Part I
Mathematical Preliminaries

The first part of the notes begins with an elementary and detailed exposition of the notion of a conformal transformation in the case of the flat spaces $\mathbb{R}^{p,q}$ (Chap. 1) and a thorough investigation of the conformal groups, that is the groups of all conformal transformations on the corresponding compactified spaces $N^{p,q}$ (Chap. 2). As a result, the conformal groups are finite-dimensional Lie groups except for the case of the Minkowski plane. In the case of the Minkowski plane one obtains (two copies of) the infinite dimensional Witt algebra as a complexified Lie algebra of infinitesimal conformal transformations.

Chapters 3 and 4 deal with central extensions of groups and Lie algebras. Central extensions occur in a natural way if one studies projective representations and wants to compare them with true representation in the linear space to which the projective space is associated. Since quantization represents observables as linear operators in a linear (mostly Hilbert) space W and the space of quantum states is the associated projectivation $\mathbb{P}(W)$, it is unavoidable that central extensions of Lie groups and Lie algebras naturally appear as the quantization of classical symmetries.

The first part of the notes concludes with an elementary description of the Virasoro algebra as the only nontrivial central extension of the Witt algebra (Chap. 5).

As a consequence, in a two-dimensional conformally invariant quantum field theory the Virasoro algebra shall be a symmetry algebra providing the theory with an infinite collection of invariants of motion.

Chapter 1
Conformal Transformations and Conformal Killing Fields

This chapter presents the notion of a conformal transformation on general semi-Riemannian manifolds and gives a complete description of all conformal transformations on an open connected subset $M \subset \mathbb{R}^{p,q}$ in the flat spaces $\mathbb{R}^{p,q}$. Special attention is given to the two-dimensional cases, that is to the Euclidean plane $\mathbb{R}^{2,0}$ and to the Minkowski plane $R^{1,1}$.

1.1 Semi-Riemannian Manifolds

Definition 1.1. A *semi-Riemannian manifold* is a pair (M,g) consisting of a smooth[1] manifold M of dimension n and a smooth tensor field g which assigns to each point $a \in M$ a nondegenerate and symmetric bilinear form on the tangent space T_aM:

$$g_a : T_aM \times T_aM \to \mathbb{R}.$$

In local coordinates $x^1,\dots,x^n$ of the manifold M (given by a chart $\phi : U \to V$ on an open subset U in M with values in an open subset $V \subset \mathbb{R}^n$, $\phi(a) = (x^1(a),\dots,x^n(a))$, $a \in M$) the bilinear form g_a on T_aM can be written as

$$g_a(X,Y) = g_{\mu\nu}(a)X^\mu Y^\nu.$$

Here, the tangent vectors $X = X^\mu\partial_\mu$, $Y = Y^\nu\partial_\nu \in T_aM$ are described with respect to the basis

$$\partial_\mu := \frac{\partial}{\partial x^\mu}, \quad \mu = 1,\dots,n,$$

of the tangent space T_aM which is induced by the chart ϕ.

By assumption, the matrix

$$(g_{\mu\nu}(a))$$

is nondegenerate and symmetric for all $a \in U$, that is one has

[1] We restrict our study to smooth (that is to $\mathscr{C}^\infty$ or infinitely differentiable) mappings and manifolds.

Schottenloher, M.: *Conformal Transformations and Conformal Killing Fields*. Lect. Notes Phys. **759**, 7–21 (2008)
DOI 10.1007/978-3-540-68628-6_1

$$\det(g_{\mu\nu}(a)) \neq 0 \quad \text{and} \quad (g_{\mu\nu}(a))^T = (g_{\mu\nu}(a)).$$

Moreover, the differentiability of g implies that the matrix $(g_{\mu\nu}(a))$ depends differentiably on a. This means that in its dependence on the local coordinates x^j the coefficients $g_{\mu\nu} = g_{\mu\nu}(x)$ are smooth functions.

In general, however, the condition $g_{\mu\nu}(a)X^\mu X^\nu > 0$ does not hold for all $X \neq 0$, that is the matrix $(g_{\mu\nu}(a))$ is not required to be positive definite. This property distinguishes Riemannian manifolds from general semi-Riemannian manifolds. The Lorentz manifolds are specified as the semi-Riemannian manifolds with $(p,q) = (n-1,1)$ or $(p,q) = (1,n-1)$.

Examples:

- $\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, g^{p,q})$ for $p,q \in \mathbb{N}$ where

$$g^{p,q}(X,Y) := \sum_{i=1}^{p} X^i Y^i - \sum_{i=p+1}^{p+q} X^i Y^i.$$

  Hence

$$(g_{\mu\nu}) = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1).$$

- $\mathbb{R}^{1,3}$ or $\mathbb{R}^{3,1}$: the usual *Minkowski space*.
- $\mathbb{R}^{1,1}$: the two-dimensional Minkowski space (the *Minkowski plane*).
- $\mathbb{R}^{2,0}$: the *Euclidean plane*.
- $\mathbb{S}^2 \subset \mathbb{R}^{3,0}$: compactification of $\mathbb{R}^{2,0}$; the structure of a Riemannian manifold on the 2-sphere $\mathbb{S}^2$ is induced by the inclusion in $\mathbb{R}^{3,0}$.
- $\mathbb{S} \times \mathbb{S} \subset \mathbb{R}^{2,2}$: compactification of $\mathbb{R}^{1,1}$. More precisely, $\mathbb{S} \times \mathbb{S} \subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,2} \cong \mathbb{R}^{2,2}$ where the first circle $\mathbb{S} = \mathbb{S}^1$ is contained in $\mathbb{R}^{2,0}$, the second one in $\mathbb{R}^{0,2}$ and where the structure of a semi-Riemannian manifold on $\mathbb{S} \times \mathbb{S}$ is induced by the inclusion into $\mathbb{R}^{2,2}$.
- Similarly, $\mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1,0} \times \mathbb{R}^{0,q+1} \cong \mathbb{R}^{p+1,q+1}$, with the p-sphere $\mathbb{S}^p = \{X \in \mathbb{R}^{p+1} : g^{p+1,0}(X,X) = 1\} \subset \mathbb{R}^{p+1,0}$ and the q-sphere $\mathbb{S}^q \subset \mathbb{R}^{0,q+1}$, as a generalization of the previous example, yields a compactification of $\mathbb{R}^{p,q}$ for $p,q \geq 1$. This compact semi-Riemannian manifold will be denoted by $\mathbb{S}^{p,q}$ for all $p,q \geq 0$.

