TWO CHARACTERIZATIONS OF FINITE QUASI-HOPF ALGEBRAS

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Abstract. Let H be a finite-dimensional quasibialgebra. We show that H is a quasi-Hopf algebra if and only if the monoidal category of its finite-dimensional left modules is rigid, if and only if a structure theorem for Hopf modules over H holds. We also show that a dual structure theorem for Hopf modules over a coquasibialgebra H holds if and only if the category of finite-dimensional right H-comodules is rigid; this is not equivalent to H being a coquasi-Hopf algebra.

1. Introduction

Let H be a bialgebra over the field k. It was shown by Ulbrich [12] that H is a Hopf algebra if and only if the monoidal category $\mathcal{M}_f^H$ of finite-dimensional right H-comodules is rigid, that is, if every finite-dimensional H-comodule has a dual object within the category $\mathcal{M}_f^H$. In particular, if H is a finite-dimensional bialgebra, then H is a Hopf algebra if and only if the category ${}_H\mathcal{M}_f$ of finite-dimensional left H-modules is rigid.

It is a natural question whether the same holds for quasibialgebras: Drinfeld's definition of a quasibialgebra H ensures that the category ${}_H\mathcal{M}$ is, just like in the bialgebra case, a monoidal category. And Drinfeld's definition of a quasiantipode is motivated by the fact that the category of finite-dimensional modules over a quasi-Hopf algebra is a rigid monoidal category. However, if we try to prove the converse, then we run into difficulties. The key problem is that the underlying functor ${}_H\mathcal{M} \to \mathcal{M}_k$ to the category of k-vector spaces is monoidal if H is a bialgebra, and monoidal functors automatically preserve dual objects. If H is only a quasibialgebra, then the underlying functor ${}_H\mathcal{M} \to \mathcal{M}_k$ is still compatible with tensor products, but not coherent in the sense of the definition of a monoidal functor. Thus it is not clear that the functor preserves duals. That the problem is really serious was shown in [9] by an example based on a construction of Yongchang Zhu [13]: There is a coquasibialgebra H such that the category $\mathcal{M}_f^H$ is rigid, although H is not a coquasi-Hopf algebra. The existence of a coquasiantipode is ruled out quite drastically by the fact that a finite-dimensional H-comodule and its dual object may have different dimensions. One result of this paper will be that all is well in the finite-dimensional case: A finite-dimensional quasibialgebra H is a quasi-Hopf algebra if and only if ${}_H\mathcal{M}_f$ is rigid.

Another well-known criterion says that a bialgebra H is a Hopf algebra if and only if the structure theorem for Hopf modules holds, that is, if the obvious functor $\mathcal{M}_k \to \mathcal{M}_H^H$ mapping a vector space V to $V \otimes H^{\boldsymbol{\cdot}}_{\boldsymbol{\cdot}}$ is a category equivalence. If

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we try to establish a version of this criterion for quasibialgebras, the first problem is that there are no Hopf modules: If H is a quasibialgebra, then it is not a coassociative coalgebra, so one does not know what a comodule should be. This first problem was solved by Hausser and Nill [4], who observed that one can still define a Hopf (bi)module category ${}_H\mathcal{M}_H^H$; we may say briefly that it is the category of H-comodules over the coassociative coalgebra H *within the monoidal category* ${}_H\mathcal{M}_H$. Moreover, Hausser and Nill prove a structure theorem for Hopf bimodules: A certain functor ${}_H\mathcal{M} \to {}_H\mathcal{M}_H^H$ is a category equivalence if H is a quasi-Hopf algebra. If we try to prove a converse, we run into difficulties once again. In the case of ordinary bialgebras, the proof is based on another criterion: H is a Hopf algebra if the canonical map $H \otimes H \ni g \otimes h \mapsto gh_{(1)} \otimes h_{(2)} \in H \otimes H$ is a bijection. If the structure theorem for Hopf modules holds, it is very easy to check that the canonical map is bijective. For a quasi-Hopf algebra H, Drinfeld's paper [3] contains an analog of the canonical map $H \otimes H \to H \otimes H$. However, the proper anolog is given by a more complicated formula; in particular, one already needs a quasiantipode to even write down the map (or its inverse), and it seems to have no analog for quasibialgebras. We shall show that the problem is just as serious as that with the first criterion mentioned above: Given a coquasibialgebra H we will prove in Section 2 that the structure theorem for Hopf modules holds — that is, a certain functor $\mathcal{M}^H \to {}_H^H\mathcal{M}^H$ is a a category equivalence —, if and only if the category $\mathcal{M}_f^H$ is rigid. For a finite-dimensional quasi-Hopf algebra this provides a new and rather conceptual proof for the structure theorem of Hausser and Nill — in fact our proof needs hardly any unpleasant calculations with the quasibialgebra structure and its axioms, and none at all with the quasiantipode. Of course the result also provides examples of coquasibialgebras that satisfy the structure theorem for Hopf modules, while they do not have a coquasiantipode.

