SoSe14. Math. Seminar "ΨDO's" Intro 1st. Lecture

Formally, ~~the~~ Fourier transform is defined as (1)

(*) $\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)\,dx$

(comments on 2π's)

Then (!)

(**) $f(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x,\xi\rangle} \hat{f}(\xi)\,d\xi$

(***) Let $e_\xi(x) := e^{i\langle x,\xi\rangle} \notin L^p(\mathbb{R}^n) \quad \forall p \in [1,\infty)$ (!).

(*) is formally $\hat{f}(\xi) = \langle e_\xi, f\rangle$ (scalar product in $L^2(\mathbb{R}^n)$ - but see (***)!)

and (**) is formally

$$f = \frac{1}{(2\pi)^n} \int e_\xi \langle e_\xi, f\rangle\, d\xi$$

$$\sim \sum_\xi \langle e_\xi, f\rangle e_\xi$$

"f expressed in the basis $\{e_\xi\}_{\xi\in\mathbb{R}^n}$".

(2)

From (**): Formally

$$(\partial_x^\alpha f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (i\xi)^\alpha \hat{f}(\xi) e^{i\langle x,\xi\rangle} d\xi \qquad (\square)$$

I.e. for PDO $P = \sum_{|\alpha|\le m} a_\alpha(x) \partial^\alpha$

(i.e. $(Pf)(x) = \sum_{|\alpha|\le m} a_\alpha(x) (\partial^\alpha f)(x)$), we have

$$(Pf)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \Big(\sum_{|\alpha|\le m} a_\alpha(x) (i\xi)^\alpha\Big) \hat{f}(\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\, \hat{f}(\xi) d\xi$$

with $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x) (i\xi)^\alpha \qquad (\square\square)$

This seminar is about <u>how</u> to gene<u>ralize</u> this formula, to ~~[illegible]~~ more <u>general</u> functions $p(x,\xi)$ than the one in $(\square)$, to obtain more general operators than PDO. – <u>Why</u> one ~~a~~ wants to do this, and what it is good for (mostly, to study PDE's), I will tell more about <u>next</u> time.

<u>Note</u>: $(\square\square)$ is $Pf \sim \sum_\xi \langle e_\xi, f\rangle Pe_\xi$ – and Pe_ξ is "easy", just mult. by p: P is <u>diag.</u> in basis e_ξ.

Today, I will give overview of the themes / topics / talks, and we will distribute them - according to who signed up first (need: volunteers to give two talks → Hauptseminar!).

First: Need to study class of functions, f, for which $(*)$ + $(**)$ makes sense

- rather, one class.

Note: $(\square)$ says " $\partial_x f \leftrightarrow (i\xi)\hat{f}(\xi)$ "

- similarly, " $\partial_\xi \hat{f} \leftarrow (ix)\hat{f}(x)$ "

Turns out: Natural to ask $\partial_x^\alpha f$ exists for all α

- and still decays such that can integrate

⇝ Schwartz-functions:

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \;\middle|\; \forall \alpha, \beta \in \mathbb{N}_0^n : \exists C = C_{\alpha,\beta} : |x^\alpha \partial_x^\beta f(x)| \le C_{\alpha,\beta} \quad \forall x \in \mathbb{R}^n \right\}.$$

Shall define topology on $\mathcal{S}$ - in fact, metric - and study the dual space of $\mathcal{S}$: space of linear functionals on $\mathcal{S}$, cont. in this top.:

$$\mathcal{S}'(\mathbb{R}^n) = \{ u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} \mid u \text{ is linear \& cont.} \}$$

(4)

⌈ For normed spaces X, $u \in X'$ $\Leftrightarrow$

(✱) $\exists c > 0 \quad |u(x)| \leq c\|x\| \quad \forall x \in X$

– for the topology on $\mathcal{S}(\mathbb{R}^n)$, given by family of semi-norms

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

– (✱) gets replaced by other ineq.'s

Not essential to know – is distribution theory / theory of local convex top. vector spaces. ⌋

$\mathcal{S}'(\mathbb{R}^n)$: tempered distribution

Shall study Fourier-transform on $\mathcal{S}$ and $\mathcal{S}'$ (via duality):

(i.e. define & study properties)

~~As seen in ... there is relation~~

① | Fourier-transf. & distr. in $\mathbb{R}^n$. |

As seen in ~~...~~ there is relation

(□) (& analog $\partial_\xi \hat{f}$)

smoothness of f $\longleftrightarrow$ decay of $\hat{f}$

($(i\xi)^\alpha \hat{f}(\xi)$ needs to be in $L^1(\mathbb{R}^n)$, so $\hat{f}$ must get small at $\pm\infty$)

This is studied in detail in

② | Payley-Wiener-Schwartz Theorem. |

One then studies certain subspaces of $\mathcal{S}'(\mathbb{R}^n)$, namely those when $\hat{u}$ has good integrability properties: $u \in H^s(\mathbb{R}^n) \Leftrightarrow (1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$

(note: related – somehow! – to regularity of u – since about decay of $\hat{u}$!!

This is

(3) Sobolev spaces

Having studied the spaces of functions (and distr.) that the op.'s should work on, we go on to study the symbols: the generalization of the function p in (III): We will – eventually! – define ~~[illegible]~~ Pseudodiff. op's as

$$(\Delta)\quad [a(x,\partial) f](x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\hat{f}(\xi)\,d\xi$$

for certain a (and, first, for $f \in \mathcal{S}$, then $f \in \mathcal{S}'$).

First:

(4) Def. and approx. of symbols. (the a's in (Δ))

To study (Δ) in big generality, we need to study non-absolutely conv. integrals – i.e. some, when $|a(x,\cdot)\hat{f}| \notin L^1_\xi(\mathbb{R}^n)$. They will converge in some sense as improper integrals

(think of Fresnel's integrals $\int_0^\infty \sin(t^2)\,dt$:
$= \lim_{x \to \infty} \int_0^x \sin(t^2)\,dt$)

- the oscillations of $e^{i\langle x,\xi\rangle}$

($= \cos(\langle x,\xi\rangle) + i \sin\langle x,\xi\rangle$)

gives cancellation, hence convergence - this is in:

(5) Oscillatory integrals.

A main idea of the study of ΨDO's is that (complicated) operations on operators should be replaced
?
by "simple" operations on their symbols:

Obvious, $a(x,D) + b(x,D)$ is an op with symbol $a+b$.

- But how to find symbol of adjoint $a^*(x\partial)$?
  - and of composition of op.'s $a(x,\partial)\circ b(x,\partial)$,

This is

(6) $\boxed{\text{Operations on symbols}}$ (to get a ring of operators)

For symbols, a last important operator is inversion

- this is related to existence & uniqueness of sol.'s to $a(x,\partial)u = f$ (more next time)

This is in

(7) $\boxed{\text{Ellipticity.}}$

Finally, we are ready to study / define the operators:

8 | ΨDO's : Action on $\mathcal{S}$ and $\mathcal{S}'$ |

9 | Action in Sobolev spaces |

(i.e. define the op.'s & show continuous - i.e. "bounded op's").

Finally, a last relation between ~~sa~~ op's & their symbols is:

If a symbol is positive (a positive fct) do we get a positive operator? ($A \geq 0 \Leftrightarrow \langle x, Ax \rangle \geq 0$)

Not quite - but almost - this is studied in

10 | Gårding's inequality. |

In the book there is more stuff - but we have no more time! - If strong wish, I can possibly, in the last week of semester, give an overview of some of the applications discussed in Chap. 4.

Until next time: read some (!) of the overviews, to get an idea "what all good for".