Chapter 2
The Conformal Group

Definition 2.1. The *conformal group* $Conf(\mathbb{R}^{p,q})$ is the connected component containing the identity in the group of conformal diffeomorphisms of the conformal compactification of $\mathbb{R}^{p,q}$.

In this definition, the group of conformal diffeomorphisms is considered as a topological group with the topology of compact convergence, that is the topology of uniform convergence on the compact subsets. More precisely, the topology of compact convergence on the space $\mathscr{C}(X,Y)$ of continuous maps $X \to Y$ between topological spaces X,Y is generated by all the subsets

$$\{f \in \mathscr{C}(X,Y) : f(K) \subset V\},$$

where $K \subset X$ is compact and $V \subset Y$ is open.

First of all, to understand the definition we have to introduce the concept of conformal compactification. The conformal compactification as a hyperquadric in five-dimensional projective space has been used already by Dirac [Dir36*] in order to study conformally invariant field theories in four-dimensional spacetime. The concept has its origin in general geometric principles.

2.1 Conformal Compactification of $\mathbb{R}^{p,q}$

To study the collection of all conformal transformations on an open connected subset $M \subset \mathbb{R}^{p,q}$, $p+q \geq 2$, a conformal compactification $N^{p,q}$ of $\mathbb{R}^{p,q}$ is introduced, in such a way that the conformal transformations $M \to \mathbb{R}^{p,q}$ become everywhere-defined and bijective maps $N^{p,q} \to N^{p,q}$. Consequently, we search for a "minimal" compactification $N^{p,q}$ of $\mathbb{R}^{p,q}$ with a natural semi-Riemannian metric, such that every conformal transformation $\varphi : M \to \mathbb{R}^{p,q}$ has a continuation to $N^{p,q}$ as a conformal diffeomorphism $\widehat{\varphi} : N^{p,q} \to N^{p,q}$ (cf. Definition 2.7 for details).

Note that conformal compactifications in this sense do only exist for $p+q > 2$. We investigate the two-dimensional case in detail in the next two sections below. We show that the spaces $N^{p,q}$ still can be defined as compactifications of $\mathbb{R}^{p,q}, p+q = 2$, with a natural conformal structure inducing the original conformal structure on $\mathbb{R}^{p,q}$.

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However, the spaces $N^{p,q}$ do not possess the continuation property mentioned above in full generality: there exist many conformal transformations $\varphi : M \to \mathbb{R}^{p,q}$ which do not have a conformal continuation to all of $N^{p,q}$.

Let $n = p+q \geq 2$. We use the notation $\langle x\rangle_{p,q} := g^{p,q}(x,x)$, $x \in \mathbb{R}^{p,q}$. For short, we also write $\langle x\rangle = \langle x\rangle_{p,q}$ if p and q are evident from the context. $\mathbb{R}^{p,q}$ can be embedded into the $(n+1)$-dimensional projective space $\mathbb{P}_{n+1}(\mathbb{R})$ by the map

$$\begin{aligned} \iota : \mathbb{R}^{p,q} &\to \mathbb{P}_{n+1}(\mathbb{R}), \\ x = (x^1,\ldots,x^n) &\mapsto \left(\frac{1-\langle x\rangle}{2} : x^1 : \ldots : x^n : \frac{1+\langle x\rangle}{2}\right). \end{aligned}$$

Recall that $\mathbb{P}_{n+1}(\mathbb{R})$ is the quotient

$$(\mathbb{R}^{n+2} \setminus \{0\}) / \sim$$

with respect to the equivalence relation

$$\xi \sim \xi' \iff \xi = \lambda \xi' \text{ for a } \lambda \in \mathbb{R} \setminus \{0\}.$$

$\mathbb{P}_{n+1}(\mathbb{R})$ can also be described as the space of one-dimensional subspaces of $\mathbb{R}^{n+2}$. $\mathbb{P}_{n+1}(\mathbb{R})$ is a compact $(n+1)$-dimensional smooth manifold (cf. for example [Scho95]). If $\gamma : \mathbb{R}^{n+2} \setminus \{0\} \to \mathbb{P}_{n+1}(\mathbb{R})$ is the quotient map, a general point $\gamma(\xi) \in \mathbb{P}_{n+1}(\mathbb{R})$, $\xi = (\xi^0,\ldots,\xi^{n+1}) \in \mathbb{R}^{n+2}$, is denoted by $(\xi^0 : \ldots : \xi^{n+1}) := \gamma(\xi)$ with respect to the so-called *homogeneous coordinates*. Obviously, we have

$$(\xi^0 : \cdots : \xi^{n+1}) = (\lambda\xi^0 : \cdots : \lambda\xi^{n+1}) \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\}.$$

We are looking for a suitable compactification of $\mathbb{R}^{p,q}$. As a candidate we consider the closure $\overline{\iota(\mathbb{R}^{p,q})}$ of the image of the smooth embedding $\iota : \mathbb{R}^{p,q} \to \mathbb{P}_{n+1}(\mathbb{R})$.

Remark 2.2. $\overline{\iota(\mathbb{R}^{p,q})} = N^{p,q}$, where $N_{p,q}$ is the *quadric*

$$N^{p,q} := \{(\xi^0 : \cdots : \xi^{n+1}) \in \mathbb{P}_{n+1}(\mathbb{R}) \big| \langle \xi \rangle_{p+1,q+1} = 0\}$$

in the real projective space $\mathbb{P}_{n+1}(\mathbb{R})$.

Proof. By definition of ι we have $\langle \iota(x)\rangle_{p+1,q+1} = 0$ for $x \in \mathbb{R}^{p,q}$, that is $\overline{\iota(\mathbb{R}^{p,q})} \subset N^{p,q}$.

For the converse inclusion, let $(\xi^0 : \cdots : \xi^{n+1}) \in N^{p,q} \setminus \iota(\mathbb{R}^{p,q})$. Then $\xi^0 + \xi^{n+1} = 0$, since

$$\iota(\lambda^{-1}(\xi^1,\ldots,\xi^n)) = (\xi^0 : \cdots : \xi^{n+1}) \in \iota(\mathbb{R}^{p,q})$$

for $\lambda := \xi^0 + \xi^{n+1} \neq 0$. Given $(\xi^0 : \cdots : \xi^{n+1}) \in N^{p,q}$ there always exist sequences $\epsilon_k \to 0$, $\delta_k \to 0$ with $\epsilon_k \neq 0 \neq \delta_k$ and $2\xi^1\epsilon_k + \epsilon_k^2 = 2\xi^{n+1}\delta_k + \delta_k^2$. For $p \geq 1$ we have

$$P_k := (\xi^0 : \xi^1 + \epsilon_k : \xi^2 : \cdots : \xi^n : \xi^{n+1} + \delta_k) \in N^{p,q}.$$

Moreover, $\xi^0 + \xi^{n+1} + \delta_k = \delta_k \neq 0$ implies $P_k \in \iota(\mathbb{R}^{p,q})$. Finally, since $P_k \to (\xi^0 : \ldots : \xi^{n+1})$ for $k \to \infty$ it follows that $(\xi^0 : \cdots : \xi^{n+1}) \in \overline{\iota(\mathbb{R}^{p,q})}$, that is $N^{p,q} \subset \overline{\iota(\mathbb{R}^{p,q})}$. □

We therefore choose $N^{p,q}$ as the underlying manifold of the conformal compactification. $N^{p,q}$ is a regular quadric in $\mathbb{P}_{n+1}(\mathbb{R})$. Hence it is an n-dimensional compact submanifold of $\mathbb{P}_{n+1}(\mathbb{R})$. $N^{p,q}$ contains $\iota(\mathbb{R}^{p,q})$ as a dense subset.

