Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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Extra exercises: For your own preparation for the make-up/Wiederholungsklausur Oct 20 2012.
Solutions: *NO* written solutions will be provided.
However, there will be two Exercise classes, Oct 15th and 16th, 18:15 - 20:00, in B 138 and B 005, respectively (**to be confirmed**).

> **Disclaimer:** *Neither the themes nor the length nor the difficulty of these exercises give any hint on the exercises on the make-up exam/Wiederholungsklausur Oct 20 2012.*

Exercise 1. Let $p \in (1,\infty]$, $\alpha := 1 - \frac{1}{p}$, $I = [a,b] \subset \mathbf{R}$. Prove: $\exists C > 0 : \forall f \in C^1(I), \forall x_0 \in I$:

$$\|f\|_{C^{0,\alpha}(I)} \leq |f(x_0)| + C\|f'\|_{L^p(I)}\,.$$

Exercise 2. Let X and Y be Banach spaces and let $B : X \times Y \to \mathbb{K}$ be bilinear. Assume that B is *partially* continuous in both entries, i.e. $\forall\, x \in X$: $B(x,\cdot) \in Y'$ and $\forall\, y \in Y$: $B(\cdot,y) \in X'$. Prove that $B : X \times Y \to \mathbb{K}$ is continuous, i.e. B is *jointly* continuous in both entries.

Remark. If $B : X \times Y \to \mathbb{C}$ is sesquilinear and partially continuous in both entries, then it also follows that B is continuous on $X \times Y$.

Exercise 3. Let X be a Banach space, Y a normed space, and $T \in B(X,Y)$. Assume that T is not surjective, but that $T(X) \subseteq Y$ is dense.

(i) Prove that $T(X)$ is of first category in Y.

(ii) Prove that $T(X)$ is not no-where dense in Y.

Exercise 4. Let $(X, \|\cdot\|)$ be a normed real or complex vector space and let $\varphi : X \to \mathbf{K}$ be some bounded linear functional, i.e.

$$\|\varphi\| = \sup\left\{|\varphi(x)| \,\middle|\, x \in X,\ \|x\| \leq \right\} < \infty\,. \tag{1}$$

(i) Explain why $V := \ker(\varphi)$ is a closed subspace of X.
(ii) Prove that $\|\varphi\|\operatorname{dist}(x,V) = |\varphi(x)|$, for every $x \in X$.
(iii) Prove that, if $0 < \|\varphi\| < \infty$, then the following conditions are equivalent:

(a) The supremum in (1) is not attained.

(b) There is no $x \in X$ such that $\|x\| = 1$ and $\operatorname{dist}(x,V) = 1$.

(c) For all $x \in X \setminus V$ and $v \in V$ we have $\|x - v\| > \operatorname{dist}(x,V)$.

Exercise 5. Let $c_0 = \{x = (x_j)_{j\in\mathbf{N}} \in \ell_\infty \,|\, x_j \to 0,\ j \to \infty\}$ be the space of null sequences which is a Banach space with norm $\|\cdot\|_\infty$. Prove that the following formula defines a continuous linear functional on c_0,

$$\varphi(x) := \sum_{j=1}^{\infty} \frac{x_j}{2^j}\,, \qquad x = (x_j)_{j\in\mathbf{N}} \in c_0\,.$$

Compute its norm $\|\varphi\|$ and show that it fulfills one of the equivalent conditions (a), (b), or (c) of Exercise 4(iii).

Exercise 6.

(i) Let $\mathcal{H}$ be a Hilbert space, and $(x_n)_{n\in\mathbf{N}} \subseteq \mathcal{H}$ an orthonormal system. Prove that $x_n \to 0$ weakly (i.e., $x_n \rightharpoonup 0$).

(ii) Let $A \subseteq [0, 2\pi]$ be a Lebesgue measurable set. Prove that

$$\lim_{n\to\infty} \int_A \sin(nt)\,dt = \lim_{n\to\infty} \int_A \cos(nt)\,dt = 0.$$

Exercise 7. Let X, Y, Z be Banach spaces, $T : X \to Y$ linear, $J : Y \to Z$ linear, bounded, and injective, and assume that $JT := J \circ T : X \to Z$ is bounded. Prove that T is also bounded.

