbf{X})) \to \max \quad \text{subject to} \quad \mathbf{X} \in \Gamma_X, \tag{1}$$

[1] As usual, we exclude the case of infinite utility, whereas infinite disutility cannot be excluded *a priori*.

where Γ_X is the set of all allocations $\mathbf{X} \in \mathcal{X}^n$ with the property $X_1 + ... + X_n = X$, whereas $\mathfrak{U}(\mathbf{X}) := (\mathfrak{U}_1(X_1), ..., \mathfrak{U}_n(X_n))$ denotes the vector of individual utilities resulting for the agents from the sharing $\mathbf{X}$. These individual utilities are aggregated to a single quantity using the aggregation function

$$\Lambda(y) := \sum_{i=1}^{n} w_i y_i, \quad -\infty \leq y_i < \infty, \tag{2}$$

and the optimal value is finite. As the aggregation function Λ in problem (1) may be chosen freely, it introduces substantial flexibility which we shall exploit in Section 5. Given a parameter $0 < \alpha \leq 1$, we will use the aggregation function

$$\Lambda_\alpha(y) := \alpha \min_{1 \leq i \leq n} y_i + (1-\alpha) \max_{1 \leq i \leq n} y_i, \quad -\infty \leq y_i < \infty,$$

to obtain *(biased) weakly Pareto efficient* allocations as maximisers for problem (1). Similarly, if we choose the aggregation function

$$\Xi_\alpha(y) := \sum_{\emptyset \neq S \subset \{1,...,n\}} \alpha \min_{1 \leq i \leq n} y_i + (1-\alpha) \max_{i \in S} y_i, \quad -\infty \leq y_i < \infty,$$

maximisers will be so-called *core allocations*, which reflect certain notions of fairness in game theory. The reader should keep in mind that both the optimal value and the optimisation problem itself will be of secondary importance.

We shall assume throughout our study that all goods are risky future quantities or *Savage acts*, i.e. real-valued random variables contingent on a measurable space $(\Omega, \mathcal{F})$ of future states of the world. One may also think of them in the interpretation of Mas-Colell & Zame [29] as consumption patterns. Riskiness in the realisation of the states $\omega \in \Omega$ is assumed to be governed by a reference probability measure $\mathbb{P}$ such that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *non-atomic*.[2] As usual, we shall identify two Savage acts X and Y if the event $\{\omega \in \Omega \mid X(\omega) = Y(\omega)\}$ has full $\mathbb{P}$-probability. Substantial results have been achieved solving problem (1) with aggregation function Λ chosen as in (2) in a framework of Savage acts and involving *law-invariant* preferences, c.f. [1, 6, 10, 15, 17, 22, 31].

We adopt the assumption that the agents involved have law-invariant preferences, i.e. the values of the utility functions $\mathfrak{U}_i$, $i \in \{1, ..., n\}$, only depend on the *distribution* of the commodity under the reference probability measure $\mathbb{P}$: if two Savage acts X and Y induce the same lottery over the real line under $\mathbb{P}$, i.e. if the Borel probability measures $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ on $\mathbb{R}$ are identical, then $\mathfrak{U}_i(X) = \mathfrak{U}_i(Y)$ holds for all $i = 1, ..., n$, the reasoning being that utility only depends on statistical properties of the commodity. Along the lines of Dana [15] and Jouni et al. [22], we shall refer to such utility assessments law-invariant. Under the name of *probabilistic sophistication* it is a well-known property of preference relations which was introduced by Machina & Schmeidler [28]; we refer to Cerreia-Vioglio et al. [8] as well as the references in [8, footnote 2]; however, these references typically study preference relations in an Anscombe-Aumann framework with general sets of consequences. Strzalecki [33], on

[2] That is, there is a random variable U whose distribution function $\mathbb{R} \ni x \mapsto \mathbb{P}(U \leq x)$ is a continuous function.

the other hand, studies probabilistic sophistication for general finitely additive reference probabilities. We use the term law-invariance to emphasise that we are working in a Savage setting with a numerical representation of a preference relation whose values only depend on the law under a countably additive reference probability measure.

Normatively, law-invariance can be interpreted as a form of *consequentialism* of the agents in that they are indifferent between two Savage acts yielding the same *consequences* — by inducing the same lottery under the reference probability measure $\mathbb{P}$. Practically, this consequentialism mostly relies on the fact that Savage acts can be grasped only in terms of empirical distributions of certain quantities, an observation which also explains the requirement of non-atomicity of the state space $(\Omega, \mathcal{F}, \mathbb{P})$. There is a one-to-one correspondence between law-invariant utility functions over Savage acts contingent on a non-atomic space and preference relations on (suitable sets of) lotteries on the real line.

Preferences expressed by law-invariant utility functions have another economically appealing feature. Under a mild continuity assumption and quasi-concavity of the utility function $\mathfrak{U}_i$ — that is, convexity of the preference relation expressed by $\mathfrak{U}_i$ — law-invariance of $\mathfrak{U}_i$ is equivalent to two standard notions of risk aversion: (i) monotonicity in the concave order which was introduced to the economics literature by Rothschild & Stiglitz [32]: if every risk averse expected utility agent weakly prefers X to Y, agent i with utility $\mathfrak{U}_i$ weakly prefers X to Y; (ii) dilatation monotonicity: if Π is a finite measurable partition of the state space, agent i weakly prefers the act associated to more information encoded by Π, i.e. the conditional expectation $\mathbb{E}_{\mathbb{P}}[X|\sigma(\Pi)]$, to X itself. This will be discussed in detail in Theorem 18, to the best of our knowledge the most general version of this result in the literature and one of the main results of the paper.

Our established equivalence between law-invariance and concave order monotonicity has the important consequence that, in many situations, *comonotone* maximisers for (1) can be found. An allocation $\mathbf{X}$ of $X \in \mathcal{X}$ is comonotone if there are n non-decreasing functions $f_i : \mathbb{R} \to \mathbb{R}$ summing up to the identity — $f_1(x) + ... + f_n(x) = x$ holds for all $x \in \mathbb{R}$ — such that $X_i = f_i(X)$, $i = 1, ..., n$. The commodity $f_i(X)$ can be interpreted as a contract contingent on the common risk driver X. Such comonotone allocations are desirable and have been widely studied. Empirical investigations of comonotonicity as a property of (optimal) allocations can be found in Attanasio & Davis [4] and Townsend [36]. According to Carlier et al. [7], who study multivariate comonotonicity, it is a property which — statistically — is "testable and tractable". Key to solving (1) are so-called *comonotone improvement* results as given by Landsberger & Meilijson [25], Ludkovski & Rüschendorf [26], Carlier et al. [7], and Filipović & Svindland [17]. For comonotonicity in a multivariate setting we refer to Carlier et al. [7] and the references therein. For its use beyond the risk sharing problem, see Cheung et al. [12] and Jouini & Napp [21] as well as the references therein.

Before we outline our main contributions, we give a brief overview of the rich existing literature of sharing problems as described above. For its treatment in general equilibrium theory, we refer to the survey articles by Debreu [16] and Mas-Colell & Zame [29] as well as the monograph Khan & Yannelis [23]. More closely related and involving law-invariant criteria are the problems studied by Carlier & Dana [6], which focuses on Rank Dependent Expected Utility agents and uses additional conditions, and Dana [15], which studies optimal allocations and equilibria for concave, monotone and law-invariant preferences with strong order continuity properties on bounded wealths. Jouini et al. [22] and Acciaio [1] study the problem for law-invariant utility functions under the additional assumption of cash-additivity. The comonotonicity of solutions to such sharing problems has been subject of, e.g., Chateauneuf et al. [9] for Choquet expected utility agents, Strzalecki & Werner [34] in the case of more general ambiguity averse preferences, and Ravanelli & Svindland [31] who study agents with *variational preferences* as axiomatised by Maccheroni et al. [27]. There is also a rich strand of literature on sharing problems when the objective is not to maximise utility, but to minimise risk. The functionals involved are thus not utility functions, but *risk measures*. The case of agents with convex law-invariant and cash-additive risk measures has been studied by Filipović & Svindland [17] on Lebesgue spaces, and by Chen et al. [10] on general rearrangement invariant spaces. While Acciaio & Svindland [2] treat the case of law-invariance for *different* reference probability measures, Liebrich & Svindland [24] consider the problem for convex risk measures beyond law-invariance of the involved functionals. Finally, Mastrogiacomo & Rosazza Gianin [30] study weak Pareto optima involving quasi-convex risk measures.

Our main contribution is to prove the existence of comonotone maximisers in (1), and thus of economically desirable allocations, in a wide range of situations by laying the groundwork in clear-cut meta results and then applying these in concrete cases which encompass, but go beyond Pareto efficiency, such as the application in game theory mentioned above. We prove that maximisers in problem (1) exist for agents with heterogeneous preferences as long as their utilities are law-invariant with respect to the reference probability measure $\mathbb{P}$ and suitable bounds hold on the one-dimensional subspace of riskless commodities. This approach distinguishes it from other contributions in this direction which restrict preferences to certain classes of law-invariant utilities. It therefore qualifies as unifying. We would like to point out a few noteworthy directions in which we were able to obtain general results:

- *The range of applications:* By making suitable choices for the aggregation function Λ in (1), we show the existence of comonotone biased weakly Pareto efficient, Pareto efficient, and individually rational Pareto efficient allocations under mild assumptions. Moreover, we discuss applications in game theory and the systemic fairness of allocations.
- *Concavity assumptions*: Apart from Mastrogiacomo & Rosazza Gianin [30], preceding studies of instances of the optimisation problem (1) assume full concavity of the utility functions $\mathfrak{U}_i$.[3] However, this requirement is a very strong form of convexity of the preference relation $\preceq_i$ on $\mathcal{X}$, which means that diversification does (comparatively) not *decrease* utility.[4] Convexity of the preference relation is equivalently characterised

[3] That is, for all $i = 1, ..., n$, for all $X, Y \in \mathcal{X}$, and all $0 < \lambda < 1$, $\mathfrak{U}_i(\lambda X + (1-\lambda)Y) \geq \lambda \mathfrak{U}_i(X) + (1-\lambda)\mathfrak{U}_i(Y)$ holds.

[4] That is, for all $X, Y, Z \in \mathcal{X}$ and $0 < \lambda < 1$, $X \preceq_i Y$ and $X \preceq_i Z$ together imply $X \preceq_i \lambda Y + (1-\lambda)Z$.

by *quasi-concavity* of the numerical representations $\mathfrak{U}_i$: all upper level sets $\{X \in \mathcal{X} \mid \mathfrak{U}_i(X) \geq c\}$, $c \in \mathbb{R}$, are convex sets.

- *Monotonicity assumptions:* Particularly in the financial context, law-invariant utilities are widely assumed to be *monotone in the* $\mathbb{P}$*-a.s. order*: if for two Savage acts X and Y the event $\{\omega \in \Omega \mid X(\omega) \leq Y(\omega)\}$ has $\mathbb{P}$-probability 1, the individual utility assessments satisfy $\mathfrak{U}_i(X) \leq \mathfrak{U}_i(Y)$. This assumption of "more is better" is not always convincing, in particular in light of finiteness of ressources as well as the adverse collateral and ecological effects of economic activity. Our analysis does therefore not rely on monotonicity whatsoever. In all our applications, we only assume that the utility of strictly negative riskless commodities satisfies

$$\lim_{c \downarrow -\infty} \mathfrak{U}_i(c) = -\infty.$$

  Such an assumption of loss aversion does not seem far-fetched.
  Non-monotonicity of individual preferences is also the reason why we distinguish between sharing *with* and *without* free disposal. In the first case, the aggregated good $X \in \mathcal{X}$ has to be shared without any remainder, i.e. one considers allocations $\mathbf{X} = (X_1, ..., X_n) \in \mathcal{X}^n$ with the property $X_1 + ... + X_n = X$. In the second case, a unanimously rejected remainder term may be left aside in the sharing scheme, i.e. relevant allocations have the property $X_1 + ... + X_n \leq X$ with $\mathbb{P}$-probability 1. In other words, we will study the problem

$$\Lambda(\mathfrak{U}(\mathbf{X})) \to \max \quad \text{subject to} \quad \mathbf{X} \in \widehat{\Gamma}_X := \{\mathbf{X} \in \mathcal{X}^n \mid \mathbb{P}(X_1 + ... + X_n \leq X) = 1\}.$$

- *Choice of the commodity space*: Most applications — apart from Chen et al. [10] — assume order continuity of the model space norm, or, even more specifically, the commodity space being an L^p-space of random variables. We show that any rearrangement invariant Banach lattice of $\mathbb{P}$-integrable random variables containing all riskless commodities may be considered. This is a consequence of law-invariance in conjunction with slight regularity assumptions, a combination which *implies* that the problem in question may be viewed as the *localised* version of a problem posed on the space L^1 of all integrable random variables. To phrase this differently, the problem can be solved on the level of all integrable random variables if, and only if, it can be solved on any rearrangement invariant commodity space. In this sense, Section 4 and Proposition 20 contain some of the main results of our investigations.
- *Methodology:* Throughout our investigations, we will solve problem (1) in the most classical fashion: we show that a maximising sequence of allocations has a subsequence which converges to a maximiser. This seems interesting against the backdrop of general equilibrium theory in infinite dimensional spaces. As elaborated in Mas-Colell & Zame [29], infinite dimensionality poses multiple challenges which are usually overcome using fixed point arguments. Whereas it is not clear if the fixed point argument works in our setting, the methodology we use is a powerful addition to the toolkit of general equilibrium theory.

Structure of the paper. In Section 2, we thoroughly describe the setting in which we shall study the sharing problem. In Section 3, we study the problem on the commodity space L^1 of all $\mathbb{P}$-integrable random variables. We isolate the core difficulties of the problem, find powerful meta results applicable in a range of situations as wide as possible, and give a set of straightforward criteria guaranteeing that problems of the shape (1) have maximisers. Section 4 has two parts: Section 4.1 collects the main contributions of our paper on the structural properties of quasi-concave functions on general rearrangement invariant Banach lattices of integrable functions. These findings are of interest beyond the existence of efficient allocations. In Section 4.2 we provide suitable local versions of the results in Section 3 on such general rearrangement invariant commodity spaces. Section 5 illustrates the range of economically relevant allocations which can be obtained with our method. Technical but straightforward estimates necessary for the applications are relegated to Appendix A.

2. Preliminaries

We begin with a few crucial pieces of terminology in use throughout our investigations and introduce the setting of the paper.

Given a non-empty set $\mathcal{X}$, a function $f : \mathcal{X} \to [-\infty, \infty] := \mathbb{R} \cup \{-\infty, \infty\}$ and a level $c \in \mathbb{R}$, the UPPER LEVEL SET of f at level c is the set

$$E_c(f) := \{x \in \mathcal{X} \mid f(x) \geq c\}.$$

If $\mathcal{X}$ is endowed with a topology τ, f is UPPER SEMICONTINUOUS with respect to τ if the sets $E_c(f)$ are τ-closed, $c \in \mathbb{R}$. If $\mathcal{X}$ is a real vector space, f is called QUASI-CONCAVE if each set $E_c(f)$, $c \in \mathbb{R}$, is convex, i.e. for all choices of $x, y \in E_c(f)$ and all $0 < \lambda < 1$, we have $\lambda x + (1 - \lambda)y \in E_c(f)$.

The effective domain of f is defined by

$$\mathrm{dom}(f) := \{x \in \mathcal{X} \mid f(x) > -\infty\}.$$

If $f^{-1}(\{\infty\}) = \emptyset$ and $\mathrm{dom}(f) \neq \emptyset$, f is called PROPER.

Throughout the text, bolded symbols will refer to vectors of objects. Hence, whenever $\mathcal{X}$ is a set and $n \in \mathbb{N}$ is a dimension, objects in $\mathcal{X}^n$ will be denoted by $\mathbf{x} = (x_1, ..., x_n)$. If $f : \mathcal{X} \to [-\infty, \infty]$ is a function, we denote by $f(\mathbf{x}) := (f(x_1), ..., f(x_n))$ the vector in $[-\infty, \infty]^n$ arising from a coordinatewise evaluation with f. Similarly, if $\mathbf{g} : \mathcal{X}^n \to [-\infty, \infty]^n$ is a vector-valued function, $\mathbf{g}(\mathbf{x})$ is defined as $\mathbf{g}(\mathbf{x}) := (g_1(x_1), ..., g_n(x_n))$.

We shall fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. there is a random variable U with continuous cumulative distribution function $\mathbb{R} \ni x \mapsto \mathbb{P}(U \leq x)$. Given a real-valued random variable $X : \Omega \to \mathbb{R}$, $\mathbb{P} \circ X^{-1}$ denotes its distribution or law under $\mathbb{P}$, i.e. the probability measure $\mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in \cdot\})$ on Borel sets of the real line.