We get another description of $N^{p,q}$ using the quotient map γ on $\mathbb{R}^{p+1,q+1}$ restricted to $\mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1,q+1}$.

Lemma 2.3. *The restriction of γ to the product of spheres*

$$\mathbb{S}^p \times \mathbb{S}^q := \left\{ \xi \in \mathbb{R}^{n+2} : \sum_{j=0}^{p} (\xi^j)^2 = 1 = \sum_{j=p+1}^{n+1} (\xi^j)^2 \right\} \subset \mathbb{R}^{n+2}$$

gives a smooth 2*-to-*1 *covering*

$$\pi := \gamma|_{\mathbb{S}^p \times \mathbb{S}^q} : \mathbb{S}^p \times \mathbb{S}^q \longrightarrow N^{p,q}.$$

Proof. Obviously $\gamma(\mathbb{S}^p \times \mathbb{S}^q) \subset N^{p,q}$. For $\xi, \xi' \in \mathbb{S}^p \times \mathbb{S}^q$ it follows from $\gamma(\xi) = \gamma(\xi')$ that $\xi = \lambda \xi'$ with $\lambda \in \mathbb{R} \setminus \{0\}$. $\xi, \xi' \in \mathbb{S}^p \times \mathbb{S}^q$ implies $\lambda \in \{1, -1\}$. Hence, $\gamma(\xi) = \gamma(\xi')$ if and only if $\xi = \xi'$ or $\xi = -\xi'$. For $P = (\xi^0 : \ldots : \xi^{n+1}) \in N^{p,q}$ the two inverse images with respect to π can be specified as follows: $P \in N^{p,q}$ implies $\langle \xi \rangle = 0$, that is $\sum_{j=0}^{p} (\xi^j)^2 = \sum_{j=p+1}^{n+1} (\xi^j)^2$. Let

$$r := \left(\sum_{j=0}^{p} (\xi^j)^2 \right)^{\frac{1}{2}}$$

and $\eta := \frac{1}{r}(\xi^0, \ldots, \xi^{n+1}) \in \mathbb{S}^p \times \mathbb{S}^q$. Then η and $-\eta$ are the inverse images of ξ. Hence, π is surjective and the description of the inverse images shows that π is a local diffeomorphism. □

With the aid of the map $\pi : \mathbb{S}^p \times \mathbb{S}^q \to N^{p,q}$, which is locally a diffeomorphism, the metric induced on $\mathbb{S}^p \times \mathbb{S}^q$ by the inclusion $\mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1,q+1}$, that is the semi-Riemannian metric of $\mathbb{S}^{p,q}$ described in the examples of Sect. 1.1 on page 8, can be carried over to $N^{p,q}$ in such a way that $\pi : \mathbb{S}^{p,q} \to N^{p,q}$ becomes a (local) isometry.

Definition 2.4. $N^{p,q}$ with this semi-Riemannian metric will be called the *conformal compactification* of $\mathbb{R}^{p,q}$.

In particular, it is clear what the conformal transformations $N^{p,q} \to N^{p,q}$ are. In this way, $N^{p,q}$ obtains a *conformal structure* (that is the equivalence class of semi-Riemannian metrics).

We know that $\iota : \mathbb{R}^{p,q} \to N^{p,q}$ is an embedding (injective and regular) and that $\iota(\mathbb{R}^{p,q})$ is dense in the compact manifold $N^{p,q}$. In order to see that this embedding is conformal we compare ι with the natural map $\tau : \mathbb{R}^{p,q} \to \mathbb{S}^p \times \mathbb{S}^q$ defined by

$$\tau(x) = \frac{1}{r(x)}\left(\frac{1-\langle x\rangle}{2}, x^1, \dots, x^n, \frac{1+\langle x\rangle}{2}\right),$$

where

$$r(x) = \frac{1}{2}\sqrt{1 + 2\sum_{j=1}^{n}(x^j)^2 + \langle x\rangle^2} \geq \frac{1}{2}.$$

τ is well-defined because of

$$r(x)^2 = \left(\frac{1-\langle x\rangle}{2}\right)^2 + \sum_{j=1}^{p}(x^j)^2 = \sum_{j=p+1}^{n}(x^j)^2 + \left(\frac{1+\langle x\rangle}{2}\right)^2,$$

and we have

Proposition 2.5. $\tau : \mathbb{R}^{p,q} \to \mathbb{S}^p \times \mathbb{S}^q$ *is a conformal embedding with* $\iota = \pi \circ \tau$.

Proof. For the proof we only have to confirm that τ is indeed a conformal map. This can be checked in a similar manner as in the case of the stereographic projection on p. 12 in Chap. 1. We denote the factor $\frac{1}{r}$ by ρ and will observe that the result is independent of the special factor in question. For an index $1 \leq i \leq n$ we denote by τ_i, ρ_i the partial derivatives with respect to the coordinate x^i of $\mathbb{R}^{p,q}$. We have for $i \leq p$

$$\tau_i = \left(\rho_i \frac{1-\langle x\rangle}{2} - \rho x^i, \rho_i x^1, \dots \rho_i x^i + \rho, \dots, \rho_i x^n, \rho_i \frac{1+\langle x\rangle}{2} + \rho x^i\right)$$

and a similar formula for $j > p$ with only two changes in signs. For $i \leq p$ we obtain in $\mathbb{R}^{p+1,q+1}$

$$\begin{aligned}
\langle \tau_i, \tau_i\rangle &= \left(\rho_i \frac{1-\langle x\rangle}{2} - \rho x^i\right)^2 + (\rho_i x^1)^2 + \dots + (\rho_i x^i + \rho)^2 + \\
&\quad + \dots - (\rho_i x^n)^2 - \left(\rho_i \frac{1+\langle x\rangle}{2} + \rho x^i\right)^2 \\
&= -2\rho_i\left(\rho_i \frac{\langle x\rangle}{2} + \rho x^i\right) + (\rho_i x^1)^2 + \dots + (\rho_i x^i)^2 + 2\rho_i x^1 \rho + \\
&\quad + \rho^2 - (\rho_i x^{p+1})^2 \dots - (\rho_i x^n)^2 \\
&= -\rho_i^2 \langle x\rangle + \rho_i^2 \langle x\rangle - 2\rho_i x^1 \rho + 2\rho_i x^1 \rho \\
&= \rho^2,
\end{aligned}$$

and for $j > p$ we obtain $\langle \tau_j, \tau_j\rangle = -\rho^2$ in the same way. Similarly, one checks $\langle \tau_i, \tau_j\rangle = 0$ for $i \neq j$. Hence, $\langle \tau_i, \tau_j\rangle = \rho^2 \eta_{ij}$ where $\eta = \mathrm{diag}(1, \dots 1, -1, \dots, -1)$ is the diagonal matrix of the standard Minkowski metric of $\mathbb{R}^{p,q}$. This property is equivalent to τ being a conformal map. $\square$

We now want to describe the collection of all conformal transformations $N_{p,q} \to N_{p,q}$.