On the other hand, if H is a finite-dimensional quasibialgebra, Theorem 3.1 shows that all is well, that is, the structure theorem for Hopf modules is equivalent to the existence of a quasiantipode.

In Section 4 we will show that the quasiantipode of a finite-dimensional quasi-Hopf algebra is a bijection. This was first proved by Bulacu and Caenepeel [2]. We will give a rather different proof.

Throughout the paper, we work over a base field k.

2. Duality and the structure of Hopf modules

Throughout this section, we let H be a coquasibialgebra. That is, (H, Δ, ε) is a coassociative coalgebra, endowed with a (nonassociative) multiplication $\nabla\colon H \otimes H \to H$ which is a coalgebra map, a grouplike unit element $1 \in H$, and a convolution invertible form $\phi\colon H \otimes H \otimes H \to k$, the associator, satisfying the identities $\phi(g \otimes 1 \otimes h) = \varepsilon(g)\varepsilon(h)$,

$$(f_{(1)}g_{(1)})h_{(1)}\phi(f_{(2)} \otimes g_{(2)} \otimes h_{(2)}) = \phi(f_{(1)} \otimes g_{(1)} \otimes h_{(1)})f_{(2)}(g_{(2)}h_{(2)}),$$

and

$$\begin{aligned}\phi(d_{(1)}f_{(1)} \otimes g_{(1)} \otimes h_{(1)})\phi(d_{(2)} \otimes f_{(2)} \otimes g_{(2)}h_{(2)})\\ = \phi(d_{(1)} \otimes f_{(1)} \otimes g_{(1)})\phi(d_{(2)} \otimes f_{(2)}g_{(2)} \otimes h_{(1)})\phi(f_{(3)} \otimes g_{(3)} \otimes h_{(2)})\end{aligned}$$

for $d, f, g, h \in H$. We have used Sweedler notation in the form $\Delta(h) = h_{(1)} \otimes h_{(2)}$. We will also use Sweedler notations $V \ni v \mapsto v_{(0)} \otimes v_{(1)} \in V \otimes H$ for right, and $V \ni v \mapsto v_{(-1)} \otimes v_{(0)} \in H \otimes V$ for left comodule structures.

The axioms ensure that the category $\mathcal{M}^H$ of right H-comodules is a monoidal category in the following way: For $V, W \in \mathcal{M}^H$, the tensor product $V \otimes W$ over k is an H-comodule with the codiagonal comodule structure induced by multiplication in H; the associativity isomorphism $\Phi\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given by $\Phi(u \otimes v \otimes w) = u_{(0)} \otimes v_{(0)} \otimes w_{(0)} \phi(u_{(1)} \otimes v_{(1)} \otimes w_{(1)})$. Since the opposite of a coquasibialgebra and the tensor product of two coquasibialgebras are coquasibialgebras as well, the category ${}^H\mathcal{M}^H$ of H-H-bicomodules is also a monoidal category. This time the associativity isomorphism $\Phi\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given by $\Phi(u \otimes v \otimes w) = \phi^{-1}(u_{(-1)} \otimes v_{(-1)} \otimes w_{(-1)}) u_{(0)} \otimes v_{(0)} \otimes w_{(0)} \phi(u_{(1)} \otimes v_{(1)} \otimes w_{(1)})$. It is a key observation that H (which is not associative as a k-algebra) is an associative algebra within the monoidal category ${}^H\mathcal{M}^H$, that is, we have

$$\nabla(\nabla \otimes H) = \nabla(H \otimes \nabla)\Phi\colon (H \otimes H) \otimes H \to H \otimes (H \otimes H).$$

Thus we can use the general theory of algebras and modules in monoidal categories, see Pareigis [6, 7], to do (or rather avoid) calculations with the multiplicative structure of H.

