REAL ANALYSIS

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M. Griesemer

Throughout these notes m denotes Lebesgue measure.

1. Abstract Integration

σ-Algebras. A σ-algebra in X is a non-empty collection of subsets of X which is closed under taking complements and countable unions. The pair $(X, \mathcal{M})$ is called a measurable space and the subsets of X which belong to $\mathcal{M}$ are called measurable sets.

Theorem. Given a collection $\mathfrak{F}$ of subsets of X there exists a smallest σ-algebra which contains $\mathfrak{F}$. (The σ-algebra generated by $\mathfrak{F}$.)

Borel sets. The σ-algebra generated by the topology of a topological space is called the *Borel σ-algebra*. Its elements are called *Borel sets*. Two types of Borel sets have special names. Sets of type F_σ are countable unions of closed sets and sets of type G_δ are countable intersections of open sets.

Measurable functions. Let X be a measurable space and Y a topological space. Then $f : X \to Y$ is said to *measurable* if one of the following equivalent definitions is satisfied.

1. preimages of Borel sets are measurable,
2. preimages of open sets are measurable,
3. $f^{-1}((\alpha, \infty])$ is measurable for all $\alpha \in \mathbb{R}$ ($Y = \bar{\mathbb{R}}$).

If X is in addition a topological space and the measurable sets are the Borel sets then a measurable function is called a Borel function. In this case every continuous function is measurable.

Theorem. Let $f, g : X \to \mathbb{C}$ where X is a measurable space.

1. f is measurable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.
2. If f and g are measurable then so are $f+g$, $f \cdot g$, and $|f|$.

Theorem. If $f, g, f_n : X \to \bar{\mathbb{R}}$ are measurable then so are $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf_n f_n$, $\lim f_n$ (if it exists), $\max(f, g)$, $\min(f, g)$ and $f_\pm$. If $f, g : X \to [0, \infty]$ are measurable then so are $f + g$ and $f \cdot g$.

A simple function $s : X \to \mathbb{C}$ is a function of the form $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where the sets A_i are disjoint and $n < \infty$. For every function $f : X \to [0, \infty]$ there exists a sequence s_n of simple functions s.t. $0 \le s_n \le s_{n+1} \le f$ and $s_n(x) \to f(x)$ for all x and s.t. s_n is measurable if f is.

Measures. A (positive) measure on a measurable space $(X, \mathcal{M})$ is a map $\mu : \mathcal{M} \to [0, \infty]$, $\mu \not\equiv +\infty$, which is countably additive:

$$\mu(\bigcup A_n) = \sum \mu(A_n)$$

on disjoint sets A_n. The triple $(X, \mathcal{M}, \mu)$ is called a measure space. A measure space is *complete* if all subsets of sets of measure zero are measurable. The measure μ is *σ-finite* if X is a countable union of sets of finite measure.

Theorem. Every measure space $(X, \mathcal{M}, \mu)$ has a smallest complete extension $(X, \mathcal{M}^*, \mu^*)$. A set $E \subset X$ belongs to $\mathcal{M}^*$ if and only if there exist $A, B \in \mathcal{M}$, s.t. $A \subset E \subset B$ and $\mu(B \backslash A) = 0$, then $\mu^*(E) := \mu(A)$.

Positive measures have the following properties

1. If $A_n \nearrow A$ then $\mu(A_n) \to \mu(A)$
2. If $A_n \searrow A$ and $\mu(A_1) < \infty$ then $\mu(A_n) \to \mu(A)$
3. $\mu(\bigcup A_n) \le \sum \mu(A_n)$

Theorem (Egorov). Suppose f_n is a sequence of complex measurable functions on a measure space (X, μ) with $\mu(X) < \infty$. If $f_n \to f$ a.e. and f is finite then for each $\varepsilon > 0$ there exists a $E \subset X$ with $\mu(X \backslash E) < \varepsilon$ such that $f_n \to f$ *uniformly* on E.