Theorem 2.6. *For every matrix* $\Lambda \in \mathrm{O}(p+1,q+1)$ *the map* $\psi = \psi_\Lambda : N^{p,q} \to N^{p,q}$ *defined by*

$$\psi_\Lambda(\xi^0 : \ldots : \xi^{n+1}) := \gamma(\Lambda\xi), \quad (\xi^0 : \ldots : \xi^{n+1}) \in N^{p,q}$$

is a conformal transformation and a diffeomorphism. The inverse transformation $\psi^{-1} = \psi_{\Lambda^{-1}}$ *is also conformal. The map* $\Lambda \mapsto \psi_\Lambda$ *is not injective. However,* $\psi_\Lambda = \psi_{\Lambda'}$ *implies* $\Lambda = \Lambda'$ *or* $\Lambda = -\Lambda'$.

Proof. For $\xi \in \mathbb{R}^{n+2} \setminus \{0\}$ with $\langle x \rangle = 0$ and $\Lambda \in \mathrm{O}(p+1,q+1)$ we have $\langle \Lambda\xi \rangle = g(\Lambda\xi,\Lambda\xi) = g(\xi,\xi) = \langle \xi \rangle = 0$, that is $\gamma(\Lambda\xi) \in N^{p,q}$. $\gamma(\Lambda\xi)$ does not depend on the representative ξ as we can easily check: $\xi \sim \xi'$, that is $\xi' = r\xi$ with $r \in \mathbb{R} \setminus \{0\}$, implies $\Lambda\xi' = r\Lambda\xi$, that is $\Lambda\xi' \sim \Lambda\xi$. Altogether, $\psi : N^{p,q} \to N^{p,q}$ is well-defined. Because of the fact that the metric on $\mathbb{R}^{p+1,q+1}$ is invariant with respect to Λ, ψ_Λ turns out to be conformal. For $P \in N^{p,q}$ one calculates the conformal factor $\Omega^2(P) = \sum_{j=0}^{n+1} \left(\Lambda^j_k \xi^k\right)^2$ if P is represented by $\xi \in \mathbb{S}^p \times \mathbb{S}^q$. (In general, $\Lambda(\mathbb{S}^p \times \mathbb{S}^q)$ is not contained in $\mathbb{S}^p \times \mathbb{S}^q$, and the (punctual) deviation from the inclusion is described precisely by the conformal factor $\Omega(P)$:

$$\frac{1}{\Omega(P)}\Lambda(\xi) \in \mathbb{S}^p \times \mathbb{S}^q \text{ for } \xi \in \mathbb{S}^p \times \mathbb{S}^q \text{ and } P = \gamma(\xi).$$

Obviously, $\psi_\Lambda = \psi_{-\Lambda}$ and $\psi_\Lambda^{-1} = \psi_{\Lambda^{-1}}$. In the case $\psi_\Lambda = \psi_{\Lambda'}$ for $\Lambda, \Lambda' \in \mathrm{O}(p+1,q+1)$ we have $\gamma(\Lambda\xi) = \gamma(\Lambda'\xi)$ for all $\xi \in \mathbb{R}^{n+2}$ with $\langle \xi \rangle = 0$. Hence, $\Lambda = r\Lambda'$ with $r \in \mathbb{R} \setminus \{0\}$. Now $\Lambda, \Lambda' \in \mathrm{O}(p+1,q+1)$ implies $r = 1$ or $r = -1$. □

The requested continuation property for conformal transformations can now be formulated as follows:

Definition 2.7. Let $\varphi : M \to \mathbb{R}^{p,q}$ be a conformal transformation on a connected open subset $M \subset \mathbb{R}^{p,q}$. Then $\widehat{\varphi} : N^{p,q} \to N^{p,q}$ is called a *conformal continuation* of φ, if $\widehat{\varphi}$ is a conformal diffeomorphism (with conformal inverse) and if $\imath(\varphi(x)) = \widehat{\varphi}(\imath(x))$ for all $x \in M$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & \mathbb{R}^{p,q} \\ \downarrow \imath & & \downarrow \imath \\ N^{p,q} & \xrightarrow{\widehat{\varphi}} & N^{p,q} \end{array}$$

Remark 2.8. In a more conceptual sense the notion of a conformal compactification should be defined and used in the following general formulation. A *conformal compactification* of a connected semi-Riemannian manifold X is a compact semi-Riemannian manifold N together with a conformal embedding $\imath : X \to N$ such that

1. $\iota(X)$ is dense in N.
2. Every conformal transformation $\varphi : M \to X$ (that φ is injective and conformal) on an open and connected subset $M \subset X, M \neq \emptyset$, has a conformal continuation $\widehat{\varphi} : N \to N$.

A conformal compactification is unique up to isomorphism if it exists.

In the case of $X = \mathbb{R}^{p,q}$ the construction of $\iota : \mathbb{R}^{p,q} \to N^{p,q}$ so far together with Theorem 2.9 asserts that $N^{p,q}$ is indeed a conformal compactification in this general sense.

2.2 The Conformal Group of $\mathbb{R}^{p,q}$ for $p+q > 2$

Theorem 2.9. *Let $n = p+q > 2$. Every conformal transformation on a connected open subset $M \subset \mathbb{R}^{p,q}$ has a unique conformal continuation to $N^{p,q}$. The group of all conformal transformations $N^{p,q} \to N^{p,q}$ is isomorphic to $\mathrm{O}(p+1,q+1)/\{\pm 1\}$. The connected component containing the identity in this group – that is, by Definition 2.1 the conformal group $\mathrm{Conf}(\mathbb{R}^{p,q})$ – is isomorphic to $\mathrm{SO}(p+1,q+1)$ (or $\mathrm{SO}(p+1,q+1)/\{\pm 1\}$ if -1 is in the connected component of $\mathrm{O}(p+1,q+1)$ containing 1, for example, if p and q are odd.)*

Here, $\mathrm{SO}(p+1,q+1)$ is defined to be the connected component of the identity in $\mathrm{O}(p+1,q+1)$. $\mathrm{SO}(p+1,q+1)$ is contained in

$$\{\Lambda \in \mathrm{O}(p+1,q+1) \,|\, \det \Lambda = 1\}.$$

However, it is, in general, different from this subgroup, e.g., for the case $(p,q) = (2,1)$ or $(p,q) = (3,1)$.