Exercise 8. Assume that V and W are closed subspaces of the Hilbert space $(\mathcal{H}, \langle .,.\rangle)$. Given any closed subspace $U \subseteq \mathcal{H}$ let P_U denote the orthogonal projection onto U. Prove the following statements:

(i) $\forall\, x, y \in \mathcal{H}\colon\ \langle x, P_V\, y\rangle = \langle P_V\, x, y\rangle$.

(ii) $V \subseteq W \Rightarrow P_V = P_V\, P_W = P_W\, P_V$.

(iii) $V \perp W \Rightarrow P_{V\oplus W} = P_V + P_W$.

(iv) Assume $V \subseteq W$. Then $R := W \ominus V := W \cap V^\perp$ is closed and $P_R = P_W - P_V$.

Exercise 9. Let $\Omega \subseteq \mathbf{R}^n$ be measurable, $\mathbf{K} = \mathbf{R}$. Assume $k \in L^2(\Omega \times \Omega)$ satisfies: $\exists c_0 > 0$:

$$k(x, y) \geq c_0 > 0 \ \text{ for a.e. } x, y \in \Omega\,.$$

Prove that for all $f \in L^2(\Omega)$ there exists a unique solution $u \in L^2(\Omega)$ to the equation

$$\int_\Omega k(x, y)u(y)\,dy = f(x) \ \text{ for a.e. } x \in \Omega\,.$$

Exercise 10. Let $(f_n)_{n\in\mathbf{N}} \subseteq L^3(\mathbf{R})$ satisfy

$$\Big|\int_{\mathbf{R}} g(x) f_n(x)\,dx\Big| \leq \big(1 + \|g\|_{L^{3/2}(\mathbf{R})}\big)^{2012} \quad \forall n \in \mathbf{N}, \forall g \in L^{3/2}(\mathbf{R}).$$

Prove: There exists $C > 0$:

$$\Big|\int_{\mathbf{R}} g(x) f_n(x)\,dx\Big| \leq C\|g\|_{L^{3/2}(\mathbf{R})} \quad \forall n \in \mathbf{N}, \forall g \in L^{3/2}(\mathbf{R}).$$

Hint: Consider the maps $g \mapsto \int_{\mathbf{R}} g(x) f_n(x)\,dx$.

Exercise 11. Consider the real (!) Hilbert space ℓ_2, and let

$$A = \Big\{ \sum_{i=1}^{n} \alpha_i e_i \,\Big|\, n \in \mathbf{N}, \alpha_i \in \mathbf{R}, i = 1, \ldots, n, \alpha_n > 0 \Big\}$$

where $(e_n)_{n\in\mathbf{N}}$ is the standard basis from ℓ_2. Let $B = -A$. Prove that A and B are convex disjoint sets, and that for any $x^* \in (\ell_2)^* \setminus \{0\}$ we have $x^*(A) = x^*(B) = \mathbf{R}$. What can we deduce?

Exercise 12. Let X be some real or complex vector space and $\|\cdot\|$ a norm on X. Prove that the following statements are equivalent:

(i) X is *strictly normed*, that is, for all $x, y \in X$ we have

$$\|x + y\| = \|x\| + \|y\| \quad \Rightarrow \quad x \text{ and } y \text{ are linearly dependent.}$$

(ii) The closed unit ball is strictly convex, that is, for all $x, y \in X$ we have

$$\big(\, \|x\| = \|y\| = 1 \ \wedge \ x \neq y \,\big) \quad \Rightarrow \quad \big\|(1/2)(x+y)\big\| < 1 \,.$$

(iii) Every closed convex set $K \subseteq X$ has at most one element minimizing the norm on K, that is,

$$\big(\, x, y \in K \ \wedge \ \|x\| = \|y\| = \inf_{v\in K} \|v\| \,\big) \quad \Rightarrow \quad x = y\,.$$

Exercise 13. Let X be a complete metric space, and $f_n : X \to \mathbf{K}, n \in \mathbf{N}$, be a sequence of continuous functions for which there is a function $f : X \to \mathbf{K}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$, i,e., $f_n \to f$ pointwise on X.