Regularity. A measure μ whose σ-algebra contains all Borel sets is *inner regular* if

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{compact}\}$$

and *outer regular* if

$$\mu(E) = \inf\{\mu(V) : V \supset E,\ V \text{open}\}.$$

If μ is inner and outer regular then μ is said to be regular. Lebesgue measure on $\mathbb{R}^k$ is regular.

Convergence in measure. Let μ be a positive measure on X. A sequence (f_n) of complex measurable functions on X is said to *converge in measure* to a measurable function f ($f_n \xrightarrow{\mu} f$) if for each $\varepsilon > 0$

$$\lim_{n \to \infty} \mu\{x : |f_n(x) - f(x)| > \varepsilon\} = 0.$$

Theorem.

(i) If $f_n(x) \to f(x)$ a.e. (μ) and $\mu(X) < \infty$ then $f_n \xrightarrow{\mu} f$.
(ii) If $f_n \xrightarrow{\mu} f$ then there exists a subsequence f_{n_j} such that $f_{n_j}(x) \to f(x)$ a.e. (μ).

Integration. Let $s = \sum_{i=1}^n \alpha_i \chi_{A_i} \geq 0$ be a measurable simple function and let $f \geq 0$ be measurable. Then

$$\int_E s\, d\mu := \sum_i^n \alpha_i \mu(A_i \cap E)$$
$$\int_E f\, d\mu := \sup_{0 \leq s \leq f} \int_E s\, d\mu$$

where $E \in \mathcal{M}$. The *sup* is taken over measurable simple functions. One has $\int_E f\, d\mu = \int \chi_E f\, d\mu$.

Theorems. Let $f_n : X \to \bar{\mathbb{R}}$ be a sequence of measurable functions with $f_n \geq 0$.

(1) (*Monotone convergence*) If $f_n \nearrow f$ then $\int f_n\, d\mu \to \int f\, d\mu$.
(2) $\int \sum f_n\, d\mu = \sum \int f_n\, d\mu$.
(3) (*Fatou*) $\int \liminf f_n\, d\mu \leq \liminf \int f_n\, d\mu$.

Integration of complex functions. A measurable function $f : X \to \mathbb{C}$ is said to be (Lebesgue-) integrable if $\int |f|\, d\mu < \infty$. The set of these functions is denoted $L^1(X)$. Suppose f is measurable and real-valued. Then $|f| = f_+ + f_-$ and hence $f \in L^1(X)$ if and only if $f_\pm \in L(X)$. If at least one of the functions $f_\pm$ is integrable then the integral

$$\int f\, d\mu := \int f_+\, d\mu - \int f_-\, d\mu$$

exists. If $f \in L(X)$ is complex valued one defines

$$\int f\, d\mu := \int \mathrm{Re}(f)\, d\mu + i \int \mathrm{Im}(f)\, d\mu.$$

A property $P(x)$ is said to hold almost everywhere w.r.t. μ (a.e. (μ)) if it hold everywhere except for a set of measure zero. For instance, if $f \geq 0$ is measurable then $\int f\, d\mu = 0$ if and only if $f = 0$ almost everywhere. This example shows that integration is not sensitive for sets of measure zero, which allows one to strengthen the above theorems.

Theorems. Suppose $f_n,\ n = 1, 2, \ldots$ are measurable functions on a measure space X and $\phi \in L(X)$.

(1) (*Monotone convergence*) If $f_n \nearrow f$ a.e. and $f_n \geq \phi$ for all n then $\int f_n\, d\mu \to \int f\, d\mu$.
(2) (*Fatou*) If $f_n \geq \phi$ for all n then $\int \liminf f_n\, d\mu \leq \liminf \int f_n\, d\mu$.
(3) (*Dominated convergence*) If $f_n \to f$ a.e. and if $|f_n| \leq \phi$ for all n then $f \in L(X)$ and $\int f_n\, d\mu \to \int f\, d\mu$.
(4) If $\sum \int |f_n|\, d\mu < \infty$ then $\sum f_n$ converges a.e., is integrable and $\int \sum f_n\, d\mu = \sum \int f_n\, d\mu$.