Proof. It suffices to find conformal continuations $\widehat{\varphi}$ to $N^{p,q}$ (according to Definition 2.7) of all the conformal transformations φ described in Theorem 1.9 and to represent these continuations by matrices $\Lambda \in \mathrm{O}(p+1,q+1)$ according to Lemma 2.3:

1. Orthogonal transformations. The easiest case is the conformal continuation of an orthogonal transformation $\varphi(x) = \Lambda' x$ represented by a matrix $\Lambda' \in \mathrm{O}(p,q)$ and defined on all of $\mathbb{R}^{p,q}$. For the block matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda' & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

one obviously has $\Lambda \in \mathrm{O}(p+1,q+1)$, because of $\Lambda^T \eta \Lambda = \eta$, where $\eta = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ is the matrix representing $g^{p+1,q+1}$. Furthermore,

$$\Lambda \in \mathrm{SO}(p+1,q+1) \Longleftrightarrow \Lambda' \in \mathrm{SO}(p,q).$$

We define a conformal map $\widehat{\varphi} : N^{p,q} \to N^{p,q}$ by $\widehat{\varphi} := \psi_\Lambda$, that is

$$\widehat{\varphi}(\xi^0 : \ldots : \xi^{n+1}) = (\xi^0 : \Lambda'\xi : \xi^{n+1})$$

for $(\xi^0 : \ldots : \xi^{n+1}) \in N^{p,q}$ (cf. Theorem 2.6). For $x \in \mathbb{R}^{p,q}$ we have

$$\begin{aligned}\widehat{\varphi}(\iota(x)) &= \left(\frac{1-\langle x\rangle}{2} : \Lambda' x : \frac{1+\langle x\rangle}{2}\right)\\ &= \left(\frac{1-\langle \Lambda' x\rangle}{2} : \Lambda' x : \frac{1+\langle \Lambda' x\rangle}{2}\right),\end{aligned}$$

since $\Lambda' \in \mathrm{O}(p,q)$ implies $\langle x\rangle = \langle \Lambda' x\rangle$. Hence, $\widehat{\varphi}(\iota(x)) = \iota(\varphi(x))$ for all $x \in \mathbb{R}^{p,q}$.

2. Translations. For a translation $\varphi(x) = x + c$, $c \in \mathbb{R}^n$, one has the continuation

$$\begin{aligned}\widehat{\varphi}(\xi^0 : \ldots : \xi^{n+1}) := (&\xi^0 - \langle \xi', c\rangle - \xi^+\langle c\rangle : \xi' + 2\xi^+ c\\ &: \xi^{n+1} + \langle \xi', c\rangle + \xi^+\langle c\rangle)\end{aligned}$$

for $(\xi^0 : \ldots : \xi^{n+1}) \in N^{p,q}$. Here,

$$\xi^+ = \frac{1}{2}(\xi^{n+1} + \xi^0) \quad \text{and} \quad \xi' = (\xi^1, \ldots, \xi^n).$$

We have

$$\widehat{\varphi}(\iota(x)) = \left(\frac{1-\langle x\rangle}{2} - \langle x, c\rangle - \frac{\langle c\rangle}{2} : x + c : \frac{1+\langle x\rangle}{2} + \langle x, c\rangle + \frac{\langle c\rangle}{2}\right),$$

since $\iota(x)^+ = \frac{1}{2}$, and therefore

$$\widehat{\varphi}(\iota(x)) = \left(\frac{1-\langle x+c\rangle}{2} : x + c : \frac{1+\langle x+c\rangle}{2}\right) = \iota(\varphi(x)).$$

Since $\widehat{\varphi} = \psi_\Lambda$ with $\Lambda \in \mathrm{SO}(p+1, q+1)$ can be shown as well, $\widehat{\varphi}$ is a well-defined conformal map, that is a conformal continuation of φ. The matrix we look for can be found directly from the definition of $\widehat{\varphi}$. It can be written as a block matrix:

$$\Lambda_c = \begin{pmatrix} 1 - \frac{1}{2}\langle c\rangle & -(\eta' c)^T & -\frac{1}{2}\langle c\rangle \\ c & E_n & c \\ \frac{1}{2}\langle c\rangle & (\eta' c)^T & 1 + \frac{1}{2}\langle c\rangle \end{pmatrix}.$$

Here, E_n is the $(n \times n)$ unit matrix and

$$\eta' = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$$

is the $(n \times n)$ diagonal matrix representing $g^{p,q}$. The proof of $\Lambda_c \in \mathrm{O}(p+1, q+1)$ requires some elementary calculation. $\Lambda_c \in \mathrm{SO}(p+1, q+1)$ can be shown by looking at the curve $t \mapsto \Lambda_{tc}$ connecting E_{n+2} and Λ_c.

3. Dilatations. The following matrices belong to the dilatations $\varphi(x) = rx$, $r \in \mathbb{R}_+$:

$$\Lambda_r = \begin{pmatrix} \frac{1+r^2}{2r} & 0 & \frac{1-r^2}{2r} \\ 0 & E_n & 0 \\ \frac{1-r^2}{2r} & 0 & \frac{1+r^2}{2r} \end{pmatrix}$$

($\Lambda_r \in \mathrm{O}(p+1,q+1)$ requires a short calculation again).
$\Lambda_r \in \mathrm{SO}(\mathrm{p}+1,\mathrm{q}+1)$ follows as above using the curve $t \mapsto \Lambda_{tr}$. The conformal transformation $\widehat{\varphi} = \psi_\Lambda$ actually is a conformal continuation of φ, as can be seen by substitution:

$$\begin{aligned}
&\widehat{\varphi}(\xi^0 : \ldots : \xi^{n+1}) \\
&= \left(\frac{1+r^2}{2r}\xi^0 + \frac{1-r^2}{2r}\xi^{n+1} : \xi' : \frac{1+r^2}{2r}\xi^{n+1} + \frac{1-r^2}{2r}\xi^0\right) \\
&= \left(\frac{1+r^2}{2}\xi^0 + \frac{1-r^2}{2}\xi^{n+1} : r\xi' : \frac{1+r^2}{2}\xi^{n+1} + \frac{1-r^2}{2}\xi^0\right).
\end{aligned}$$

For $\xi = \iota(x)$, that is $\xi' = x$, $\xi^0 = \frac{1}{2}(1 - \langle x \rangle)$, $\xi^{n+1} = \frac{1}{2}(1 + \langle x \rangle)$, one has

$$\begin{aligned}
\widehat{\varphi}(\iota(x)) &= \left(\frac{1-\langle x \rangle r^2}{2} : rx : \frac{1+\langle x \rangle r^2}{2}\right) \\
&= \left(\frac{1-\langle rx \rangle}{2} : rx : \frac{1+\langle rx \rangle}{2}\right) = \iota(\varphi(x)).
\end{aligned}$$

