Chapter 7
String Theory as a Conformal Field Theory

We give an exposition of the classical system of a bosonic string and its quantization.

In bosonic string theory as a classical field theory we have the flat semi-Riemannian manifold

$$(\mathbb{R}^D, \eta) \text{ with } \eta = \mathrm{diag}(-1, 1, \dots, 1)$$

as background space and a *world sheet* in this space, that is a C^∞-parameterization

$$x : Q \to \mathbb{R}^D$$

of a surface $W = x(Q) \subset \mathbb{R}^D$, where $Q \subset \mathbb{R}^2$ is an open or closed rectangle. This corresponds to the idea of a one-dimensional object, the *string*, which moves in the space $\mathbb{R}^D$ and wipes out the two-dimensional surface $W = x(Q)$. The classical fields (that is the kinematic variables of the theory) are the components $x^\mu : Q \to \mathbb{R}$ of the parameterization $x = (x^0, x^1, \dots, x^{D-1}) : Q \to \mathbb{R}^D$ of the surface $W = x(Q) \subset \mathbb{R}^D$.

7.1 Classical Action Functionals and Equations of Motion for Strings

In classical string theory the admissible parameterizations, that is the dynamic variables of the world sheet, are those for which a given action functional is stationary. A natural action of the classical field theory uses the "area" of the world sheet. One defines the so-called *Nambu–Goto action*:

$$S_{NG}(x) := -\kappa \int_Q \sqrt{-\det g}\, dq^0 dq^1,$$

with a constant $\kappa \in \mathbb{R}$ (the "string tension", cf. [GSW87]). Here,

$$g := x^*\eta, (x^*\eta)_{\mu\nu} = \eta_{ij}\partial_\mu x^i \partial_\nu x^j,$$

is the metric on Q induced by $x : Q \to \mathbb{R}^D$ and the variation is taken only over those parameterizations x, for which g is a *Lorentz metric* (at least in the interior of Q), that is

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$$\det(g_{\mu\nu}) < 0.$$

Hence, (Q,g) is a two-dimensional *Lorentz manifold*, that is a two-dimensional semi-Riemannian manifold with a Lorentz metric g.

From the action principle

$$\frac{d}{d\varepsilon} S_{NG}(x+\varepsilon y)|_{\varepsilon=0} = 0$$

with suitable boundary conditions, one derives the equations of motion. Since it is quite difficult to make calculations with respect to the action S_{NG}, one also uses a different action, which leads to the same equations of motion. The *Polyakov action*

$$S_P(x,h) := -\frac{\kappa}{2}\int_Q \sqrt{-\det\, h}\, h^{ij}\, g_{ij}\, dq^0\, dq^1$$

depends, in addition, on a (Lorentz) metric h on Q. A separate variation of S_P with respect to h only leads to the former action S_{NG}:

Lemma 7.1.

$$\frac{d}{d\varepsilon} S_P(x,h+\varepsilon f)|_{\varepsilon=0} = 0$$

holds precisely for those Lorentzian metrics h on Q which satisfy $g = \lambda h$, where $\lambda : Q \to \mathbb{R}_+$ is a smooth function. Substitution of $h = \frac{1}{\lambda} g$ into S_P yields the original action S_{NG}.

Proof. In order to show the first statement let $(\widetilde{h}^{ij})$ be the matrix satisfying

$$2\det h = \widetilde{h}^{ij} h_{ij}, \quad h^{ij} = (\det h)^{-1}\widetilde{h}^{ij}.$$

Then $\widetilde{h}^{00} = h_{11}$, $\widetilde{h}^{11} = h_{00}$, and $\widetilde{h}^{01} = -h_{10}$. Hence,

$$\sqrt{-\det(h+\varepsilon f)}(h+\varepsilon f)^{ij} = -(\sqrt{-\det(h+\varepsilon f)})^{-1}\widetilde{(h+\varepsilon f)}^{ij}$$

for symmetric $f = (f_{ij})$ with $\det(h+\varepsilon f) < 0$, and it follows

$$S_P(x,h+\varepsilon f) = \frac{\kappa}{2}\int_Q (\sqrt{-\det(h+\varepsilon f)})^{-1}(\widetilde{h}^{ij} + \varepsilon \widetilde{f}^{ij}) g_{ij} dq^0 dq^1.$$

Since $h^{ij} = -(-\det h)^{-1}\widetilde{h}^{ij}$ and $\widetilde{h}^{\alpha\beta} f_{\alpha\beta} = \widetilde{f}^{\alpha\beta} h_{\alpha\beta}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon} S_P(x,h+\varepsilon f)\Big|_{\varepsilon=0} \\
&= \frac{\kappa}{2}\int_Q \left(\frac{\widetilde{f}^{ij}}{\sqrt{-\det h}} + \frac{\widetilde{h}^{ij}\widetilde{f}^{\alpha\beta} h_{\alpha\beta}}{2\sqrt{-\det h}^3} \right) g_{ij} dq^0 dq^1 \\
&= \frac{\kappa}{2}\int_Q \frac{\widetilde{f}^{ij}}{\sqrt{-\det h}} \left(g_{ij} - \frac{1}{2} h^{\alpha\beta} g_{\alpha\beta} h_{ij} \right) dq^0 dq^1.
\end{aligned}$$

This implies that $\delta S_P(x,h) = 0$ for fixed x leads to the "equation of motion"

$$g_{ij} - \frac{1}{2} h^{\alpha\beta} g_{\alpha\beta} h_{ij} = 0 \tag{7.1}$$

for h. Equivalently, the *energy–momentum tensor*

$$T_{ij} := g_{ij} - \frac{1}{2} h^{\alpha\beta} g_{\alpha\beta} h_{ij} \tag{7.2}$$

has to vanish. The solution h of (7.1) is $g = \lambda h$ with

$$\lambda = \frac{1}{2} h^{\alpha\beta} g_{\alpha\beta} > 0$$

($\lambda > 0$ follows from $\det g < 0$ and $\det h < 0$).

Substitution of the solution $h = \frac{1}{\lambda} g$ of the equation $T = 0$ in the action $S_P(x,h)$ yields the original action $S_{NG}(x)$. □

Invariance of the Action. It is easy to show that the action S_P is invariant with respect to

- Poincaré transformations,
- Reparameterizations of the world sheet, and
- Weyl rescalings: $h \mapsto h' := \Omega^2 h$.

S_{NG} is invariant with respect to Poincaré transformations and reparameterizations only.