2. Lebesgue Measure and Outer Measure

Intervals. An interval in $\mathbb{R}^n$ is a set of the form $I = \{x \in \mathbb{R}^n | a_i \leq x_i \leq b_i\}$ where $-\infty < a_i \leq b_i < \infty$. Its volume is by definition $v(I) = \prod_i^n (b_i - a_i)$. Two intervals are said to be non-overlapping if there intersection has no inner points.

Parallelepipeds. $n+1$ vectors $x_0, e_1, \ldots, e_n$ in $\mathbb{R}^n$ span a parallelepiped P with edges parallel to $e_1, \ldots, e_n$ and one corner at x_0. Its volume $v(P)$ is by definition $v(P) = \det E$ where E is the $n \times n$ matrix with components $E_{ik} = (e_k)_i$ (= ith component of e_k w. r. to the standard basis). This generalizes the above definitions for intervals.

Basic facts about volume. We take the following basic facts about volume for granted without proof. If P is a parallelepiped and $I_k,\ k = 1, \ldots, N < \infty$ are intervals then

(1) $P \subset \bigcup I_k$ implies $v(P) \leq \sum v(I_k)$.
(2) If $\bigcup I_k \subset P$ and the intervals are non-overlapping then $\sum v(I_k) \leq v(P)$.

Lebesgue outer measure. Given an arbitrary subset $E \subset \mathbb{R}^n$ its (Lebesgue-) outer measure is defined as $m^*(E) = \inf \sum v(I_k)$ where the infimum is take over all *countable* collections of intervals I_k such that $E \subset \bigcup I_k$.

Theorem. m^* is an outer measure on $\mathbb{R}^n$, i.e.,

(1) $m^*(\emptyset) = 0$.
(2) If $A \subset B$ then $m^*(A) \leq m^*(B)$.
(3) $m^*(\bigcup_{n \geq 1} E_n) \leq \sum_{n \geq 1} m^*(E_n)$.

For a *finite* collection $\{I_n\}_{n=1}^N$ of non-overlapping intervals it follows from the Heine-Borel theorem and basic facts about volume that

$$m^*(\bigcup I_n) = \sum v(I_n).$$

It is also straight forward to prove that given $E \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists an open set $G \supset E$ such that

$$m^*(G) \leq m^*(E) + \varepsilon. \tag{1}$$

Lebesgue measurable. A set $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable if given $\varepsilon > 0$ there exists an open set $G \supset E$ such that $m^*(G \backslash E) \leq \varepsilon$.

Theorem. The collection of all Lebesgue measurable subsets of $\mathbb{R}^n$ is a σ-algebra (containing al Borel sets) and m^* restricted to this σ-algebra is a (positive) measure, called Lebesgue measure.

We use m to denote Lebesgue measure.

From (1) it follows that every set of outer measure zero is Lebesgue measurable. I.e. Lebesgue measure is complete. In fact, Lebesgue measure on $\mathbb{R}^n$ is the completion of Lebesgue measure restricted to the Borel-sets of $\mathbb{R}^n$.

Characterizations of measurability. The following three statements are equivalent.

(i) E is Lebesgue measurable.
(ii) There exists a set H of type G_δ with $H \supset E$ and $m(H\backslash E) = 0$.
(iii) There exists a set H of type F_σ with $H \subset E$ and $m(E\backslash H) = 0$.

Translation invariance. It is easy to show that Lebesgue measure is translation invariant. This is a property which characterizes Lebesgue measure up to an over all constant. More precisely, if μ is a positive Borel measure on $\mathbb{R}^k$ which is translation invariant and finite on compact sets then there exists a constant $c \geq 0$ such that $\mu = cm$.

Theorem (Vitali). There exists a subset of $\mathbb{R}$ which is not Lebesgue measurable.

The proof is based on the axiom of choice, the translation invariance of Lebesgue measure and the fact that $E \cap (E+d) \neq \emptyset$ for $|d|$ small enough if $E \subset \mathbb{R}$ is measurable and has positive measure.