4. Special conformal transformations. Let $b \in \mathbb{R}^n$ and

$$\varphi(x) = \frac{x - \langle x \rangle b}{1 - 2\langle x,b \rangle + \langle x \rangle \langle b \rangle}, \quad x \in M_1 \subsetneqq \mathbb{R}^{p,q}.$$

With $N = N(x) = 1 - 2\langle x,b \rangle + \langle x \rangle \langle b \rangle$ the equation $\langle \varphi(x) \rangle = \frac{\langle x \rangle}{N}$ implies

$$\begin{aligned}
\iota(\varphi(x)) &= \left(\frac{1-\langle \varphi(x) \rangle}{2} : \frac{x - \langle x \rangle b}{N} : \frac{1+\langle \varphi(x) \rangle}{2}\right) \\
&= \left(\frac{N - \langle x \rangle}{2} : x - \langle x \rangle b : \frac{N + \langle x \rangle}{2}\right).
\end{aligned}$$

This expression also makes sense for $x \in \mathbb{R}^{p,q}$ with $N(x) = 0$. It furthermore leads to the continuation

$$\begin{aligned}
\widehat{\varphi}(\xi^0 : \ldots : \xi^{n+1}) = (&\xi^0 - \langle \xi', b \rangle + \xi^- \langle b \rangle : \xi' - 2\xi^- b \\
&: \xi^{n+1} - \langle \xi', b \rangle + \xi^- \langle b \rangle),
\end{aligned}$$

where $\xi^- = \frac{1}{2}(\xi^{n+1} - \xi^0)$. Because of $\iota(x)^- = \frac{1}{2}\langle x \rangle$, one finally gets

$$\widehat{\varphi}(\iota(x)) = \left(\frac{N-\langle x\rangle}{2} : x-\langle x\rangle b : \frac{N+\langle x\rangle}{2}\right) = \iota(\varphi(x))$$

for all $x \in \mathbb{R}^{p,q}$, $N(x) \neq 0$. The mapping $\widehat{\varphi}$ is conformal, since $\widehat{\varphi} = \psi_\Lambda$ with

$$\Lambda = \begin{pmatrix} 1-\frac{1}{2}\langle b\rangle & -(\eta' b)^T & \frac{1}{2}\langle b\rangle \\ b & E_n & -b \\ -\frac{1}{2}\langle b\rangle & -(\eta' b)^T & 1+\frac{1}{2}\langle b\rangle \end{pmatrix} \in \mathrm{SO}(p+1,q+1).$$

In particular, $\widehat{\varphi}$ is a conformal continuation of φ.

To sum up, for all conformal transformations φ on open connected $M \subset \mathbb{R}^{p,q}$ we have constructed conformal continuations in the sense of Definition 2.7 $\widehat{\varphi} : N^{p,q} \to N^{p,q}$ of the type $\widehat{\varphi}(\xi^0 : \ldots : \xi^{n+1}) = \gamma(\Lambda\xi)$ with $\Lambda \in \mathrm{SO}(p+1,q+1)$ having a conformal inverse $\widehat{\varphi}^{-1} = \psi_{\Lambda^{-1}}$. The map $\varphi \mapsto \widehat{\varphi}$ turns out to be injective (at least if φ is conformally continued to a maximal domain M in $\mathbb{R}^{p,q}$, that is $M = \mathbb{R}^{p,q}$ or $M = M_1$, cf. Theorem 1.9). Conversely, every conformal transformation $\psi : N^{p,q} \to N^{p,q}$ is of the type $\psi = \widehat{\varphi}$ with a conformal transformation φ on $\mathbb{R}^{p,q}$, since there exist open nonempty subsets $U,V \subset \iota(\mathbb{R}^{p,q})$ with $\psi(U) = V$ and the map

$$\varphi := \iota^{-1} \circ \psi \circ \iota : \iota^{-1}(U) \to \iota^{-1}(V)$$

is conformal, that is φ has a conformal continuation $\widehat{\varphi}$, which must be equal to ψ. Furthermore, the group of conformal transformations $N^{p,q} \to N^{p,q}$ is isomorphic to $\mathrm{O}(p+1,q+1)/\{\pm 1\}$, since $\widehat{\varphi}$ can be described by the uniquely determined set $\{\Lambda, -\Lambda\}$ of matrices in $\mathrm{O}(p+1,q+1)$. This is true algebraically in the first place, but it also holds for the topological structures. Finally, this implies that the connected component containing the identity in the group of all conformal transformations $N^{p,q} \to N^{p,q}$, that is the conformal group Conf $(\mathbb{R}^{p,q})$, is isomorphic to $\mathrm{SO}(p+1,q+1)$. This completes the proof of the theorem. □

2.3 The Conformal Group of $\mathbb{R}^{2,0}$

By Theorem 1.11, the orientation-preserving conformal transformations $\varphi : M \to \mathbb{R}^{2,0} \cong \mathbb{C}$ on open subsets $M \subset \mathbb{R}^{2,0} \cong \mathbb{C}$ are exactly those holomorphic functions with nowhere-vanishing derivative. This immediately implies that a conformal compactification according to Remark 2.2 and Definition 2.7 cannot exist, because there are many noninjective conformal transformations, e.g.,

$$\mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto z^k, \quad \text{for} \quad k \in \mathbb{Z} \setminus \{-1,0,1\}.$$

There are also many injective holomorphic functions without a suitable holomorphic continuation, like

$$z \mapsto \sqrt{z}, \quad z \in \{w \in \mathbb{C} : \mathrm{Re}\, w > 0\},$$

or the principal branch of the logarithm on the plane that has been slit along the negative real axis $\mathbb{C} \setminus \{-x : x \in \mathbb{R}_+\}$. However, there is a useful version of the ansatz from Sect. 2.3 for the case $p = 2, q = 0$, which leads to a result similar to Theorem 2.9.

Definition 2.10. A *global* conformal transformation on $\mathbb{R}^{2,0}$ is an injective holomorphic function, which is defined on the entire plane $\mathbb{C}$ with at most one exceptional point.

The analysis of conformal Killing factors (cf. Sect. 1.4.2) shows that the global conformal transformations and all those conformal transformations, which admit a (necessarily unique) continuation to a global conformal transformation are exactly the transformations which have a linear conformal Killing factor or can be written as a composition of a transformation having a linear conformal Killing factor with a reflection $z \mapsto \bar{z}$. Using this result, the following theorem can be proven in the same manner as Theorem 2.9.

Theorem 2.11. *Every global conformal transformation φ on $M \subset \mathbb{C}$ has a unique conformal continuation $\widehat{\varphi} : N^{2,0} \to N^{2,0}$, where $\widehat{\varphi} = \varphi_\Lambda$ with $\Lambda \in \mathrm{O}(3,1)$. The group of conformal diffeomorphisms $\psi : N^{2,0} \to N^{2,0}$ is isomorphic to* $\mathrm{O}(3,1)/\{\pm 1\}$ *and the connected component containing the identity is isomorphic to* $\mathrm{SO}(3,1)$.