In particular, there is a well-defined notion of (say, left) H-module within the monoidal category ${}^H\mathcal{M}^H$. We denote the category of such modules by ${}^H_H\mathcal{M}^H$, and call its objects Hopf modules. We note that for any $M \in {}^H_H\mathcal{M}^H$ and $P \in {}^H\mathcal{M}^H$ we have $M \otimes P \in {}^H_H\mathcal{M}^H$ with the "obvious" left module structure

$$H \otimes (M \otimes P) \xrightarrow{\Phi^{-1}} (H \otimes M) \otimes P \xrightarrow{\mu \otimes P} M \otimes P,$$

where μ denotes the module structure of M. We will abbreviate this module structure by a dot attached to the tensorand M, i.e. write ${}^{\cdot}_{\cdot}M^{\cdot} \otimes {}^{\cdot}P^{\cdot}$ for it, with the upper dots indicating on which tensor factors we have a codiagonal coaction, and the upper dot indicating where the action takes place; note, though, that the actual formula for the action involves both tensorands through the action of the associators.

Taking $M = H$ as a special case we obtain the left adjoint $P \mapsto {}^{\cdot}_{\cdot}H^{\cdot} \otimes {}^{\cdot}P^{\cdot}$ to the underlying functor ${}^H_H\mathcal{M}^H \to {}^H\mathcal{M}^H$.

As a particular case, we can consider a right H-comodule V as a bicomodule with the trivial comodule structure on the left, and apply the above construction to obtain a functor

$$\mathcal{L}\colon \mathcal{M}^H \ni V \mapsto {}^{\cdot}_{\cdot}H^{\cdot} \otimes V^{\cdot} \in {}^H_H\mathcal{M}^H.$$

The formally dual version of this functor (for a quasibialgebra) was studied by Hausser and Nill [4], who also proved that it is an equivalence in case H is a quasi-Hopf algbebra. Moreover, ${}^H_H\mathcal{M}^H$ is a monoidal category, and $\mathcal{L}$ is a monoidal functor. Hausser and Nill show this in the dual case using the assumption that H is quasi-Hopf, the quasibialgebra case is treated in [10]. We shall say for short that the structure theorem for Hopf modules holds if $\mathcal{L}$ is a category equivalence. In this section we shall give a different proof of the structure theorem for Hopf modules than the ones in [4, 10], under the weaker assumption that $\mathcal{M}^H_f$ is a rigid monoidal category.

We start by stating several facts on the functor $\mathcal{L}$ and the category ${}^H_H\mathcal{M}^H$ that are formally dual to facts proved and used in [4] and [10]. We will not give the formally dual proofs, but will indicate how Proposition 2.4 follows from more abstract reasons without any work.

Lemma 2.1 (dual to part of [10, Prop.3.6])**.** *The functor $\mathcal{L}\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$ is exact, fully faithful, and commutes with arbitrary colimits. In particular, colimits and equalizers of diagrams whose objects are in the image of $\mathcal{L}$ are also in that image.*

The dual statement of the following is observed between Corollary 3.9 and Lemma 3.10 of [4]. See also [10, Lem.and Def.3.2].

Lemma 2.2. *The category ${}^H_H\mathcal{M}^H$ is a monoidal category in the following way:*
The tensor product of $M, N \in {}^H_H\mathcal{M}^H$ is their cotensor product $M \,\square_H\, N$ equipped with the module structure given by $h(m \otimes n) = h_{(1)}m \otimes h_{(2)}n$.
In particular the underlying functor ${}^H_H\mathcal{M}^H \to {}^H\mathcal{M}^H$ is a strict monoidal functor with the monoidal category structure on the target given by cotensor product.