Lipschitz transformations. If $E \subset \mathbb{R}^k$ is Lebesgue measurable and $T : \mathbb{R}^k \to \mathbb{R}^k$ is continuous then $T(E)$ does not need to be measurable (the Cantor-Lebesgue function provides a counter example). However if T is locally Lipschitz then $T(E)$ is Lebesgue measurable. If T is even linear then

$$m(TE) = |\det(T)| m(E).$$

It follows that Lebesgue measure is invariant under arbitrary motions in $\mathbb{R}^k$. The following theorem thus implies the existence of non-measurable sets in $\mathbb{R}^k$, $k \geq 3$.

Theorem (Banach & Tarski 1924). Suppose $k \geq 3$ and $A, B \subset \mathbb{R}^k$ are bounded sets with non-empty interior. Then there exist disjoint subsets $C_1, \ldots, C_m$ of A and disjoint subsets $D_1, \ldots, D_m$ of B, m being finite, such that $A = \bigcup_{i=1}^m C_i$, $B = \bigcup_{i=1}^m D_i$ and C_i is congruent to D_i for all $i = 1, \ldots, m$.

3. Riemann versus Lebesgue Integral

Theorem. Suppose $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a bounded, Riemann integrable function and let $\int_a^b f(x)\,dx$ denote its Riemann integral. Then $f \in L^1([a,b])$ and

$$\int_{[a,b]} f\,dm = \int_a^b f(x)\,dx.$$

4. Integration on Product Spaces

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Then $\mathcal{S} \times \mathcal{T}$ denotes the smallest σ-algebra in $X \times Y$ witch contains all measurable rectangles. In this way $X \times Y$ becomes a measurable space. It is important that $\mathcal{S} \times \mathcal{T}$ is also the smallest monotone class which contains all finite unions of disjoint measurable rectangles (elementary sets).

If $E \subset X \times Y$ let

$$E_x = \{y \in Y : (x,y) \in E\}$$
$$E^y = \{x \in X : (x,y) \in E\}.$$

Theorems.

(i) All sections E_x and E^y of a measurable set $E \subset X \times Y$ are measurable.
(ii) If $f : X \times Y \to \mathbb{C}$ is measurable then the maps $x \mapsto f^y(x) = f(x,y)$ and $y \mapsto f_x(y) = f(x,y)$ are measurable for all $x \in X$ and all $y \in Y$.

Theorem. If (X, μ) and (Y, λ) are σ-finite measure spaces and $Q \subset X \times Y$ is measurable then the mappings $x \mapsto \lambda(Q_x)$ and $y \mapsto \mu(Q^y)$ are measurable and

$$\int \lambda(Q_x) d\mu = \int \mu(Q^y) d\lambda =: (\mu \times \lambda)(Q). \tag{2}$$

Theorem. Let (X, μ) and (Y, λ) be σ-finite measure spaces.

(i) (*Tonelli*) If $f : X \times Y \to [0, \infty]$ is measurable then the maps $x \mapsto \int f_x\,d\lambda$ and $y \mapsto \int f^y\,d\mu$ are measurable and

$$\int d\lambda \int d\mu\, f^y = \int d\mu \int d\lambda\, f_x = \int d(\mu \times \lambda) f. \tag{3}$$

In particular if $f : X \times Y \to \mathbb{C}$ is measurable then $f \in L^1(X \times Y)$ if and only if one of these three integrals for $|f|$ is finite.

(ii) (*Fubini*) If $f \in L^1(X \times Y)$ then $f^y \in L^1(X)$ for almost every y, $f_x \in L^1(Y)$ for almost every x, $x \mapsto \int f_x\,d\lambda$ and $y \mapsto \int f^y\,d\mu$ are integrable and (3) holds.

Let $\mathcal{B}_k$ and $\mathcal{M}_k$ denote the σ-algebras of the Borel and the Lebesgue measurable sets in $\mathbb{R}^k$ respectively. Then $\mathcal{B}_r \times \mathcal{B}_s = \mathcal{B}_{r+s}$. However $\mathcal{M}_r \times \mathcal{M}_s \subset \mathcal{M}_{r+s}$ and equality does not hold.

Theorem. If m_k denotes the Lebesgue measure in $\mathbb{R}^k$, then m_{r+s} is the completion of $m_r \times m_s$.