In view of this result, it is justified to call the connected component containing the identity the conformal group $\mathrm{Conf}(\mathbb{R}^{2,0})$ of $\mathbb{R}^{2,0}$. Another reason for this comes from the impossibility of enlarging this group by additional conformal transformations discussed below.

A comparison of Theorems 2.9 and 2.11 shows the following exceptional situation of the case $p+q > 2$: every conformal transformation, which is defined on a connected open subset $M \subset \mathbb{R}^{p,q}$, is injective and has a unique continuation to a global conformal transformation. (A global conformal transformation in the case of $\mathbb{R}^{p,q}$, $p+q > 2$, is a conformal transformation $\varphi : M \to \mathbb{R}^{p,q}$, which is defined on the entire set $\mathbb{R}^{p,q}$ with the possible exception of a hyperplane. By the results of Sect. 1.4.2, the domain M of definition of a global conformal transformation is $M = \mathbb{R}^{p,q}$ or $M = M_1$, see (1.3).)

Now, $N^{2,0}$ is isometrically isomorphic to the 2-sphere $\mathbb{S}^2$ (in general, one has $N^{p,0} \cong \mathbb{S}^p$, since $\mathbb{S}^p \times \mathbb{S}^0 = \mathbb{S}^p \times \{1, -1\}$) and hence $N^{2,0}$ is conformally isomorphic to the Riemann sphere $\mathbb{P} := \mathbb{P}_1(\mathbb{C})$.

Definition 2.12. A Möbius transformation is a holomorphic function φ, for which there is a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \quad \text{such that} \quad \varphi(z) = \frac{az+b}{cz+d}, cz+d \neq 0.$$

The set Mb of these Möbius transformations is precisely the set of all orientation-preserving global conformal transformations (in the sense of Definition 2.10). Mb forms a group with respect to composition (even though it is not a subgroup of the bijections of $\mathbb{C}$). For the exact definition of the group multiplication of φ and ψ one usually needs a continuation of $\varphi \circ \psi$ (cf. Lemma 2.13). This group operation coincides with the matrix multiplication in $\mathrm{SL}(2,\mathbb{C})$. Hence, Mb is also isomorphic to the group $\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$. Moreover, by Theorem 2.11, Mb is isomorphic to the group of orientation-preserving and conformal diffeomorphisms of $N^{2,0} \cong \mathbb{P}$, that is Mb is isomorphic to the group $\mathrm{Aut}(\mathbb{P})$ of all biholomorphic maps $\psi : \mathbb{P} \to \mathbb{P}$ of the Riemann sphere $\mathbb{P}$. This transition from the group Mb to $\mathrm{Aut}(\mathbb{P})$ using the compactification $\mathbb{C} \to \mathbb{P}$ has been used as a model for the compactification $N^{p,q}$ of $\mathbb{R}^{p,q}$ and the respective Theorem 2.9. Theorem 2.11 says even more: Mb is also isomorphic to the proper Lorentz group $\mathrm{SO}(3,1)$. An interpretation of the isomorphism $\mathrm{Aut}(\mathbb{P}) \cong \mathrm{SO}(3,1)$ from a physical viewpoint was given by Penrose, cf., e.g., [Scho95, p. 210]. In summary, we have

$$\mathrm{Mb} \cong \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Aut}(\mathbb{P}) \cong \mathrm{SO}(3,1) \cong \mathrm{Conf}(\mathbb{R}^{2,0}).$$

2.4 In What Sense Is the Conformal Group Infinite Dimensional?

We have seen in the preceding section that from the point of view of mathematics the conformal group of the Euclidean plane or the Euclidean 2-sphere is the group $\mathrm{Mb} \cong \mathrm{SO}(3,1)$ of Möbius transformations.

However, throughout physics texts on two-dimensional conformal field theory one finds the claim that the group $\mathscr{G}$ of conformal transformations on $\mathbb{R}^{2,0}$ is infinite dimensional, e.g.,

> "The situation is somewhat better in two dimensions. The main reason is that the conformal group is infinite dimensional in this case; it consists of the conformal analytical transformations..." and later "...the conformal group of the 2-dimensional space consists of all substitutions of the form
>
> $$z \mapsto \xi(z), \quad \bar{z} \mapsto \overline{\xi}(\bar{z}),$$
>
> where ξ and $\overline{\xi}$ are arbitrary analytic functions." [BPZ84, p. 335]

> "Two dimensions is an especially promising place to apply notions of conformal field invariance, because there the group of conformal transformations is infinite dimensional. Any analytical function mapping the complex plane to itself is conformal." [FQS84, p. 420]

> "The conformal group in 2-dimensional Euclidean space is infinite dimensional and has an algebra consisting of two commuting copies of the Virasoro algebras." [GO89, p. 333]

At first sight, the statements in these citations seem to be totally wrong. For instance, the class of all holomorphic (that is analytic) and injective functions $z \mapsto \xi(z)$ does not form a group – in contradiction to the first citation – since for two general holomorphic functions $f : U \to V$, $g : W \to Z$ with open subsets $U,V,W,Z \subset \mathbb{C}$, the composition $g \circ f$ can be defined at best if $f(U) \cap W \neq \emptyset$. Moreover, the non injective holomorphic functions are not invertible. If we restrict ourselves to the set J of all injective holomorphic functions the composition cannot define a group structure on

J because of the fact that $f(U) \subset W$ will, in general, be violated; even $f(U) \cap W = \emptyset$ can occur. Of course, J contains groups, e.g., Mb and the group of biholomorphic $f : U \to U$ on an open subset $U \subset \mathbb{C}$. However, these groups $\mathrm{Aut}(U)$ are not infinite dimensional, they are finite-dimensional Lie groups. If one tries to avoid the difficulties of $f(U) \cap W = \emptyset$ and requires – as the second citation [FQS84] seems to suggest – the transformations to be global, one obtains the finite-dimensional Möbius group. Even if one admits more than 1-point singularities, this yields no larger group than the group of Möbius transformations, as the following lemma shows:

Lemma 2.13. *Let $f : \mathbb{C} \setminus S \to \mathbb{C}$ be holomorphic and injective with a discrete set of singularities $S \subset \mathbb{C}$. Then, f is a restriction of a Möbius transformation. Consequently, it can be holomorphically continued on $\mathbb{C}$ or $\mathbb{C} \setminus \{p\}$, $p \in S$.*

Proof. By the theorem of Casorati–Weierstraß, the injectivity of f implies that all singularities are poles. Again from the injectivity it follows by the Riemann removable singularity theorem that at most one of these poles is not removable and this pole is of first order. □

The omission of larger parts of the domain or of the range also yields no infinite-dimensional group: doubtless, Mb should be a subgroup of the conformal group $\mathscr{G}$. For a holomorphic function $f : U \to V$, such that $\mathbb{C} \setminus U$ contains the disc D and $\mathbb{C} \setminus V$ contains the disc D', there always exists a Möbius transformation h with $h(V) \subset D'$ (inversion with respect to the circle $\partial D'$). Consequently, there is a Möbius transformation g with $g(V) \subset D$. But then $\mathrm{Mb} \cup \{f\}$ can generate no group, since f cannot be composed with $g \circ f$ because of $(g \circ f(U)) \cap U = \emptyset$. A similar statement is true for the remaining $f \in J$.