Because of the invariance with respect to reparameterizations, the action S_P can be simplified by a suitable choice of parameterization. To achieve this, we need the following theorem:

Theorem 7.2. *Every two-dimensional Lorentz manifold (M,g) is* conformally flat*, that is there are local parameterizations ψ, such that for the induced metric g one has*

$$\psi^* g = \Omega^2 \eta = \Omega^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.3}$$

with a smooth function Ω. Coordinates for which the metric tensor is of this form are called isothermal coordinates.

For a positive definite metric g (on a surface) the existence of isothermal coordinates can be derived from the solution of the Beltrami equation (cf. [DFN84, p. 110]). In the Lorentzian case the existence of isothermal coordinates is much easier to prove. Since the issue of existence of isothermal coordinates has been neglected in the respective literature and since it seems to have no relation to the Beltrami equation, a proof shall be provided in the sequel. A proof can also be found in [Dic89].

Proof. [1] Let $x \in M$ and let $\psi : \mathbb{R}^2 \supset U \to M$ be a chart for M with $x \in \psi(U)$. We denote the matrix representing $\psi^* g$ by $g_{\mu\nu} \in C^\infty(U,\mathbb{R})$. If we choose a suitable linear map $A \in \mathrm{GL}(\mathbb{R}^2)$ and replace ψ with $\psi \circ A : A^{-1}(U) \to M$, we can assume that

[1] By A. Jochens

$$(g_{\mu\nu}(\xi)) = \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\xi := \psi^{-1}(x)$. We also have

$$\det(g_{\mu\nu}) = g_{11}g_{22} - g_{12}^2 < 0$$

since g is a Lorentz metric. We define

$$a := \sqrt{g_{12}^2 - g_{11}g_{22}} \in C^\infty(U,\mathbb{R}).$$

By our choice of the chart ψ we have $g_{22}(\xi) = 1$. The continuity of g_{22} implies that there is an open neighborhood $V \subset U$ of ξ with $g_{22}(\xi') > 0$ for $\xi' \in V$.

Now, there are two positive integrating factors $\lambda, \mu \in C^\infty(V', \mathbb{R}^+)$ and two functions $F, G \in C^\infty(V', \mathbb{R})$ on an open neighborhood $V' \subset V$ of ξ, so that

$$\partial_1 F = \lambda\sqrt{g_{22}}, \qquad \partial_2 F = \lambda \frac{g_{12} + a}{\sqrt{g_{22}}},$$

$$\partial_1 G = \mu\sqrt{g_{22}}, \qquad \partial_2 G = \mu \frac{g_{12} - a}{\sqrt{g_{22}}}.$$

The existence of F and λ can be shown as follows: we apply to the function $f \in C^\infty(V, \mathbb{R})$ defined by

$$f(t,x) := (g_{12}(x,t) + a(x,t))/g_{22}(x,t)$$

a theorem of the theory of ordinary differential equations, which guarantees the existence of a family of solutions depending differentiably on the initial conditions (cf. [Die69, 10.8.1 and 10.8.2]). By this theorem, we get an open interval $J \subset \mathbb{R}$ and open subsets $U_0, U \subset \mathbb{R}$ with $\xi \in U_0 \times J \subset U \times J \subset V$, as well as a map $\phi \in C^\infty(J \times J \times U_0, U)$, so that for all $t, s \in J$ and $x \in U_0$ we have

$$\frac{d}{dt}\phi(t,s,x) = f(t, \phi(t,s,x)) \qquad \text{and} \qquad \phi(t,t,x) = x. \tag{7.4}$$

Using the uniqueness theorem for ordinary differential equations, it can be shown that $\partial_3 \phi$ is positive and that

$$\phi(\tau,t,x) \in U_0 \Rightarrow \phi(s,\tau,\phi(\tau,t,x)) = \phi(s,t,x)$$

for $t, s, \tau \in J$ and $x \in U_0$. Defining

$$F(x,t) := \phi(t_0,t,x) \qquad \text{and} \qquad \lambda(x,t) := \frac{\partial_1 F(x,t)}{\sqrt{g_{22}(x,t)}}$$

for $(x,t) \in U_0 \times J$ and a fixed $t_0 \in J$ we obtain functions $F, \lambda \in C^\infty(U_0 \times J, \mathbb{R})$ with the required properties. By the same argument we also obtain the functions G and μ. The open subset $V' \subset V$ is the intersection of the domains of F and G.

For the map $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} := \begin{pmatrix} F - G \\ F + G \end{pmatrix} \in C^\infty(V', \mathbb{R}^2)$ we have

$$\partial_1 \varphi^1 = (\lambda - \mu)\sqrt{g_{22}}, \qquad \partial_2 \varphi^1 = \lambda \frac{g_{12} + a}{\sqrt{g_{22}}} - \mu \frac{g_{12} - a}{\sqrt{g_{22}}},$$
$$\partial_1 \varphi^2 = (\lambda + \mu)\sqrt{g_{22}}, \qquad \partial_2 \varphi^2 = \lambda \frac{g_{12} + a}{\sqrt{g_{22}}} + \mu \frac{g_{12} - a}{\sqrt{g_{22}}}.$$

After a short calculation we get

$$\partial_\mu \varphi^\rho \partial_\nu \varphi^\sigma \eta_{\rho\sigma} = \partial_\mu \varphi^1 \partial_\nu \varphi^1 - \partial_\mu \varphi^2 \partial_\nu \varphi^2 = 4\lambda \mu g_{\mu\nu},$$

that is $\varphi^* \eta = 4\lambda \mu \psi^* g$. Furthermore,

$$\det D\varphi = \partial_1 \varphi^1 \partial_2 \varphi^2 - \partial_1 \varphi^2 \partial_2 \varphi^1 = -4\lambda \mu a \neq 0.$$

Hence, by the inverse mapping theorem there exists an open neighborhood $W \subset V'$ of ξ, so that $\widetilde{\varphi} := \varphi|_W : W \to \varphi(W)$ is a C^∞ diffeomorphism. $\varphi^* \eta = 4\lambda \mu \psi^* g$ implies

$$\eta = \left(\widetilde{\varphi}^{-1}\right)^* \varphi^* \eta = 4\lambda \mu \left(\widetilde{\varphi}^{-1}\right)^* \psi^* g = 4\lambda \mu \left(\psi \circ \widetilde{\varphi}^{-1}\right)^* g.$$

Now $\widetilde{\psi} := \psi \circ \widetilde{\varphi}^{-1} : \varphi(W) \to M$ is a chart for M with $x \in \widetilde{\psi}(\varphi(W))$ and we have

$$\widetilde{\psi}^* g = \Omega^2 \eta$$

with $\Omega := 1/(2\sqrt{\lambda\mu})$. □

By Theorem 7.2 one can choose a local parameterization of the world sheet in such a way that

$$h = \Omega^2 \eta = \Omega^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This fixing of h is called *conformal gauge*. Even after conformal gauge fixing a residual symmetry remains: it is easy to see that $S_P(x)$ in conformal gauge is invariant with respect to conformal transformations on the world sheet. In this manner, the conformal group $\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$ turns out to be a symmetry group of the system, even if this holds only on the level of "constraints". In any case, the classical field theory of the bosonic string can be viewed as a conformally invariant field theory.