In the following, we will use the above examples of semi-Riemannian manifolds and their open subspaces only—except for the quadrics $N^{p,q}$ occurring in Sect. 2.1. (These quadrics are locally isomorphic to $\mathbb{S}^{p,q}$ from the point of view of conformal geometry.)

1.2 Conformal Transformations

Definition 1.2. Let (M,g) and (M',g') be two semi-Riemannian manifolds of the same dimension n and let $U \subset M, V \subset M'$ be open subsets of M and M', respectively. A smooth mapping $\varphi : U \to V$ of maximal rank is called a *conformal transformation*, or *conformal map*, if there is a smooth function $\Omega : U \to \mathbb{R}_+$ such that

$$\varphi^* g' = \Omega^2 g\,,$$

where $\varphi^* g'(X,Y) := g'(T\varphi(X), T\varphi(Y))$ and $T\varphi : TU \to TV$ denotes the tangent map (derivative) of φ. Ω is called the *conformal factor* of φ. Sometimes a conformal transformation $\varphi : U \to V$ is additionally required to be bijective and/or orientation preserving.

In local coordinates of M and M'

$$(\varphi^* g')_{\mu\nu}(a) = g'_{ij}(\varphi(a))\partial_\mu \varphi^i \partial_\nu \varphi^j.$$

Hence, φ is conformal if and only if

$$\Omega^2 g_{\mu\nu} = (g'_{ij} \circ \varphi)\partial_\mu \varphi^i \partial_\nu \varphi^j \tag{1.1}$$

in the coordinate neighborhood of each point.

Note that for a conformal transformation φ the tangent maps $T_a\varphi : T_aM \to T_{\varphi(a)}M'$ are bijective for each point $a \in U$. Hence, by the inverse mapping theorem a conformal transformation is always locally invertible as a smooth map.

Examples:

- Local isometries, that is smooth mappings φ with $\varphi^* g' = g$, are conformal transformations with conformal factor $\Omega = 1$.
- In order to study conformal transformations on the Euclidean plane $\mathbb{R}^{2,0}$ we identify $\mathbb{R}^{2,0} \cong \mathbb{C}$ and write $z = x + iy$ for $z \in \mathbb{C}$ with "real coordinates" $(x,y) \in \mathbb{R}$. Then a smooth map $\varphi : M \to \mathbb{C}$ on a connected open subset $M \subset \mathbb{C}$ is conformal according to (1.1) with conformal factor $\Omega : M \to \mathbb{R}_+$ if and only if for $u = \operatorname{Re}\varphi$ and $v = \operatorname{Im}\varphi$

$$u_x^2 + v_x^2 = \Omega^2 = u_y^2 + v_y^2 \neq 0\,,\, u_x u_y + v_x v_y = 0. \tag{1.2}$$

  These equations are, of course, satisfied by the holomorphic (resp. antiholomorphic) functions from M to $\mathbb{C}$ because of the Cauchy–Riemann equations $u_x = v_y, u_y = -v_x$ (resp. $u_x = -v_y, u_y = v_x$) if $u_x^2 + v_x^2 \neq 0$. For holomorphic or antiholomorphic functions, $u_x^2 + v_x^2 \neq 0$ is equivalent to $\det D\varphi \neq 0$ where $D\varphi$ denotes the Jacobi matrix representing the tangent map $T\varphi$ of φ.

  Conversely, for a general conformal transformation $\varphi = (u,v)$ the equations (1.2) imply that (u_x, v_x) and (u_y, v_y) are perpendicular vectors in $\mathbb{R}^{2,0}$ of equal length $\Omega \neq 0$. Hence, $(u_x, v_x) = (-v_y, u_y)$ or $(u_x, v_x) = (v_y, -u_y)$, that is φ is holomorphic or antiholomorphic with nonvanishing $\det D\varphi$.

As a **first important result**, we have shown that the conformal transformations $\varphi : M \to \mathbb{C}$ with respect to the Euclidean structure on $M \subset \mathbb{C}$ are the locally invertible holomorphic or antiholomorphic functions. The conformal factor of φ is $|\det D\varphi|$.