We will usually identify random variables if they agree $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.). The space of (equivalence classes of) $\mathbb{P}$-integrable random variables is, as usual, denoted by L^1. Similarly, L^∞ is the space of equivalence classes of $\mathbb{P}$-a.s. bounded random variables. By $\mathbb{E}[\cdot] := \mathbb{E}_{\mathbb{P}}[\cdot]$ and $\mathbb{E}[\cdot|\mathcal{H}] := \mathbb{E}_{\mathbb{P}}[\cdot|\mathcal{H}]$ we abbreviate the (conditional) expectation operator (with respect to a sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$) under $\mathbb{P}$. The following notions will be of the utmost importance for our investigations:

Definition 1. A set $\mathcal{C} \subset L^1$ is called REARRANGEMENT INVARIANT with respect to $\mathbb{P}$ if $X \in \mathcal{C}$ and Y being equal to X in law under $\mathbb{P}$, i.e. $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$, implies $Y \in \mathcal{C}$.
Given a rearrangement invariant set $\mathcal{C}$ and a function $f : \mathcal{C} \to [-\infty, \infty]$, f is LAW-INVARIANT with respect to $\mathbb{P}$ if $f(X) = f(Y)$ whenever $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$.

Whenever we speak of law-invariance in the following, we mean law-invariance with respect to the underlying reference probability measure $\mathbb{P}$ unless specified otherwise. The term is also widely used in the theory of risk measures; c.f. Föllmer & Schied [18].
Economically, we shall model all appearing goods as random and contingent on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. They are *Savage acts* and represent *state-dependent wealth*.
The set of all goods towards which the agents in question have preferences will be assumed to be an ideal $\mathcal{X}$ of L^1 with respect to the $\mathbb{P}$-a.s. order between random variables[5] which contains all bounded random variables, i.e.

$$L^\infty \subset \mathcal{X} \subset L^1.$$

Although the precise formal properties of these commodity spaces will be elaborated later in Section 4, we remark at this point already that $\mathcal{X}$ is assumed to be rearrangement invariant and normed by a law-invariant lattice norm $\|\cdot\|$. Hence, the commodity spaces cover a very general range of spaces of random variables. One of the crucial messages of our investigations, however, is that we can treat the problem in the setting $\mathcal{X} = L^1$, and we shall do so in Section 3. The general case follows by means of localisation as elaborated in Section 4.
We close this section by recalling the *concave order*, a crucial notion of risk aversion.

Definition 2. Let $X, Y \in L^1$. Y dominates X in the CONCAVE ORDER ($X \preceq_c Y$) if, and only if, $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for all concave $u : \mathbb{R} \to \mathbb{R}$. Given a subset $\mathcal{C} \subset L^1$, a function $f : \mathcal{C} \to [-\infty, \infty]$ is non-decreasing in the concave order if $f(X) \leq f(Y)$ holds for all $X, Y \in \mathcal{C}$ such that $X \preceq_c Y$.

3. The cornerstones of optimisation involving law-invariance

In the following, for a natural number $n \in \mathbb{N}$, $[n]$ denotes the set of the first n natural numbers, i.e. $[n] = \{1, 2, ..., n\}$. Throughout this section, we assume the commodity space $\mathcal{X}$ to be given by L^1 endowed with its natural norm $\|\cdot\|_1$, that is

$$\|X\|_1 := \mathbb{E}[|X|], \quad X \in L^1.$$

We identify each agent with some $i \in [n]$ and assume that their preferences over L^1 are represented by a vector $\mathfrak{U} = (\mathfrak{U}_i)_{i \in [n]}$ of functions $\mathfrak{U}_i : L^1 \to [-\infty, \infty)$, $i \in [n]$, with the following properties:

[5] That is, if $X, Y \in L^1$ satisfy $|X| \leq |Y|$ a.s. and $Y \in \mathcal{X}$, then also $X \in \mathcal{X}$.

Assumption 3. For each $i \in [n]$, the function $\mathfrak{U}_i : L^1 \to [-\infty, \infty)$ is proper, quasi-concave, upper semicontinuous, and law-invariant.

As discussed in the introduction, law-invariance is a consequentialist assumption which means that two commodities produce the same utility if their distribution under the reference measure $\mathbb{P}$ is the same; that is, $\mathfrak{U}_i(X) = \mathfrak{U}_i(Y)$, $i \in [n]$, if the two distributions $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ on the real line agree. Alternatively, we will see later in Theorem 18(iii) that law-invariance of $\mathfrak{U}_i$ is equivalent to risk aversion in the sense of $\mathfrak{U}_i$ being non-decreasing in the concave order. We remark here that without law-invariance, the existence of solutions to the optimisation problems studied in this paper cannot be guaranteed.
Recall that we abbreviate $\mathfrak{U} : (L^1)^n \to [-\infty, \infty)^n$, $\mathfrak{U}(\mathbf{X}) = (\mathfrak{U}_1(X_1), ...\mathfrak{U}_n(X_n))$. As mentioned above, vectors in $(L^1)^n$ will be denoted by $\mathbf{X}$, whereas their individual coordinates are denoted by X_i, $i \in [n]$.
As our aim is to maximise aggregated utility within a system arising from distributing a good $X \in L^1$, we first have to clarify what a feasible distribution scheme, an *allocation*, is. To this end, we introduce two types of ATTAINABLE SETS relevant throughout the remainder of the paper. For $X \in L^1$, we consider

$$\Gamma_X := \{\mathbf{X} \in (L^1)^n \mid X_1 + ... + X_n = X\}$$

and

$$\widehat{\Gamma}_X := \{\mathbf{X} \in (L^1)^n \mid X_1 + ... + X_n \leq X\}.$$

The vectors $\mathbf{X}$ in Γ_X or $\widehat{\Gamma}_X$, respectively, are called ALLOCATIONS of X. $\mathbf{X} \in \Gamma_X$ allocates X without free disposal, whereas $\mathbf{X} \in \widehat{\Gamma}_X$ is an allocation of X when free disposal is allowed. It is apparent from the definition that we study an economy *without production.* Due to potential non-monotonicity of utilities in the almost sure order, $\widehat{\Gamma}_X$ is more relevant in situations in which the economy is not subject to external constraints and a unanimously rejected remainder of X may thus be left aside. Second, we need to introduce the notion of an aggregation function.

Definition 4. A function

$$\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$$

is an AGGREGATION FUNCTION if it is non-decreasing with respect to the pointwise order on $[-\infty, \infty)^n$, i.e. $\Lambda(\mathbf{y}) \leq \Lambda(\mathbf{z})$ for all $\mathbf{y}, \mathbf{z} \in [-\infty, \infty)^n$ such that $y_i \leq z_i$ for all $i \in [n]$.

Given a vector $\mathfrak{U}$ of utility functions satisfying Assumption 3 and an aggregation function Λ, we will hence be interested in the quantity

$$\Lambda\left(\mathfrak{U}(\mathbf{X})\right) = \Lambda\left(\mathfrak{U}_1(X_1), ..., \mathfrak{U}_n(X_n)\right)$$

representing the aggregated individual utilities in the system given by an allocation $\mathbf{X} \in \Gamma_X$ or $\mathbf{X} \in \widehat{\Gamma}_X$ of a commodity $X \in L^1$.

3.1. **Comonotone allocations.** The first step towards tackling the optimisation problem (1) would be to reduce the attention to a well-behaved subset of the attainable set. This will turn out to be the set of *comonotone allocations*.

Definition 5. Given $n \in \mathbb{N}$, the set of COMONOTONE n-PARTITIONS OF THE IDENTITY (or COMONOTONE FUNCTIONS) is the set $\mathbb{C}(n)$ of functions $\mathbf{f} = (f_1, ..., f_n) : \mathbb{R} \to \mathbb{R}^n$ such that each coordinate f_i is non-decreasing and $\sum_{i=1}^n f_i = id_{\mathbb{R}}$ holds.
For $\gamma > 0$, we set $\mathbb{C}(n)_\gamma$ to be the subset of $\mathbf{f} \in \mathbb{C}(n)$ which satisfy $\sum_{i=1}^n |f_i(0)| \leq \gamma$. Moreover, for $\mathbf{f} \in \mathbb{C}(n)$, we set $\tilde{\mathbf{f}} := \mathbf{f} - \mathbf{f}(0) \in \mathbb{C}(n)$.

For each $\mathbf{f} \in \mathbb{C}(n)$, $i \in [n]$, and $x, y \in \mathbb{R}$, the equality

$$\sum_{i=1}^n |f_i(x) - f_i(y)| = |x - y|, \tag{3}$$

holds.[6] In particular, it entails that each f_i, $i \in [n]$, is a Lipschitz continuous function with Lipschitz constant 1. As a consequence, $f_i(X) \in L^1$ holds for all $X \in L^1$. Given $X \in L^1$, recall the abbreviation $\mathbf{f}(X) := (f_1(X), ..., f_n(X)) \in (L^1)^n$. Clearly, $\mathbf{f}(X) \in \Gamma_X$ holds by definition. Moreover, if $\mathbf{f} \in \mathbb{C}(n+1)$ and $f_{n+1}(X) \geq 0$ a.s., $(f_1(X), ..., f_n(X)) \in \widehat{\Gamma}_X$ holds.
The following results on comonotone functions are essential; statement (ii) is usually referred to as *comonotone order improvement*.

Proposition 6. *(i) For every $\gamma > 0$, $\mathbb{C}(n)_\gamma \subset (\mathbb{R}^n)^{\mathbb{R}}$ is sequentially compact in the topology of pointwise convergence. That is, for each sequence $(\mathbf{f}^k)_{k\in\mathbb{N}} \subset \mathbb{C}(n)_\gamma$ there is a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and $\mathbf{f} \in \mathbb{C}(n)_\gamma$ such that for all $x \in \mathbb{R}$*

$$\mathbf{f}^{k_\lambda}(x) \to \mathbf{f}(x), \quad n \to \infty.$$

(ii) Let $\mathbf{X} \in (L^1)^n$ and set $X := X_1 + ... + X_n$. Then there is $\mathbf{f} \in \mathbb{C}(n)$ such that $X_i \preceq_c f_i(X)$ holds for all $i \in [n]$.

Proof. For (i), note that $\mathbb{C}(n)_\gamma$ is a closed subset of the set $\{\mathbf{f} \in \mathbb{C}(n) \mid \mathbf{f}(0) \in [-\gamma, \gamma]^n\}$ and the latter is sequentially compact in the topology of pointwise convergence by [17, Lemma B.1]. (ii) is proved in [17, Proposition 5.1]. In the case $\mathbf{X} \in (L^\infty)^n$, it is [7, Theorem 3.1]. We also refer to Landsberger and Meilijson [25] and Ludkovski and Rüschendorf [26]. □

The next two result are essential for optimisation with law-invariant inputs: Proposition 7 shows that in the optimisation problems we consider an optimal allocation can be found if, and only if, an optimal comonotone allocation can be found. Proposition 8 shows that, under further mild conditions, such optimal comonotone allocations actually exist because the set of comonotone allocations is particularly well-behaved.

[6] Each f_i is non-decreasing. Hence, for $x, y \in \mathbb{R}$, $x \geq y$, we have,

$$\sum_{i=1}^n |f_i(x) - f_i(y)| = \sum_{i=1}^n f_i(x) - f_i(y) = x - y = |x - y|.$$

Proposition 7. *Suppose $\mathfrak{U}$ checks Assumption 3, Λ is an aggregation function, and $X \in L^1$ is arbitrary. Then the identities*

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right), \tag{4}$$

and

$$\sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \sup\{\Lambda(\mathfrak{U}(f_1(X),...,f_n(X))) \mid \mathbf{f}\in\mathbb{C}(n+1),\ f_{n+1}(X)\geq 0\}. \tag{5}$$

hold.

Proof. Fix an arbitrary $X \in L^1$. In order to prove (4), let $\mathbf{Y} \in \Gamma_X$ be arbitrary. By Proposition 6(ii), there is $\mathbf{g} \in \mathbb{C}(n)$ such that $Y_i \preceq_c g_i(X)$ holds for all $i \in [n]$. By Theorem 18, $\mathfrak{U}(\mathbf{Y}) \leq \mathfrak{U}(\mathbf{g}(X))$ holds in the pointwise order on $[-\infty,\infty)^n$. As Λ is non-decreasing by assumption,

$$\Lambda(\mathfrak{U}(\mathbf{Y})) \leq \Lambda(\mathfrak{U}(\mathbf{g}(X))) \leq \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda(\mathfrak{U}(\mathbf{f}(X))),$$

and thus

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) \leq \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right).$$

The converse inequality, however, follows from the observation that $\mathbf{f}(X) \in \Gamma_X$ holds for all $\mathbf{f} \in \mathbb{C}(n)$, and (4) is proved.

For the second assertion, consider the slightly altered aggregation function

$$\Xi : [-\infty,\infty)^{n+1} \to [-\infty,\infty), \quad \Xi(x_1,...,x_{n+1}) = \Lambda(x_1,...,x_n) + x_{n+1},$$

which, indeed, is non-decreasing in the pointwise order on $[-\infty,\infty)^{n+1}$. Moreover, let

$$\mathfrak{U}_{n+1} = \delta(\cdot|L^1_+) : L^1 \to [-\infty,\infty), \quad X \mapsto \begin{cases} 0, & X \in L^1_+, \\ -\infty, & X \notin L^1_+, \end{cases}$$

be the concave indicator of $L^1_+ := \{Y \in L^1 \mid Y \geq 0 \text{ a.s.}\}$, the positive cone of L^1 in the almost sure order. The function $\mathfrak{U}_{n+1}$ is proper, concave, upper semicontinuous, and law-invariant. We set

$$\bar{\mathfrak{U}} : (L^1)^{n+1} \to [-\infty,\infty)^{n+1}, \quad \mathbf{X} \mapsto (\mathfrak{U}_1(X_1),...,\mathfrak{U}_n(X_n),\mathfrak{U}_{n+1}(X_{n+1})),$$

and obtain

$$\begin{aligned}\sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) &= \left\{\Xi(\bar{\mathfrak{U}}(\mathbf{Z})) \mid \mathbf{Z}\in (L^1)^{n+1},\ Z_1+...+Z_{n+1} = X\right\} \\ &= \sup_{\mathbf{f}\in\mathbb{C}(n+1)} \Xi\left(\bar{\mathfrak{U}}(\mathbf{f}(X))\right).\end{aligned}$$

The last equality here is due to (4). In the last supremum only vectors $\mathbf{f}(X)$, $\mathbf{f} \in \mathbb{C}(n+1)$, are relevant for which $f_{n+1}(X) \in L^1_+$. This in turn implies $(f_1(X), ..., f_n(X)) \in \widehat{\Gamma}_X$ for these $\mathbf{f} \in \mathbb{C}(n+1)$. Since they also satisfy $\mathfrak{U}_{n+1}(f_{n+1}(X)) = 0$, we obtain (5) as follows:

$$\sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \sup\left\{\Xi\left(\bar{\mathfrak{U}}(\mathbf{f}(X))\right) \mid \mathbf{f} \in \mathbb{C}(n+1),\ f_{n+1}(X) \geq 0\right\}$$
$$= \sup\left\{\Lambda\left(\mathfrak{U}_1(f_1(X)), ..., \mathfrak{U}_n(f_n(X))\right) \mid \mathbf{f} \in \mathbb{C}(n+1),\ f_{n+1}(X) \geq 0\right\}.$$

□

Proposition 8. *In the situation of Proposition 7 assume* Λ *is additionally upper semicontinuous. For* $X \in L^1$ *we define the quantities*

$$\eta(X) := \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) \quad \text{and} \quad \widehat{\eta}(X) := \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})).$$

(i) If $\eta(X) < \infty$ *and there is a constant* $\gamma(X) > 0$ *such that for* $\mathbf{f} \in \mathbb{C}(n)$,

$$\Lambda(\mathfrak{U}(\mathbf{f}(X))) \geq \eta(X) - 1 \quad \Longrightarrow \quad \mathbf{f} \in \mathbb{C}(n)_{\gamma(X)},$$

there is a $\mathbf{g} \in \mathbb{C}(n)_{\gamma(X)}$ *such that*

$$\eta(X) = \Lambda\left(\mathfrak{U}(\mathbf{g}(X))\right).$$

(ii) If $\widehat{\eta}(X) < \infty$ *and there is some constant* $\widehat{\gamma}(X) > 0$ *such that for* $\mathbf{f} \in \mathbb{C}(n+1)$ *with* $f_{n+1}(X) \geq 0$,

$$\Lambda(\mathfrak{U}_1(f_1(X)), ..., \mathfrak{U}_n(f_n(X))) \geq \widehat{\eta}(X) - 1 \quad \Longrightarrow \quad \mathbf{f} \in \mathbb{C}(n+1)_{\widehat{\gamma}(X)},$$

there is a $\mathbf{g} \in \mathbb{C}(n+1)_{\widehat{\gamma}(X)}$ *such that* $g_{n+1}(X) \geq 0$ *a.s. and*

$$\widehat{\eta}(X) = \Lambda\left(\mathfrak{U}_1(g_1(X)), ..., \mathfrak{U}_n(g_n(X))\right).$$

Proof. (i) By Proposition 7, we may choose a maximising sequence $(\mathbf{f}^k)_{k \in \mathbb{N}} \subset \mathbb{C}(n)$, i.e.

$$\Lambda\left(\mathfrak{U}(\mathbf{f}^k(X))\right) \uparrow \eta(X) < \infty.$$

Combining the assumption and Proposition 6(i) there is a subsequence $(k_\lambda)_{\lambda \in \mathbb{N}}$ and $\mathbf{g} \in \mathbb{C}(n)_{\gamma(X)}$ such that

$$\forall x \in \mathbb{R}: \ \mathbf{f}^{k_\lambda}(x) \to \mathbf{g}(x), \quad \lambda \to \infty.$$

Moreover, by 1-Lipschitz continuity, for all $\lambda \in \mathbb{N}$ and $i \in [n]$,

$$|f_i^{k_\lambda}(X) - g_i(X)| \leq |\tilde{f}_i^{k_\lambda}(X)| + |\tilde{g}_i(X)| + |f_i^{k_\lambda}(0) - g_i(0)| \leq 2|X| + 2\gamma(X) \ \mathbb{P}\text{-a.s.}$$

By the Dominated Convergence Theorem,

$$\mathbf{f}^{k_\lambda}(X) \to \mathbf{g}(X) \text{ in } (L^1)^n, \quad \lambda \to \infty.$$

As the *limes superior* is realised as limit along a subsequence, we may, after potentially passing to another subsequence, assume without loss of generality that for each $\mathfrak{U}_i$, $i \in [n]$, the identity $\lim_{\lambda \to \infty} \mathfrak{U}_i(f_i^{k_\lambda}(X)) = \limsup_{\lambda \to \infty} \mathfrak{U}_i(f_i^{k_\lambda}(X))$ holds. As $\mathfrak{U}_i$ is upper semicontinuous, we also obtain $\lim_{\lambda\to\infty}\mathfrak{U}_i(f_i^{k_\lambda}(X)) \le \mathfrak{U}_i(g_i(X))$. Λ being non-decreasing in the pointwise order implies

$$\Lambda(\mathfrak{U}(\mathbf{g}(X))) \ge \Lambda\left(\limsup_{\lambda\to\infty}\mathfrak{U}_1(f_1^{k_\lambda}(X)), ..., \limsup_{\lambda\to\infty}\mathfrak{U}_n(f_n^{k_\lambda}(X))\right)$$
$$= \Lambda\left(\lim_{\lambda\to\infty}\mathfrak{U}_1(f_1^{k_\lambda}(X)), ..., \lim_{\lambda\to\infty}\mathfrak{U}_n(f_n^{k_\lambda}(X))\right).$$

Moreover, upper semicontinuity of Λ shows

$$\Lambda\left(\lim_{\lambda\to\infty}\mathfrak{U}_1(f_1^{k_\lambda}(X)), ..., \lim_{\lambda\to\infty}\mathfrak{U}_n(f_n^{k_\lambda}(X))\right) \ge \limsup_{\lambda\to\infty}\Lambda\left(\mathfrak{U}(\mathbf{f}^{k_\lambda}(X))\right) = \eta(X).$$

Together the inequalities read as $\Lambda\left(\mathfrak{U}(\mathbf{g}(X))\right) \ge \eta(X)$. As the converse inequality holds *a priori*, the proof is complete.