In this context it is also useful to know that given a Lebesgue measurable function $f : \mathbb{R}^k \to \mathbb{C}$ there exists a Borel function $g : \mathbb{R}^k \to \mathbb{C}$ such that $f = g$ a.e.

The Distribution Function. Let (X, μ) be a σ-finite measure space and suppose $f : X \to [0, \infty]$ is measurable. Then the set $E = \{(x,t) \in X \times [0, \infty) : f(x) > t\}$ is measurable in $X \times [0, \infty)$, where $[0, \infty)$ is equipped with the Borel σ-algebra. Hence by (2)

$$\int f\,d\mu = (\mu \times m)(E) = \int_0^\infty \mu\{x : f(x) > t\} dt.$$

Theorem. Let (X, μ) and f be as above and suppose $\varphi : [0, \infty] \to [0, \infty]$ is non-decreasing, AC on compact intervals, $\varphi(0) = 0$ and $\varphi(t) \to \varphi(\infty)$ as $t \to \infty$. Then

$$\int \varphi \circ f \, d\mu = \int_0^\infty \mu\{x : f(x) > t\} \varphi'(t) \, dt.$$

A nice application of this theorem is the proof of the next theorem.

Theorem (Hardy-Littlewood). If $f \in L^p(\mathbb{R}^k)$ and $1 < p < \infty$ then the Hardy-Littlewood maximal function Mf of f is also in $L^p(\mathbb{R}^k)$.

5. L^p Spaces

Let (X, μ) be a measure space. For each $p > 0$ we denote by $L^p(X, \mu)$ the set of all measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$ where

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p} \quad (p < \infty)$$

$$\|f\|_\infty = \operatorname{esssup}|f|$$

Suppose $f, g : X \to \mathbb{C}$ are measurable, $1 \le p \le \infty$ and $p^{-1} + q^{-1} = 1$. Then we have Hölder's inequality

$$\|fg\|_1 \le \|f\|_p \|f\|_q$$

and Minkowski's inequality

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

In particular $fg \in L^1$ if $f \in L^p$ and $g \in L^q$, and L^p is a complex linear space. Once one identifies functions in L^p which differ only on a sets of measure zero $(L^p, \|\cdot\|_p)$ for $p \ge 1$ becomes a normed linear space.

Theorem (Fischer-Riesz). If $1 \le p \le \infty$ then $L^p(X, \mu)$ is a complete normed space. Moreover, if $f_n \to f$ in L^p then there exists a subsequence which converges pointwise almost everywhere.

Lemma. The set of complex-valued, measurable, simple functions on X with support of finite measure is dense in $L^p(X, \mu)$ if $1 \le p < \infty$.

This theorem together with Urysohn's lemma implies following theorem.

Theorem. Let μ be a measure on a locally compact Hausdorff space X. If Borel sets are measurable, μ is regular and compact sets have finite measure, then the set $C_c(X; \mathbb{C})$ of compactly supported continuous functions on X is dense in $L^p(X, \mu)$ if $1 \le p < \infty$.

In particular $C_c(\mathbb{R}^n; \mathbb{C})$ is dense in $L^p(\mathbb{R}^n, m)$ for $1 \le p < \infty$.

Convolutions. The convolution $f * g$ of two measurable functions f and g on $\mathbb{R}^k$ is defined by

$$(f * g)(x) = \int f(x - y) g(y) \, dm$$

provided that $y \mapsto f(x - y)g(y)$ is integrable. In this case $f * g = g * f$. If f and g are integrable then so is $f * g$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

Theorem (Young). Suppose $f \in L^p(\mathbb{R}^k)$, $g \in L^q(\mathbb{R}^k)$, $1 \le p, q \le \infty$ and $1/p + 1/q \ge 1$. Define $r \ge 1$ by $1/p + 1/q = 1 + 1/r$. Then $f * g \in L^r$ and

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

In particular if $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^k)$ and $g \in L^1(\mathbb{R}^k)$ then $f * g \in L^p$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1.$$

6. Convexity

If $\varphi : (a, b) \subset \bar{\mathbb{R}} \to \mathbb{R}$ is convex then φ is continuous and for each $x \in (a, b)$ there exists a constant $c \in \mathbb{R}$ such that

$$\varphi(y) \ge \varphi(x) + c(y - x)$$

for all $y \in (a, b)$. This elementary fact immediately implies Jensen's inequality.