As a result, there can be no infinite dimensional conformal group $\mathscr{G}$ for the Euclidean plane.

What do physicists mean when they claim that the conformal group is infinite dimensional? The misunderstanding seems to be that physicists mostly think and calculate infinitesimally, while they write and talk globally. Many statements become clearer, if one replaces "group" with "Lie algebra" and "transformation" with "infinitesimal transformation" in the respective texts.

If, in the case of the Euclidean plane, one looks at the conformal Killing fields instead of conformal transformations (cf. Sect. 1.4.2), one immediately finds many infinite dimensional Lie algebras within the collection of conformal Killing fields. In particular, one finds the *Witt algebra*. In this context, the Witt algebra W is the complex vector space with basis $(L_n)_{n \in \mathbb{Z}}$, $L_n := -z^{n+1}\frac{d}{dz}$ or $L_n := z^{1-n}\frac{d}{dz}$ (cf. Sect. 5.2), and the Lie bracket

$$[L_n, L_m] = (n-m)L_{n+m}.$$

The Witt algebra will be studied in detail in Chap. 5 together with the Virasoro algebra.

In two-dimensional conformal field theory usually only the infinitesimal conformal invariance of the system under consideration is used. This implies the existence of an infinite number of independent constraints, which yields the exceptional feature of two-dimensional conformal field theory.

In this context the question arises whether there exists an abstract Lie group $\mathscr{G}$ such that the corresponding Lie algebra Lie $\mathscr{G}$ is essentially the algebra of infinitesimal conformal transformations. We come back to this question in Sect. 5.4 after having introduced and studied the Witt algebra and the Virasoro algebra in Chap. 5.

Another explanation for the claim that the conformal group is infinite dimensional can perhaps be given by looking at the Minkowski plane instead of the Euclidean plane. This is not the point of view in most papers on conformal field theory, but it fits in with the type of conformal invariance naturally appearing in string theory (cf. Chap. 2). Indeed, conformal symmetry was investigated in string theory, before the actual work on conformal field theory had been done. For the Minkowski plane, there is really an infinite dimensional conformal group, as we will show in the next section. The associated complexified Lie algebra is again essentially the Witt algebra (cf. Sect. 5.1).

Hence, on the infinitesimal level the cases $(p,q) = (2,0)$ and $(p,q) = (1,1)$ seem to be quite similar. However, in the interpretation and within the representation theory there are differences, which we will not discuss here in detail. We shall just mention that the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ belongs to the Witt algebra in the Euclidean case since it agrees with the Lie algebra of Mb generated by $L_{-1}, L_0, L_1 \in \mathsf{W}$, while in the Minkowski case $\mathfrak{sl}(2,\mathbb{C})$ is generated by complexification of $\mathfrak{sl}(2,\mathbb{R})$ which is a subalgebra of the infinitesimal conformal transformations of the Minkowski plane.

2.5 The Conformal Group of $\mathbb{R}^{1,1}$

By Theorem 1.13 the conformal transformations $\varphi : M \to \mathbb{R}^{1,1}$ on domains $M \subset \mathbb{R}^{1,1}$ are precisely the maps $\varphi = (u,v)$ with

$$u_x = v_y, u_y = v_x \text{ or } u_x = -v_y, u_y = -v_x,$$

and, in addition,

$$u_x^2 > v_x^2.$$

For $M = \mathbb{R}^{1,1}$ the global orientation-preserving conformal transformations can be described by using light cone coordinates $x_\pm = x \pm y$ in the following way:

Theorem 2.14. *For $f \in C^\infty(\mathbb{R})$ let $f_\pm \in C^\infty(\mathbb{R}^2,\mathbb{R})$ be defined by $f_\pm(x,y) := f(x \pm y)$. The map*

$$\begin{aligned} \Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}^2,\mathbb{R}^2), \\ (f,g) &\longmapsto \frac{1}{2}(f_+ + g_-, f_+ - g_-) \end{aligned}$$

has the following properties:

1. $\operatorname{im}\Phi = \{(u,v) \in C^\infty(\mathbb{R}^2,\mathbb{R}^2) : u_x = v_y, u_y = v_x\}$.
2. $\Phi(f,g)$ *conformal* $\iff f' > 0, g' > 0$ *or* $f' < 0, g' < 0$.

3. $\Phi(f,g)$ *bijective* $\Longleftrightarrow$ f *and* g *bijective.*
4. $\Phi(f\circ h, g\circ k) = \Phi(f,g)\circ\Phi(h,k)$ *for* $f,g,h,k \in C^\infty(\mathbb{R})$.

Hence, the group of orientation-preserving conformal diffeomorphisms

$$\varphi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$$

is isomorphic to the group

$$\big(\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})\big) \cup (\mathrm{Diff}_-(\mathbb{R}) \times \mathrm{Diff}_-(\mathbb{R})).$$

The group of all conformal diffeomorphisms $\varphi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$, endowed with the topology of uniform convergence of φ and all its derivatives on compact subsets of $\mathbb{R}^2$, consists of four components. Each component is homeomorphic to $\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$. Here, $\mathrm{Diff}_+(\mathbb{R})$ denotes the group of orientation-preserving diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ with the topology of uniform convergence of f and all its derivatives on compact subsets $K \subset \mathbb{R}$.

Proof.

1. Let $(u,v) = \Phi(f,g)$. From

$$u_x = \frac{1}{2}(f_+' + g_-'), u_y = \frac{1}{2}(f_+' - g_-'),$$
$$v_x = \frac{1}{2}(f_+' - g_-'), v_y = \frac{1}{2}(f_+' + g_-'),$$

it follows immediately that $u_x = v_y, u_y = v_x$. Conversely, let

$$(u,v) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$$

with $u_x = v_y, u_y = v_x$. Then $u_{xx} = v_{yx} = u_{yy}$. Now, a solution of the one-dimensional wave equation u has the form $u(x,y) = \frac{1}{2}(f_+(x,y) + g_-(x,y))$ with suitable $f,g \in C^\infty(\mathbb{R})$. Because of $v_x = u_y = \frac{1}{2}(f_+' - g_-')$ and $v_y = u_x = \frac{1}{2}(f_+' + g_-')$, we have $v = \frac{1}{2}(f_+ - g_-)$ where f and g possibly have to be changed by a constant.
2. For $(u,v) = \Phi(f,g)$ one has $u_x^2 - v_x^2 = f_+' g_-'$. Hence

$$u_x^2 - v_x^2 > 0 \Longleftrightarrow f_+' g_-' > 0 \Longleftrightarrow f'g' > 0.$$