- With the same identification $\mathbb{R}^{2,0} \cong \mathbb{C}$ a linear map $\varphi : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ with representing matrix
$$A = A_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
is conformal if and only if $a^2 + c^2 \neq 0$ and $a = d, b = -c$ or $a = -d, b = c$. As a consequence, for $\zeta = a + ic \neq 0$, φ is of the form $z \mapsto \zeta z$ or $z \mapsto \zeta \bar{z}$.
These conformal linear transformations are *angle preserving* in the following sense: for points $z, w \in \mathbb{C} \setminus \{0\}$ the number
$$\omega(z,w) := \frac{z\overline{w}}{|zw|}$$
determines the (Euclidean) angle between z and w up to orientation. In the case of $\varphi(z) = \zeta z$ it follows that
$$\omega(\varphi(z), \varphi(w)) = \frac{\zeta z \overline{\zeta w}}{|\zeta z \zeta w|} = \omega(z,w),$$
and the same holds for $\varphi(z) = \zeta \bar{z}$.
Conversely, the linear maps φ with $\omega(\varphi(z), \varphi(w)) = \omega(z,w)$ for all $z, w \in \mathbb{C} \setminus \{0\}$ or $\omega(\varphi(z), \varphi(w)) = -\omega(z,w)$ for all $z, w \in \mathbb{C} \setminus \{0\}$ are conformal transformations. We conclude that an $\mathbb{R}$-linear map $\varphi : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ is a conformal transformation for the Euclidean plane if and only if it is angle preserving.
- We have shown that an orientation-preserving $\mathbb{R}$-linear map $\varphi : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ is a conformal transformation for the Euclidean plane if and only if it is the multiplication with a complex number $\zeta \neq 0$: $z \mapsto \zeta z$. In the case of $\zeta = r \exp i\alpha$ with $r \in \mathbb{R}_+$ and with $\alpha \in]0, 2\pi]$, we obtain the following interpretation: α induces a rotation with angle α and $z \mapsto (\exp i\alpha)z$ is an isometry, while r induces a dilatation $z \mapsto rz$.
Consequently, the group of orientation-preserving $\mathbb{R}$-linear and conformal maps $\mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ is isomorphic to $\mathbb{R}_+ \times \mathbb{S} \cong \mathbb{C} \setminus \{0\}$. The group of orientation-preserving $\mathbb{R}$-linear isometries is isomorphic to $\mathbb{S}$ while the group of dilatations is isomorphic to $\mathbb{R}_+$ (with the multiplicative structure) and therefore isomorphic to the additive group $\mathbb{R}$ via $t \to r := \exp t,\ t \in \mathbb{R}$.
- The above considerations also show that the conformal transformations $\varphi : M \to \mathbb{C}$, where M is an open subset of $\mathbb{R}^{2,0}$, can also be characterized as those mappings which preserve the angles infinitesimally: let $z(t), w(t)$ be smooth curves in M with $z(0) = w(0) = a$ and $\dot{z}(0) \neq 0 \neq \dot{w}(0)$, where $\dot{z}(0) = \frac{d}{dt} z(t)|_{t=0}$ is the derivative of $z(t)$ at $t = 0$. Then $\omega(\dot{z}(0), \dot{w}(0))$ determines the angle between the

curves $z(t)$ and $w(t)$ at the common point a. Let $z_\varphi = \varphi \circ z$ and $w_\varphi = \varphi \circ w$ be the image curves. By definition, φ is called to preserve angles infinitesimally if and only if $\omega(\dot{z}(0), \dot{w}(0)) = \omega(\dot{z_\varphi}(0), \dot{w_\varphi}(0))$ for all points $a \in M$ and all curves $z(t), w(t)$ in M through $a = z(0) = w(0)$ with $\dot{z}(0) \neq 0 \neq \dot{w}(0)$. Note that $\dot{z_\varphi}(0) = D\varphi(a)(\dot{z}(0))$ by the chain rule. Hence, by the above characterization of the linear conformal transformations, φ preserves angles infinitesimally if and only if $D\varphi(a)$ is a linear conformal transformation for all $a \in M$ which by (1.2) is equivalent to φ being a conformal transformation.

- Again in the case of $\mathbb{R}^{2,0} \cong \mathbb{C}$ one can deduce from the above results that the conformal, orientation-preserving, and bijective transformations $\mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ are the entire holomorphic functions $\varphi : \mathbb{C} \to \mathbb{C}$ with holomorphic inverse functions $\varphi^{-1} : \mathbb{C} \to \mathbb{C}$, that is the biholomorphic functions $\varphi : \mathbb{C} \to \mathbb{C}$. These functions are simply the complex-linear affine maps of the form

$$\varphi(z) = \zeta z + \tau,\ z \in \mathbb{C},$$

with $\zeta, \tau \in \mathbb{C}$, $\zeta \neq 0$.

The group of all conformal, orientation-preserving invertible transformations $\mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$ of the Euclidean plane can thus be identified with $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, where the group law is given by

$$(\zeta, \tau)(\zeta', \tau') = (\zeta\zeta', \zeta\tau' + \tau).$$

In particular, this group is a four-dimensional real manifold.

This is an example of a semidirect product of groups. See Sect. 3.1 for the definition.

- The orientation-preserving and $\mathbb{R}$-linear conformal transformations $\psi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ can be identified by elementary matrix multiplication. They are represented by matrices of the form

$$A = A_\psi = A(s,t) = \exp t \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$$

with $(s,t) \in \mathbb{R}^2$ (see Corollary 1.14 for details).

- Consider $\mathbb{R}^2$ endowed with the metric on $\mathbb{R}^2$ given by the bilinear form

$$\langle (x,y), (x',y') \rangle := \frac{1}{2}(xy' + yx').$$

This is a Minkowski metric g on $\mathbb{R}^2$, for which the coordinate axes coincide with the light cone

$$L = \{(x,y) : \langle (x,y), (x,y) \rangle = 0\}$$

in $0 \in \mathbb{R}^2$. With this metric, $(\mathbb{R}^2, g)$ is isometrically isomorphic to $\mathbb{R}^{1,1}$ with respect to the isomorphism $\psi : \mathbb{R}^{1,1} \to \mathbb{R}^2$,

$$(x,y) \mapsto (x+y, x-y).$$

- The *stereographic projection*

$$\begin{aligned} \pi : \mathbb{S}^2 \setminus \{(0,0,1)\} &\to \mathbb{R}^{2,0}, \\ (x,y,z) &\mapsto \frac{1}{1-z}(x,y) \end{aligned}$$

  is conformal with $\Omega = \frac{1}{1-z}$. In order to prove this it suffices to show that the inverse map $\varphi := \pi^{-1} : \mathbb{R}^{2,0} \to \mathbb{S}^2 \subset \mathbb{R}^{3,0}$ is a conformal transformation. We have

$$\varphi(\xi,\eta) = \frac{1}{1+r^2}(2\xi, 2\eta, r^2 - 1),$$

  for $(\xi,\eta) \in \mathbb{R}^2$ and $r = \sqrt{\xi^2+\eta^2}$. For the tangent vectors $X_1 = \frac{\partial}{\partial \xi}, X_2 = \frac{\partial}{\partial \eta}$ we get

$$\begin{aligned} T\varphi(X_1) &= \frac{d}{dt}\varphi(\xi+t,\eta)|_{t=0} \\ &= 2\left(\frac{1}{1+r^2}\right)^2 (r^2+1-2\xi^2, -2\xi\eta, 2\xi), \\ T\varphi(X_2) &= 2\left(\frac{1}{1+r^2}\right)^2 (-2\xi\eta, r^2+1-2\eta^2, 2\eta). \end{aligned}$$