(ii) This is proved in complete analogy with (i).

□

Proposition 8 may appear technical at first sight, but it is precisely the instrument which allows us to prove the existence of efficient allocations in Theorem 12. This, however requires to characterise when the interplay between the aggregation function Λ and the individual utilities $(\mathfrak{U}_i)_{i\in[n]}$ is such that the additional assumptions are met, i.e. the bounds $\gamma(X)$ and $\widehat{\gamma}(X)$, respectively, can be found. To this end we suggest the notion of *coercive aggregation functions.*

3.2. **Coercive aggregation rules.** Recall that for a vector $\mathbf{u}$ of scalar functions $u_i : \mathbb{R} \to [-\infty,\infty)$, we define $\mathbf{u} : \mathbb{R}^n \to [-\infty,\infty)^n$ by $\mathbf{u}(\mathbf{y}) = (u_1(y_1), ..., u_n(y_n))$, $\mathbf{y} \in \mathbb{R}^n$.

Definition 9. Let $n \in \mathbb{N}$, $\mathbf{u} : \mathbb{R}^n \to [-\infty,\infty)^n$ be a vector-valued function, and $\Lambda : [-\infty,\infty)^n \to [-\infty,\infty)$ be an aggregation function. For $x, m \in \mathbb{R}$, we define the set

$$S(x,m) := \left\{\mathbf{y} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n y_i \le x,\ \Lambda(\mathbf{u}(\mathbf{y})) \ge m\right\}. \tag{6}$$

We say Λ is COERCIVE[7] for $\mathbf{u}$ if there are functions $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ such that for all $x, m \in \mathbb{R}$, the condition $S(x,m) \neq \emptyset$ implies

$$\sum_{i=1}^n |y_i| \le G(x,m), \quad \mathbf{y} \in S(x,m), \tag{7}$$

and

$$m \le H(x). \tag{8}$$

[7] We remark that our use of the term *coercive* is not canonical, however, it does not have a unique meaning in the literature anyway. For instance, a function $f : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ between two normed spaces is called coercive if $\|x\|_{\mathcal{X}} \to \infty$ implies $\|f(x)\|_{\mathcal{Y}} \to \infty$. As coercive functions in optimisation usually play a similar role as coercive aggregation functions in our setting, we decided to use this suggestive terminology.

From an economic point of view, we can interpret a vector $\mathbf{y} \in \mathbb{R}^n$ as a collection of (deterministic) endowments of agents $i = 1, ..., n$. The sum $\sum_{i=1}^n y_i$ then is the total endowment of the system $[n]$. If we think of the vector-valued function $\mathbf{u}$ as the individual utility assessments, the quantity $\Lambda(\mathbf{u}(\mathbf{y}))$ is the aggregated utility in the system. Suppose the aggregation function Λ is coercive for $\mathbf{u}$. Condition (8) means that a bounded total endowment x cannot lead to arbitrarily large aggregated utility. Regarding condition (7), consider a fixed total endowment x allocated over the system such that $y_i \to \infty$ and $y_j \to -\infty$ for at least two agents $i, j \in [n]$. This implies substantial disutility for agent j, and condition (7) ensures that such allocations will eventually not lead to optimal utility if the spread $y_i - y_j$ is too large. However, as the functions G and H do not have to fulfil any specific requirements, they only pose very soft constraints.
We will use in Section 5 and show in Appendix A that the following aggregation functions are coercive for suitable vector-valued functions $\mathbf{u} : \mathbb{R}^n \to [-\infty, \infty)^n$:

- $\Lambda_\alpha(\mathbf{y}) := \alpha \min_{i\in[n]} y_i + (1-\alpha) \max_{i\in[n]} y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, where $0 < \alpha \leq 1$ is a fixed parameter;
- $\Xi_\alpha(\mathbf{y}) := \sum_{\emptyset\neq S\subset[n]} \alpha \min_{i\in[n]} y_i + (1-\alpha) \max_{i\in S} y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, where $0 < \alpha < 1$ is a fixed parameter;
- $\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, where $\mathbf{w} = (w_1, ..., w_n) \in (0, \infty)^n$ is a family of positive weights.

Let us also give an example of a vector-valued function $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$ and an aggregation function $\Lambda : [-\infty, \infty)^2 \to [0, \infty)$ such that Λ is *not* coercive for $\mathbf{u}$.

Example 10. Let $A_1, A_2 \in \mathbb{R}$ and $B_1, B_2 > 0$. We set $\mathbf{u}(\mathbf{y}) := (A_1 + B_1 y_1, A_2 + B_2 y_2)$, $\mathbf{y} \in \mathbb{R}^2$. Moreover, we consider the aggregation function $\Lambda(\mathbf{z}) := e^{z_1} + e^{z_2}$, $\mathbf{z} \in [-\infty, \infty)^2$ (here, $e^{-\infty} := 0$). Let $x, m \in \mathbb{R}$ be arbitrary. As

$$\Lambda(\mathbf{u}(x - n, n)) = e^{A_1+B_1x-B_1n} + e^{A_2+B_2n} \to \infty, \quad n \to \infty,$$

and $x - n + n = x$, we have for all $m \in \mathbb{R}$ and all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ depending on x and m that $(x - n, n) \in S(x, m)$. This implies

$$\forall\, m > 0 : \sup_{\mathbf{y}\in S(x,m)} |y_1| + |y_2| \geq \sup_{n\geq n_0} |x - n| + |n| = \infty.$$

Therefore the function G as in (7) cannot exist in this situation. Similarly, as $S(x, m) \neq \emptyset$ holds for all $m \in \mathbb{R}$, the function H in (8) cannot exist either. Note that Λ not being coercive for $\mathbf{u}$ is a result of the very different and conflicting nature of utility assessment and aggregation.

The class of coercive aggregation functions is usually rich though and closed under certain algebraic and order operations:

Proposition 11. *Suppose $n \in \mathbb{N}$ and $\mathbf{u} : \mathbb{R}^n \to [-\infty, \infty)^n$ is a vector-valued function. Moreover, assume Λ, Ξ, $\Gamma : [-\infty, \infty)^n \to [-\infty, \infty)$ are aggregation functions such that Λ*

and Ξ are coercive for $\mathbf{u}$ *and such that* $\Gamma \leq \Lambda$. *Then the following functions are coercive for* $\mathbf{u}$, *as well:*

(i) $\alpha\Lambda$, $\alpha > 0$ *arbitrary;*
(ii) $\Lambda + \Xi$*;*
(iii) Γ.

Proof. Let $G, G' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $H, H' : \mathbb{R} \to \mathbb{R}$ be functions as in Definition 9 for Λ or Ξ, respectively.

(i) The functions $G_\alpha(m, x) := G(x, \frac{m}{\alpha})$ and $H_\alpha := \alpha H$ satisfy (7) and (8).
(ii) If $\mathbf{y} \in \mathbb{R}^n$ satisfies $(\Lambda + \Xi)(\mathbf{u}(\mathbf{y})) \geq m$, this is only possible if $\max\{\Lambda(\mathbf{u}(\mathbf{y})), \Xi(\mathbf{u}(\mathbf{y}))\} \geq \frac{m}{2}$. Hence, the function $G_+(x, m) := \max\{G(x, \frac{m}{2}), G'(x, \frac{m}{2})\}$ satisfies (7). Similarly, the function $H_+ := 2\max\{H, H'\}$ satisfies (8).
(iii) As $\Gamma \leq \Lambda$, the inclusion

$$\{\mathbf{y} \in \mathbb{R}^n \mid \textstyle\sum_{i=1}^n y_i \leq x,\ \Gamma(\mathbf{u}(\mathbf{y})) \geq m\} \subset \{\mathbf{y} \in \mathbb{R}^n \mid \textstyle\sum_{i=1}^n y_i \leq x,\ \Lambda(\mathbf{u}(\mathbf{y})) \geq m\}$$

holds. Hence, the same functions G and H work for Γ in (7) and (8), as well.

□

3.3. **The existence theorem.** The aim of this section is to combine the results obtained above and give a unifying criterion for the existence of comonotone solutions to optimisation problems involving agents with law-invariant preferences.

Beforehand, we define the regions of relevance for the optimisation problem in question. Given a vector of utilities $\mathfrak{U} : (L^1)^n \to [-\infty, \infty)^n$ and an aggregation function $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$, the relevant region corresponding to the attainable set Γ_X is

$$\Delta := \{X \in L^1 \mid \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty\},$$

whereas the region corresponding to the attainable set $\widehat{\Gamma}_X$ is

$$\widehat{\Delta} := \{X \in \mathcal{X} \mid \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty\}.$$

Theorem 12. *Suppose* $\mathfrak{U}$ *checks Assumption 3. Let* $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$ *be an upper semicontinuous aggregation function which is coercive for* $\mathbf{u} := (u_1, ..., u_n)$, *where* $u_i := \mathfrak{U}_i|_{\mathbb{R}}$, $i \in [n]$. *Then:*

(i) For all $X \in L^1$, *the optimal values satisfy*

$$\eta(X) := \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) < \infty \quad \textit{and} \quad \widehat{\eta}(X) := \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) < \infty. \tag{9}$$

(ii) There is a function $\gamma : \Delta \to \mathbb{R}$ *such that for all* $X \in \Delta$ *and all* $\mathbf{f} \in \mathbb{C}(n)$,

$$\Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right) \geq \eta(X) - 1 \quad \Longrightarrow \quad \mathbf{f} \in \mathbb{C}(n)_{\gamma(X)}.$$

Moreover, the first supremum in (9) *is attained by* $\mathbf{g}(X) \in \Gamma_X$, $\mathbf{g} \in \mathbb{C}(n)$ *suitably chosen.*

(iii) *There is a function* $\widehat{\gamma} : \widehat{\Delta} \to \mathbb{R}$ *such that for all* $X \in \widehat{\Delta}$ *and all* $\mathbf{f} \in \mathbb{C}(n+1)$ *with* $f_{n+1}(X) \geq 0$

$$\Lambda\left(\mathfrak{U}_1(f_1(X)), ..., \mathfrak{U}_n(f_n(X))\right) \geq \widehat{\eta}(X) - 1 \quad \Longrightarrow \quad f \in \mathbb{C}(n)_{\widehat{\gamma}(X)}.$$

Moreover, the second supremum in (9) *is attained by* $(g_1(X), ..., g_n(X)) \in \widehat{\Gamma}_X$*,* $\mathbf{g} \in \mathbb{C}(n+1)$ *suitably chosen.*

(iv) *If the function* $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *in* (7) *is non-decreasing in the first coordinate and non-increasing in the second, the optimal value mappings* $X \mapsto \eta(X)$ *and* $X \mapsto \widehat{\eta}(X)$ *are upper semicontinuous on* Δ *and* $\widehat{\Delta}$*, respectively.*

Proof. Recall that for $\mathbf{Y} \in \mathcal{X}^n$ we abbreviate $\mathbb{E}[\mathbf{Y}] := (\mathbb{E}[Y_1], ..., \mathbb{E}[Y_n]) \in \mathbb{R}^n$.

(i) By Corollary 19 and the assumptions on Λ, for all $X \in \mathcal{X}$ and all $\mathbf{Y} \in \widehat{\Gamma}_X$ we have

$$\Lambda(\mathfrak{U}(\mathbf{Y})) \leq \Lambda(\mathfrak{U}(\mathbb{E}[\mathbf{Y}])) = \Lambda(\mathbf{u}(\mathbb{E}[\mathbf{Y}])).$$

Moreover, $\sum_{i=1}^n Y_i \leq X$ implies $\sum_{i=1}^n \mathbb{E}[X_i] \leq \mathbb{E}[X]$. Hence, coercivity of Λ for $\mathbf{u}$ yields

$$\eta(X) = \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) \leq \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \widehat{\eta}(X) \leq \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathbf{u}(\mathbb{E}[\mathbf{Y}])) \leq H(\mathbb{E}[X]) < \infty.$$

(ii) Let $X \in \Delta$ and suppose $\mathbf{f} \in \mathbb{C}(n)$ is such that $\eta(X) - 1 \leq \Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right)$. By Corollary 19, $\eta(X) - 1 \leq \Lambda\left(\mathbf{u}(\mathbb{E}[\mathbf{f}(X)])\right)$, which means that

$$\sum_{i=1}^n |f_i(0)| - |\mathbb{E}[\tilde{f}_i(X)]| \leq \sum_{i=1}^n |\mathbb{E}[f_i(X)]| \leq G(\mathbb{E}[X], \eta(X) - 1).$$

Rearranging this inequality yields

$$\begin{aligned}\sum_{i=1}^n |f_i(0)| &\leq G(\mathbb{E}[X], \eta(X) - 1) + \sum_{i=1}^n |\mathbb{E}[\tilde{f}_i(X)]| \\ &\leq G(\mathbb{E}[X], \eta(X) - 1) + \sum_{i=1}^n \mathbb{E}[\tilde{f}_i(|X|)] \\ &\leq G(\mathbb{E}[X], \eta(X) - 1) + \mathbb{E}[|X|] =: \gamma(X).\end{aligned}$$

The existence of a maximiser $\mathbf{g}(X)$, $\mathbf{g} \in \mathbb{C}(n)$, follows with Proposition 8.

(iii) Let $X \in \widehat{\Delta}$. If $\mathbf{f} \in \mathbb{C}(n+1)$ is such that $f_{n+1}(X) \geq 0$ and

$$\widehat{\eta}(X) - 1 \leq \Lambda(\mathfrak{U}_1(f_1(X)), ..., \mathfrak{U}_n(f_n(X))),$$

using the same arguments as in (ii) yields

$$\sum_{i=1}^n |f_i(0)| \leq G(\mathbb{E}[X], \widehat{\eta}(X) - 1) + \mathbb{E}[|X|].$$

As $f_{n+1}(0) = -\sum_{i=1}^n f_i(0)$,

$$\sum_{i=1}^{n+1} |f_i(0)| \leq 2 \sum_{i=1}^n |f_i(0)| \leq 2G(\mathbb{E}[X], \widehat{\eta}(X) - 1) + 2\mathbb{E}[|X|] =: \widehat{\gamma}(X).$$

The existence of a maximiser $\mathbf{g}(X)$, $\mathbf{g} \in \mathbb{C}(n+1)$, follows with Proposition 8.

(iv) This will be proved in the context of Theorem 23.

□

4. Commodity spaces and law-invariance

As mentioned above, we will now demonstrate how the results in the preceding section can be generalised — or rather *localised* — to general rearrangement invariant commodity spaces $\mathcal{X}$ with the property $L^\infty \subset \mathcal{X} \subset L^1$. For the terminology concerning ordered vector spaces, we refer to Aliprantis & Burkinshaw [3]. The space $\mathcal{X}$ is assumed to have the following properties:

(a) As a subset of L^1, $\mathcal{X}$ is rearrangement invariant;
(b) with respect to the $\mathbb{P}$-a.s. order on L^1, it is a solid Riesz subspace;
(c) $\mathcal{X}$ carries a lattice norm $\|\cdot\|$ which makes it into a Banach lattice and is law-invariant as a function $\|\cdot\| : \mathcal{X} \to [0, \infty)$.