Dually to [10, Lem.3.4] one can check that for any $M \in {}^H_H\mathcal{M}^H$ and $V \in \mathcal{M}^H$ the canonical isomorphism

$$\begin{aligned}\hat{\xi}\colon M \underset{H}{\square} (H \otimes V) &\to M \otimes V\\ m_{(0)} \otimes m_{(1)} \otimes v &\mapsto m \otimes v\\ m \otimes h \otimes v &\hookleftarrow m\varepsilon(h) \otimes v\end{aligned}$$

is a morphism in ${}^H_H\mathcal{M}^H$. This follows from the following more general statement:

Lemma 2.3. *Let $M, N \in {}^H_H\mathcal{M}^H$, and $V \in \mathcal{M}^H$. The canonical isomorphism ("the identity")*

$$(M \underset{H}{\square} N) \otimes V \cong M \underset{H}{\square} (N \otimes V)$$

is an isomorphism in ${}^H_H\mathcal{M}^H$. If we identify $(M \,\square_H\, N) \otimes V = M \,\square_H\, (N \otimes V) = M \,\square_H\, N \otimes V$, then

$$\Phi_{M\square_H N,V,W} = M \underset{H}{\square} \Phi_{N,V,W} : (M \underset{H}{\square} N \otimes V) \otimes W \to M \underset{H}{\square} N \otimes (V \otimes W).$$

Proof. It is obvious that the isomorphism is left and right H-colinear. H-linearity is a small calculation: Denoting the respective actions by $h((m \otimes n) \otimes v)$ and $h(m \otimes (n \otimes v))$ for $m \otimes n \in M \,\square_H\, N$, $v \in V$, and $h \in H$, we find

$$\begin{aligned}h((m \otimes n) \otimes v) &= h_{(1)}(m \otimes n)_{(0)} \otimes v_{(0)}\phi(h_{(2)} \otimes (m \otimes n)_{(1)} \otimes v_{(1)})\\ &= h_{(1)}(m \otimes n_{(0)}) \otimes v_{(0)}\phi(h_{(2)} \otimes n_{(1)} \otimes v_{(1)})\\ &= h_{(1)}m \otimes h_{(2)}n_{(0)} \otimes v_{(0)}\phi(h_{(3)} \otimes n_{(1)} \otimes v_{(1)})\\ &= h_{(1)}m \otimes h_{(2)}(n \otimes v) = h(m \otimes (n \otimes v))\end{aligned}$$

The two associativity isomorphisms both map $m \otimes n \otimes v \otimes w$ to $m \otimes n_{(0)} \otimes v_{(0)} \otimes w_{(0)}\phi(n_{(1)} \otimes v_{(1)} \otimes w_{(1)})$. □

We can restate the Lemma as follows: The category ${}^H_H\mathcal{M}^H$ of left H-modules in ${}^H\mathcal{M}^H$ is naturally a right ${}^H\mathcal{M}^H$-category, hence a right $\mathcal{M}^H$-category. On the other hand ${}^H_H\mathcal{M}^H$ is a monoidal category, so it is naturally a left ${}^H_H\mathcal{M}^H$-category. The Lemma says that the right $\mathcal{M}^H$-category structure is compatible with the left ${}^H_H\mathcal{M}^H$-category structure (or we have an ${}^H_H\mathcal{M}^H$-$\mathcal{M}^H$-bicategory). It follows as in [8, Thm.3.3] that a monoidal functor $(\mathcal{L},\xi)\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$ is given by $\mathcal{L}(V) = H \otimes V$, and

$$\xi\colon (H \otimes V) \mathop{\Box}_H (H \otimes W) \xrightarrow{\hat{\xi}} (H \otimes V) \otimes W \xrightarrow{\Phi} H \otimes (V \otimes W).$$

We shall repeat briefly the abstract argument: The right action of $\mathcal{M}^H$ on ${}^H_H\mathcal{M}^H$ gives a monoidal functor (here actually an antimonoidal functor) from $\mathcal{M}^H$ to the monoidal category of endofunctors of ${}^H_H\mathcal{M}^H$. Since the action is compatible with the action of ${}^H_H\mathcal{M}^H$, the antimonoidal functor has its image in the category of ${}^H_H\mathcal{M}^H$-endofunctors of ${}^H_H\mathcal{M}^H$, which is (anti)monoidally equivalent to the monoidal category ${}^H_H\mathcal{M}^H$ itself (compare to the fact that the endomorphism ring of a ring R considered as an R-module, is isomorphic to the ring itself). Thus we have a monoidal functor fron $\mathcal{M}^H$ to ${}^H_H\mathcal{M}^H$.