  Hence

$$g'(T\varphi(X_i), T\varphi(X_j)) = \left(\frac{2}{1+r^2}\right)^2 (\delta_{ij}),$$

  that is $\Lambda = \frac{2}{1+r^2}$ is the conformal factor of φ. Thus, $\pi = \varphi^{-1}$ has the conformal factor $\Omega = \Lambda^{-1} = \frac{1}{2}(1+r^2) = \frac{1}{1-z}$.
  Similarly, the stereographic projection of the n-sphere,

$$\begin{aligned} \pi : \mathbb{S}^n \setminus \{(0,\ldots,0,1)\} &\to \mathbb{R}^{n,0}, \\ (x^0,\ldots,x^n) &\mapsto \frac{1}{1-x^n}(x^0,\ldots,x^{n-1}), \end{aligned}$$

  is a conformal map.
- In Proposition 2.5 we present another natural conformal map in detail, the conformal embedding

$$\tau : \mathbb{R}^{p,q} \to \mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1,q+1}$$

  into the non-Riemannian version of $\mathbb{S}^p \times \mathbb{S}^q$. $\mathbb{S}^p \times \mathbb{S}^q$ has been described in the preceding section.
- The composition of two conformal maps is conformal.
- If $\varphi : M \to M'$ is a bijective conformal transformation with conformal factor Ω then φ is a diffeomorphism (that is φ^{-1} is smooth) and, moreover, $\varphi^{-1} : M' \to M$ is conformal with conformal factor $\frac{1}{\Omega}$. This property has been used in the investigation of the above example on the stereographic projection.

1.3 Conformal Killing Fields

In the following, we want to study the conformal maps $\varphi : M \to M'$ between open subsets $M, M' \subset \mathbb{R}^{p,q}$, $p+q=n>1$. To begin with, we will classify them by an infinitesimal argument:

Let $X : M \subset \mathbb{R}^{p,q} \to \mathbb{R}^n$ be a smooth vector field. Then

$$\dot{\gamma} = X(\gamma)$$

for smooth curves $\gamma = \gamma(t)$ in M is an autonomous differential equation. The *local one-parameter group* $(\varphi_t^X)_{t\in\mathbb{R}}$ corresponding to X satisfies

$$\frac{d}{dt}(\varphi^X(t,a)) = X(\varphi^X(t,a))$$

with initial condition $\varphi^X(0,a) = a$. Moreover, for every $a \in U$, $\varphi^X(\cdot,a)$ is the unique maximal solution of $\dot{\gamma} = X(\gamma)$ defined on the maximal interval $]t_a^-, t_a^+[$. Let $M_t := \{a \in M : t_a^- < t < t_a^+\}$ and $\varphi_t^X(a) := \varphi^X(t,a)$ for $a \in M_t$. Then $M_t \subset M$ is an open subset of M and $\varphi_t^X : M_t \to M_{-t}$ is a diffeomorphism. Furthermore, we have $\varphi_t^X \circ \varphi_s^X(a) = \varphi_{s+t}^X(a)$ if $a \in M_{t+s} \cap M_s$ and $\varphi_s^X(a) \in M_t$, and, of course, $\varphi_0^X = \mathrm{id}_M, M_0 = M$. In particular, the local one-parameter group $(\varphi_t^X)_{t\in\mathbb{R}}$ satisfies the *flow equation*

$$\frac{d}{dt}(\varphi_t^X)|_{t=0} = X.$$

Definition 1.3. A vector field X on $M \subset \mathbb{R}^{p,q}$ is called a *conformal Killing field* if φ_t^X is conformal for all t in a neighborhood of 0.

Theorem 1.4. *Let $M \subset \mathbb{R}^{p,q}$ be open, $g = g^{p,q}$ and X a conformal Killing field with coordinates*

$$X = (X^1, \dots, X^n) = X^\nu \partial_\nu$$

with respect to the canonical cartesian coordinates on $\mathbb{R}^n$. Then there is a smooth function $\kappa : M \to \mathbb{R}$, so that

$$X_{\mu,\nu} + X_{\nu,\mu} = \kappa g_{\mu\nu}.$$

Here we use the notation: $f_{,\nu} := \partial_\nu f$, $X_\mu := g_{\mu\nu}X^\nu$.

Proof. Let X be a conformal Killing field, (φ_t) the associated local one-parameter group, and $\Omega_t : M_t \to \mathbb{R}^+$, such that

$$(\varphi_t^* g)_{\mu\nu}(a) = g_{ij}(\varphi_t(a))\partial_\mu \varphi_t^i \partial_\nu \varphi_t^j = (\Omega_t(a))^2 g_{\mu\nu}(a).$$

By differentiation with respect to t at $t=0$ we get (g_{ij} is constant!)

$$
\begin{aligned}
\frac{d}{dt}(\Omega_t^2(a)\, g_{\mu\nu}(a))|_{t=0} &= \frac{d}{dt}\,(g_{ij}(\varphi_t(a))\partial_\mu \varphi_t^i\,\partial_\nu \varphi_t^j)\Big|_{t=0} \\
&= g_{ij}\partial_\mu \dot{\varphi}_0^i\,\partial_\nu \varphi_0^j + g_{ij}\partial_\mu \varphi_0^i\,\partial_\nu \dot{\varphi}_0^j \\
&= g_{ij}\,\partial_\mu X^i(a)\,\delta_\nu^j + g_{ij}\,\delta_\mu^i\,\partial_\nu X^j(a) \\
&= \partial_\mu X_\nu(a) + \partial_\nu X_\mu(a).
\end{aligned}
$$

Hence, the statement follows with $\kappa(a) = \dfrac{d}{dt}\,\Omega_t^2(a)\Big|_{t=0}$. □

If $g_{\mu\nu}$ is not constant, we have

$$(L_X g)_{\mu\nu} = X_{\mu;\nu} + X_{\nu;\mu} = \kappa g_{\mu\nu}.$$

Here, L_X is the *Lie derivative* and a semicolon in the index denotes the covariant derivative corresponding to the Levi-Civita connection for g.