(i) Prove that there exist $V \subseteq X$, open and non-empty, and $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in V$ and all $n \in \mathbf{N}$.

(ii) Prove that for any $\varepsilon > 0$ there exist $V \subseteq X$, open and non-empty, and $p \in \mathbf{N}$ such that $|f(x) - f_n(x)| \leq \varepsilon$ for all $x \in V$ and all $n \geq p$.

Exercise 14. Let $f \in C^1(\mathbf{R})$ be 2π-periodic and real valued. Assume $\int_0^{2\pi} f(t)\,dt = 0$. Prove that

$$\int_0^{2\pi} |f(t)|^2\,dt < \int_0^{2\pi} |f'(t)|^2\,dt$$

except if $f(t) = a\cos t + b\sin t$ for some $a, b \in \mathbf{R}$.

Exercise 15. Let X be a Banach space and let $\{T_n\}_{n\in\mathbf{N}}$ and $\{S_n\}_{n\in\mathbf{N}}$ be sequences in $B(X)$ such that $\forall x \in X$: $T_n\,x \to T\,x \wedge S_n\,x \to S\,x$, $n \to \infty$, where $T, S \in B(X)$. Prove that $\forall x \in X$: $T_n\,S_n\,x \to T\,S\,x$, $n \to \infty$.

Exercise 16. Prove that the following maps T, defined on $X := C([0,1], \mathbf{R})$, are linear and bounded (with the supremum norm on X). Then compute their norm.

(i) $T : X \to X$, $Tf(x) := f(x)g(x)$ for some (given, fixed) $g \in C[0,1]$.

(ii) For $p \in \mathbf{N}$, $0 = t_0 < t_1 < \cdots < t_p = 1$, $\alpha_0, \ldots, \alpha_p \in \mathbf{R} \setminus \{0\}$ (all fixed), let $T : X \to \mathbf{R}$, $Tf := \sum_{i=0}^{p} \alpha_i f(t_i)$.

(iii) $T : X \to \mathbf{R}$, $Tf := \int_0^1 f(t)\, dt - \sum_{i=0}^{p} \alpha_i f(t_i)$, with the t_i's and α_i's as in (ii).

Exercise 17. (i) Let $V \subseteq X$ be a closed subspace of the normed vector space X and let $x \in X \setminus V$. Prove that there exists $f \in X'$ such that $f{\restriction_V} = 0$ and $f(x) = \operatorname{dist}(x, V)$.
(ii) Let $V \subseteq X$ be some subspace of the normed vector space X and assume that $\forall\, f \in X'$: $f{\restriction_V} = 0 \Rightarrow f = 0$. Prove that V is dense in X.

Exercise 18. We consider the following two subspaces of ℓ_1,

$$U := \left\{ (t_n)_{n\in\mathbf{N}} \in \ell_1 \,\middle|\, \forall\, n \in \mathbf{N} :\, t_{2n} = 0 \right\},$$
$$V := \left\{ (s_n)_{n\in\mathbf{N}} \in \ell_1 \,\middle|\, \forall\, n \in \mathbf{N} :\, s_{2n-1} = n\, s_{2n} \right\}.$$

Prove that U and V are closed but that their direct sum $U \oplus V$ is not closed.

Exercise 19. Consider the space $C^1[0,1]$, endowed with the uniform norm from the space $C[0,1]$, and consider the differentiation operator $D : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ defined by $Df = f'$.

(i) Prove that D is linear but not continuous.

(ii) Prove that D has closed graph. What can we deduce?

Exercise 20. Let X be a normed vector space and assume that X' is separable. Show that X is also separable.

Hint: If $\{f_n | n \in \mathbf{N}\}$ is dense in X', pick $x_n \in X$ such that $|f_n(x_n)| \geq \|f_n\|/2$.

Exercise 21. Prove that the set $P = \{(x_n)_{n\in\mathbf{N}} \in \ell_2 \,|\, |x_n| < 1 \forall n \in \mathbf{N}\}$ is an open subset of ℓ_2.