To simplify the equations of motion and, furthermore, to present solutions as certain Fourier series, we need a generalization of Theorem 7.2, stating that (in the case of closed strings, to which we restrict our discussion here) there exists a conformal gauge not only in a neighborhood of any given point, but also in a neighborhood of a closed injective curve (as a starting curve for the "time $\tau = 0$"). The existence of such isothermal coordinates can be shown by the same argumentation as Theorem 7.2. Finally, for the variation in the conformal gauge, it can be assumed that isothermal coordinates exist on the rectangle

$$Q = [0, 2\pi] \times [0, 2\pi]$$

and that $\sigma \mapsto x(0, \sigma)$, $\sigma \in [0, 2\pi]$ describes a simple closed curve. This is possible at least up to an irrelevant distortion factor (cf. [Dic89]).

Theorem 7.3. *The variation of S_{NG} or S_P in the conformal gauge leads to the equations of motion on $Q = [0, 2\pi] \times [0, 2\pi]$: These are the two-dimensional wave equations*

$$\partial_0^2 x - \partial_1^2 x = 0 \quad \textit{resp.} \quad x_{\tau\tau} - x_{\sigma\sigma} = 0$$

with the constraints

$$\langle x_\sigma, x_\tau \rangle = 0 = \langle x_\sigma, x_\sigma \rangle + \langle x_\tau, x_\tau \rangle, \quad \langle x_\tau, x_\tau \rangle < 0,$$

imposed by the conformal gauge.

By x_σ we denote the partial derivative of $x = x(\tau, \sigma)$ with respect to σ (that is $\tau := q^0, \sigma := q^1$), and $\langle v, w \rangle$ is the inner product $\langle v, w \rangle = v^\mu w^\nu \eta_{\mu\nu}$ for $v, w \in \mathbb{R}^D$.

Proof. To derive the equations of motion and the constraints we start by writing S_P in the conformal gauge $h = \Omega^2 \eta$ with $\sqrt{-\det h} = \Omega^2$ and $h^{ij} g_{ij} = \Omega^2(-g_{00} + g_{11})$:

$$S_P(x) = S_P(x, \Omega^2 \eta) = \frac{\kappa}{2} \int_Q (\langle \partial_0 x, \partial_0 x \rangle - \langle \partial_1 x, \partial_1 x \rangle) dq^0 dq^1.$$

For $y : Q \to \mathbb{R}^D$ and suitable boundary conditions $y|_{\partial Q} = 0$ we have

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} S_P(x + \varepsilon y) \bigg|_{\varepsilon = 0} &= \kappa \int_Q (\langle \partial_0 x, \partial_0 y \rangle - \langle \partial_1 x, \partial_1 y \rangle) dq^0 dq^1 \\ &= \kappa \int_Q \langle \partial_{11} x - \partial_{00} x, y \rangle dq^0 dq^1 \end{aligned}$$

(integration by parts). This yields

$$\partial_{11} x - \partial_{00} x = 0$$

as the equations of motion in the conformal gauge.

Because of the description of the metric h by $h = \frac{1}{\lambda} g$ with $\lambda > 0$, that is

$$\lambda h = \lambda(h_{ij}) = \begin{pmatrix} \langle x_\tau, x_\tau \rangle & \langle x_\sigma, x_\tau \rangle \\ \langle x_\tau, x_\sigma \rangle & \langle x_\sigma, x_\sigma \rangle \end{pmatrix},$$

the gauge fixing $h = \Omega^2 \eta$ implies the conditions

$$\langle x_\sigma, x_\tau \rangle = 0, \quad \langle x_\sigma, x_\sigma \rangle = -\langle x_\tau, x_\tau \rangle > 0. \qquad \square$$

The constraints are equivalent to the vanishing of the energy–momentum T, which is given by

$$T_{ij} = \langle x_i, x_j \rangle - \frac{1}{2} h_{ij} h^{kl} \langle x_k, x_l \rangle, \quad i,j,k,l \in \{\tau, \sigma\}$$

(see (7.2) and cf. [GSW87, p. 62ff]).

The solutions of the two-dimensional wave equations are

$$x(\tau,\sigma) = x_R(\tau - \sigma) + x_L(\tau + \sigma)$$

with two arbitrary differentiable maps x_R and x_L on Q with values in $\mathbb{R}^D$. For the closed string we get on $Q := [0,2\pi] \times [0,2\pi]$ (that is $x(\tau,\sigma) = x(\tau, \sigma + 2\pi)$) the following Fourier series expansion:

$$\begin{aligned} x_R^\mu(\tau - \sigma) &= \frac{1}{2} x_0^\mu + \frac{1}{4\pi\kappa} p_0^\mu (\tau - \sigma) + \frac{i}{\sqrt{4\pi\kappa}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau - \sigma)}, \\ x_L^\mu(\tau + \sigma) &= \frac{1}{2} x_0^\mu + \frac{1}{4\pi\kappa} p_0^\mu (\tau + \sigma) + \frac{i}{\sqrt{4\pi\kappa}} \sum_{n \neq 0} \frac{1}{n} \overline{\alpha}_n^\mu e^{-in(\tau + \sigma)}. \end{aligned} \tag{7.5}$$

x_0 and p_0 can be interpreted as the center of mass and the center of momentum, respectively, while α_n^μ, $\overline{\alpha}_n^\nu$ are the oscillator modes of the string. x_L and x_R are viewed as "left movers" and "right movers". We have $x_0^\mu, p_0^\mu \in \mathbb{R}$ and $\alpha_n^\mu, \overline{\alpha}_m^\nu \in \mathbb{C}$. $\overline{\alpha}_m^\nu$ is not the complex conjugate of α_m^ν, but completely independent of α_m^ν. For x_R and x_L to be real, it is necessary that