While this abstract argument is considerably simpler than the calculations needed in either [4] or [10] to show that $\mathcal{L}$ is monoidal, the reader might not want to get into $\mathcal{C}$-category theory. In this case one may dualize the proof of [10, Prop.3.6] to obtain:

Proposition 2.4. *Define*

$$\xi = \xi_{VW} = \left((H \otimes V) \mathop{\Box}_H (H \otimes W) \xrightarrow{\hat{\xi}} (H \otimes V) \otimes W \xrightarrow{\Phi} H \otimes (V \otimes W)\right)$$

for $V, W \in \mathcal{M}^H$. Then $(\mathcal{L},\xi)\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$ is a monoidal functor.

Let $\mathcal{C}$ be a monoidal category. Recall that a dual object of $V \in \mathcal{C}$ is a triple $(V^\vee, \mathrm{ev}, \mathrm{db})$ in which $V^\vee \in \mathcal{C}$, and $\mathrm{ev}\colon V^\vee \otimes V \to I$ and $\mathrm{db}\colon I \to V \otimes V^\vee$ are morphisms such that the two compositions

$$V \xrightarrow{\mathrm{db}\otimes V} (V \otimes V^\vee) \otimes V \xrightarrow{\Phi} V \otimes (V^\vee \otimes V) \xrightarrow{V\otimes \mathrm{ev}} V$$
$$V^\vee \xrightarrow{V^\vee\otimes \mathrm{db}} V^\vee \otimes (V \otimes V^\vee) \xrightarrow{\Phi^{-1}} (V^\vee \otimes V) \otimes V^\vee \xrightarrow{\mathrm{ev}\otimes V^\vee} V^\vee$$

are identities. If $(\mathcal{F},\xi)\colon \mathcal{C} \to \mathcal{D}$ is a monoidal functor, and $(V^\vee, \mathrm{ev}, \mathrm{db})$ is a dual object of V in $\mathcal{C}$, then $\mathcal{F}(V^\vee)$ is a dual object of $\mathcal{F}(V)$ in $\mathcal{D}$, with evaluation and coevaluation

$$\mathcal{F}(V^\vee) \otimes \mathcal{F}(V) \xrightarrow{\xi} \mathcal{F}(V^\vee \otimes V) \xrightarrow{\mathcal{F}(\mathrm{ev})} \mathcal{F}(I) \cong I$$
$$I \cong \mathcal{F}(I) \xrightarrow{\mathcal{F}(\mathrm{db})} \mathcal{F}(V \otimes V^\vee) \xrightarrow{\xi^{-1}} \mathcal{F}(V) \otimes \mathcal{F}(V^\vee).$$

Let $V \in \mathcal{M}^H$ be finite-dimensional. We can endow the dual vector space V^* with a canonically corresponding left H-comodule structure defined by $\varphi_{(-1)}\varphi_{(0)}(v) = \varphi(v_{(0)})v_{(1)}$ for all $\varphi \in V^*$ and $v \in V$. Equivalently, $v_{i(0)} \otimes v_{i(1)} \otimes v^i = v_i \otimes v^i{}_{(-1)} \otimes v^i{}_{(0)} \in V \otimes H \otimes V^*$, that is $v_i \otimes v^i \in V \mathbin{\Box}_H V^*$. Note that the map

$$E'\colon {}^{\bullet}(V^*) \otimes V^{\bullet} \ni \varphi \otimes v \mapsto \varphi(v_{(0)})v_{(1)} = \varphi_{(-1)}\varphi_{(0)}(v) \in H$$

is an H-bimodule map.

Lemma 2.5. *Let $V \in \mathcal{M}^H_f$. Then a dual object of $\mathcal{L}(V)$ in the monoidal category ${}^H_H\mathcal{M}^H$ is given by*

$$({}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}(V^*), E, D)$$

with

$$E = \left(({}_{\bullet}^{\bullet}H^{\bullet} \otimes {}^{\bullet}(V^*)) \mathop{\square}_{H} ({}_{\bullet}^{\bullet}H^{\bullet} \otimes {}^{\bullet}V) \xrightarrow{\hat{\xi}} H \otimes V^* \otimes V \xrightarrow{H \otimes E'} H \otimes H \xrightarrow{\nabla} H \right)$$

$$D\colon H \ni h \mapsto h_{(1)} \otimes v_i \otimes h_{(2)} \otimes v^i \in ({}_{\bullet}^{\bullet}H^{\bullet} \otimes V^{\bullet}) \mathop{\square}_{H} ({}_{\bullet}^{\bullet}H^{\bullet} \otimes {}^{\bullet}(V^*))$$