The preceding assumptions entail that the embeddings $L^\infty \hookrightarrow \mathcal{X} \hookrightarrow L^1$ are continuous, i.e. there are positive constants $\kappa, K > 0$ such that for all $X \in L^\infty$ and all $Y \in \mathcal{X}$ the estimates

$$\|Y\| \le \kappa \|Y\|_\infty \quad \text{and} \quad \mathbb{E}[|Y|] \le K\|Y\| \tag{10}$$

hold. For the aforementioned facts on rearrangement invariant function spaces, we refer to [10, Appendix A] and the references therein.

Let $\mathcal{X}^*$ denote the dual space of $\mathcal{X}$. A linear functional $\phi \in \mathcal{X}^*$ is *order continuous* if $\lim_{n\to\infty} \phi(X_n) = 0$ holds for every sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $X_n \downarrow 0$ $\mathbb{P}$-a.s.[8] The space $\mathcal{X}_n^\sim$ of all order continuous functionals may be identified with a subspace of L^1. More precisely, for every $\phi \in \mathcal{X}_n^\sim$ there is a unique $Q \in L^1$ such that $\mathbb{E}[|QX|] < \infty$ holds for all $X \in \mathcal{X}$ and $\phi(X) = \mathbb{E}[QX]$. Moreover, for each $Q \in L^\infty$, $X \mapsto \mathbb{E}[QX]$ defines an order continuous bounded linear functional by (10). Using the Hardy-Littlewood inequalities as stated in [13, Theorem 13.4], one can prove that $\mathcal{X}_n^\sim \subset L^1$ is rearrangement invariant, as well.

4.1. Structural properties of law-invariant functions.

We assume that $(\mathcal{X}, \|\cdot\|)$ is either $(L^1, \|\cdot\|_1)$ or a rearrangement invariant Banach lattice $L^\infty \subset \mathcal{X} \subsetneq L^1$ as introduced above.

Before we can generalise the results on the existence of efficient allocations to general commodity spaces $\mathcal{X}$, we point out that the potential lack of order continuity of the norm $\|\cdot\|$ is the main problem which needs to be overcome. It results in the fact that $\mathcal{X}_n^\sim \subsetneq \mathcal{X}^*$ is possible. Hence, many of the structural properties of (quasi-)concave and law-invariant functions do not transfer directly.

This necessitates to study structural properties of law-invariant functions on general commodity spaces more closely. We shall see that the localisation procedure works if the individual

[8] As the $\mathbb{P}$-a.s. order on L^1 (and also $\mathcal{X}$) renders *super Dedekind complete* spaces, order convergent sequences suffice to characterise order continuity.

utilities in question have minimal order continuity properties on $\mathcal{X}$. For the (strong) Fatou property introduced in the following we particularly refer to the recent contributions of Chen et al. [10] and Gao et al. [19].

Definition 13. Let $L^\infty \subset \mathcal{X} \subset L^1$ be a rearrangement invariant Banach lattice as elaborated above. A function $f : \mathcal{X} \to [-\infty, \infty)$ is said to have

- the FATOU PROPERTY if every order convergent sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ with limit $X \in \mathcal{X}$ satisfies[9]
$$f(X) \geq \limsup_{n\to\infty} f(X_n);$$
- the STRONG FATOU PROPERTY if the preceding estimate holds for every norm bounded sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ which converges to $X \in \mathcal{X}$ a.s.

Note that the space $\mathcal{X}$ is closed under suitable conditional expectations: if a σ-algebra $\mathcal{H}$ is finitely generated, i.e. $\mathcal{H} = \sigma(\pi)$ for some finite measurable partition $\pi = \{A_1, ..., A_n\} \subset \mathcal{F}$ of Ω, and $X \in \mathcal{X}$, then $\mathbb{E}[X|\mathcal{H}]$ is well-defined and a simple — and thus bounded — function.

Lemma 14. *Given $X \in \mathcal{X}$ and a finitely generated sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$, the conditional expectation $\mathbb{E}[X|\mathcal{H}]$ lies again in $\mathcal{X}$ and satisfies $\|\mathbb{E}[X|\mathcal{H}]\| \leq \|X\|$.*

Proof. Fix arbitrary $X \in \mathcal{X}$ and a finitely generated sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$. Moreover, let $A_k := \{|X| \leq k\} \in \mathcal{F}$. For all $k \in \mathbb{N}$, $\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}] \in L^\infty \subset \mathcal{X}$, and the set $\{Y \in L^\infty \mid \|Y\| \leq \|X\mathbf{1}_{A_k}\|\}$, is $\|\cdot\|_\infty$-closed by (10). Applying [35, Lemma 1.3] for the first and the lattice norm property for the second inequality yields
$$\|\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}]\| \leq \|X\mathbf{1}_{A_k}\| \leq \|X\|.$$
As $\mathcal{H}$ is finitely generated, $\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}] \to \mathbb{E}[X|\mathcal{H}]$ in L^∞. Again by (10),
$$\|\mathbb{E}[X|\mathcal{H}]\| = \lim_{k\to\infty} \|\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}]\| \leq \|X\|.$$
□

This observation allows us to define *dilatation monotonicity* of a function on $\mathcal{X}$.

Definition 15. A function $f : \mathcal{X} \to [-\infty, \infty]$ is DILATATION MONOTONE if for every $X \in \mathcal{X}$ and every finite measurable partition π we have
$$f(X) \leq f\left(\mathbb{E}[X|\sigma(\pi)]\right).$$

By Jensen's inequality, $\mathbb{E}[X|\sigma(\pi)]$ dominates X in the concave order. As an immediate consequence, a function $f : \mathcal{X} \to [-\infty, \infty]$ is dilatation monotone if it is non-decreasing in the concave order.

Our main goal here, however, is to link law-invariance of a quasi-concave function to monotonicity properties such as dilatation monotonicity or being non-decreasing in the concave order, c.f. Definition 2, and mild order continuity properties such as the Fatou property, strong Fatou property, and $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuity.

[9] That is, $\mathbb{P}(X_n \to X) = 1$ and there is some $X_0 \in \mathcal{X}_+$ such that $\sup_{n\in\mathbb{N}} |X_n| \leq X_0$ holds a.s.

As a first step, we recall the version of [10, Proposition 2.11] suited to our purposes.

Proposition 16. *Suppose $\mathcal{X} \subsetneq L^1$ and $\|\cdot\|$ is order continuous. For a proper, quasi-concave, and law-invariant function $f : \mathcal{X} \to [-\infty, \infty)$, the following are equivalent:*

(i) f has the strong Fatou property;
(ii) f has the Fatou property;
(iii) f is $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous.

Hence, in the situation of the preceding proposition, the three aforementioned order continuity properties agree.
In the following, we shall denote the *left-continuous quantile function* of $X \in L^1$ by q_X, i.e.

$$q_X(s) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) \ge s\}, \quad 0 < s < 1.$$

The concave order can be characterised in terms of quantiles; c.f. [14, Lemma 2.2]:

Lemma 17. *For $X, Y \in L^1$, $X \preceq_c Y$ if, and only if, the estimate*

$$\int_0^1 q_X(s)g(s)ds \le \int_0^1 q_Y(s)g(s)ds$$

holds for any choice of a non-increasing bounded function $g : (0, 1) \to \mathbb{R}$.

The following theorem — which is furthermore of independent interest — encompasses all relevant structural properties needed for the utility inputs in the optimisation problems in question. We will discuss the relation to dilatation monotonicity of $f : \mathcal{X} \to [-\infty, \infty]$ for the sake of completeness.

Theorem 18. *Let $f : \mathcal{X} \to [-\infty, \infty]$ be a function.*

(i) Suppose f is quasi-concave, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous, and law-invariant. Then it is non-decreasing in the concave order.
(ii) If f does not attain the value $+\infty$, is dilatation monotone and has the strong Fatou property, then it is law-invariant. The same assertion holds if $\|\cdot\|$ is order continuous and f is $\|\cdot\|$-upper semicontinuous.
If f is additionally quasi-concave, it is $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous in both cases.
(iii) Suppose a quasi-concave function $f : \mathcal{X} \to [-\infty, \infty)$ has the strong Fatou property, or, if $\|\cdot\|$ is order continuous, is $\|\cdot\|$-upper semicontinuous. Then the following statements are equivalent:
(a) f is non-decreasing in the concave order;
(b) f is dilatation monotone;
(c) f is law-invariant.
Under any of the equivalent conditions (a)-(c), f is $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous.

Proof. (i) Suppose $f : \mathcal{X} \to [-\infty, \infty]$ is quasi-concave, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous, and law-invariant. For $r \in \mathbb{R}$, we set $\sigma_r : \mathcal{X}^* \to [-\infty, \infty]$ to be the support function of the superlevel set $E_r(f) = \{Y \in \mathcal{X} \mid f(Y) \ge r\}$, i.e.

$$\sigma_r(\phi) = \inf_{Y \in E_r(f)} \phi(Y), \quad \phi \in \mathcal{X}^*.$$

Suppose $E_r(f) \neq \emptyset$. The Hahn-Banach Theorem and $\sigma(\mathcal{X}, L^\infty)$-closedness or the superlevel sets shows that $Y \in E_r(f)$ holds if, and only if,

$$\forall\, Q \in \mathrm{dom}(\sigma_r) \cap L^\infty : \ \mathbb{E}[QW] \geq \sigma_r(Q). \tag{11}$$

Moreover, the superlevel sets of f are rearrangement invariant. This property transfers to law-invariance of $\sigma_r|_{L^\infty}$ and rearrangement invariance of $\mathrm{dom}(\sigma_r) \cap L^\infty$.
Let now $X, Y \in \mathcal{X}$ be arbitrary with the property $X \preceq_c Y$. We have to show that $f(X) \leq f(Y)$. This inequality holds trivially if $f(X) = -\infty$. Otherwise, if $f(X) > -\infty$, there is $r \in \mathbb{R}$ such that $X \in E_r(f)$. Pick any such r and let $Q \in \mathrm{dom}(\sigma_r) \cap L^\infty$ be arbitrary. By [13, Theorem 13.4],

$$\mathbb{E}[QY] \geq \inf_{\tilde{Q} \sim Q} \mathbb{E}[\tilde{Q}Y] = -\sup_{Q' \sim -Q} \mathbb{E}[Q'Y] = \int_0^1 (-q_{-Q}(s)) q_Y(s) ds. \tag{12}$$

As $-Q$ is bounded, $-q_{-Q} : (0,1) \to \mathbb{R}$ is a non-increasing bounded function. Lemma 17 yields the estimate

$$\int_0^1 (-q_{-Q}(s)) q_X(s) ds \leq \int_0^1 (-q_{-Q}(s)) q_Y(s) ds. \tag{13}$$

Combining (12) and (13) yields

$$\mathbb{E}[QY] \geq \int_0^1 (-q_{-Q}(s)) q_Y(s) ds \geq \int_0^1 (-q_{-Q}(s)) q_X(s) ds = \inf_{\tilde{Q} \sim Q} \mathbb{E}[\tilde{Q}X].$$

Using law-invariance of σ_r on L^∞, we obtain

$$\mathbb{E}[QY] \geq \inf_{\tilde{Q} \sim Q} \sigma_r(\tilde{Q}) = \sigma_r(Q).$$

As $Q \in \mathrm{dom}(\sigma_r) \cap L^\infty$ was chosen arbitrarily, $f(Y) \geq r$ whenever $f(X) \geq r$, which in turn implies $f(X) \leq f(Y)$.

(ii) In a first step, we show that f is law-invariant on the level of simple functions. To this end, suppose two simple functions X and Y are equal in law, i.e. $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$. By [11, Lemma 2.4], for every $\varepsilon > 0$ there is a $K_\varepsilon \in \mathbb{N}$ and finitely generated sub-σ-algebras $\mathcal{H}_1, ..., \mathcal{H}_{K_\varepsilon} \subset \mathcal{F}$ such that

$$\|X - \mathbb{E}[\mathbb{E}[...\mathbb{E}[Y|\mathcal{H}_1]|\mathcal{H}_2]...|\mathcal{H}_{K_\varepsilon}]\|_\infty < \varepsilon.$$

Setting $X_n := \mathbb{E}[\mathbb{E}[...\mathbb{E}[Y|\mathcal{H}_1]|\mathcal{H}_2]...|\mathcal{H}_{K_{\varepsilon_n}}]$ for a sequence $\varepsilon_n \downarrow 0$, we infer that X is approximated by a sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ uniformly which is bounded in norm $\|\cdot\|$ and converges a.s.
If f has the strong Fatou property, this yields

$$f(X) \geq \limsup_{n\to\infty} f(X_n) \geq \liminf_{n\to\infty} f(X_n) \geq f(Y),$$

where the last inequality is due to dilatation monotonicity applied to each $n \in \mathbb{N}$. The argument is symmetric in the roles of X and Y, hence $f(X) = f(Y)$.

In the second case, i.e. $\|\cdot\|$ is order continuous and f is upper semicontinuous, using $\lim_{n\to\infty}\|X_n - X\| = 0$ yields the same assertion.
In a second step, let X and Y be arbitrary in $\mathcal{X}$ with the property of being equal in law. Note that there are two sequences of finitely generated sub-σ-algebras $(\mathcal{H}_n)_{n\in\mathbb{N}}$ and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}[X|\mathcal{H}_n] \to X$ and $\mathbb{E}[Y|\mathcal{G}_n] \to Y$ a.s. and $\mathbb{P}\circ\mathbb{E}[X|\mathcal{H}_n]^{-1} = \mathbb{P}\circ\mathbb{E}[Y|\mathcal{G}_n]^{-1}$, $n\in\mathbb{N}$. Moreover, by Lemma 14,

$$\sup_{n\in\mathbb{N}}\|\mathbb{E}[X|\mathcal{H}_n]\| \le \|X\| \quad \text{and} \quad \sup_{n\in\mathbb{N}}\|\mathbb{E}[Y|\mathcal{G}_n]\| \le \|Y\|.$$

Combining dilatation monotonicity and the strong Fatou property yields

$$f(X) = \lim_{n\to\infty} f(\mathbb{E}[X|\mathcal{H}_n]) \quad \text{and} \quad f(Y) = \lim_{n\to\infty} f(\mathbb{E}[Y|\mathcal{G}_n]). \tag{14}$$

In the second case, order continuity of $\|\cdot\|$ yields $\mathbb{E}[X|\mathcal{H}_n] = X$ and $\lim_{n\to\infty}\mathbb{E}[Y|\mathcal{G}_n] = Y$, whence the statement of (14) follows by upper semicontinuity and dilatation monotonicity. Finally, in both cases, the already proved law-invariance on the level of simple functions combined with $\mathbb{P}\circ\mathbb{E}[X|\mathcal{H}_n]^{-1} = \mathbb{P}\circ\mathbb{E}[Y|\mathcal{G}_n]^{-1}$, $n\in\mathbb{N}$ proves the assertion.
$\sigma(\mathcal{X}, L^\infty)$-upper semicontinuity of f is implied by our assumptions and law-invariance by [10, Theorem 2.6] in the case of the strong Fatou property. If $\|\cdot\|$ is order continuous, upper semicontinuity of f entails that it enjoys the Fatou property. If $\mathcal{X} \neq L^1$, f is hence both $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous and has the strong Fatou Property by [10, Proposition 2.11]. If $\mathcal{X} = L^1$, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuity is weak upper semicontinuity and hence already captured by the assumption of $\|\cdot\|$-upper semicontinuity.

(iii) Combine (i) and (ii).

□

From the preceding proposition, we immediately obtain the following corollary which allows to control the global behaviour of a quasi-concave function f as in Theorem 18(i) or (iii) in terms of the behaviour on deterministic random variables.

Corollary 19. *In the situation of Theorem 18(i) or (iii),*

$$\forall\, X \in \mathcal{X}:\ f(X) \le f(\mathbb{E}[X]).$$

We conclude this interlude on the structural properties of law-invariant functions section with an important generalisation of [10, Theorem 2.6]. It will be the key to localising Theorem 12 to general commodity spaces in Theorem 23.