Definition 1.5. A smooth function $\kappa : M \subset \mathbb{R}^{p,q} \to \mathbb{R}$ is called a *conformal Killing factor* if there is a conformal Killing field X, such that

$$X_{\mu,\nu} + X_{\nu,\mu} = \kappa g_{\mu\nu}.$$

(Similarly, for general semi-Riemannian manifolds on coordinate neighborhoods:

$$X_{\mu;\nu} + X_{\nu;\mu} = \kappa g_{\mu\nu}.)$$

Theorem 1.6. *$\kappa : M \to \mathbb{R}$ is a conformal Killing factor if and only if*

$$(n-2)\kappa_{,\mu\nu} + g_{\mu\nu}\Delta_g \kappa = 0,$$

where $\Delta_g = g^{kl}\partial_k\partial_l$ is the Laplace–Beltrami operator *for $g = g^{p,q}$.*

Proof. "⇒": Let $\kappa : M \to \mathbb{R}$ and $X_{\mu,\nu} + X_{\nu,\mu} = \kappa g_{\mu\nu}$ ($M \subset \mathbb{R}^{p,q}, g = g^{p,q}$). Then from

$$\partial_k\partial_l(X_{\mu,\nu}) = \partial_\nu\partial_k(X_{\mu,l}), \quad \text{etc.},$$

it follows that

$$
\begin{aligned}
0 = {} & \partial_k\partial_l(X_{\mu,\nu} + X_{\nu,\mu}) - \partial_l\partial_\mu(X_{k,\nu} + X_{\nu,k}) \\
& + \partial_\mu\partial_\nu(X_{k,l} + X_{l,k}) - \partial_\nu\partial_k(X_{\mu,l} + X_{l,\mu}).
\end{aligned}
$$

Since κ is a conformal Killing factor, one can deduce

$$\partial_k\partial_l(X_{\mu,\nu} + X_{\nu,\mu}) = \kappa_{,kl}\, g_{\mu\nu}, \quad \text{etc.}$$

Hence

$$0 = g_{\mu\nu}\,\kappa_{,kl} - g_{k\nu}\,\kappa_{,l\mu} + g_{kl}\,\kappa_{,\mu\nu} - g_{\mu l}\,\kappa_{,\nu k}.$$

By multiplication with g^{kl} (defined by $g^{\mu\lambda}g_{\lambda\nu} = \delta_\nu^\mu$) we get

$$\begin{aligned} 0 &= g^{kl} g_{\mu\nu}\, \kappa_{,kl} - g^{kl} g_{k\nu}\, \kappa_{,l\mu} + g^{kl} g_{kl}\, \kappa_{,\mu\nu} - g^{kl} g_{\mu l}\, \kappa_{,\nu k} \\ &= g^{kl}(g_{\mu\nu}\, \kappa_{,kl}) - \delta^l_\nu\, \kappa_{,l\mu} + n\, \kappa_{,\mu\nu} - \delta^k_\mu\, \kappa_{,l\mu} \\ &= g_{\mu\nu} \Delta_g \kappa + (n-2)\kappa_{,\mu\nu}. \end{aligned}$$

The reverse implication "$\Leftarrow$" follows from the discussion in Sect. 1.4. □

The theorem also holds for open subsets M in semi-Riemannian manifolds with ";" instead of ",".

Important Observation. In the case $n = 2$, κ is conformal if and only if $\Delta_g \kappa = 0$. For $n > 2$, however, there are many additional conditions. More precisely, these are

$$\begin{aligned} \kappa_{,\mu\nu} &= 0 \text{ for } \mu \neq \nu, \\ \kappa_{,\mu\mu} &= \pm(n-2)^{-1} \Delta_g \kappa. \end{aligned}$$

1.4 Classification of Conformal Transformations

With the help of the implication "$\Rightarrow$" of Theorem 1.6, we will determine all conformal Killing fields and hence all conformal transformations on connected open sets $M \subset \mathbb{R}^{p,q}$.

1.4.1 Case 1: $n = p + q > 2$

From the equations $g_{\mu\mu}(n-2)\kappa_{,\mu\mu} + \Delta_g \kappa = 0$ for a conformal Killing factor κ we get $(n-2)\Delta_g \kappa + n\Delta_g \kappa = 0$ by summation, hence $\Delta_g \kappa = 0$ (as in the case $n = 2$). Using again $g_{\mu\mu}(n-2)\kappa_{,\mu\mu} + \Delta_g \kappa = 0$, it follows that $\kappa_{,\mu\mu} = 0$. Consequently, $\kappa_{,\mu\nu} = 0$ for all μ, ν. Hence, there are constants $\alpha_\mu \in \mathbb{R}$ such that

$$\kappa_{,\mu}(q^1, \ldots, q^n) = \alpha_\mu\,, \quad \mu = 1, \ldots, n.$$

It follows that the solutions of $(n-2)\kappa_{,\mu\nu} + g_{\mu\nu}\Delta_g \kappa = 0$ are the affine-linear maps

$$\kappa(q) = \lambda + \alpha_\nu q^\nu, \quad q = (q^\nu) \in M \subset \mathbb{R}^n,$$

with $\lambda, \alpha_\nu \in \mathbb{R}$.