Proof. From the definition of E it is clear that E is a well-defined morphism in ${}^H_H\mathcal{M}^H$. Note that we have $\hat{\xi}^{-1}(h \otimes \varphi \otimes v) = h_{(1)} \otimes \varphi \otimes h_{(2)} \otimes v$, and $E(h_{(1)} \otimes \varphi \otimes h_{(2)} \otimes v) = h\varphi(v_{(0)})v_{(1)} = h\varphi_{(-1)}\varphi_{(0)}(v)$, hence $\varepsilon E(h_{(1)} \otimes \varphi \otimes h_{(2)} \otimes v) = \varepsilon(h)\varphi(v)$. The map D is well-defined since $v_i \otimes v^i \in V \square_H V^*$ and $h_{(1)} \otimes h_{(2)} \in H \square_H H$. It is obviously left and right H-colinear, and it is H-linear by the calculation

$$\begin{aligned} gD(h) = g(h_{(1)} \otimes v_i \otimes h_{(2)} \otimes v^i) &= g_{(1)}(h_{(1)} \otimes v_i) \otimes g_{(2)}(h_{(2)} \otimes v^i) \\ &= g_{(1)}h_{(1)} \otimes v_{i(0)}\phi(g_{(2)} \otimes h_{(2)} \otimes v_{i(1)}) \otimes g_{(3)}(h_{(3)} \otimes v^i) \\ &= g_{(1)}h_{(1)} \otimes v_i \otimes \phi(g_{(2)} \otimes h_{(2)} \otimes v^i{}_{(-1)})g_{(2)}(h_{(2)} \otimes v^i{}_{(0)}) \\ &= g_{(1)}h_{(1)} \otimes v_i \otimes g_{(2)}h_{(2)} \otimes v^i = D(gh) \end{aligned}$$

for $g, h \in H$. To check the identities for a dual object, we have to bear in mind the canonical identifications

$$\begin{array}{cc} H \mathop{\square}_{H} M \cong M & M \mathop{\square}_{H} H \cong M \\ \sum h_i \otimes m_i \mapsto \sum \varepsilon(h_i)m_i & \sum m_i \otimes h_i \mapsto \sum m_i\varepsilon(h_i) \\ m_{(-1)} \otimes m_{(0)} \leftarrow\!\shortmid m & m_{(0)} \otimes m_{(1)} \leftarrow\!\shortmid m, \end{array}$$

and can calculate

$$\begin{aligned} ((H \otimes V) \mathop{\square}_{H} E)(D \mathop{\square}_{H} (H \otimes V))(h \otimes v) &= ((H \otimes V) \mathop{\square}_{H} E)(D(h_{(1)}) \otimes (h_{(2)} \otimes v)) \\ = h_{(1)} \otimes v_i\varepsilon E(h_{(2)} \otimes v^i \otimes h_{(3)} \otimes v) &= h_{(1)} \otimes v_i\varepsilon(h_{(2)}v^i(v_{(0)})v_{(1)}) \\ &= h \otimes v_iv^i(v) = h \otimes v \end{aligned}$$

for $h \in H$ and $v \in V$, and

$$\begin{aligned} (E \mathop{\square}_{H} (H \otimes V^*))((H \otimes V^*) \mathop{\square}_{H} D)(h \otimes \varphi) &= (E \mathop{\square}_{H} (H \otimes V^*)(h_{(1)} \otimes \varphi \otimes D(h_{(2)})) \\ &= \varepsilon E(h_{(1)} \otimes \varphi \otimes h_{(2)} \otimes v_i)(h_{(3)} \otimes v^i) = h \otimes \varphi(v_i)v^i = h \otimes \varphi \end{aligned}$$

for $h \in H$ and $\varphi \in V^*$. □

Theorem 2.6. *Let H be a coquasibialgebra. The following are equivalent:*

1. *The functor $\mathcal{L}\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$ is an equivalence.*
2. *The category $\mathcal{M}^H_f$ is rigid.*