$$(\alpha_n^\mu)^* = \left(\alpha_{-n}^\mu\right) \quad \text{and} \quad (\overline{\alpha}_n^\mu)^* = \left(\overline{\alpha}_{-n}^\mu\right) \tag{7.6}$$

hold for all $\mu \in \{0, \ldots, D-1\}$ and $n \in \mathbb{Z} \setminus \{0\}$, where $c \mapsto c^*$ denotes the complex conjugation. We let $\alpha_0^\mu := \overline{\alpha}_0^\mu := \frac{1}{\sqrt{4\pi\kappa}} p_0^\mu$. The $x = x_L + x_R$ with (7.5) can be written as

$$x(\sigma,\tau) = x_0 + \frac{2}{\sqrt{4\pi\kappa}} \alpha_0 \tau + \frac{i}{\sqrt{4\pi\kappa}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n e^{-in(\tau - \sigma)} + \overline{\alpha}_n e^{-in(\tau + \sigma)} \right).$$

Hence, arbitrary $\alpha_n, \overline{\alpha}_n, x_0, p_0$ with (7.6) yield solutions of the one-dimensional wave equation. In order that these solutions are, in fact, solutions of the equations of motion for the actions S_{NG} or S_P, they must, in addition, respect the conformal gauge. Using

$$L_n := \frac{1}{2} \sum_{k \in \mathbb{Z}} \langle \alpha_k, \alpha_{n-k} \rangle \quad \text{and} \quad \overline{L}_n := \frac{1}{2} \sum_{k \in \mathbb{Z}} \langle \overline{\alpha}_k, \overline{\alpha}_{n-k} \rangle \quad \text{for } n \in \mathbb{Z}, \tag{7.7}$$

the gauge condition can be expressed as follows:

Lemma 7.4. *A parameterization $x(\tau,\sigma) = x_L(\tau-\sigma) + x_R(\tau+\sigma)$ of the world sheet with x_R, x_L as in (7.5) and (7.6) gives isothermal coordinates if and only if $L_n = \overline{L}_n = 0$ for all $n \in \mathbb{Z}$.*

Proof. We have isothermal coordinates if and only if

$$\langle x_\tau + x_\sigma, x_\tau + x_\sigma\rangle = \langle x_\tau - x_\sigma, x_\tau - x_\sigma\rangle = 0.$$

Using the identities

$$x_\tau - x_\sigma = \frac{2}{\sqrt{4\pi\kappa}} \sum_{n\in\mathbb{Z}} \alpha_n e^{-in(\tau-\sigma)} \quad \text{and}$$
$$x_\tau + x_\sigma = \frac{2}{\sqrt{4\pi\kappa}} \sum_{n\in\mathbb{Z}} \overline{\alpha}_n e^{-in(\tau+\sigma)},$$

we get

$$\begin{aligned}
&\langle x_\tau - x_\sigma, x_\tau - x_\sigma\rangle = 0\\
\Longleftrightarrow\ & 0 = \left\langle \sum_{n\in\mathbb{Z}} \alpha_n e^{-in(\tau-\sigma)}, \sum_{n\in\mathbb{Z}} \alpha_n e^{-in(\tau-\sigma)} \right\rangle\\
\Longleftrightarrow\ & 0 = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} e^{-i(n+k)(\tau-\sigma)} \langle \alpha_n, \alpha_k\rangle\\
\Longleftrightarrow\ & 0 = \sum_{m\in\mathbb{Z}}\sum_{n+k=m} e^{-im(\tau-\sigma)} \langle \alpha_n, \alpha_k\rangle\\
\Longleftrightarrow\ & \forall m \in \mathbb{Z} : \sum_{n+k=m} \langle \alpha_n, \alpha_k\rangle = 0\\
\Longleftrightarrow\ & \forall m \in \mathbb{Z} : \sum_{k\in\mathbb{Z}} \langle \alpha_{m-k}, \alpha_k\rangle = 0\\
\Longleftrightarrow\ & \forall m \in \mathbb{Z} : L_m = 0.
\end{aligned}$$

The same argument holds for $x_\tau + x_\sigma$ and $\overline{L}_m$. □

Altogether, we have the following:

Theorem 7.5. *The solutions of the string equations of motion are the functions*

$$x(\tau,\sigma) = x_0 + \frac{2}{\sqrt{4\pi\kappa}}\alpha_0\tau + \frac{i}{\sqrt{4\pi\kappa}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n e^{-in(\tau-\sigma)} + \overline{\alpha}_n e^{-in(\tau+\sigma)}\right),$$

for which the conditions (7.6) and $L_n = \overline{L}_n = 0$ hold.

For a connection of the energy–momentum tensor T of a conformal field theory with the Virasoro generators L_n and $\overline{L}_n$ we refer to (9.3) and to Sect. 10.5 in the context of conformal vertex operators.

The oscillator modes α_n^μ and $\overline{\alpha}_m^\nu$ are observables of the classical system. Obviously, they are constants of motion. Hence, one should try to quantize the $\alpha_n^\mu, \overline{\alpha}_m^\nu$.

In order to quantize the classical field theory of the bosonic string one needs the Poisson brackets of the classical system:

$$\{\alpha_m^\mu, \alpha_n^\nu\} = im\eta^{\mu\nu}\delta_{m+n} = \{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\}, \tag{7.8}$$

$$\{\alpha_m^\mu, \overline{\alpha}_n^\nu\} = 0, \tag{7.9}$$

$$\{p_0^\mu, x_0^\nu\} = \eta^{\mu\nu}, \tag{7.10}$$

$$\{x_0^\mu, x_0^\nu\} = \{x_0^\mu, \alpha_m^\nu\} = \{x_0^\mu, \overline{\alpha}_m^\nu\} = 0, \tag{7.11}$$

for all $\mu, \nu \in \{0, \dots, D-1\}$ and $m, n \in \mathbb{Z}$ (here and in the following we set $4\pi\kappa = 1$).

Observe that for each single index ν the collection of the observables $\alpha_n^\nu, n \in \mathbb{Z}$, define a Lie algebra with respect to the Poisson bracket which is isomorphic to the Heisenberg algebra.