To begin with a complete description of all conformal Killing fields on connected open subsets $M \subset \mathbb{R}^{p,q}$, $p + q > 2$, we first determine the conformal Killing fields X with conformal Killing factor $\kappa = 0$ (that is the proper Killing fields, which belong to local isometries). $X_{\mu,\mu} + X_{\mu,\mu} = 0$ means that X^μ does not depend on q^μ. $X_{\mu,\nu} + X_{\nu,\mu} = 0$ implies $X^\mu_{,\nu} = 0$. Thus X^μ can be written as

$$X^\mu(q) = c^\mu + \omega^\mu_\nu q^\nu$$

with $c^\mu \in \mathbb{R}$, $\omega^\mu_\nu \in \mathbb{R}$.

If all the coefficients ω^μ_ν vanish, the vector field $X^\mu(q) = c^\mu$ determines the differential equation

$$\dot{q} = c,$$

with the (global) one-parameter group $\varphi^X(t,q) = q + tc$ as its flow. The associated conformal transformation ($\varphi^X(t,q)$ for $t = 1$) is the *translation*

$$\varphi_c(q) = q + c.$$

For $c = 0$ and general $\omega = (\omega^\mu_\nu)$ the equations

$$X_{\mu,\nu} + X_{\nu,\mu} = g_{\mu\nu}\kappa = 0$$

imply

$$g_{\nu\rho}\,\omega^\rho_\mu + g_{\mu\rho}\,\omega^\rho_\nu = 0,$$

that is $\omega^T g + g\,\omega = 0$. Hence, these solutions are given by the elements of the Lie algebra $\mathfrak{o}(p,q) := \{\omega : \omega^T g^{p,q} + g^{p,q}\omega = 0\}$. The associated conformal transformations ($\varphi^X(t,q) = e^{t\omega}q$ for $t = 1$) are the *orthogonal transformations*

$$\varphi_\Lambda : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}, \quad q \mapsto \Lambda q,$$

with

$$\Lambda = e^\omega \in \mathrm{O}(p,q) := \{\Lambda \in \mathbb{R}^{n\times n} : \Lambda^T g^{p,q}\Lambda = g^{p,q}\}$$

(equivalently, $\mathrm{O}(p,q) = \{\Lambda \in \mathbb{R}^{n\times n} : \langle \Lambda x, \Lambda x'\rangle = \langle x, x'\rangle\}$ with the symmetric bilinear form $\langle\cdot,\cdot\rangle$ given by $g^{p,q}$).

We have thus determined all local isometries on connected open subsets $M \subset \mathbb{R}^{p,q}$. They are the restrictions of maps

$$\varphi(q) = \varphi_\Lambda(q) + c, \quad \Lambda \in \mathrm{O}(p,q), \quad c \in \mathbb{R}^n,$$

and form a finite-dimensional Lie group, the group of motions belonging to $g^{p,q}$. This group can also be described as a semidirect product (cf. Sect. 3.1) of $\mathrm{O}(p,q)$ and $\mathbb{R}^n$.

The constant conformal Killing factors $\kappa = \lambda \in \mathbb{R}\setminus\{0\}$ correspond to the conformal Killing fields $X(q) = \lambda q$ belonging to the conformal transformations

$$\varphi(q) = e^\lambda q, \quad q \in \mathbb{R}^n,$$

which are the *dilatations*.

All the conformal transformations on $M \subset \mathbb{R}^{p,q}$ considered so far have a unique conformal continuation to $\mathbb{R}^{p,q}$. Hence, they are essentially conformal transformations on all of $\mathbb{R}^{p,q}$ associated to global one-parameter groups (φ_t). This is no longer true for the following conformal transformations.

In view of the preceding discussion, every conformal Killing factor $\kappa \neq 0$ without a constant term is linear and thus can be written as

$$\kappa(q) = 4\langle q, b\rangle, \quad q \in \mathbb{R}^n,$$

with $b \in \mathbb{R}^n \setminus \{0\}$ and $\langle q, b\rangle = g^{p,q}_{\mu\nu} q^\mu b^\nu$. A direct calculation shows that

$$X^\mu(q) := 2\langle q, b\rangle q^\mu - \langle q, q\rangle b^\mu, \quad q \in \mathbb{R}^n,$$

is a solution of $X_{\mu,\nu} + X_{\nu,\mu} = \kappa g_{\mu\nu}$. (This proves the implication "$\Leftarrow$" in Theorem 1.6 for $n > 2$.) As a consequence, for every conformal Killing field X with conformal Killing factor

$$\kappa(q) = \lambda + x_\mu q^\mu = \lambda + 4\langle q, b\rangle,$$

the vector field $Y(q) = X(q) - 2\langle q, b\rangle q^\mu - \langle q, q\rangle b^\mu - \lambda q$ is a conformal Killing field with conformal Killing factor 0. Hence, by the preceding discussion, it has the form $Y(q) = c + \omega q$. To sum up, we have proven

Theorem 1.7. *Every conformal Killing field X on a connected open subset M of $\mathbb{R}^{p,q}$ (in case of $p+q = n > 2$) is of the form*

$$X(q) = 2\langle q, b\rangle q^\mu - \langle q, q\rangle b^\mu + \lambda q + c + \omega q$$

with suitable $b, c \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\omega \in \mathfrak{o}(p, q)$.

Exercise 1.8. The Lie bracket of two conformal Killing fields is a conformal Killing field. The Lie algebra of all the conformal Killing fields is isomorphic to $\mathfrak{o}(p+1, q+1)$ (cf. Exercise 1.10).

