Introduction

Conformal field theory in two dimensions has its roots in statistical physics (cf. [BPZ84] as a fundamental work and [Gin89] for an introduction) and it has close connections to string theory and other two-dimensional field theories in physics (cf., e.g., [LPSA94]). In particular, all massless fields are conformally invariant.

The special feature of conformal field theory in two dimensions is the existence of an infinite number of independent symmetries of the system, leading to corresponding invariants of motion which are also called conserved quantities. This is the content of Noether's theorem which states that a symmetry of a physical system given by a local one-parameter group or by an infinitesimal version thereof induces an invariant of motion of the system. Any collection of invariants of motion simplifies the system in question up to the possibility of obtaining a complete solution. For instance, in a typical system of classical mechanics an invariant of motion reduces the number of degrees of freedom. If the original phase space has dimension $2n$ the application of an invariant of motion leads to a system with a phase space of dimension $2(n-1)$. In this way, an independent set of n invariants of motion can lead to a zero-dimensional phase space that means, in general, to a complete solution.

Similarly, in the case of conformal field theory the invariants of motion which are induced by the infinitesimal conformal symmetries reduce the infinite dimensional system completely. As a consequence, the structure constants which determine the system can be calculated explicitly, at least in principle, and one obtains a complete solution. This is explained in Chap. 9, in particular in Proposition 9.12.

These symmetries in a conformal field theory can be understood as infinitesimal conformal symmetries of the Euclidean plane or, more generally, of surfaces with a conformal structure, that is Riemann surfaces. Since conformal transformations on an open subset U of the Euclidean plane are angle preserving, the conformal orientation-preserving transformations on U are holomorphic functions with respect to the natural complex structure induced by the identification of the Euclidean plane with the space $\mathbb{C}$ of complex numbers. As a consequence, there is a close connection between conformal field theory and function theory. A good portion of conformal field theory is formulated in terms of holomorphic functions using many results of function theory. On the other hand, this interrelation between conformal field theory and function theory yields remarkable results on moduli spaces of vector bundles

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over compact Riemann surfaces and therefore provides an interesting example of how physics can be applied to mathematics.

The original purpose of the lectures on which the present text is based was to describe and to explain the role the Virasoro algebra plays in the quantization of conformal symmetries in two dimensions. In view of the usual difficulties of a mathematician reading research articles or monographs on conformal field theory, it was an essential concern of the lectures not to rely on background knowledge of standard methods in physics. Instead, the aim was to try to present all necessary concepts and methods on a purely mathematical basis. This explains the adjective "mathematical" in the title of these notes. Another motivation was to discuss the sometimes confusing use of language by physicists, who for example emphasize that the group of holomorphic maps of the complex plane is infinite dimensional – which is not true. What is meant by this statement is that a certain Lie algebra closely related to conformal symmetry, namely the Witt algebra or its central extension, the Virasoro algebra, is infinite dimensional.

Clearly, with these objectives the lectures could hardly cover an essential part of actual conformal field theory. Indeed, in the course of the present text, conformally invariant quantum field theory does not appear before Chap. 6, which treats the representation theory of the Virasoro algebra as a first topic of conformal field theory. These notes should therefore be seen as a preparation for or as an introduction to conformal field theory for mathematicians focusing on some background material in geometry and algebra. Physicists may find the detailed investigation in Part I useful, where some elementary geometric and algebraic prerequisites for conformal field theory are studied, as well as the more advanced mathematical description of fundamental structures and principles in the context of quantum field theory in Part II.

In view of the above-mentioned tasks, it makes sense to start with a detailed description of the conformal transformations in arbitrary dimensions and for arbitrary signatures (Chap. 1) and to determine the associated conformal groups (Chap. 2) with the aid of the conformal compactification of spacetime. In particular, the conformal group of the Minkowski plane turns out to be infinite dimensional, it is essentially isomorphic to $\text{Diff}_+(\mathbb{S}^1) \times \text{Diff}_+(\mathbb{S}^1)$, while the conformal group of the Euclidean plane is finite-dimensional, it is the group of Möbius transformations isomorphic to $\text{SL}(2,\mathbb{C})/\{\pm 1\}$.

The next two chapters (Chaps. 3 and 4) are concerned with central extensions of groups and Lie algebras and their classification by cohomology. These two chapters contain several examples appearing in physics and mathematics. Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of (the universal covering of) the classical symmetry group, and in the same way the infinitesimal symmetry algebra of the quantum system is, in general, a central extension of the classical symmetry algebra.

Chapter 5 leads to the Virasoro algebra as the unique nontrivial central extension of the Witt algebra. The Witt algebra is the essential component of the classical infinitesimal conformal symmetry in two dimensions for the Euclidean plane as well as for the Minkowski plane. This concludes the first part of the text which is comparatively elementary except for some aspects in the examples.

The second part presents several different approaches to conformal field theory. We start this program with the representation theory of the Virasoro algebra including the Kac formula (Chap. 6) in order to describe the unitary representations.

In Chap. 7 we give an elementary introduction into the quantization of the bosonic string and explain how the conformal symmetry is present in classical and in quantized string theory. The quantization induces a natural representation of the Virasoro algebra on the Fock space of the Heisenberg algebra which is of interest in later considerations concerning examples of vertex algebras.

The next two chapters are dedicated to axiomatic quantum field theory. In Chap. 8 we provide an exposition of the relativistic case in any dimension by presenting the Wightman axioms for the field operators as well as the equivalent axioms for the correlation functions called Wightman distributions. The Wightman distributions are boundary values of holomorphic functions which can be continued analytically into a large domain in complexified spacetime and thereby provide the correlation functions of a Euclidean version of the axioms, the Osterwalder–Schrader axioms. In Chap. 9 we concentrate on the two-dimensional Euclidean case with conformal symmetry. We aim to present an axiomatic approach to conformal field theory along the suggestion of [FFK89] and the postulates of the groundbreaking paper of Belavin, Polyakov, and Zamolodchikov [BPZ84].

Many papers on conformal field theory nowadays use the language of vertex operators and vertex algebras. Chapter 10 gives a brief introduction to the basic concepts of vertex algebras and some fundamental results. Several concepts and constructions reappear in this chapter – sometimes in a slightly different form – so that one has a common view of the different approaches to conformal field theory presented in the preceding chapters.

Finally we discuss the Verlinde formula as an application of conformal field theory to mathematics (Chap. 11).

References

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