Proposition 20. *If $f : \mathcal{X} \to [-\infty, \infty]$ is quasi-concave, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous and law-invariant, there exists a unique extension $f^\sharp : L^1 \to [-\infty, \infty]$ which is quasi-concave, upper semicontinuous with respect to $\|\cdot\|_1$, and law-invariant. Moreover, $f^\sharp$ is non-decreasing in the concave order, and properness of f implies properness of $f^\sharp$.*

Proof. By assumption on f, each upper level set $E_r(f)$ is $\sigma(\mathcal{X}, L^\infty)$-closed and (11) holds. Define

$$\mathcal{A}_r := \mathrm{cl}_{L^1}(E_r(f)).$$

If $E_r(f) \neq \emptyset$, the representation

$$\mathcal{A}_r = \{W \in L^1 \mid \forall\, Q \in \mathrm{dom}(\sigma_r) \cap L^\infty : \ \mathbb{E}[QW] \geq \sigma_r(Q)\}$$

holds, where σ_r is as in (11), and

$$f^\sharp(X) = \sup\{r \in \mathbb{R} \mid X \in \mathcal{A}_r\}, \quad X \in L^1.$$

$f^\sharp$ is law-invariant and quasi-concave. Indeed, for law-invariance, note that $f^\sharp(X) \geq r$ is equivalent to $\mathbb{E}[QX] \geq \sigma_r(Q)$ for all Q in the rearrangement invariant set $L^\infty \cap \mathrm{dom}(\sigma_r)$ and in terms of the law-invariant function σ_r. For quasi-concavity, let $X, Y \in L^1$ and $\lambda \in (0,1)$, $X, Y \in \mathcal{A}_r$ implies $\lambda X + (1-\lambda)Y \in \mathcal{A}_r$ by convexity of the latter set. As $\min\{f^\sharp(X), f^\sharp(Y)\} = \sup\{r \in \mathbb{R} \mid \{X, Y\} \subset \mathcal{A}_r\}$, the inequality $f^\sharp(\lambda X + (1-\lambda)Y) \geq \min\{f^\sharp(X), f^\sharp(Y)\}$ follows.
Moreover, $f^\sharp$ is upper semicontinuous. Indeed, suppose $X_n \to X$ in L^1. Without loss of generality, we may assume that $s := \lim_{n\to\infty} f^\sharp(X_n) \in [-\infty, \infty]$ exists. Suppose $r \in \mathbb{R}$ is such that $r < s$. $X_n \in \mathcal{A}_r$ has to hold for all n large enough, hence, for all $Q \in L^\infty \cap \mathrm{dom}(\sigma_r)$,

$$\mathbb{E}[QX] = \lim_{n\to\infty} \mathbb{E}[QX_n] \geq \sigma_r(Q),$$

which means $X \in \mathcal{A}_r$. This shows upper semicontinuity.
We now show that $f^\sharp$ extends f. Clearly, $f^\sharp|_{\mathcal{X}} \geq f$, and we hence assume for contradiction the existence of some $X \in \mathcal{X}$ such that $f^\sharp(X) > f(X)$. This allows us to find some $r \in \mathbb{R}$ such that $X \in \mathcal{A}_r$, whereas $X \notin E_r(f)$. The latter set is $\sigma(\mathcal{X}, L^\infty)$-closed. We can thus find some $Q \in L^\infty$ which gives a separating hyperplane in that

$$\mathbb{E}[QX[< \inf_{Y \in E_r(f)} \mathbb{E}[QY] = \inf_{W \in \mathcal{A}_r} \mathbb{E}[QW],$$

where we have used $\mathcal{A}_r = \mathrm{cl}_{L^1}(E_r(f))$ in the last equality. This contradicts $X \in \mathcal{A}_r$. $f^\sharp(X) > f(X)$ has to be absurd.
$f^\sharp$ is the unique extension of f to L^1 which is quasi-concave, upper semicontinuous, and law-invariant. Indeed, let $\hat{f} : L^1 \to [-\infty, \infty]$ be any extension of f with these properties. As $L^\infty \subset \mathcal{X}$, the restrictions $f^\sharp|_{L^\infty}$ and $\hat{f}|_{L^\infty}$ agree. Let $X \in L^1$ and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of finitely generated sub-σ-algebras such that $\lim_{n\to\infty} \mathbb{E}[X|\mathcal{G}_n] = X$ holds in L^1. $f^\sharp$ and $\hat{f}$ being non-decreasing in the concave order follows from Theorem 18(i). Together with upper semicontinuity, we obtain

$$\hat{f}(X) = \lim_{n\to\infty} \hat{f}\left(\mathbb{E}[X|\mathcal{G}_n]\right) = \lim_{n\to\infty} f\left(\mathbb{E}[X|\mathcal{G}_n]\right) = \lim_{n\to\infty} f^\sharp\left(\mathbb{E}[X|\mathcal{G}_n]\right) = f^\sharp(X).$$

Thus, both extensions $f^\sharp$ and $\hat{f}$ agree.
It remains to prove that properness of f implies properness of $f^\sharp$. By Corollary 19, $f^\sharp(\mathbb{E}[X]) = \infty$ whenever $X \in L^1$ satisfies $f^\sharp(X) = \infty$. As $f^\sharp(\mathbb{E}[X]) = f(\mathbb{E}[X])$, $f^\sharp$ only attains the value $+\infty$ if f does.

□

4.2. **The existence theorem for general commodity spaces.** We shall assume $\mathcal{X} \subsetneq L^1$ here. If $\|\cdot\|$ is order continuous, Proposition 16 states that for a quasi-concave and law-invariant function $f : \mathcal{X} \to [-\infty, \infty)$, the Fatou property, the strong Fatou property, and upper semicontinuity with respect to the $\sigma(\mathcal{X}, L^\infty)$-topology are all equivalent. This leads to following assumption:

Assumption 21. The commodity space satisfies $\mathcal{X} \subsetneq L^1$ as well as properties (a)-(c) above. For each agent $i \in [n]$, its individual utility assessment is given by a function $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$, which is proper, quasi-concave, law-invariant, and has the strong Fatou property.

By Proposition 20, each $\mathfrak{U}_i$ has a canonical extension $\mathfrak{U}_i^\sharp : L^1 \to [-\infty, \infty)$ which is proper, quasi-concave, upper semicontinuous, and law-invariant. Consequently, the family of functions $\mathfrak{U}^\sharp := (\mathfrak{U}_i^\sharp)_{i\in[n]}$ checks Assumption 3.
The following two results prove that we can extend the optimisation problem to L^1 and solve it in the larger space. Note that we extend the definition of the attainable sets Γ_X and $\widehat{\Gamma}_X$, $X \in \mathcal{X}$, by

$$\Gamma_X^{\mathcal{X}} = \Gamma_X \cap \mathcal{X}^n \quad \text{and} \quad \widehat{\Gamma}_X^{\mathcal{X}} = \widehat{\Gamma}_X \cap \mathcal{X}^n.$$

Lemma 22. *Suppose $\mathfrak{U} := (\mathfrak{U}_i)_{i\in[n]} : \mathcal{X}^n \to [-\infty, \infty)^n$ is a vector of utility functions satisfying Assumption 21. Let $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$ be an aggregation function and $X \in \mathcal{X}$. Then:*

(i) $\mathbf{f}(X) \in \mathcal{X}^n$ *holds for every comonotone function* $\mathbf{f} \in \mathbb{C}(n)$.
(ii) The identities

$$\sup_{\mathbf{X}\in\Gamma_X^{\mathcal{X}}} \Lambda(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y}))$$

and

$$\sup_{\mathbf{X}\in\widehat{\Gamma}_X^{\mathcal{X}}} \Lambda(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y}))$$

hold.
(iii) Λ *is coercive for* $(\mathfrak{U}_i|_{\mathbb{R}})_{i\in[n]}$ *if, and only if, it is coercive for* $(\mathfrak{U}_i^\sharp|_{\mathbb{R}})_{i\in[n]}$.

Proof. (i) By (3), the estimate $|f_i(X)| \le |f_i(0)| + |X|$ holds for all $i \in [n]$. The right-hand side is an element of $\mathcal{X}$ and this space is solid as a subset of L^1. We infer that the left-hand side has to be an element of $\mathcal{X}$ as well.

(ii) As none of the functions $\mathfrak{U}_i^\sharp$ attains the value $+\infty$, the expression

$$\Lambda(\mathfrak{U}^\sharp(\mathbf{Y})) = \Lambda(\mathfrak{U}_1^\sharp(Y_1), ..., \mathfrak{U}_n^\sharp(Y_n))$$

is well-defined for all $\mathbf{Y} \in (L^1)^n$. As the family $\mathfrak{U}^\sharp = (\mathfrak{U}_i^\sharp)_{i\in[n]}$ checks Assumption 3, Proposition 7 together with (i) implies

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y})) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}^\sharp(\mathbf{f}(X))\right) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right).$$

As for $\mathbf{f} \in \mathbb{C}(n)$ the vector $\mathbf{f}(X)$ lies in $\Gamma_X^{\mathcal{X}}$ we obtain

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y})) \leq \sup_{\mathbf{X}\in\Gamma_X^{\mathcal{X}}} \Lambda\left(\mathfrak{U}(\mathbf{X})\right).$$

The converse inequality holds *a priori*. Note that the second equality is derived from Proposition 7 in an analogous way.

(iii) This follows from the equality $\mathfrak{U}(r) = \mathfrak{U}^\sharp(r)$, which holds for all $r \in \mathbb{R}$.

□

The following local version of the existence theorem, Theorem 12, is an immediate consequence of the preceding lemma.

Theorem 23. *Theorem 12 holds true* verbatim *if we replace L^1 by $\mathcal{X}$, Γ_X by $\Gamma_X^{\mathcal{X}}$, and $\widehat{\Gamma}_X$ by $\widehat{\Gamma}_X^{\mathcal{X}}$.*

Proof. It only remains to verify (iv) from Theorem 12. Note that the following proof works in L^1 as well as in the setting of a general commodity space $\mathcal{X}$ of this section.
Let $(X_k)_{k\in\mathbb{N}} \subset \Delta$ be a sequence and $X_\infty \in \Delta$ such that $\lim_{k\to\infty} \|X_k - X_\infty\| = 0$. We shall prove that

$$\eta(X_\infty) \geq \limsup_{k\to\infty} \eta(X_k).$$

The proof for $\widehat{\Delta}$ and $\widehat{\eta}$ is completely analogous. We proceed similarly to the proof of Proposition 8, however under the additional problem that not a fixed X is considered, but a sequence thereof.
First of all, we may assume that $\lim_{k\to\infty} \eta(X_k) = \limsup_{k\to\infty} \eta(X_k)$ up to passing to a subsequence and that $\limsup_k \eta(X_k) > -\infty$ — otherwise, the desired inequality is trivial. Now, for all $k \in \mathbb{N}$ choose $\mathbf{g}^k \in \mathbb{C}(n)$ such that $\Lambda\left(\mathfrak{U}(\mathbf{g}^k(X_k))\right) = \eta(X_k)$. The proof of Theorem 12(ii) together with (10) yields

$$\begin{aligned}\sum_{i=1}^n |g_i^k(0)| &\leq G(\mathbb{E}[X_k], \eta(X_k) - 1) + \mathbb{E}[|X_k|] \leq G(\mathbb{E}[X_k], \eta(X_k) - 1) + K\|X_k\| \\ &\leq G(\mathbb{E}[X_k], \eta(X_k) - 1) + K \sup_{k\in\mathbb{N}} \|X_k\|.\end{aligned}$$

By the convergence of $\eta(X_k)$, $L := \inf_{k\in\mathbb{N}} \eta(X_k) > -\infty$. The estimate $\mathbb{E}[X_k] \leq K \sup_{k\in\mathbb{N}} \|X_k\|$ holds for all $k \in \mathbb{N}$. In conjunction with G being non-decreasing in the first coordinate and non-increasing in the second, we obtain the bound

$$\sum_{i=1}^n |g_i^k(0)| \leq G(K \sup_{k\in\mathbb{N}} \|X_k\|, L - 1) + K \sup_{k\in\mathbb{N}} \|X_k\| =: \rho < \infty,$$

a constant which is in particular independent of k. We conclude $(\mathbf{g}^k)_{k\in\mathbb{N}} \subset \mathbb{C}(n)_\rho$.
Recall that $X_k \to X_\infty$ holds in L^1, as well. After passing to subsequences twice, we may hence infer by Proposition 6(i) that for a suitable subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and a suitable $\mathbf{g} \in \mathbb{C}(n)_\rho$, we have

$$\mathbf{g}^{k_\lambda}(x) \to \mathbf{g}(x), \ \lambda \to \infty,$$

for all $x \in \mathbb{R}$, and

$$\mathbb{P}(\lim_{\lambda\to\infty} X_{k_\lambda} = X_\infty) = 1.$$

Hence,

$$\mathbf{g}_{k_\lambda}(X_{k_\lambda}) \to \mathbf{g}(X_\infty) \text{ a.s.}$$

If the norm $\|\cdot\|$ is order continuous, this convergence holds in norm as well. At last, choosing the constant κ as in (10), for all $\lambda \in \mathbb{N}$ and all $i \in [n]$ we can estimate

$$\|g_i^{k_\lambda}(X_{k_\lambda})\| \leq \kappa|g_i^{k_\lambda}(0)| + \|X_{k_\lambda}\| \leq \kappa\rho + \sup_{k\in\mathbb{N}} \|X_k\|.$$

This allows us to reason as in the proof of Proposition 8, however invoking the strong Fatou property of the individual utility functions if necessary. We obtain

$$\begin{aligned}\limsup_{k\to\infty} \eta(X_k) &= \limsup_{\lambda\to\infty} \Lambda\left(\mathfrak{U}(\mathbf{g}^{k_\lambda}(X_{k_\lambda})\right) \leq \Lambda\left(\mathfrak{U}(\mathbf{g}(X_\infty))\right) \\ &\leq \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X_\infty))\right) = \eta(X_\infty).\end{aligned}$$

□

The crucial message of this section is that in a situation of law-invariant utilities with minimal order continuity poperties it does not matter on which commodity space we solve the optimisation problem. Without loss of generality, it may be solved on the canonical commodity space L^1 as the solution automatically localises to the commodity space in question. This is due to the *homogeneity* of the only solutions which are guaranteed to exist, being comonotone transformations of the aggregate wealth.

5. Applications

In this section, the main existence theorem, Theorem 12, will be applied to a number of optimisation problems involving individual preferences within a system of agents and an aggregation thereof. We shall see that the solutions have various economic interpretations.

For the sake of clarity, we we assume that a system of $n \geq 2$ agents $i \in [n]$ is given who all have preferences over the commodity space L^1. These can be represented numerically by a vector of utility functions $\mathfrak{U} : (L^1)^n \to [-\infty, \infty)^n$ satisfying Assumption 3. The localisation procedure discussed in Section 4 shows that we are simultaneously solving the problem in all spaces $\mathcal{X}$ and for all individual utility functions $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$ which satisfy the assumptions of Section 4, in particular Assumption 21.

Before we can discuss the promised applications, we need to introduce more notation: Given a vector $\mathbf{x} \in [-\infty, \infty]^n$, $\mathbf{x}^*$ denotes the maximum and $\mathbf{x}_*$ the minimum of the entries of $\mathbf{x}$, respectively;

$$\mathbf{x}^* := \max_{i\in[n]} x_i \quad \text{and} \quad \mathbf{x}_* := \min_{i\in[n]} x_i.$$

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y}$ denotes the Hadamard product of the two vectors, i.e. $\mathbf{x} \cdot \mathbf{y} = (x_i y_i)_{i\in[n]}$. Also, in order to make fruitful use of the concept of a coercive aggregation function, we shall focus on utilities whose behaviour on riskless commodities can be controlled in the following way:

Definition 24. A function $u : \mathbb{R} \to [-\infty, \infty)$ is an (A, B, C)-FUNCTION, $(A, B, C) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$, if

$$u(x) \leq A + Bx^+ - Cx^-, \quad x \in \mathbb{R},$$

where $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$.

Remark 25. (i) $u : \mathbb{R} \to [-\infty, \infty)$ is an (A, B, C)-function for some $(A, B, C) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ if, and only if, there are $\alpha_\pm \in \mathbb{R}$ and $\beta_\pm > 0$ such that for all $x \in \mathbb{R}$

$$u(x) \leq \begin{cases} \alpha_+ + \beta_+ x, \ x \geq 0, \\ \alpha_- - \beta_- |x|, \ x < 0, \end{cases}$$

that is, u can be controlled from above by affine functions.

(ii) Any proper concave function $u : \mathbb{R} \to [-\infty, \infty)$ is an (A, B, C)-function.

(iii) Suppose (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are elements of $\mathbb{R} \times (0, \infty) \times (0, \infty)$ with the property that $A \leq \hat{A}$, $B \leq \hat{B}$ and $\hat{C} \leq C$. One easily sees that every (A, B, C)-function is also a $(\hat{A}, \hat{B}, \hat{C})$-function.

In this section, we will work under the following assumption:

Assumption 26. The vector of utility functions $\mathfrak{U} = (\mathfrak{U}_i)_{i \in [n]}$ satisfies Assumption 3. Moreover, setting $u_i(x) := \mathfrak{U}_i(x)$, $x \in \mathbb{R}$, we impose that each u_i is an (A_i, B_i, C_i)-function, $i \in [n]$.

We remark that in some of the case studies below, the additional control of the behaviour on riskless commodities rules out cash-additivity of the corresponding utility functions.[10]

5.1. **(Biased) weak Pareto efficiency.** In this section, we prove the existence of (biased) weak Pareto optima as solutions to a suitable optimisation problem.

Definition 27. Let $\Gamma \subset (L^1)^n$ be an attainable set. $\mathbf{X} \in \Gamma$ is called WEAKLY PARETO EFFICIENT or a WEAK PARETO OPTIMUM, if there is no $\mathbf{Y} \in \Gamma$ with $\mathfrak{U}_i(Y_i) > \mathfrak{U}_i(X_i)$, $i \in [n]$.

The aggregation function we shall be interested in,

$$\Lambda_\alpha(\mathbf{y}) := \alpha \min_{i \in [n]} y_i + (1 - \alpha) \max_{i \in [n]} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

where $0 < \alpha \leq 1$ is a parameter, is easily seen to be upper semicontinuous. We remark that the function

$$L^1 \ni X \mapsto \sup_{\mathbf{Y} \in \Gamma_X} \Lambda_1(\mathfrak{U}(\mathbf{Y}))$$

is the *quasi-concave sup-convolution* of the individual utilities; c.f. Mastrogiacomo & Rosazza Gianin [30].