The conformal Killing field $X(q) = 2\langle q, b\rangle q - \langle q, q\rangle b$, $b \neq 0$, has no global one-parameter group of solutions for the equation $\dot{q} = X(q)$. Its solutions form the following local one-parameter group

$$\varphi_t(q) = \frac{q - \langle q, q\rangle tb}{1 - 2\langle q, tb\rangle + \langle q, q\rangle\langle tb, tb\rangle}, \quad t \in]t_q^-, t_q^+[\,,$$

where $]t_q^-, t_q^+[$ is the maximal interval around 0 contained in

$$\{t \in \mathbb{R} \mid 1 - 2\langle q, tb\rangle + \langle q, q\rangle\langle tb, tb\rangle \neq 0\}.$$

Hence, the associated conformal transformation $\varphi := \varphi_1$

$$\varphi(q) = \frac{q - \langle q, q\rangle b}{1 - 2\langle b, q\rangle + \langle q, q\rangle\langle b, b\rangle}$$

– which is called a *special conformal transformation* – has (as a map into $\mathbb{R}^{p,q}$) a continuation at most to M_t at $t = 1$, that is to

$$M = M_1 := \{ q \in \mathbb{R}^{p,q} | 1 - 2\langle b,q\rangle + \langle q,q\rangle\langle b,b\rangle \neq 0 \}. \tag{1.3}$$

In summary, we have

Theorem 1.9. *Every conformal transformation $\varphi : M \to \mathbb{R}^{p,q}$, $n = p+q \geq 3$, on a connected open subset $M \subset \mathbb{R}^{p,q}$ is a composition of*

- *a translation $q \mapsto q+c$, $c \in \mathbb{R}^n$,*
- *an orthogonal transformation $q \mapsto \Lambda q$, $\Lambda \in \mathrm{O}(p,q)$,*
- *a dilatation $q \mapsto e^{\lambda} q$, $\lambda \in \mathbb{R}$, and*
- *a special conformal transformation*

$$q \mapsto \frac{q - \langle q,q\rangle b}{1 - 2\langle q,b\rangle + \langle q,q\rangle\langle b,b\rangle}, \quad b \in \mathbb{R}^n.$$

To be precise, we have just shown that every conformal transformation $\varphi : M \to \mathbb{R}^{p,q}$ on a connected open subset $M \subset \mathbb{R}^{p,q}$, $p+q > 2$, which is an element $\varphi = \varphi_{t_0}$ of a one-parameter group (φ_t) of conformal transformations, is of the type stated in the theorem. (Then Λ is an element of $\mathrm{SO}(p,q)$, where $\mathrm{SO}(p,q)$ is the component containing the identity $1 = \mathrm{id}$ in $\mathrm{O}(p,q)$.) The general case can be derived from this.

Exercise 1.10. The conformal transformations described in Theorem 1.9 form a group with respect to composition (in spite of the singularities, it is not a subgroup of the bijections $\mathbb{R}^n \to \mathbb{R}^n$), which is isomorphic to $O(p+1,q+1)/\{\pm 1\}$ (cf. Theorem 2.9).

1.4.2 Case 2: Euclidean Plane $(p = 2,\ q = 0)$

This case has already been discussed as an example (cf. 1.2).

Theorem 1.11. *Every holomorphic function*

$$\varphi = u + iv : M \to \mathbb{R}^{2,0} \cong \mathbb{C}$$

on an open subset $M \subset \mathbb{R}^{2,0}$ with nowhere-vanishing derivative is an orientation-preserving conformal mapping with conformal Killing factor $\Omega^2 = u_x^2 + u_y^2 = \det D\varphi = |\varphi'|^2$. Conversely, every conformal and orientation-preserving transformation $\varphi : M \to \mathbb{R}^{2,0} \cong \mathbb{C}$ is such a holomorphic function.

This follows immediately from the Cauchy–Riemann differential equations (cf. 1.2). Of course, a corresponding result holds for the antiholomorphic functions. In the case of a connected open subset M of the Euclidean plane the collection of all the holomorphic and antiholomorphic functions exhausts the conformal transformations on M.

We want to describe the conformal transformations again by analyzing conformal Killing fields and conformal Killing factors: Every conformal Killing field $X = (u,v) : M \to \mathbb{C}$ on a connected open subset M of $\mathbb{C}$ with conformal Killing factor κ satisfies $\Delta\kappa = 0$ as well as $u_y + v_x = 0$ and $u_x = \frac{1}{2}\kappa = v_y$. In particular, X fulfills the Cauchy–Riemann equations and is a holomorphic function.

In the special case of a conformal Killing field corresponding to a vanishing conformal Killing factor $\kappa = 0$, one gets

$$X(z) = c + i\theta z, \quad z \in M,$$

with $c \in \mathbb{C}$ and $\theta \in \mathbb{R}$. Here we again use the notation $z = x+iy \in \mathbb{C} \cong \mathbb{R}^{2,0}$. The respective conformal transformations are the Euclidean motions (that is the isometries of $\mathbb{R}^{2,0}$)

$$\varphi(z) = c + e^{i\theta} z.$$

For constant conformal Killing factors $\kappa \neq 0$, $\kappa = \lambda \in \mathbb{R}$, one gets the dilatations

$$X(z) = \lambda z \quad \text{with} \quad \varphi(z) = e^{\lambda} z.$$

Moreover, for $\mathbb{R}$-linear κ in the form $\kappa = 4\mathrm{Re}(z\bar{b}) = 4(xb_1 + yb_2)$ one gets the "inversions". For instance, in the case of $b = (b_1, b_2) = (1,0)$ we obtain

$$\begin{aligned}\varphi(z) &= \frac{z - |z|^2}{1 - 2x + |z|^2} = \frac{-1 + 2x - |z|^2 - x + 1 + iy}{|z-1|^2} \\ &= -1 - \frac{\overline{z-1}}{|z-1|^2} = -\frac{z}{z-1}.\end{aligned}$$

We conclude

Proposition 1.12. *The linear conformal Killing factors κ describe precisely the Möbius transformations (cf. 2.12).*

For general conformal Killing factors $\kappa \neq 0$ on a connected open subset M of the complex plane, the equation $\Delta\kappa = 0$ implies that locally there exist holomorphic $X = (u,v)$ with $u_y + v_x = 0$, $u_x = \frac{1}{2}\kappa = v_y$, that is

$$u_x = v_y\,,\; u_y = -v_x.$$

(This proves the implication "$\Leftarrow$" in Theorem 1.6 for $p = 2, q = 0$, if one localizes the definition of a conformal Killing field.) In this situation, the one-parameter groups (φ_t) for X are also holomorphic functions with nowhere-vanishing derivative.