3. Let f and g be injective. For $\varphi = \Phi(f,g)$ we have as follows:
$\varphi(x,y) = \varphi(x',y')$ implies

$$f(x+y) + g(x-y) = f(x'+y') + g(x'-y'),$$
$$f(x+y) - g(x-y) = f(x'+y') - g(x'-y').$$

Hence, $f(x+y) = f(x'+y')$ and $g(x-y) = g(x'-y')$, that is $x+y = x'+y'$ and $x-y = x'-y'$. This implies $x = x', y = y'$. So φ is injective if f and g are injective. From the preceding discussion one can see that if φ is injective then f and g are injective too. Let now f and g be surjective and $\varphi = \Phi(f,g)$. For $(\xi,\eta) \in \mathbb{R}^2$

there exist $s,t \in \mathbb{R}$ with $f(s) = \xi + \eta$, $g(t) = \xi - \eta$. Then $\varphi(x,y) = (\xi, \eta)$ with $x := \frac{1}{2}(s+t)$, $y := \frac{1}{2}(s-t)$. Conversely, the surjectivity of f and g follows from the surjectivity of φ.

4. With $\varphi = \Phi(f,g)$, $\psi = \Phi(h,k)$ we have $\varphi \circ \psi = \frac{1}{2}(f_+ \circ \psi + g_- \circ \psi, f_+ \circ \psi - g_- \circ \psi)$ and $f_+ \circ \psi = f(\frac{1}{2}(h_+ + k_-) + \frac{1}{2}(h_+ - k_-)) = f \circ h_+ = (f \circ h)_+$, etc. Hence

$$\varphi \circ \psi = \frac{1}{2}((f \circ h)_+ + (g \circ k)_-, (f \circ h)_+ - (g \circ k)_-) = \Phi(f \circ h, g \circ k). \qquad \square$$

As in the case $p = 2, q = 0$, there is no theorem similar to Theorem 2.9. For $p = q = 1$, the global conformal transformations need no continuation at all, hence a conformal compactification is not necessary. In this context it would make sense to define the conformal group of $\mathbb{R}^{1,1}$ simply as the connected component containing the identity of the group of conformal transformations $\mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$. This group is very large; it is by Theorem 2.14 isomorphic to $\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$.

However, for various reasons one wants to work with a group of transformations on a compact manifold with a conformal structure. Therefore, one replaces $\mathbb{R}^{1,1}$ with $\mathbb{S}^{1,1}$ in the sense of the conformal compactification of the Minkowski plane which we described at the beginning (cf. page 8):

$$\mathbb{R}^{1,1} \to \mathbb{S}^{1,1} = \mathbb{S} \times \mathbb{S} \subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,2}.$$

In this manner, one defines the conformal group $\mathrm{Conf}(\mathbb{R}^{1,1})$ as the connected component containing the identity in the group of all conformal diffeomorphisms $\mathbb{S}^{1,1} \to \mathbb{S}^{1,1}$. Of course, this group is denoted by $\mathrm{Conf}(\mathbb{S}^{1,1})$ as well.

In analogy to Theorem 2.14 one can describe the group of orientation-preserving conformal diffeomorphisms $\mathbb{S}^{1,1} \to \mathbb{S}^{1,1}$ using $\mathrm{Diff}_+(\mathbb{S})$ and $\mathrm{Diff}_-(\mathbb{S})$ (one simply has to repeat the discussion with the aid of 2π-periodic functions). As a consequence, the group of orientation-preserving conformal diffeomorphisms $\mathbb{S}^{1,1} \to \mathbb{S}^{1,1}$ is isomorphic to the group

$$(\mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})) \cup (\mathrm{Diff}_-(\mathbb{S}) \times \mathrm{Diff}_-(\mathbb{S})).$$

Corollary 2.15. $\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$.

In the course of the investigation of classical field theories with conformal symmetry $\mathrm{Conf}(\mathbb{R}^{1,1})$ and its quantization one is therefore interested in the properties of the group $\mathrm{Diff}_+(\mathbb{S})$ and even more (cf. the discussion of the preceding section) in its associated Lie algebra of infinitesimal transformations.

Now, $\mathrm{Diff}_+(\mathbb{S})$ turns out to be a Lie group with models in the Fréchet space of smooth $\mathbb{R}$-valued functions $f : \mathbb{S} \to \mathbb{R}$ endowed with the uniform convergence on $\mathbb{S}$ of f and all its derivatives. The corresponding Lie algebra $\mathrm{Lie}(\mathrm{Diff}_+(\mathbb{S}))$ is the Lie algebra of smooth vector fields $\mathrm{Vect}(\mathbb{S})$. The complexification of this Lie algebra contains the Witt algebra W (mentioned at the end of the preceding section 2.4) as a dense subspace.

For the quantization of such classical field theories the symmetry groups or algebras $\mathrm{Diff}_+(\mathbb{S})$, $\mathrm{Lie}(\mathrm{Diff}_+(\mathbb{S}))$, and W have to be replaced with suitable central extensions. We will explain this procedure in general for arbitrary symmetry algebras and groups in the following two chapters and introduce after that the Virasoro algebra Vir as a nontrivial central extension of the Witt algebra W in Chap. 5.

Remark 2.16. Recall that in the case of $n = p+q, p,q \geq 1$, but $(p,q) \neq (1,1)$, the conformal group has been identified with the group $\mathrm{SO}(p+1,q+1)$ or $\mathrm{SO}(p+1,q+1)/\{\pm 1\}$ using the natural compactifications of $\mathbb{R}^{p,q}$ described above. To have a finite dimensional counterpart to these conformal groups also in the case of $(p,q) = (1,1)$ one could call the group $\mathrm{SO}(2,2)/\{\pm 1\} \subset \mathrm{Conf}(\mathbb{S}^{1,1})$ the *restricted conformal group* of the (compactified) Minkowski plane and use it instead of the full infinite dimensional conformal group $\mathrm{Conf}(\mathbb{S}^{1,1})$.

The restricted conformal group is generated by the translations and the Lorentz transformations, which form a three-dimensional subgroup, and moreover by the dilatations and the special transformations.

Introducing again light cone coordinates replacing $(x,y) \in \mathbb{R}^2$ by

$$x_+ = x+y, \quad x_- = x-y,$$

the restricted conformal group $\mathrm{SO}(2,2)/\{\pm 1\}$ acts in the form of two copies of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. For $\mathrm{SL}(2,\mathbb{R})$-matrices

$$A_+ = \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix}, \quad A_- = \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix}$$

the action decouples in the following way:

$$(A_+,A_-)(x_+,x_-) = \left(\frac{a_+x_+ + b_+}{c_+x_+ + d_+}, \frac{a_-x_- + b_-}{c_-x_- + d_-} \right).$$

Proposition 2.17. *The action of the restricted conformal group decouples with respect to the light cone coordinates into two separate actions of* $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$*:*

$$\mathrm{SO}(2,2)/\{\pm 1\} \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}).$$

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