The following observation is immediate:

[10] $\mathfrak{U}_i$ would be cash-additive if $\mathfrak{U}_i(X + r) = \mathfrak{U}_i + r$ holds for all $X \in L^1$ and all $r \in \mathbb{R}$. In that case, quasi-concavity automatically implies concavity.

Lemma 28. *Suppose* $0 < \alpha \leq 1$. *If* $\mathbf{X} \in \Gamma_X$ *is such that*

$$\Lambda_\alpha(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y} \in \Gamma_X} \Lambda_\alpha(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R},$$

then $\mathbf{X}$ *is a weakly Pareto efficient allocation of* X *within* Γ_X. *The analogous result holds when* Γ_X *is replaced by* $\widehat{\Gamma}_X$.

Theorem 29. *Suppose* $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ *fulfils Assumption 26 and let* $0 \leq \alpha < 1$. *If* $\alpha > 0$, *assume that* $\mathbf{B} = (B_i)_{i\in[n]}$ *and* $\mathbf{C} = (C_i)_{i\in[n]}$ *additionally satisfy*

$$\frac{B_2}{C_1} < \frac{\alpha}{1-\alpha} < \frac{C_2}{B_1} \quad \text{if } n = 2, \tag{15}$$

or

$$\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \frac{\alpha}{1-\alpha} \quad \text{if } n \geq 3. \tag{16}$$

(i) For all $X \in \sum_{i=1}^n \operatorname{dom}(\mathfrak{U}_i)$ *there is* $\mathbf{g} \in \mathbb{C}(n)$ *such that*

$$\Lambda_\alpha\left(\mathfrak{U}(\mathbf{g}(X))\right) = \sup_{\mathbf{Y} \in \Gamma_X} \Lambda_\alpha(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$\mathbf{g}(X)$ *is a weakly Pareto efficient allocation of* X *in case that free disposal is not allowed.*

(ii) For all $X \in \sum_{i=1}^n \operatorname{dom}(\mathfrak{U}_i) + L^1_+$ *there is* $\mathbf{g} \in \mathbb{C}(n+1)$ *such that* $g_{n+1}(X) \geq 0$ *and*

$$\Lambda_\alpha\left(\mathfrak{U}_1(g_1(X)), ..., \mathfrak{U}_n(g_n(X))\right) = \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda_\alpha(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$(g_1(X), ..., g_n(X))$ *is a weakly Pareto efficient allocation of* X *in case that free disposal is allowed.*

Proof. (i) is an immediate consequence of Lemmas A.1 and A.2 and Theorem 12(ii) if one notices that $X \in \Delta$ if, and only if, $X \in \sum_{i=1}^n \operatorname{dom}(\mathfrak{U}_i)$.
(ii) follows from Lemmas A.1 and A.2 and Theorem 12(iii) as $X \in \widehat{\Delta}$ if, and only if, $X \geq Y$ for some $Y \in \Delta = \sum_{i=1}^n \operatorname{dom}(\mathfrak{U}_i)$, or equivalently, $X \in \sum_{i=1}^n \operatorname{dom}(\mathfrak{U}_i) + L^1_+$. □

The reason not only to look at the quasi-concave sup-convolution given by Λ_1, but also at the weighted aggregation functions Λ_α for $0 < \alpha < 1$, is the following: Λ_1 only depends on the worst utility achieved by redistribution. It is unaffected by positive deviations from the worst utility other agents may achieve. Therefore it has a bias to overemphasise and sanction negative deviations from a systemic mean utility. The smaller one chooses $\alpha \in (0,1)$, the more the optimal value under aggregation Λ_α depends on the best utility achieved by redistribution. Within the set of weak Pareto optima, Λ_α therefore has a bias towards those allowing for well-performing agents, while the situation of the worst-performing agents is not too dire at the same time.

5.2. **Game theory and core allocations.** For this application, we consider the case $n \geq 3$ for the sake of non-triviality. The following notions are adopted from Aliprantis & Burkinshaw [3, Section 8.10].

Definition 30. Given a vector $\mathbf{W} \in (L^1)^n$ of initial endowments set $W := W_1 + ... + W_n$. A CORE ALLOCATION of $\mathbf{W}$ is a vector $\mathbf{X} \in \Gamma_W$ such that no $\emptyset \neq S \subset [n]$ and $\mathbf{Y} \in \Gamma_W$ with the following properties can be found:

- $\sum_{i \in S} Y_i = \sum_{i \in S} W_i$;
- for all $i \in S$, $\mathfrak{U}_i(Y_i) > \mathfrak{U}_i(X_i)$.

The set of all core allocations of $\mathbf{W}$ is denoted by $\mathrm{core}(\mathbf{W})$.

Core allocations are fair redistributions of a vector of initial endowments: no subsystem $S \subset [n]$ of agents is disadvantaged in that they would be better off by withdrawing their ressources from the larger system $[n]$ and distributing them among themselves.
We are interested in the closely related question whether an aggregated quantity $W \in L^1$ can be split into initial endowments such that, relative to these, the allocation is already perceived as fair in the sense of core allocations. More precisely, we ask if there is an allocation $\mathbf{W} \in \Gamma_W$ such that $\mathbf{W} \in \mathrm{core}(\mathbf{W})$. We prove that solutions to a suitable optimisation problem of type (1) do exactly satisfy this. For $0 < \alpha < 1$ consider the aggregation function

$$\begin{aligned}\Xi_\alpha(\mathbf{y}) &:= \sum_{\emptyset \neq S \subset [n]} \alpha \min_{i \in [n]} y_i + (1-\alpha) \max_{i \in S} y_i \\ &= (2^n - 1)\alpha \min_{i \in [n]} y_i + (1-\alpha) \sum_{\emptyset \neq S \subset [n]} \max_{i \in S} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,\end{aligned}$$

Ξ_α is easily seen to be upper semicontinuous.

Theorem 31. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i \in [n]}$ satisfies Assumption 26. Then there is $0 < \alpha < 1$ such that for any $W \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ there is $\mathbf{g} \in \mathbb{C}(n)$ with the property*

$$\Xi_\alpha\left(\mathfrak{U}(\mathbf{g}(W))\right) = \sup_{\mathbf{W} \in \Gamma_W} \Xi_\alpha(\mathfrak{U}(\mathbf{W})) \in \mathbb{R}.$$

In particular, $\mathbf{g}(W) \in \mathrm{core}(\mathbf{g}(W))$ holds.

Proof. Let $W \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$. As $\lim_{\alpha \uparrow 1} \frac{\alpha}{1-\alpha} = \infty$, we may choose $\alpha \in (0,1)$ with the property that

$$\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \frac{\alpha}{1-\alpha},$$

i.e. (16) is satisfied. As for all $\emptyset \neq S \subset [n]$ and all $\mathbf{y} \in [-\infty, \infty)^n$, we have

$$\alpha \min_{i \in [n]} y_i + (1-\alpha) \max_{i \in S} y_i \leq \Lambda_\alpha(\mathbf{y})$$

and the latter function is coercive for (A_i, B_i, C_i)-functions satisfying (16) by Lemma A.1, the mapping

$$[-\infty, \infty)^n \ni \mathbf{y} \mapsto \alpha \min_{i \in [n]} y_i + (1-\alpha) \max_{i \in S} y_i$$

is coercive for $\mathbf{u} = (u_1, ..., u_n)$ by Proposition 11(iii). By Proposition 11(ii), Ξ_α is coercive for $\mathbf{u}$. By Theorem 12(ii), there is $\mathbf{g} \in \mathbb{C}(n)$ with the claimed properties.
It remains to prove that $\mathbf{g}(W) \in \text{core}(\mathbf{g}(W))$. To this end, assume there is $\emptyset \neq S^* \subset [n]$ and $\mathbf{Y} \in \Gamma_W$ such that

$$\sum_{i\in S^*} Y_i = \sum_{i\in S^*} g_i(W) \quad \text{and} \quad \forall\, i \in S^* : \ \mathfrak{U}_i(g_i(W)) < \mathfrak{U}_i(Y_i).$$

Without loss of generality, we may assume $Y_i = g_i(W)$ for all $i \notin S^*$. This implies

$$\min_{i\in[n]} \mathfrak{U}_i(Y_i) \geq \min_{i\in[n]} \mathfrak{U}_i(g_i(W)) \quad \text{and} \quad \max_{i\in S} \mathfrak{U}_i(Y_i) \geq \max_{i\in S} \mathfrak{U}_i(g_i(W)), \ \emptyset \neq S \subset [n].$$

As furthermore $\max_{i\in S^*} \mathfrak{U}_i(Y_i) > \max_{i\in S^*} \mathfrak{U}_i(g_i(W))$, we obtain $\Xi_\alpha\left(\mathfrak{U}(\mathbf{g}(W))\right) < \Xi_\alpha(\mathfrak{U}(\mathbf{Y}))$ which is a CONTRADICTION to $\mathbf{g}(W)$ being a maximiser. Hence, $\mathbf{g}(W)$ has to be a core allocation of itself. □

5.3. **Pareto efficiency with and without free disposal.** In this section we turn to the more restrictive and, compared to weak Pareto efficiency, economically more desirable property of *Pareto efficiency.*

Definition 32. Let $\Gamma \subset (L^1)^n$ be a attainable set. $\mathbf{X} \in \Gamma$ is called PARETO EFFICIENT or a PARETO OPTIMUM, if $\mathbf{Y} \in \Gamma$ and $\mathfrak{U}_i(Y_i) \geq \mathfrak{U}_i(X_i)$, $i \in [n]$, implies $\mathfrak{U}_i(X_i) = \mathfrak{U}_i(Y_i)$, $i \in [n]$.

Clearly, every Pareto efficient allocation is weakly Pareto efficient. Suppose now $\mathbf{w} := (w_1, ..., w_n) \in (0, \infty)^n$ is a vector of positive weights. We define the upper semicontinuous aggregation function

$$\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i, \quad \mathbf{y} \in [-\infty, \infty)^n.$$

As elaborated in the introduction, if $\mathbf{X} \in \Gamma_X$ satisfies

$$\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) = \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R},$$

then $\mathbf{X}$ is Pareto efficient within Γ_X. Analogously, $\mathbf{X} \in \widehat{\Gamma}_X$ is Pareto efficient within $\widehat{\Gamma}_X$ whenever $\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}$. A natural question is whether for a particular choice of $\mathbf{w} \in (0, \infty)^n$ the function $\Lambda_{\mathbf{w}}$ checks the assumptions of Theorem 12.

Theorem 33. *Suppose* $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ *fulfils Assumption 26. If* $\mathbf{w} \in (0, \infty)^n$, $\mathbf{B} = (B_i)_{i\in[n]}$ *and* $\mathbf{C} = (C_i)_{i\in[n]}$ *satisfy*

$$\frac{B_2}{C_1} < \frac{w_1}{w_2} < \frac{C_2}{B_1} \quad \text{if } n = 2, \tag{17}$$

or

$$(\mathbf{w} \cdot \mathbf{B})^* < (\mathbf{w} \cdot \mathbf{C})_* \quad \text{if } n \geq 3, \tag{18}$$

the following assertions hold:

(i) If $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ *there is* $\mathbf{g} \in \mathbb{C}(n)$ *such that*

$$\Lambda_{\mathbf{w}}\left(\mathfrak{U}(\mathbf{g}(X))\right) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

Consequently, $\mathbf{g}(X)$ *is a Pareto efficient allocation of* X *in case free disposal is not allowed.*

(ii) If $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + L^1_+$ *there is* $\mathbf{g} \in \mathbb{C}(n+1)$ *such that* $g_{n+1}(X) \geq 0$ *and*

$$\Lambda_{\mathbf{w}}\left(\mathfrak{U}_1(g_1(X)), ..., \mathfrak{U}_n(g_n(X))\right) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

Consequently, $(g_1(X), ..., g_n(X))$ *is a Pareto efficient allocation of* X *in case free disposal is allowed.*

Proof. Both (i) and (ii) follow from Lemma A.3 and Theorem 12 if one notices that $\Delta = \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ and $\widehat{\Delta} = \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + \mathcal{X}_+$. □

Example 34. In this example, we consider two agents with different law-invariant utility assessments. First, given some fixed constant $\beta > 1$, we set $\mathcal{Q} := \{Q \in L^\infty_+ \mid \mathbb{E}[Q] = 1,\, Q \leq \beta \ \mathbb{P}\text{-a.s.}\}$. The preferences of agent 1 over L^1 are of Yaari type and given by the concave law-invariant and positively homogeneous utility function

$$\mathfrak{U}_1(X) := \inf_{Q\in\mathcal{Q}} \mathbb{E}[QX], \quad X \in L^1,$$

for which $\mathfrak{U}_1|_{\mathbb{R}}$ is a $(0, 1, 1)$-function; c.f. Yaari [37]. Regarding agent 2, we assume that she has law-invariant variational preferences. More precisely, assume that $u_2 : \mathbb{R} \to [-\infty, \infty)$ is a utility function, i.e. u_2 is concave, right-continuous, non-decreasing, $\mathrm{dom}(u_2) \neq \emptyset$ and such that there are $x, y \in \mathrm{dom}(u_2)$ such that $u_2(x) \neq u_2(y)$. Moreover, let $\mathcal{Q}_2 \subset L^\infty_+$ be law-invariant with the property that $\mathbb{E}[Q] = 1$ for all $Q \in \mathcal{Q}_2$, i.e. $\mathcal{Q}_2$ is a set of probability densities with respect to $\mathbb{P}$. We furthermore suppose a convex and law-invariant function $\alpha_2 : \mathcal{Q}_2 \to \mathbb{R}$ with the property $\iota := \inf_{Q\in\mathcal{Q}_2} \alpha_2(Q) > -\infty$ is given. Eventually, the preferences of agent 2 are given by the utility function

$$\mathfrak{U}_2(X) = \inf_{Q\in\mathcal{Q}_2} \mathbb{E}[Qu_2(X)] + \alpha_2(Q).$$

The right derivative $u_2'(x) := \lim_{y\downarrow x} \frac{u_2(y)-u_2(x)}{y-x} \in [0, \infty]$, $x \in \mathbb{R}$, exists and is non-decreasing. Let us assume we can find $z_- < z_+$ such that $\infty > u_2'(z_-) > u_2'(z_+)$. Then $\mathfrak{U}_2|_{\mathbb{R}}$ is an $(A, u_2'(z_+), u_2'(z_-))$-function, where

$$A := \max\{u_2(z_+) - u_2'(z_+)z_+ + \iota, u_2(z_-) - u_2'(z_-)z_- + \iota\}.$$

If we choose $w_1, w_2 > 0$ such that

$$u_2'(z_+) < \frac{w_1}{w_2} < u_2'(z_-),$$

(17) is satisfied. By Theorem 33(i), for every $X \in \mathrm{dom}(\mathfrak{U}_1) + \mathrm{dom}(\mathfrak{U}_2) = L^1 + \mathrm{dom}(\mathfrak{U}_2)$ there is $\mathbf{g} \in \mathbb{C}(2)$ such that

$$\infty > w_1\mathfrak{U}_1(g_1(X)) + w_2\mathfrak{U}_2(g_2(X)) = \sup_{\mathbf{X}\in\Gamma_X} w_1\mathfrak{U}_1(X_1) + w_2\mathfrak{U}_2(X_2).$$

If we want to say more about the concrete shape of $\mathbf{g}$, we need to make further assumptions on $\mathfrak{U}_2$. Hence, let us assume

- $\mathrm{dom}(\mathfrak{U}_2) = \mathrm{dom}(\mathfrak{U}_2) + \mathbb{R}$;
- $\mathfrak{U}_2$ is strictly monotone with respect to the a.s. order, i.e. $X \le Y$ a.s. and $\mathbb{P}(X < Y) > 0$ implies $\mathfrak{U}_2(X) < \mathfrak{U}_2(Y)$;
- $\mathfrak{U}_2$ is *strictly risk averse conditional on lower tail events*, that is,

  $$\mathfrak{U}_2(X) < \mathfrak{U}_2\left(X\mathbf{1}_{A^c} + \tfrac{\mathbb{E}[X\mathbf{1}_A]}{\mathbb{P}(A)}\mathbf{1}_A\right)$$

  whenever A is a lower tail event for X. The latter means that $\mathbb{P}(A) > 0$ and

  $$\begin{aligned}\sup\{m \in \mathbb{R} \mid \mathbb{P}(\{X \le m\} \cap A) = 0\} &< \inf\{m \in \mathbb{R} \mid \mathbb{P}(\{X \le m\} \cap A) = 1\}\\ &\le \sup\{m \in \mathbb{R} \mid \mathbb{P}(\{X \le m\} \cap A^c) = 0\}.\end{aligned}$$

  Interpretationally, the infimal value X attains on A is strictly less than the supremal value it attains on A, which is bounded from above by the infimal value attained on A^c. As an illustrating example, assume that for some $m \in \mathbb{R}$ and some $\delta > 0$ the three probabilities $\mathbb{P}(X \le m)$, $\mathbb{P}(m < X \le m+\delta)$, and $\mathbb{P}(X > m+\delta)$ are all positive. Then $\{X \le m + \delta\}$ is a lower tail event for X.