1.4.3 Case 3: Minkowski Plane (p = q = 1)

In analogy to Theorem 1.11 we have

Theorem 1.13. *A smooth map $\varphi = (u,v) : M \to \mathbb{R}^{1,1}$ on a connected open subset $M \subset \mathbb{R}^{1,1}$ is conformal if and only if*

$$u_x^2 > v_x^2, \quad \text{and} \quad u_x = v_y, u_y = v_x \text{ or } u_x = -v_y, u_y = -v_x.$$

Proof. The condition $\varphi^* g = \Omega^2 g$ for $g = g^{1,1}$ is equivalent to the equations

$$u_x^2 - v_x^2 = \Omega^2, \quad u_x u_y - v_x v_y = 0, \quad u_y^2 - v_y^2 = -\Omega^2, \quad \Omega^2 > 0.$$

"$\Leftarrow$" : these three equations imply $u_x^2 = \Omega^2 + v_x^2 > v_x^2$ and

$$0 \;=\; \Omega^2 + 2u_x u_y - 2v_x v_y - \Omega^2 \;=\; (u_x + u_y)^2 - (v_x + v_y)^2.$$

Hence $u_x + u_y = \pm(v_x + v_y)$. In the case of the sign "+" it follows that

$$\begin{aligned} 0 &= u_x^2 - u_x^2 + v_x v_y - u_x u_y \\ &= u_x^2 - u_x(u_x + u_y) + v_x v_y \\ &= u_x^2 - u_x(v_x + v_y) + v_x v_y \\ &= (u_x - v_x)(u_x - v_y), \end{aligned}$$

that is $u_x = v_x$ or $u_x = v_y$. $u_x = v_x$ is a contradiction to $u_x^2 - v_x^2 = \Omega^2 > 0$. Therefore we have $u_x = v_y$ and $u_y = v_x$.

Similarly, the sign "$-$" yields $u_x = -v_y$ and $u_y = -v_x$.

"$\Rightarrow$" : with $\Omega^2 := u_x^2 - v_x^2 > 0$ we get by substitution

$$u_y^2 - v_y^2 \;=\; v_x^2 - u_x^2 \;=\; -\Omega^2 \qquad \text{and} \qquad u_x u_y - v_x v_y \;=\; 0.$$

Hence φ is conformal. In the case of $u_x = v_y, u_y = v_x$ it follows that

$$\det D\varphi \;=\; u_x v_y - u_y v_x \;=\; u_x^2 - v_x^2 \;>\; 0,$$

that is φ is orientation preserving. In the case of $u_x = -v_y, u_y = -v_x$ the map φ reverses the orientation. □

The solutions of the wave equation $\Delta\kappa = \kappa_{xx} - \kappa_{yy} = 0$ in 1 + 1 dimensions can be written as

$$\kappa(x,y) = f(x+y) + g(x-y)$$

with smooth functions f and g of one real variable in the light cone variables $x_+ = x + y$, $x_- = x - y$. Hence, any conformal Killing factor κ has this form in the case of $p = q = 1$. Let F and G be integrals of $\frac{1}{2}f$ and $\frac{1}{2}g$, respectively. Then

$$X(x,y) = (F(x_+) + G(x_-), F(x_+) - G(x_-))$$

is a conformal Killing field with $X_{\mu,\nu} + X_{\nu,\mu} = g_{\mu\nu}\kappa$. (This eventually completes the proof of the implication "$\Leftarrow$" in Theorem 1.6.) The associated one-parameter group (φ_t) of conformal transformations consists of orientation-preserving maps with $u_x = v_y$, $u_y = v_x$ for $\varphi_t = (u,v)$.

Corollary 1.14. *The orientation-preserving linear and conformal maps* $\psi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ *have matrix representations of the form*

$$A = A_\psi = A_+(s,t) = \exp t \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$$

or

$$A = A_\psi = A_-(s,t) = \exp t \begin{pmatrix} -\cosh s & \sinh s \\ \sinh s & -\cosh s \end{pmatrix}$$

with $(s,t) \in \mathbb{R}^2$.

Proof. Let A_ψ be the matrix representing $\psi = (u,v)$ with respect to the standard basis in $\mathbb{R}^2$:

$$A_\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $u = ax + by$, $v = cx + dy$, hence $u_x = a, u_y = b, v_x = c, v_y = d$. Our Theorem 1.13 implies $a^2 > c^2$ and $a = d, b = c$ (the choice of the sign comes from $\det A > 0$). There is a unique $t \in \mathbb{R}$ with $\exp 2t = a^2 - c^2$ and also a unique $s \in \mathbb{R}$ with $\sinh s = (\exp -t)c$, hence $c^2 = \exp 2t \sinh^2 s$. It follows $a^2 = \exp 2t(1 + \sinh^2 s) = (\exp t \cosh s)^2$, and we conclude $a = \exp t \cosh s = d$ or $a = -\exp t \cosh s = d$, and $b = \exp t \sinh s = d$. $\square$

There is again an interpretation of the action of t (dilatation) and s (boost) similar to the Euclidean case.

The representation in Corollary 1.14 respects the composition: The well-known identities for sinh and cosh imply $A_+(s,t)A_+(s',t') = A_+(s+s',t+t')$.

Remark 1.15. As a consequence, the identity component of the group of linear conformal mappings $\mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ is isomorphic to the additive group $\mathbb{R}^2$. Moreover, the Lorentz group $\mathrm{L} = \mathrm{L}(1,1)$ (the identity component of the linear isometries) is isomorphic to $\mathbb{R}$. The corresponding Poincaré group $\mathrm{P} = \mathrm{P}(1,1)$ is the semidirect product $\mathrm{L} \ltimes \mathbb{R}^2 \cong \mathbb{R} \ltimes \mathbb{R}^2$ with respect to the action $\mathbb{R} \to \mathrm{GL}(2,\mathbb{R})$, $s \mapsto A_+(s,0)$.