Lemma 7.6. *For $n, m \in \mathbb{Z}$ one has*

$$\{L_m, L_n\} = i(n-m)L_{m+n}, \quad \{\overline{L}_m, \overline{L}_n\} = i(n-m)\overline{L}_{m+n},$$
$$\textit{and} \quad \{\overline{L}_m, L_n\} = 0.$$

This follows from the general formula

$$\{AB, C\} = A\{B, C\} + \{A, C\}B$$

for the Poisson bracket.

7.2 Canonical Quantization

In general, quantization of a classical system shall provide quantum models reflecting the basic properties of the original classical system. A common quantization procedure is *canonical quantization*. In canonical quantization a complex Hilbert space $\mathbb{H}$ has to be constructed in order to represent the quantum mechanical states as one-dimensional subspaces of $\mathbb{H}$ and to represent the observables as self-adjoint operators in $\mathbb{H}$. (The notion of a self-adjoint operator is briefly recalled on p. 130.) Thereby the relevant classical observables $f, g, \dots$ have to be replaced with operators $\widehat{f}, \widehat{g}$ such that the Poisson bracket is preserved in the sense that it is replaced with the commutator of operators in $\mathbb{H}$

$$\{\cdot,\cdot\} \longmapsto -i[\cdot,\cdot].$$

Hence, for the relevant $f, g, \dots$ the following relations should be satisfied on a common domain of definitions of the operators

$$[\widehat{f}, \widehat{g}] = -i\widehat{\{f, g\}}.$$

In addition, some natural identities have to be satisfied. For example, in the situation of the classical phase space $\mathbb{R}^{2n}$ with its Poisson structure on the space of observables $f : \mathbb{R}^{2n} \to \mathbb{C}$ induced by the natural symplectic structure on $\mathbb{R}^{2n}$ it is natural to require the *Dirac conditions*:

1. $\widehat{1} = \mathrm{id}_{\mathbb{H}}$,
2. $[\widehat{q^\mu}, \widehat{p_\nu}] = i\delta^\mu_\nu, [\widehat{q^\mu}, \widehat{q^\nu}] = [\widehat{p_\mu}, \widehat{p_\nu}] = 0$,

with respect to the standard canonical coordinates (q^μ, p_ν) of $\mathbb{R}^{2n}$.

In general, one cannot quantize all classical observables (due to a result of van Hove) and one chooses a suitable subset $\mathscr{A}$ which can be assumed to be a Lie algebra with respect to the Poisson bracket. The canonical quantization of this subalgebra $\mathscr{A}$ of the Poisson algebra of all observables means essentially to find a representation of $\mathscr{A}$ in the Hilbert space $\mathbb{H}$.

The Harmonic Oscillator. Let us present as an elementary example a canonical quantization of the one-dimensional harmonic oscillator. The classical phase space is $\mathbb{R}^2$ with coordinates (q, p). The Poisson bracket of two classical observables f, g, that is smooth functions $f, g : \mathbb{R}^2 \to \mathbb{C}$, is

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

The hamiltonian function (that is the energy) of the harmonic oscillator is $h(q, p) = \frac{1}{2}(q^2 + p^2)$. The set of observables one wants to quantize contains at least the four functions $1, p, q, h$. Because of $\{1, f\} = 0, \{q, p\} = 1, \{h, p\} = q$, and $\{h, q\} = -p$ the vector space $\mathscr{A}$ generated by $1, q, p, h$ is a Lie algebra with respect to the Poisson bracket.

As the Hilbert space of states one typically takes the space of square integrable functions $\mathbb{H} := \mathrm{L}^2(\mathbb{R})$ in the variable q. The quantization of 1 is prescribed by the first Dirac condition. As the quantization of q one then chooses the *position operator* $\widehat{q} = Q$ defined by $\varphi(q) \mapsto q\varphi(q)$ with domain of definition $D_Q = \{\varphi \in \mathbb{H} : \int_{\mathbb{R}} |q\varphi(q)|^2 dq < \infty\}$. Q is an unbounded self-adjoint operator. This holds also for the *momentum operator* P which is the quantization of p: $P = \widehat{p}$. P is defined as $P(\varphi) = -i\frac{\partial \varphi(q)}{\partial q}$ for φ in the space D of all smooth functions on $\mathbb{R}$ with compact support and can be continued to D_P such that the continuation is self-adjoint. Observe that D is dense in $\mathbb{H}$. The second Dirac condition is satisfied on D, i.e

$$[Q, P]\varphi = i\varphi, \varphi \in D.$$

Finally, the quantization $\widehat{h}$ of the hamiltonian function h is the hamiltonian operator H, given by

$$H(\varphi) = \frac{1}{2}\left(\frac{\partial^2 \varphi}{\partial q^2}(q) + q^2\varphi(q)\right)$$

on D with domain D_H such that H is self-adjoint. It is easy to verify $[H, Q] = -iP, [H, P] = iQ$ on D from which we deduce $[\widehat{a}, \widehat{b}] = -i\widehat{\{a, b\}}$ for all $a, b \in \mathscr{A}$ on D.

Note that $\rho(a) := i\widehat{a}$ defines a representation of $\mathscr{A}$ in $\mathbb{H}$.

A different realization of a canonical quantization of the harmonic oscillator is the following. The Hilbert space is the space $\mathbb{H} = \ell^2$ of complex sequence $z = (z_\nu)_{\nu\in\mathbb{N}}$ which are square summable $\|z\|^2 = \sum_{\nu=0}^{\infty} |z_\nu|^2 < \infty$. Let $(e_n)_{\nu\in\mathbb{N}}$ be the standard (Schauder) basis of ℓ^2, that is $e_n = (\delta_n^k)$. By

$$
\begin{aligned}
H(e_n) &:= (n+\frac{1}{2})e_n, \\
A^*(e_n) &:= \sqrt{2n+2}e_{n+1}, \\
A(e_0) &:= 0, A(e_{n+1}) := \sqrt{2n+2}e_n,
\end{aligned}
$$

we define operators H, A, A^* on the subspace $D \subset \mathbb{H}$ of finite sequences, that is finite linear combinations of the e_ns. H is an essentially self-adjoint operator and A^* is the adjoint of A as the notation already suggests. (More precisely, A and A^* are the restrictions to D of operators which are adjoint to each other.)

With $Q := \frac{1}{2}(A+A^*)$ and $P := \frac{1}{2}(A-A^*)$ the operators $\mathrm{id}_{\mathbb{H}}, Q, P, H$ satisfy in D the same commutation relations

$$[Q,P] = i\,\mathrm{id}_{\mathbb{H}}, [H,Q] = -iP, [H,P] = iQ$$

as before, and therefore constitute another canonical quantization of $\mathscr{A}$. The two quantizations are equivalent.