Theorem (Jensen). If (Ω, μ) is a measure space with $\mu(\Omega) = 1$, $f \in L^1(\Omega, \mu)$ and φ is convex on $(a, b) \supset$ range f, then

$$\varphi\left(\int_\Omega f \, d\mu \right) \le \int_\Omega \varphi \circ f \, d\mu.$$

7. Complex Measures

A complex measure is a countably additive set function $\mu : \mathcal{M} \to \mathbb{C}$ defined on the measurable sets of a measurable space $(X, \mathcal{M})$. A complex measure which is actually real-valued is called a *signed measure*. The set of complex measures on a given fixed σ-algebra is a complex linear space.

Total variation. The total variation (measure) $|\mu|$ of a given complex measure μ is defined by

$$|\mu| = \sup \sum_n |\mu(E_n)|$$

where the supremum is taken over all countable collections of disjoint measurable sets $(E_n)_{n \ge 0}$.

Theorem. The total variation measure of a complex measure is a finite, positive measure.

If μ is a signed measure it follows that the positive and negative variations

$$\mu_\pm = \frac{1}{2}(|\mu| \pm \mu)$$

of μ are finite positive measures. They give rise to the *Jordan decomposition* $\mu = \mu_+ - \mu_-$ of a signed measure. Correspondingly there exists a partition of the measurable space on which μ is defined.

Theorem (Hahn-decomposition). If μ is a signed measure on X then there exist disjoint measurable sets X_+ and X_- such that $X = X_+ \cup X_-$ and

$$\mu_+ = \mu(E \cap X_+), \qquad \mu_- = -\mu(E \cap X_-).$$

Absolute Continuity. Suppose λ is an arbitrary measure defined on the same measurable space as a given positive measure μ. Then λ is *absolutely continuous* w.r. to μ ($\lambda \ll \mu$) if $\mu(E) = 0$ implies that $\lambda(E) = 0$.

If λ is a *complex* measure then this is equivalent to the condition that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies $|\lambda(E)| < \varepsilon$.

Two arbitrary measures λ_1 and λ_2 defined on the same measurable space are *mutually singular* ($\lambda_1 \perp \lambda_2$) if there exists a measurable set A such that $|\lambda_1|(A^c) = 0$ and $|\lambda_2|(A) = 0$.

Theorem (Lebesgue-Radon-Nikodym). Suppose μ is a σ-finite positive measure on a measurable space $(X, \mathcal{M})$ and λ is a complex measure on the same space. Then there exists a unique decomposition

$$\lambda = \lambda_a + \lambda_s \qquad \lambda_a \ll \mu, \; \lambda_s \perp \mu,$$

and furthermore there is a unique $h \in L^1(X, \mu)$ such that

$$\lambda_a(E) = \int_E h \, d\mu. \tag{4}$$

Remark. Note that, since $\lambda_s \perp \mu$, there exists a set Z of μ-measure zero such that $\lambda_a(E) = \lambda(E \backslash Z)$, and $\lambda_s(E) = \lambda(E \cap Z)$.

Corollary. If both λ and μ are σ-finite measures on the same measurable space, then there exists a unique pair of σ-finite measures λ_a and λ_s such that $\lambda = \lambda_a + \lambda_s$, $\lambda_a \ll \mu$ and $\lambda_s \perp \mu$. Furthermore there is a non-negative measurable function h such that (4) holds.

As a consequence from the Radon-Nikodym theorem every complex measure μ can be represented as

$$\mu(E) = \int_E h \, d|\mu|$$

where h is a measurable function with $|h| \equiv 1$. This immediately implies that Hahn-decomposition for signed measures.