Exercise 22. We consider the space $C([0,1], \mathbf{R})$ of real-valued continuous functions on $[0,1]$ with metric $d_\infty(f, g) := \|f - g\|_\infty := \sup_{x\in[0,1]} |f(x) - g(x)|$.
(i) Define the maps $\delta_0, I, J_p : C([0,1], \mathbf{R}) \to \mathbf{R}$, where $p \geq 1$, by

$$\delta_0(f) := f(0)\,, \quad I(f) := \int_0^1 f(x)\, dx\,, \quad J_p(f) := \int_0^1 |f(x)|^p\, dx\,,$$

for all $f \in C([0,1], \mathbf{R})$. Prove that δ_0, I, and J_1 are Lipschitz continuous and that J_p, $p > 1$, is continuous but not uniformly continuous.
(ii) Let K denote the set of all $f \in C([0,1], \mathbf{R})$ such that $\delta_0(f) = I(f) = 0$ and $\forall\, x, y \in [0,1]$: $|f(x) - f(y)| \leq \sqrt{|x - y|}$. Prove that the restriction of J_p, $p \geq 1$, to K attains its maximum on K.

Exercise 23. Consider $C[0,1]$ with the supremum norm. Let $E \subseteq C[0,1]$ be a closed linear subspace satisfying: $f \in E \Rightarrow f \in C^1[0,1]$. Prove that E is finite dimensional.

Hint: Prove that $D : E \to C[0,1], Df := f'$, has closed graph.

Exercise 24. Let (M,d) be a complete metric space, $a_k \geq 0$, $k \in \mathbf{N}$, with $\sum_{k=1}^{\infty} a_k < \infty$, and $T : M \to M$ a map satisfying

$$d\big(T^k(x), T^k(y)\big) \leq a_k\, d(x,y)\,,$$

for all $x, y \in M$ and $k \in \mathbf{N}$, where $T^1 := T$, $T^{k+1} := T \circ T^k$, $k \in \mathbf{N}$. Prove that T has a unique fixpoint, i.e. there is exactly one $z \in M$ such that $T(z) = z$.

Exercise 25. Let $\mathcal{E} := \{e_i | \, i \in \mathbf{N}\}$ be an orthonormal system in the (separable) Hilbert space $\mathcal{H}$ and denote the Fourier coefficients of $u \in \mathcal{H}$ by $\hat{u}(i) := \langle e_i, u\rangle$, $i \in \mathbf{N}$. Prove that $\mathcal{E}$ is an orthonormal basis if and only if

$$\forall\, x, y \in \mathcal{H} : \quad \langle x, y\rangle = \sum_{i \in \mathbf{N}} \overline{\hat{x}(i)}\, \hat{y}(i)\,. \tag{2}$$

Exercise 26. Let $\mathcal{H}$ be a Hilbert space, and assume the linear map $T : \mathcal{H} \to \mathcal{H}$ satisfies

$$\langle Tx, y\rangle = \langle x, Ty\rangle \;\text{ for all } x, y \in \mathcal{H}\,.$$

Prove that T is bounded.

Exercise 27. Let $p \in (1, \infty)$ and

$$M = \Big\{x = (x_n)_{n \in \mathbf{N}} \in \ell_p \,\Big|\, \sum_{n=1}^{\infty} x_n = 0\Big\} \subseteq \ell_p\,.$$

Prove that $G \subseteq \ell_p$ is a dense linear subspace.
Hint: Use Exercise 17(ii).

Exercise 28. Let $\mathcal{H} = L^2[\frac{1}{2}, 2]$ and $S : \mathcal{H} \to \mathcal{H}$ defined by $(Sf)(x) = x^{-1} f(x^{-1})$. Prove that S is an isometry.

Exercise 29. Let E, F be two Banach spaces with norms $\|\cdot\|_E, \|\cdot\|_F$. Let $T \in B(E,F)$ be such that $R(T)$ is closed and $\dim N(T) < \infty$. Let $|\cdot|$ denote another norm on E that is weaker than $\|\cdot\|_E$, i.e., $|x| \leq M\|x\|_E$ for all $x \in E$. Prove that there exists a constant C such that

$$\|x\|_E \leq C(\|Tx\|_F + |x|) \;\text{ for all} x \in E.$$

Hint: Argue by contradiction.