Note that D can be identified with the space of complex-valued polynomials $\mathbb{C}[T]$ by $e_n \mapsto T^n$. This opens the possibility to purely algebraic methods in quantum field theory by restricting all operations to the vector space $D = \mathbb{C}[T]$ as, e.g., in the quantization of strings (see below), in the representation of the Virasoro algebra (cf. Sect. 6.5), or in the theory of vertex operators (cf. Chap. 10).

For obvious reasons, A is called the *annihilation operator* and A^* is called the *creation operator*.

Returning to the question of quantizing a string one observes immediately that for any fixed index μ the Poisson brackets of the (α_m^μ) are those of an infinite sequence of one-dimensional harmonic oscillators (up to a constant). The corresponding *oscillator algebra* $\mathscr{A}$ generated by (α_m^μ) (with fixed μ) can therefore be interpreted as the algebra of an infinite dimensional harmonic oscillator. For a fixed index $\mu > 0$ (which we omit for the rest of this section) the relevant Poisson brackets of the oscillator algebra $\mathscr{A}$ are, according to (7.8),

$$\{\alpha_m, \alpha_n\} = im\delta_{n+m}, \{1, \alpha_n\} = 0.$$

After quantization the operators $a_n := \widehat{\alpha_n}$ generate a Lie algebra which is the complex vector space generated by $a_n, n \in \mathbb{Z}$, and Z (sometimes denoted $Z = 1$) with the Lie bracket given by

$$[a_m, a_n] = m\delta_{n+m}Z, \quad [Z, a_m] = 0.$$

We see that this Lie algebra is nothing else than the Heisenberg algebra H (cf. (4.1)).

We conclude that constructing a canonical quantization of the infinite dimensional harmonic oscillator is the same as finding a representation $\rho : \mathsf{H} \to \operatorname{End} D$ of the Heisenberg algebra H in a suitable dense subspace $D \subset \mathbb{H}$ of a Hilbert space $\mathbb{H}$ with $\rho(Z) = \mathrm{id}_{\mathbb{H}}$.

Fock Space Representation. As the appropriate Fock space (that is representation space) we choose the complex vector space

$$\mathsf{S} := \mathbb{C}[T_1, T_2, \ldots] \tag{7.12}$$

of polynomials in an infinite number of variables. We have to find a representation of the Heisenberg algebra in $\operatorname{End}_{\mathbb{C}}\mathsf{S}$. Define

$$\begin{aligned}
\rho(a_n) &:= \frac{\partial}{\partial T_n} \quad \text{for } n > 0, \\
\rho(a_0) &:= \mu \mathrm{id}_{\mathsf{S}} \quad \text{where } \mu \in \mathbb{C}, \\
\rho(a_{-n}) &:= n T_n \quad \text{for } n > 0, \quad \text{and} \\
\rho(Z) &:= \mathrm{id}_{\mathsf{S}}.
\end{aligned}$$

Then the commutation relations obviously hold and the representation is irreducible. Moreover, it is a unitary representation in the following sense:

Lemma 7.7. *For each $\mu \in \mathbb{R}$ there is a unique positive definite hermitian form on S, so that $H(1,1) = 1$ (1 stands for the vacuum vector) and*

$$H(\rho(a_n)f, g) = H(f, \rho(a_{-n})g)$$

for all $f, g \in \mathsf{S}$ and $n \in \mathbb{Z}$, $n \neq 0$.

Proof. First of all one sees that distinct monomials $f, g \in \mathsf{S}$ have to be orthogonal for such a hermitian form H on S. (The *monomials* are the polynomials of the form $T_{n_1}^{k_1} T_{n_2}^{k_2} \ldots T_{n_r}^{k_r}$ with $n_j, k_j \in \mathbb{N}$ for $j = 1, 2, \ldots, r$.) Given two distinct monomials f, g there exist an index $n \in \mathbb{N}$ and exponents $k \neq l$, $k, l \geq 0$, such that $f = T_n^k f_1, g = T_n^l g_1$ for suitable monomials f_1, g_1 which are independent of T_n. Without loss of generality let $k < l$. Then

$$\begin{aligned}
H((\rho(a_n))^{k+1} f, T_n^{l-k-1} g_1) &= H((\frac{\partial}{\partial T_n})^{k+1} T_n^k f_1, T_n^{l-k-1} g_1) \\
&= H(0, T_n^{l-k-1} g_1) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
H((\rho(a_n))^{k+1} f, T_n^{l-k-1} g_1) &= H(f, (\rho(a_{-n})^{k+1} T_n^{l-k-1} g_1)) \\
&= H(f, n^{k+1} T_n^l g_1) \\
&= H(f, g)
\end{aligned}$$

imply $H(f,g) = 0$. Moreover,

$$\begin{aligned} H(f,f) &= H(f, n^{-k}(\rho(a_n))^k f_1) \\ &= n^{-k} H(\rho(a_n)^k T_n^k f_1, f_1) \\ &= \frac{k!}{n^k} H(f_1, f_1). \end{aligned}$$

Using $H(1,1) = 1$, it follows for monomials $f = T_{n_1}^{k_1} T_{n_2}^{k_2} \dots T_{n_r}^{k_r}$ with $n_1 < n_2 < \dots < n_r$

$$H(f,f) = \frac{k_1! k_2! \dots k_r!}{n^{k_1} n^{k_2} \dots n^{k_r}}. \tag{7.13}$$

Since the monomials constitute a (Hamel) basis of S, H is uniquely determined as a positive definite hermitian form by (7.13) and the orthogonality condition. Reversing the arguments, by using (7.13) and the orthogonality condition $H(f,g) = 0$ for distinct monomials $f, g \in \mathsf{S}$ as a definition for H, one obtains a hermitian form H on S with the required properties. □

Note that $\rho(a_n)^* = \rho(a_{-n})$ by the last result and for each $n > 0$ the operator $\rho(a_n)$ is an annihilation operator while $\rho(a_n)^*$ is a creation operator.

7.3 Fock Space Representation of the Virasoro Algebra

In order to obtain a representation of the Virasoro algebra Vir on the basis of the Fock space representation $\rho : \mathsf{H} \to \mathrm{End}(\mathsf{S})$ of the Heisenberg algebra described in the last section it seems to be natural to use the definition of the Virasoro observables L_n in classical string theory, cf. (7.7),

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \alpha_{n-k} = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_{n-k} \alpha_k,$$

which satisfy the Witt relations (up to the constant i, see Lemma 7.6).