8. Differentiation

Derivatives of measures. Suppose μ is a Borel measure on $\mathbb{R}^k$, that is a measure defined in the sigma-algebra of Borel sets, and $B_r(x) = B(x, r)$ denotes the open ball in $\mathbb{R}^k$ with center at x and radius r. Then

$$D\mu(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{m(B_r(x))},$$

if it exists, is called the symmetric derivative of μ at x.

An important tool in the study of the $D\mu(x)$ is the Hardy-Littlewood maximal function Mf of a measurable function $f$on $\mathbb{R}^k$. By definition

$$Mf(x) = \sup_{r > 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f| \, dm.$$

Its distribution function satisfies the bound

$$m\{x : Mf(x) > \lambda\} \le 3^k \lambda^{-1} \|f\|_1.$$

Lebesgue points. Suppose $f \in L^1(\mathbb{R}^k)$, $x \in \mathbb{R}^k$ and

$$\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B(x,r)} |f - f(x)| dm = 0.$$

Then x is called a Lebesgue point of f. It is easy to see that if $d\mu = f dm$, $f \in L^1$ and x is a Lebesgue point of f, then $D\mu(x) = f(x)$. However much more is true.

Theorem. If $f \in L^1(\mathbb{R}^k)$ then almost every $x \in \mathbb{R}^k$ is a Lebesgue point of f.

Theorem. If μ is a complex Borel measure on $\mathbb{R}^k$ and $d\mu = f dm + d\mu_s$ is its Lebesgue decomposition w.r. to m, then $D\mu(x) = f(x)$ a.e. (m). In particular $\mu \perp m$ if and only if $D\mu = 0$ a.e.(m).

This theorem still hold if the balls in the definition of $D\mu(x)$ are replaced by Borel sets $\{E_n\}$ that *shrink to x nicely* in the sense that there exists an $\alpha > 0$ and a sequence of balls $B(x, r_n)$ such that

$$\begin{aligned} &(i) \quad r_n \to 0 \\ &(ii) \quad B(x, r_n) \supset E_n \\ &(iii) \quad m(E_n) \ge \alpha m(B(x, r_n)). \end{aligned}$$

Here α, E_n and r_n will depend on the x where $D\mu(x)$ is evaluated.

The Fundamental Theorem of Calculus. The function $f : [a, b] \to \mathbb{C}$ is said to be absolutely continuous (AC) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon$$

for every finite collection of disjoint intervals $\{(\alpha_i, \beta_i)\}_{i=1..n}$ in $[a, b]$ with $\sum(\beta_i - \alpha_i) < \delta$.

Theorem. If $f : [a, b] \to \mathbb{C}$ is absolutely continuous then f is differentiable a.e., $f' \in L^1$ and

$$f(x) - f(a) = \int_a^x f' \, dm \qquad x \in [a, b]. \tag{5}$$

Conversely, the antiderivative of every integrable function is absolutely continuous.

Functions of bounded variation. A function $f : [a, b] \to \mathbb{C}$ is said to be of bounded variation if

$$V(f) = \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty$$

where the *sup* is taken over all partitions of $[a,b]$. If f is AC the $V(f) \le \int_{[a,b]} |f'|\, dm$ by the fundamental theorem of calculus and hence f is of bounded variation. On the other hand if f is of bounded variation then f is differentiable a.e. and $f' \in L^1$. However (5) will not hold unless f is AC. The cantor Lebesgue function is a nice example.

Differentiable Transformations. A mapping $T : V \subset \mathbb{R}^m \to \mathbb{R}^n$, V open, is said to be differentiable at x if there exists a linear transformation $A : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{h\to 0} \frac{|T(x+h) - T(x) - Ah|}{|h|} = 0.$$

Then $T'(x) := A$ is called the derivative of f at x.

Theorem. If $T : V \subset \mathbb{R}^k \to \mathbb{R}^k$, V is open and T is continuous on V and differentiable at x then

$$\lim_{r\to 0} \frac{m(T(B_r(x)))}{m(B_r(x))} = |\det T'(x)|.$$

Theorem. Suppose $T : V \subset \mathbb{R}^k \to \mathbb{R}^k$, V is open and T is continuous on V and one-to-one and differentiable on a measurable subset $X \subset V$ for which $m(T(V\backslash X)) = 0$. Then for every measurable function $f : \mathbb{R}^k \to [0,\infty]$

$$\int_{T(X)} f\, dm = \int_X (f \circ T)|\det T'|\, dm.$$

9. Hilbert Space Theory

Terminology. Inner product space, unitary space, Hilbert space, linear subspace, orthogonality, continuous linear functional, isomorphism.