If these additional conditions are met, then [31, Proposition 5.2] shows that $\mathbf{g}$ is of the shape

$$\mathbf{g}(x) = (-(x-\ell)_- + k, \max\{x,\ell\} - k), \quad x \in \mathbb{R},$$

for suitable constants $k, \ell \in \mathbb{R}$.

5.4. **Pareto efficiency under individual rationality constraints.** Here we solve the problem of finding Pareto efficient allocations under individual rationality constraints as posed in Ravanelli & Svindland [31].

As in Section 5.2 we assume all agents $i \in [n]$ enter the system with an initial endowment. These are given by a vector $\mathbf{W} \in (L^1)^n$. Now, by means of redistribution, they aim to improve the aggregated situation within the system, but in that redistribution they are not willing to accept a loss in utility beyond a certain threshold compared to the utility of their initial endowment. Given these thresholds $c_i \in [-\infty,\infty)^n$ and some sensible positive weights $\mathbf{w} \in (0,\infty)^n$, we thus consider the optimisation problem

$$\sum_{i=1}^n w_i\mathfrak{U}_i(Y_i) \to \max \quad \text{subject to} \quad \mathbf{Y} \in \Gamma_W \text{ (or } \mathbf{Y} \in \widehat{\Gamma}_W),\ \mathfrak{U}_i(Y_i) \ge \mathfrak{U}_i(W_i) + c_i,\ i \in [n],$$

where $W := W_1 + ... + W_n$, depending on whether free disposal is allowed in the redistribution or not. Any solution of this optimisation problem will be a Pareto efficient allocation of the aggregated initial endowment W. We will model this situation by altering the attainable sets, which are now defined by

$$\mathbb{A}_{\mathbf{c}}(\mathbf{W}) := \{\mathbf{X} \in \Gamma_W \mid \mathfrak{U}_i(X_i) \geq \mathfrak{U}_i(W_i) + c_i\},$$

if free disposal is not allowed, or, provided free disposal is allowed,

$$\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W}) := \{\mathbf{X} \in \widehat{\Gamma}_W \mid \mathfrak{U}_i(X_i) \geq \mathfrak{U}_i(W_i) + c_i\},$$

where $\mathbf{W} \in (L^1)^n$ is the vector of initial endowments. Clearly, the inclusion $\mathbb{A}_{\mathbf{c}}(\mathbf{W}) \subset \widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})$ holds. However, it is not *a priori* clear whether for a given vector of initial endowments $\mathbf{W}$ any of these two sets is non-empty.

For the next theorem, we set $I_\infty := \{i \in [n] \mid c_i = -\infty\}$, a possibly empty set. However, we may assume without loss of generality that $I_\infty \subsetneq [n]$, as otherwise we are in the situation of Theorem 33. We also define the upper semicontinuous aggregation function

$$\Lambda(\mathbf{y}) := \sum_{i=1}^n y_i, \quad \mathbf{y} \in [-\infty, \infty)^n.$$

Theorem 35. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ fulfils Assumption 26 and assume $\mathbf{B} = (B_i)_{i\in[n]}$ and $\mathbf{C} = (C_i)_{i\in[n]}$ satisfy*

$$\max_{i\in[n]} B_i - \inf_{i\in I_\infty} C_i < 0,$$

where $\inf_{i\in I_\infty} C_i := \infty$ if $I_\infty = \emptyset$. Furthermore, let $\mathbf{W} \in \prod_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$, $W := W_1 + ... + W_n$, and let $\mathbf{c} \in [-\infty, \infty)^n$ be a vector of individual rationality constraints.

(i) If $\mathbb{A}_{\mathbf{c}}(\mathbf{W}) \neq \emptyset$ and $\sup_{\mathbf{Y}\in\mathbb{A}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty$, there is $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\Lambda\left(\mathfrak{U}(\mathbf{g}(W))\right) = \sup_{\mathbf{Y}\in\mathbb{A}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$\mathbf{g}(W)$ is a Pareto efficient allocation of W which respects the individual rationality constraints $\mathbf{c}$ in case free disposal is not allowed.

(ii) If $\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W}) \neq \emptyset$ and $\sup_{\mathbf{Y}\in\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty$ there is $\mathbf{g} \in \mathbb{C}(n+1)$ such that $g_{n+1}(W) \geq 0$ and

$$\Lambda\left(\mathfrak{U}_1(g_1(W)), ..., \mathfrak{U}_n(g_n(W))\right) = \sup_{\mathbf{Y}\in\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$(g_1(W), ..., g_n(W))$ is a Pareto efficient allocation of W which respects the individual rationality constraints $\mathbf{c}$ in case free disposal is allowed.

Proof. Both in (i) and (ii), if $(g_1(W), ..., g_n(W))$ is a maximiser, its Pareto efficiency within $\mathbb{A}_{\mathbf{c}}(\mathbf{W})$ — or $\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})$, respectively — is immediately verified.

(i) We aim to apply Theorem 33 and therefore have to verify condition (18). Consider the utility functions $\tilde{\mathfrak{U}}_i := \mathfrak{U}_i + \delta(\cdot|\mathcal{C}_i)$, where $\mathcal{C}_i := \{Y \in L^1 \mid \mathfrak{U}_i(Y) \geq \mathfrak{U}_i(W_i) + c_i\}$ is closed and $\delta(\cdot|\mathcal{C}_i)$ is the concave indicator of this set. The family $\tilde{\mathfrak{U}}$ of new utility functions $\tilde{\mathfrak{U}}_i$ checks Assumption 3. Furthermore, $\tilde{\mathfrak{U}}_i \leq \mathfrak{U}_i$. Hence, $\tilde{\mathfrak{U}}_i|_{\mathbb{R}}$ is also an (A_i, B_i, C_i)-function, and Assumption 26 is checked.

We shall now demonstrate that for all $i \in [n]$, the parameter C_i can be assumed to satisfy $\mathbf{B}^* < C_i$ after potential manipulation. This would entail that $(\tilde{\mathfrak{U}}_1, ..., \tilde{\mathfrak{U}}_n)$ checks the hypotheses of Theorem 33, namely (18) if we choose $\mathbf{w} = (1, 1, ..., 1)$.
To this end, note first that for all $i \in I_\infty$ the estimate $\mathbf{B}^* < C_i$ holds by assumption. Second, if $i \in [n] \backslash I_\infty$, assume $r \in \mathbb{R}$ satisfies $\tilde{\mathfrak{U}}_i(r) > -\infty$. This implies $\tilde{\mathfrak{U}}_i(r) = u_i(r) \geq \mathfrak{U}_i(W_i) + c_i > -\infty$. Hence, if additionally $r < 0$,

$$\mathfrak{U}_i(W_i) + c_i \leq u_i(-|r|) \leq A_i - C_i|r|,$$

which can be rearranged as

$$|r| \leq \left| \frac{\mathfrak{U}_i(W_i) + c_i - A_i}{C_i} \right| =: \sigma_i.$$

For $n_i \in \mathbb{N}$ large enough, we have for all $y \in [-\sigma_i, \infty)$ that

$$A_i + B_i y^+ - C_i y^- \leq n_i + B_i y^+ - (\mathbf{B}^* + 1) y^-.$$

Hence, $\mathfrak{U}_i$ is also a $(n_i, B_i, \mathbf{B}^* + 1)$-function.
Now we can conclude with Theorem 33 the existence of some $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\sum_{i=1}^n \tilde{\mathfrak{U}}_i(g_i(W)) = \sup_{\mathbf{Y} \in \Gamma_W} \sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) \in \mathbb{R}.$$

The left-hand side would be $-\infty$ if $\mathbf{g}(X)$ were not in the attainable set $\mathbb{A}_{\mathbf{c}}(\mathbf{W})$, whence we infer $\mathbf{g}(W) \in \mathbb{A}_{\mathbf{c}}(\mathbf{W})$. This implies $\tilde{\mathfrak{U}}_i(g_i(W)) = \mathfrak{U}_i(g_i(W))$ for all $i \in [n]$. Using again that $\sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) = -\infty$ if $\mathbf{Y} \in \Gamma_W \backslash \mathbb{A}_{\mathbf{c}}(\mathbf{W})$,

$$\sup_{\mathbf{Y} \in \Gamma_W} \sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) = \sup_{\mathbf{Y} \in \mathbb{A}_{\mathbf{c}}(\mathbf{W})} \sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) = \sup_{\mathbf{Y} \in \mathbb{A}_{\mathbf{c}}(\mathbf{W})} \sum_{i=1}^n \mathfrak{U}_i(Y_i).$$

(ii) The argument is in complete analogy with the argument for (i).

□

5.5. **Aggregation with a view towards systemic risk.** Throughout this section let $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}_+^n$ be such that $\sum_{i=1}^n p_i = 1$. We consider the upper semicontinuous aggregation function

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{y}) := \sum_{i=1}^n -p_i y_i^- + q_i (y_i - r_i)^+, \quad \mathbf{y} \in [-\infty, \infty)^n. \tag{19}$$

It was suggested by Brunnermeier & Cheridito [5] as a way to aggregate individual profits net of losses in a system of agents in a meaningful way to account for systemic risk; c.f. [20, Example 4.3].
In the present setting, we may consider the the quantity $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{X}))$, $\mathbf{X} \in (L^1)^n$, and use $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ to aggregate the individual utilities of the n agents. In such an application, it would be more appropriate to think of $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ to account for *systemic fairness*.
Note that for some $\mathbf{X} \in (L^1)^n$, $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{X}))$ has a clear-cut interpretation: $-p_i \mathfrak{U}_i(X_i)^-$, $i \in [n]$, only appears in the aggregation if i incurs a negative utility and is then weighted

according to the impact or importance of agent i. Conversely, the term $q_i(\mathfrak{U}_i(X_i) - r_i)^+$ accounts, likewise in a weighted way, for the positive utility agent i gains as far as it exceeds a certain individual threshold r_i.
One easily sees that $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ is an upper semicontinuous aggregation function.

Theorem 36. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ fulfils Assumption 26 and assume $\mathbf{B} = (B_i)_{i\in[n]}$, $\mathbf{C} = (C_i)_{i\in[n]}$, and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n_+$ satisfy*

$$(\mathbf{q}\cdot\mathbf{B})^* < (\mathbf{p}\cdot\mathbf{C})_*. \tag{20}$$

Let $\mathbf{r} \in \mathbb{R}^n_+$ be arbitrary and define $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ as in (19)*.*

(i) For all $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ there is $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}\left(\mathfrak{U}(\mathbf{g}(X))\right) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

(ii) For all $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + L^1_+$ there is $\mathbf{g} \in \mathbb{C}(n+1)$ such that $g_{n+1}(X) \geq 0$ and

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}_1(g_1(X)), ..., \mathfrak{U}_n(g_n(X))) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

Proof. Combine Lemma A.4 with Theorem 12. □

Appendix A. Coercivity results

In the following, given a vector $\mathbf{u} = (u_1, ..., u_n)$ of (A_i, B_i, C_i)-functions, we set $\mathbf{A} := (A_i)_{i\in[n]}$, $\mathbf{B} := (B_i)_{i\in[n]}$, and $\mathbf{C} := (C_i)_{i\in[n]}$.

Lemma A.1. *Assume $\mathbf{u} = (u_1, ..., u_n)$ is a vector of (A_i, B_i, C_i)-functions $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. Let $0 < \alpha < 1$ be a parameter which satisfies* (15) *if $n = 2$, or, provided $n \geq 3$,* (16)*. Then the aggregation function*

$$\Lambda_\alpha(\mathbf{y}) := \alpha \min_{i\in[n]} y_i + (1-\alpha)\max_{i\in[n]} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

is upper semicontinuous and coercive for $\mathbf{u}$.

Proof. The aggregation function Λ_α is clearly upper semicontinuous and non-decreasing in the pointwise order on $[-\infty, \infty)^n$. Fix $x, m \in \mathbb{R}$ and consider the set $S(x, m)$ defined in (6). In order to find the bound $G(x, m)$ as in (7), we choose $\mathbf{y} \in S(x, m)$ arbitrary, set $I := \{i \in [n] \mid y_i < 0\}$, and distinguish the following cases:
<u>Case 1:</u> $I = \emptyset$. Then $\sum_{i=1}^n |y_i| = \sum_{i=1}^n y_i \leq x$.
<u>Case 2:</u> $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$m \leq \alpha \min_{i\in[n]} A_i - C_i|y_i| + (1-\alpha)\max_{i\in[n]} A_i - C_i|y_i| \leq \mathbf{A}^* - \alpha\mathbf{C}_* \max_{i\in[n]} |y_i|.$$

Rearranging this inequality yields

$$\sum_{i=1}^n |y_i| \leq n \max_{i\in[n]} |y_i| \leq \frac{n(\mathbf{A}^* - m)}{\alpha\mathbf{C}_*}. \tag{A.1}$$

<u>Case 3:</u> $I \neq \emptyset$ and $J := \{i \in [n] \mid y_i > 0\} \neq \emptyset$.

Case 3.1: If $n = 2$ and $I = \{1\}$, we have $J = \{2\}$. From the property $y_1 + y_2 \le x$ we infer $|y_2| = y_2 \le x + |y_1|$. Using this and each u_i being an (A_i, B_i, C_i)-function, we can estimate

$$\begin{aligned} m &\le \alpha \min_{i=1,2} u_i(y_i) + (1-\alpha) \max_{i=1,2} u_i(y_i) \le \alpha u_1(y_1) + (1-\alpha) \max_{i=1,2} u_i(y_i) \\ &\le \alpha(A_1 - C_1|y_1|) + (1-\alpha)\max\{A_1 - C_1|y_1|, A_2 + B_2 y_2\} \\ &\le \alpha(\mathbf{A}^* - C_1|y_1|) + (1-\alpha)(\mathbf{A}^* + B_2 y_2) \\ &= \mathbf{A}^* - \alpha C_1|y_1| + (1-\alpha)B_2(x + |y_1|) = \mathbf{A}^* + (1-\alpha)B_2 x + ((1-\alpha)B_2 - \alpha C_1)\,|y_1|. \end{aligned}$$

By the first inequality in (15), $(1-\alpha)B_2 - \alpha C_1 < 0$. Hence, rearranging terms yields

$$|y_1| \le \frac{\mathbf{A}^* + (1-\alpha)B_2 x - m}{|(1-\alpha)B_2 - \alpha C_1|},$$

and eventually

$$|y_1| + |y_2| \le x + 2|y_1| \le x + \frac{2(\mathbf{A}^* + (1-\alpha)B_2 x - m)}{|(1-\alpha)B_2 - \alpha C_1|}. \tag{A.2}$$

Case 3.2: If $n = 2$ and $I = \{2\}$, we obtain completely analogously to Case 3.1 that

$$|y_1| + |y_2| \le x + 2|y_2| \le x + \frac{2(\mathbf{A}^* + \alpha B_1 x - m)}{|\alpha B_1 - (1-\alpha)C_2|}. \tag{A.3}$$

Case 3.3: $n \ge 3$. As above,

$$\max_{j\in J} |y_j| \le \sum_{j\in J} y_j \le x + (n-1)\max_{i\in I} |y_i|.$$

This allows us to infer

$$\begin{aligned} m &\le \alpha \min_{i\in[n]} A_i + B_i y_i^+ - C_i y_i^- + (1-\alpha)\max_{i\in[n]} A_i + B_i y_i^+ - C_i y_i^- \\ &\le \alpha \min_{i\in I}(\mathbf{A}^* - C_i|y_i|) + (1-\alpha)\max_{j\in J}(\mathbf{A}^* + \mathbf{B}_j y_j) \\ &\le \mathbf{A}^* - \alpha \mathbf{C}_* \max_{i\in I}|y_i| + (1-\alpha)\mathbf{B}^* \max_{j\in J}|y_i| \\ &\le \mathbf{A}^* + (1-\alpha)\mathbf{B}^* x + ((1-\alpha)(n-1)\mathbf{B}^* - \alpha \mathbf{C}_*)\max_{i\in I}|y_i|. \end{aligned}$$

Rearranging the preceding inequality and using that, by (16), $(1-\alpha)(n-1)\mathbf{B}^* - \alpha\mathbf{C}_* < 0$, we conclude

$$\max_{i\in I}|y_i| \le \frac{\mathbf{A}^* + (1-\alpha)\mathbf{B}^* x - m}{|(1-\alpha)(n-1)\mathbf{B}^* - \alpha \mathbf{C}_*|},$$

and eventually

$$\begin{aligned} \sum_{i=1}^n |y_i| &\le x + 2\sum_{i\in I}|y_i| \le x + 2(n-1)\max_{i\in I}|y_i| \\ &\le x + \frac{2(n-1)(\mathbf{A}^* + (1-\alpha)\mathbf{B}^* x - m)}{|(1-\alpha)(n-1)\mathbf{B}^* - \alpha\mathbf{C}_*|}. \end{aligned} \tag{A.4}$$

Let $\tilde{G}(x,m)$ be defined as the maximum of the bound (A.1)-(A.3) (if $n=2$) or of (A.1), and (A.4) (if $n \geq 3$). Then

$$G(x,m) := \max\{\tilde{G}(x,m), x\}, \quad (x,m) \in \mathbb{R} \times \mathbb{R},$$

gives the desired bound (7).