In a first naive attempt one could try to define the operators $L_n : \mathsf{S} \to \mathsf{S}$ by $L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k a_{n-k}$ resp. $L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \rho(a_k) \rho(a_{n-k})$. But this procedure is not well-defined on S, since

$$\rho(a_k)\rho(a_{n-k}) \neq \rho(a_{n-k})\rho(a_k),$$

in general.

However, the *normal ordering*

$$:\rho(a_i)\rho(a_j): := \begin{cases} \rho(a_i)\rho(a_j) & \text{for } i \leq j \\ \rho(a_j)\rho(a_i) & \text{for } i > j \end{cases}$$

defines operators

$$\rho(L_n) : \mathsf{S} \to \mathsf{S}, \quad \rho(L_n) := \frac{1}{2} \sum_{k \in \mathbb{Z}} :\rho(a_k)\rho(a_{n-k}):.$$

The $\rho(L_m)$ are well-defined operators, since the application to an arbitrary polynomial $P \in \mathsf{S} = \mathbb{C}[T_1, T_2, \ldots]$ yields only a finite number of nonzero terms. The normal ordering constitutes a difference compared to the classical summation for the case $n = 0$ only. This follows from

$$\rho(a_i)\rho(a_j) = \rho(a_j)\rho(a_i) \quad \text{for} \quad i + j \neq 0,$$
$$:\rho(a_k)\rho(a_{-k}): = \rho(a_{-k})\rho(a_k) \quad \text{for} \quad k \in \mathbb{N}.$$

Consequently, the operators $\rho(L_n)$ can be represented as

$$\begin{aligned}
\rho(L_0) &= \frac{1}{2}\rho(a_0)^2 + \sum_{k \in \mathbb{N}_1} \rho(a_{-k})\rho(a_k), \\
\rho(L_{2m}) &= \frac{1}{2}(\rho(a_m))^2 + \sum_{k \in \mathbb{N}_1} \rho(a_{m-k})\rho(a_{m+k}), \\
\rho(L_{2m+1}) &= \sum_{k \in \mathbb{N}_0} \rho(a_{m-k})\rho(a_{m+k+1}),
\end{aligned}$$

for $m \in \mathbb{N}_0$ (here $\mathbb{N}_k = \{n \in \mathbb{Z} : n \geq k\}$).

We encounter normal ordering as an important tool in a more general context in Chap. 10 on vertex algebras.

Theorem 7.8. *In the Fock space representation we have*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n}{12}(n^2 - 1)\delta_{n+m}\mathrm{id}$$

(with L_n instead of $\rho(L_n)$). Hence, it is a representation of the Virasoro algebra.

Proof. First of all we show

$$[L_n, a_m] = -m a_{m+n}, \tag{7.14}$$

where $m, n \in \mathbb{Z}$, using the commutation relations for the a_ns. (Here and in the following we write L_n instead of $\rho(L_n)$ and a_n instead of $\rho(a_n)$.) Let $n \neq 0$.

$$\begin{aligned}
L_n a_m &= \frac{1}{2}\sum_{k \in \mathbb{Z}} a_{n-k} a_k a_m \\
&= \frac{1}{2}\sum_{k \in \mathbb{Z}} a_{n-k}(a_m a_k + k\delta_{k+m}) \\
&= \frac{1}{2}\sum_{k \in \mathbb{Z}} ((a_m a_{n-k} + (n-k)\delta_{n+m-k})a_k + k\delta_{k+m} a_{n-k}) \\
&= a_m L_n + \frac{1}{2}(-m a_{n+m} - m a_{n+m}) \\
&= a_m L_n - m a_{n+m}.
\end{aligned}$$

The case $n = 0$ is similar. From $[L_n, a_m] = -ma_{n+m}$ one can deduce

$$[[L_n, L_m], a_k] = -k(n-m)a_{n+m+k}. \tag{7.15}$$

In fact,

$$\begin{aligned} L_n L_m a_k &= L_n(a_k L_m - k a_{m+k}) \\ &= a_k L_n L_m - k a_{n+k} L_m - k L_n a_{m+k}. \end{aligned}$$

Hence,

$$\begin{aligned} [L_n, L_m] a_k &= a_k [L_n, L_m] + k[L_m, a_{n+k}] - k[L_n, a_{m+k}] \\ &= a_k [L_n, L_m] - k(n+k) a_{m+n+k} + k(m+k) a_{m+n+k} \\ &= a_k [L_n, L_m] - k(n-m) a_{n+m+k}. \end{aligned}$$

It is now easy to deduce from (7.14) and (7.15) that for every $f \in \mathsf{S}$ with

$$[L_n, L_m] f = (n-m) L_{n+m} f + \frac{n}{12}(n^2-1)\delta_{n+m} f$$

and every $k \in \mathbb{Z}$ we have

$$[L_n, L_m](a_k f) = (n-m) L_{n+m}(a_k f) + \frac{n}{12}(n^2-1)\delta_{n+m}(a_k f).$$

As a consequence, the commutation relation we want to prove has only to be checked on the vacuum vector $\Omega = 1 \in \mathsf{S}$. The interesting case is to calculate $[L_n, L_{-n}]\Omega$. Let $n > 0$. Then $L_n \Omega = 0$. Hence $[L_n, L_{-n}]\Omega = L_n L_{-n}\Omega$. In case of $n = 2m+1$ we obtain

$$\begin{aligned} L_{-n}\Omega &= \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{-n-k} a_k \Omega \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{-n+k} a_{-k} \Omega \\ &= \frac{1}{2}\sum_{k=0}^{n} a_{-n+k} a_{-k} \Omega \\ &= \mu n T_n + \frac{1}{2}\sum_{k=1}^{n-1} k(n-k) T_k T_{n-k} \\ &= \mu n T_n + \sum_{k=1}^{m} k(n-k) T_k T_{n-k} =: P_n. \end{aligned}$$

Now, $a_l a_{n-l} P_n \neq 0$ holds for $l \in \{0, 1, \ldots n\}$ only and we infer $a_l a_{n-l} P_n = l(n-l), 1 \leq l \leq n-1$, and $a_l a_{n-l} P_n = \mu^2 n$ for $l = 0, l = n$. It follows that

$$
\begin{aligned}
[L_n, L_{-n}]\Omega &= \mu^2 n + \sum_{k=1}^{m} k(n-k) \\
&= 2nL_0\Omega + n\sum_{k=1}^{m} k - \sum_{k=1}^{m} k^2 \\
&= 2nL_0\Omega + n\frac{m}{2}(m+1) - \frac{1}{6}m(m+1)(2m+1) \\
&= 2nL_0\Omega + \frac{n}{3}m(m+1) \\
&= 2nL_0\Omega + \frac{n}{12}(n^2-1).
\end{aligned}
$$

The case $n = 2m$ can be treated in the same manner. Similarly, one checks that $[L_n, L_m]\Omega = (n-m)L_{n+m}$ for the relatively simple case $n+m \neq 0$. □

Another proof can be found, for instance, in [KR87, p. 15ff]. Here, we wanted to demonstrate the impact of the commutation relations of the Heisenberg algebra respectively the oscillator algebra $\mathscr{A}$.