The inner product (x,y) of two vectors x and y is linear in the first and antilinear in the second argument. $\|x\| = (x,x)^{1/2}$ defines a norm.

Theorem. If M is a closed linear subspace of a Hilbert space H then for each $x \in H$ there is a unique decomposition

$$x = x_1 + x_2, \qquad x_1 \in M,\ x_2 \perp M$$

and x_1 is characterized by

$$\|x - x_1\| = \min_{y\in M} \|x - y\|.$$

Theorem (Riesz). To every continuous linear functional f on a Hilbert space H there exists a unique vector x^* such that

$$f(x) = (x, x^*), \qquad \text{all } x \in H.$$

Orthonormal systems. A countable family of vectors $(u_n)_{n\ge1}$ in a inner product space E is called an orthonormal system (ONS) if $(u_n, u_k) = \delta_{nk}$. It is *maximal* if $(x, u_n)_{n\ge0}$ implies $x = 0$. An ONS $(u_n)_{n\ge1}$ is called an orthonormal basis (ONB) if $x = \sum (x, u_n)u_n$ for all $x \in E$.

Theorem. Suppose $(u_n)_{n\ge0}$ is an ONS in an inner product space E. Then for each $x \in E$ we have:

(i) (Bessel's inequality)

$$\sum_{n\ge0} |(x,u_n)|^2 \le \|x\|^2$$

and equality holds if and only if $\sum (x,u_n)u_n = x$.

(ii) If E is complete then

$$\sum_{n\ge0} (x,u_n)u_n = x_1$$

where x_1 is the orthogonal projection of x onto $\overline{\operatorname{span}\{u_n\}}$.

Parseval's Identity. If $(u_n)_{n\ge1}$ is an ONB in a Hilbert space H then for all $x, y \in H$

$$(x,y) = \sum_{n\ge1} (x,u_n)\overline{(y,u_n)}.$$

Theorems.

(i) An ONS in a Hilbert space is a ONB if and only if it is maximal.
(ii) Every separable inner product space has a countable ONB.
(iii) Every Hilbert space with a countably infinite ONB is isomorphic to l^2.

Remark. $L^2(\mathbb{R}^n)$ is separable.

10. Appendix

Let $(A_\alpha)_{\alpha\in I}$, B be subsets of a set X, where I is an arbitrary index set.

Distributive laws.

$$\begin{aligned} B \cap (\cup_{\alpha\in I} A_\alpha) &= \cup_{\alpha\in I}(B \cap A_\alpha) \\ B \cup (\cap_{\alpha\in I} A_\alpha) &= \cap_{\alpha\in I}(B \cup A_\alpha) \end{aligned}$$

De Morgan's Laws.

$$\begin{aligned} (\cup_{\alpha\in I} A_\alpha)^c &= (\cap_{\alpha\in I} A_\alpha^c) \\ (\cap_{\alpha\in I} A_\alpha)^c &= (\cup_{\alpha\in I} A_\alpha^c) \end{aligned}$$

For any map $f : X \to Y$ and sets $A_\alpha \subset X$ and $B, B_\alpha \subset Y$ where $\alpha \in I$

$$\begin{aligned} f^{-1}(B^c) &= f^{-1}(B)^c \\ f^{-1}(\cup_\alpha B_\alpha) &= \cup_\alpha f^{-1}(B_\alpha) \\ f^{-1}(\cap_\alpha B_\alpha) &= \cap_\alpha f^{-1}(B_\alpha) \\ f(\cup_\alpha A_\alpha) &= \cup_\alpha f(A_\alpha) \\ f(\cap_\alpha A_\alpha) &\subset \cap_\alpha f(A_\alpha). \end{aligned}$$

11. References

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