Note that for all $x \in \mathbb{R}$, $\tilde{G}(x,m) \to -\infty$ as $m \to \infty$. We may hence consider the real-valued function

$$H(x) := \max\{\inf\{s \in \mathbb{R} \mid \tilde{G}(x,s) \leq -1\}, \mathbf{A}^* + \mathbf{B}^* x + 1\}, \quad x \in \mathbb{R}.$$

Fix $m \geq H(x)$ and suppose we can choose $\mathbf{y} \in S(x,m)$. By construction, $\mathbf{y} \in \mathbb{R}^n_+$ has to hold. We estimate

$$m \leq \max_{i \in [n]} A_i + B_i y_i < \mathbf{A}^* + \mathbf{B}^* x + 1 \leq H(x) \leq m.$$

No such $\mathbf{y}$ can exist, and the function H has property (8). Λ_α is coercive for $\mathbf{u}$. □

Lemma A.2. *Assume* $\mathbf{u} = (u_1, ..., u_n)$ *is a vector of* (A_i, B_i, C_i)*-functions* $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. *Then the aggregation function* $\Lambda_1(\mathbf{y}) = \min_{i \in [n]} y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, *is upper semicontinuous and coercive for* $\mathbf{u}$.

Proof. Λ_1 is upper semicontinuous and non-decreasing in the pointwise order on $[-\infty, \infty)^n$. For coercivity in the case $n \geq 3$, note that $\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \infty$. Hence, for $0 < \alpha < 1$ close enough to 1, the estimate

$$\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \frac{\alpha}{1-\alpha}$$

holds. Hence, $\Lambda_1(\mathbf{y}) \leq \Lambda_\alpha(\mathbf{y})$ holds for all $\mathbf{y} \in [-\infty, \infty)^n$, and the latter function is coercive for $\mathbf{u}$ by Lemma A.1. Coercivity of Λ_1 follows with Proposition 11(iii).

It remains to treat the case $n = 2$. Let $x, m \in \mathbb{R}$ be arbitrary and suppose $\mathbf{y} \in S(x,m)$. Set $I := \{i \in [n] \mid y_i < 0\}$ and consider the following cases:

<u>Case 1:</u> $I = \emptyset$. Then $|y_1| + |y_2| = y_1 + y_2 \leq x$.

<u>Case 2:</u> $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$m \leq \Lambda_1(\mathbf{u}(\mathbf{y})) \leq \min\{A_1 - C_1|y_1|, A_2 - C_2|y_2|\} \leq \mathbf{A}^* - \mathbf{C}_* \max_{i=1,2} |y_i|.$$

From a rearrangement of this inequality, we infer

$$|y_1| + |y_2| \leq 2 \max_{i=1,2} |y_i| \leq \frac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}.$$

<u>Case 3:</u> $|I| = 1$.

<u>Case 3.1:</u> $I = \{1\}$. We use again that $|y_2| = y_2 \leq x - y_1 = x + |y_1|$. Note that

$$m \leq \Lambda_1(\mathbf{u}(\mathbf{y})) \leq \min\{A_1 - C_1|y_1|, A_2 + B_2 y_2\} \leq \mathbf{A}^* - \mathbf{C}_*|y_1|.$$

From a rearrangement of this inequality, we obtain

$$|y_1| + |y_2| \leq x + 2|y_1| \leq x + \frac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}.$$

<u>Case 3.2:</u> $I = \{2\}$. In this case, we obtain the same bound as in Case 3.1.

Consequently, the function

$$G(x,m) := \max\{x, x^+ + \tfrac{2(\mathbf{A}^*-m)}{\mathbf{C}_*}\}, \quad (x,m) \in \mathbb{R}\times\mathbb{R},$$

has the desired property (7).
Now consider the function

$$H(x) := \max\left\{\mathbf{A}^* + \mathbf{C}_*(\tfrac{1}{2}x^+ + 1), A_1 + B_1 x + 1\right\}, \quad x \in \mathbb{R}.$$

If $m \geq H(x)$, $\max\{\frac{2(\mathbf{A}^*-m)}{\mathbf{C}_*}, x + \frac{2(\mathbf{A}^*-m)}{\mathbf{C}_*}\} \leq -1$. Thus, if we could choose $\mathbf{y} \in S(x,m)$, Case 1 above would have to hold, i.e. $\mathbf{y} \in \mathbb{R}^2_+$. Using that each u_i is an (A_i, B_i, C_i)-function,

$$\Lambda_1(\mathbf{u}(\mathbf{y})) \leq u_1(y_1) \leq A_1 + B_1 y_1 < A_1 + B_1 x + 1 \leq H(x) \leq m.$$

This is a CONTRADICTION, and $S(x,m) = \emptyset$ has to hold. Hence, the function H has the desired property (8), and Λ_1 is coercive for $\mathbf{u}$. □

Lemma A.3. *Assume* $\mathbf{u} = (u_1, ..., u_n)$ *is a vector of* (A_i, B_i, C_i)*-functions* $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. *Moreover, suppose* $\mathbf{w} \in (0,\infty)^n$ *satisfies* (17) *if* $n = 2$, *or, if* $n \geq 3$, (18). *Then the aggregation function*

$$\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i, \quad \mathbf{y} \in [-\infty,\infty)^n,$$

is upper semicontinuous and coercive for $\mathbf{u}$.

Proof. The function $\Lambda_{\mathbf{w}}$ is clearly upper semicontinuous and non-decreasing in the pointwise order on $[-\infty,\infty)^n$. In order to find a function G with property (7), fix $x, m \in \mathbb{R}$ and assume $\mathbf{y} \in S(x,m)$ is arbitrarily chosen.
<u>Case 1:</u> $\mathbf{y} \in \mathbb{R}^n_+$. Then $\sum_{i=1}^n |y_i| = \sum_{i=1}^n y_i \leq x$.
<u>Case 2:</u> $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$m \leq \Lambda_{\mathbf{w}}(\mathbf{u}(\mathbf{y})) = \sum_{i=1}^n w_i u_i(y_i) \leq \sum_{i=1}^n w_i(A_i - C_i|y_i|) \leq n(\mathbf{w}\cdot\mathbf{A})^* - (\mathbf{w}\cdot\mathbf{C})_* \sum_{i=1}^n |y_i|.$$

Rearranging this inequality yields

$$\sum_{i=1}^n |y_i| \leq \frac{n(\mathbf{w}\cdot\mathbf{A})^* - m}{(\mathbf{w}\cdot\mathbf{C})_*}. \tag{A.5}$$

<u>Case 3:</u> $\mathbf{y} \in \mathbb{R}^n \backslash (\mathbb{R}^n_+ \cup \mathbb{R}^n_-)$. We set $I = \{i \in [n] \mid y_i < 0\}$.
<u>Case 3.1:</u> $n = 2$ and $I = \{1\}$. We have

$$m \leq w_1 u_1(y_1) + w_2 u_2(y_2) \leq w_1 A_1 - w_1 C_1 |y_1| + w_2 A_2 + w_2 B_2 |y_2|.$$

Using $|y_2| = y_2 \leq x + |y_1|$, one obtains

$$m \leq 2(\mathbf{w}\cdot\mathbf{A})^* + w_2 B_2 x + (w_2 B_2 - w_1 C_1)|y_1|.$$

By the first inequality in (17), $w_2 B_2 - w_1 C_1 < 0$. Hence, rearranging this inequality yields

$$|y_1| + |y_2| \leq x + 2|y_1| \leq x + \frac{4(\mathbf{w}\cdot\mathbf{A})^* + 2(\mathbf{w}\cdot\mathbf{B})^*|x| - 2m}{|w_2 B_2 - w_1 C_1|}. \tag{A.6}$$

Case 3.2: If $n = 2$ and $I = \{2\}$, we obtain completely analogously to Case 3.1 that

$$|y_1| + |y_2| \le x + 2|y_2| \le x + \frac{4(\mathbf{w}\cdot\mathbf{A})^* + 2(\mathbf{w}\cdot\mathbf{B})^*|x| - 2m}{|w_1B_1 - w_2C_2|}. \tag{A.7}$$

Combining Cases 1-3.2 implies that the function

$$G(x,m) := \max\left\{x,\ \frac{2(\mathbf{w}\cdot\mathbf{A})^* - m}{(\mathbf{w}\cdot\mathbf{C})_*},\ x + \frac{4(\mathbf{w}\cdot\mathbf{A})^* + 2(\mathbf{w}\cdot\mathbf{B})^*|x| - 2m}{\xi}\right\}$$

has property (7). Here, $\xi := \min\{|w_1B_1 - w_2C_2|, |w_2B_2 - w_1C_1|\}$.
Case 3.3: $n \ge 3$. As in preceding proofs, $\sum_{i\in[n]\setminus I} y_i \le x + \sum_{i\in I}|y_i|$. This allows us to infer

$$\begin{aligned} m &\le \sum_{i=1}^n w_iu_i(y_i) \le \sum_{i\in[n]\setminus I} w_i(A_i + B_i|y_i|) + \sum_{i\in I} w_i(A_i - C_i|y_i|) \\ &\le \sum_{i=1}^n w_iA_i + (\mathbf{w}\cdot\mathbf{B})^* \sum_{i\in[n]\setminus I}|y_i| - (\mathbf{w}\cdot\mathbf{C})_* \sum_{i\in I}|y_i| \\ &\le n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*x + ((\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*)\sum_{i\in I}|y_i| \end{aligned}$$

Rearranging this inequality using (18) yields

$$\sum_{i\in I}|y_i| \le \frac{n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*x - m}{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|},$$

and, eventually,

$$\sum_{i=1}^n |y_i| \le x + 2\sum_{i\in I}|y_i| \le x + \frac{2\left(n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*x - m\right)}{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|}. \tag{A.8}$$

Combining equations (A.5) and (A.8) shows that the function

$$G(x,m) := \max\left\{x,\ \frac{n(\mathbf{w}\cdot\mathbf{A})^* - m}{(\mathbf{w}\cdot\mathbf{C})_*},\ x + \frac{2\left(n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*x - m\right)}{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|}\right\}$$

has property (7), provided $n \ge 3$.
We now turn our attention to the existence of a function H with property (8). Fix $x \in \mathbb{R}$ and let $\tilde{H}(x) := n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*x + 1$. If $n = 2$, consider

$$H(x) := \max\left\{\tilde{H}(x),\ (\mathbf{w}\cdot\mathbf{C})_* + 2(\mathbf{w}\cdot\mathbf{A})^*,\ \tfrac{\xi(x+1)}{2} + 2(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*|x|\right\},$$

where ξ is defined as above. If $m \ge H(x)$, the right-hand sides of (A.5)-(A.7) are less or equal to -1. If $n \ge 3$, consider

$$H(x) := \max\left\{\tilde{H}(x),\ n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{C})_*,\ \tfrac{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|(x+1)}{2} + n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*x\right\}.$$

If $m \ge H(x)$ in this case, the right-hand sides of (A.5) and (A.8) are less or equal to -1.

Suppose now $m \geq H(x)$ and $S(x,m) \neq \emptyset$. Then for all $\mathbf{y} \in S(x,m)$, Case 1 from above has to hold, i.e. $\mathbf{y} \in \mathbb{R}^n_+$. Moreover,

$$\sum_{i=1}^n w_i u_i(y_i) \leq \sum_{i=1}^n w_i(A_i + B_i y_i) < n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x + 1 = \tilde{H}(x) \leq H(x) \leq m,$$

which is absurd. Hence, the function H has property (8). □

Lemma A.4. *Assume* $\mathbf{u} = (u_1, ..., u_n)$ *is a vector of* (A_i, B_i, C_i)*-functions* $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. *If* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n_+$ *satisfy* (20) *and* $\mathbf{r} \in \mathbb{R}^n_+$ *is arbitrary, the aggregation function*

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(y) := \textstyle\sum_{i=1}^n -p_i y_i^- + q_i(y_i - r_i)^+, \quad y \in [-\infty,\infty)^n,$$

is upper semicontinuous and coercive for $\mathbf{u}$.

Proof. By Remark 25(iii), we may assume without loss of generality that $A_i \geq 0$ holds for all $i \in [n]$.
As already observed, the function $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ is upper semicontinuous and non-decreasing in the pointwise order on $[-\infty,\infty)^n$. In order to find the function G, fix $x, m \in \mathbb{R}$ and let $\mathbf{y} \in S(x,m)$ be arbitrary. Again, we set $I := \{i \in [n] \mid y_i < 0\}$.
Case 1: $I = \emptyset$, i.e. $\mathbf{y} \in \mathbb{R}^n_+$. As in the preceding proofs, $\sum_{i=1}^n |y_i| \leq x$.
Case 2: $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$\begin{aligned} m &\leq \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{u}(\mathbf{y})) \leq \sum_{i=1}^n -p_i(A_i - C_i|y_i|)^- + q_i(A_i - C_i|y_i| - r_i)^+ \\ &\leq \sum_{i=1}^n p_i(A_i - C_i|y_i|) + q_i(A_i - r_i)^+ \leq n(\mathbf{p}\cdot\mathbf{A})^* - (\mathbf{p}\cdot\mathbf{C})_* \sum_{i=1}^n |y_i| + \sum_{i=1}^n q_i(A_i - r_i)^+. \end{aligned}$$

As $p_i C_i > 0$ for all $i \in [n]$, we obtain

$$\sum_{i=1}^n |y_i| \leq \frac{n(\mathbf{p}\cdot\mathbf{A})^* + \sum_{i=1}^n q_i(A_i - r_i)^+ - m}{(\mathbf{p}\cdot\mathbf{C})_*}. \tag{A.9}$$

Case 3: $I \neq \emptyset$ and $y_j > 0$ for some $j \in J := [n]\backslash I$.
Setting $I' := \{i \in I \mid A_i - C_i|y_i| > r_i\}$ and $J' := \{j \in J \mid A_j + B_j y_j > r_j\}$, we have

$$\begin{aligned} m \leq \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{u}(\mathbf{y})) &\leq \sum_{i\in I} -p_i(A_i - C_i|y_i|)^- + q_i(A_i - C_i|y_i| - r_i)^+ \\ &\quad + \sum_{j\in J} -p_j(A_j + B_j|y_j|)^- + q_j(A_j + B_j|y_j| - r_j)^+ \\ &\leq \sum_{i\in I} p_i(A_i - C_i|y_i|) + \sum_{i\in I'} q_i(A_i - C_i|y_i|) \\ &\quad + \sum_{j\in J'} q_j(A_j + B_j|y_j|). \end{aligned}$$

In the last step, we have used that for $j \in J$, our assumption $A_j \geq 0$ implies $(A_j + B_j y_j)^- = 0$.

- The estimate
$$\sum_{i\in I} p_i(A_i - C_i|y_i|) \leq n(\mathbf{p}\cdot\mathbf{A})^* - (\mathbf{p}\cdot\mathbf{C})_* \sum_{i\in I}|y_i| =: \rho_1 - (\mathbf{p}\cdot\mathbf{C})_*\sum_{i\in I}|y_i|$$
is immediate.
- We have $\sum_{i\in I'} q_i(A_i - C_i|y_i|) \leq n(\mathbf{q}\cdot\mathbf{A})^* =: \rho_2$.
- From $\sum_{j\in J'}|y_j| \leq \sum_{j\in J} y_j \leq x + \sum_{i\in I}|y_i|$, we conclude
$$\sum_{j\in J'} q_j(A_j + B_j|y_j|) \leq \rho_2 + (\mathbf{q}\cdot\mathbf{B})^*\left(x + \sum_{i\in I}|y_i|\right).$$

Combining all estimates above, we obtain
$$m \leq \rho_1 - (\mathbf{p}\cdot\mathbf{C})_*\sum_{i\in I}|y_i| + 2\rho_2 + (\mathbf{q}\cdot\mathbf{B})^*\left(x+\sum_{i\in I}|y_i|\right) =: \rho_3 + ((\mathbf{q}\cdot\mathbf{B})^* - (\mathbf{p}\cdot\mathbf{C})_*)\sum_{i\in I}|y_i|.$$
The constant $\rho_3 \in \mathbb{R}$ is independent of $\mathbf{y}$. We rearrange the inequality and use (20) in order to obtain
$$\sum_{i\in I}|y_i| \leq \frac{\rho_3 - m}{|(\mathbf{q}\cdot\mathbf{B})^* - (\mathbf{p}\cdot\mathbf{C})_*|}.$$
Consequently, the bound
$$\sum_{i\in[n]}|y_i| \leq x + 2\sum_{i\in I}|y_i| \leq x + \frac{2(\rho_3 - m)}{|(\mathbf{q}\cdot\mathbf{B})^* - (\mathbf{p}\cdot\mathbf{C})_*|} \tag{A.10}$$
holds. Let $\tilde{G}(x,m)$ be defined as the maximum of the bounds in (A.9) and (A.10). As in the preceding proofs, the function $G(x,m) := \max\{x, \tilde{G}(x,m)\}$, $(x,m)\in\mathbb{R}\times\mathbb{R}$, has property (7).
Note that for all $x\in\mathbb{R}$ we have $\tilde{G}(x,m)\to-\infty$ as $m\to\infty$. This allows us to define a function H which has property (8) by
$$H(x) := \max\{\inf\{s\in\mathbb{R}\mid \tilde{G}(x,s)\leq -1\},\, n(\mathbf{q}\cdot\mathbf{A})^* + (\mathbf{q}\cdot\mathbf{B})^*x + 1\},\quad x\in\mathbb{R}.$$
The proof of the assertion is completely analogous to the preceding cases. □

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