Corollary 7.9. *The representation of Theorem 7.8 yields a positive definite unitary highest-weight representation of the Virasoro algebra with the highest weight $c = 1, h = \frac{1}{2}\mu^2$ (cf. Chap. 6).*

Proof. For the highest-weight vector $v_0 := 1$ let

$$V := \operatorname{span}_{\mathbb{C}}\{L_n v_0 : n \in \mathbb{Z}\}.$$

Then the restrictions of $\rho(L_n)$ to the subspace $V \subset \mathsf{S}$ of S define a highest-weight representation of Vir with highest weight $(1, \frac{1}{2}\mu^2)$ and Virasoro module V. □

Remark 7.10. In most cases one has $\mathsf{S} = V$. But this does not hold for $\mu = 0$, for instance.

More unitary highest-weight representations can be found by taking tensor products: for $f \otimes g \in V \otimes V$ let

$$(\rho \otimes \rho)(L_n)(f \otimes g) := (\rho(L_n)f) \otimes g + f \otimes (\rho(L_n)g).$$

As a simple consequence one gets

Theorem 7.11. $\rho \otimes \rho : \mathsf{Vir} \to \operatorname{End}_{\mathbb{C}}(V \otimes V)$ *is a positive definite unitary highest-weight representation for the highest weight $c = 2, h = \mu^2$. By iteration of this procedure one gets unitary highest-weight representations for every weight (c, h) with $c \in \mathbb{N}_1$ and $h \in \mathbb{R}_+$.*

For the physics of strings, these representations resp. quantizations are not sufficient, since only some of the important observables are represented. It is our aim in this section, however, to present a straightforward construction of a unitary Verma module with $c > 1$ and $h \geq 0$ for the discussion in Chap. 6 based on quantization. Indeed, the starting point was the attempt of quantizing string theory. But for the construction of the Verma module only the Fock space representation of the Heisenberg algebra as the algebra of the infinite dimensional harmonic oscillator was used by restricting to one single coordinate.

We now come back to strings in taking care of all coordinates $x^\mu, \mu \in \{0, 1, \dots d-1\}$.

7.4 Quantization of Strings

In (non-compactified bosonic) string theory, the Poisson algebra

$$\mathscr{A} := \mathbb{C}1 \oplus \bigoplus_{\mu=0}^{D-1} (\mathbb{C}x_0^\mu \oplus \mathbb{C}p_0^\mu) \oplus \bigoplus_{\mu=0}^{D-1} \bigoplus_{m \neq 0} (\mathbb{C}\alpha_m^\mu)$$

of the classical oscillator modes and of the coordinates x_0^μ, p_0^ν has to be quantized. (See (7.8) for their Poisson brackets.) Equivalently, one has to find a representation of the string algebra

$$\mathscr{L} := \mathbb{C}1 \oplus \bigoplus_{\mu=0}^{D-1} (\mathbb{C}\widehat{x_0^\mu} \oplus \mathbb{C}\widehat{p_0^\mu}) \oplus \bigoplus_{\mu=0}^{D-1} \bigoplus_{m \neq 0} (\mathbb{C}a_m^\mu)$$

with the following Lie brackets

$$\begin{aligned}
\{a_m^\mu, a_n^\nu\} &= m\eta^{\mu\nu}\delta_{m+n}, \\
\{\widehat{p_0^\mu}, \widehat{x_0^\nu}\} &= -i\eta^{\mu\nu}, \\
\{\widehat{x_0^\mu}, \widehat{x_0^\nu}\} &= \{\widehat{x_0^\mu}, a_m^\nu\} = 0,
\end{aligned}$$

according to (7.8).

The corresponding Fock space is

$$\mathsf{S} := \mathbb{C}[T_n^\mu : n \in \mathbb{N}_0, \mu = 0, \dots, D-1]$$

and the respective representation is given by

$$\begin{aligned}
\rho(a_m) &:= \eta^{\mu\nu}\frac{\partial}{\partial T_m^\nu} && \text{for } m > 0, \\
P^\mu := \rho(a_0^\mu) &:= i\eta^{\mu\nu}\frac{\partial}{\partial T_0^\mu} && (\alpha_0^\mu = p_0^\mu \text{ if } 4\pi\kappa = 1), \\
\rho(a_{-m}^\mu) &:= mT_m^\mu && \text{for } m > 0, \\
Q^\mu := \rho(\widehat{x_0^\mu}) &:= T_0^\mu.
\end{aligned}$$

The natural hermitian form on S with $H(1,1) = 1$ and

$$H(\rho(\alpha_m^\mu)f,g) = H(f,\rho(\alpha_{-m}^\mu)g)$$

is no longer positive semi-definite. For instance,

$$\begin{aligned} H(T_1^0,T_1^0) &= H(\alpha_{-1}^0 1,\alpha_{-1}^0 1) = H(1,\alpha_1^0\alpha_{-1}^0 1) \\ &= H(1,[\alpha_1^0,\alpha_{-1}^0]1) = H(1,-1) \\ &= -1. \end{aligned}$$

Moreover, this representation does not respect the gauge conditions $L_n = 0$. A solution of both problems is provided by the so-called “no-ghost theorem” (cf. [GSW87]). It essentially states that taking into account the gauge conditions $L_n = 0$, $n > 0$, the representation becomes unitary for the dimension $D = 26$. This means that the restriction of the hermitian form to the space of “physical states”

$$\mathscr{P} := \{f \in \mathsf{S} : L_n f = 0 \text{ for all } n > 0, L_0 f = f\}$$

is positive semi-definite ($D = 26$). A proof of the no-ghost theorem using the Kac determinant can be found in [Tho84].

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