Proof. Assume (1), and let $V \in \mathcal{M}^H_f$. Then $\mathcal{L}(V)$ has a left dual object in ${}^H_H\mathcal{M}^H$ by Lemma 2.5. Since $\mathcal{L}$ is an equivalence, V has a left dual object $V^\vee$ in $\mathcal{M}^H$. But $V^\vee$ is necessarily finite-dimensional. For let $\mathrm{db}\colon k \to V \otimes V^\vee$ be the relevant coevaluation. Then $\mathrm{db}(1) = \sum_{i=1}^r x_i \otimes y_i$ for some $x_i \in V$ and $y_i \in V^\vee$. The latter generate a finite-dimensional subcomodule $U \subset V^\vee$, and it is straightforward to check that the map $\mathrm{db}'\colon k \to V \otimes U$ induced by db, and the restriction $\mathrm{ev}'\colon U \otimes V \to k$ of the evaluation $\mathrm{ev}\colon V^\vee \otimes V \to k$ make U a dual object of V, whence $U = V^\vee$.

Now assume (2). We need to show that $\mathcal{L}$ is essentially surjective. Let $M \in {}^H_H\mathcal{M}^H$. Then, calculating within the monoidal category ${}^H\mathcal{M}^H$, we have $M \cong H \otimes_H M$, that is, we have a coequalizer

$$H \otimes (H \otimes M) \rightrightarrows H \otimes M \longrightarrow M$$

in ${}^H_H\mathcal{M}^H$, in which the first two objects have the form ${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}P^{\bullet}$ for $P \in {}^H\mathcal{M}^H$. Since the image of $\mathcal{L}$ is closed under coequalizers, it suffices to check that objects of this form are in the image of $\mathcal{L}$. Now for $P \in {}^H\mathcal{M}^H$ we have $P \cong P \,\square_H\, H$, that is, we have an equalizer

$$P \longrightarrow {}^{\bullet}P \otimes H^{\bullet} \rightrightarrows {}^{\bullet}P \otimes H \otimes H^{\bullet}$$

in ${}^H\mathcal{M}^H$. Since the image of $\mathcal{L}$ is closed under equalizers, it suffices to check that objects of the form ${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W \otimes H^{\bullet}$ for $W \in {}^H\mathcal{M}$ are in the image of $\mathcal{L}$. Since

$${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W \otimes H^{\bullet} \cong ({}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W) \mathop{\square}_{H} ({}^{\bullet}_{\bullet}H^{\bullet} \otimes H^{\bullet}),$$

and the image of $\mathcal{L}$ is closed under cotensor product, it suffices to verify that ${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W$ is in the image of $\mathcal{L}$, and since $\mathcal{L}$ preserves colimits, we may assume that W is finite-dimensional. But then we have $W \cong V^*$ for some $V \in \mathcal{M}^H_f$, and ${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W$ is the left dual of $\mathcal{L}(V)$ in ${}^H_H\mathcal{M}^H$. Since monoidal functors preserve duals, it follows that ${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W \cong \mathcal{L}(V^\vee)$, where $V^\vee$ is a left dual of V in $\mathcal{M}^H_f$. □

Condition (2) in Theorem 2.6 is fulfilled if H is a coquasi-Hopf algebra. We will not recall the axioms of a coquasiantipode in any detail, but shall merely say that it involves an anti-coalgebra endomorphism S of H that allows us to endow $V^\vee := V^*$, the k-linear dual of $V \in \mathcal{M}^H_f$, with an H-comodule structure, and extra structure elements that make $V^\vee$ into a left dual of V in $\mathcal{M}^H_f$. If $W \cong V^*$ as at the end of the proof of Theorem 2.6, then we see that $V^\vee \cong W^S$, the right H-comodule obtained from the left H-comodule W along S. Thus we have:

Corollary 2.7. *If H is a coquasi-Hopf algebra, then the functor $\mathcal{L}\colon \mathcal{M}^H \to {}^H_H\mathcal{M}^H$ is an equivalence.*

If $W \in {}^H\mathcal{M}$, then ${}^{\bullet}_{\bullet}H^{\bullet} \otimes {}^{\bullet}W \cong {}^{\bullet}_{\bullet}H^{\bullet} \otimes (W^S)^{\bullet} \in {}^H_H\mathcal{M}^H$.

However, there are examples of coquasibialgebras H such that $\mathcal{M}^H_f$ is right and left rigid, while H does not have a coquasiantipode; see [9, Sec.4.5].

3. Finite Quasi-Hopf algebras

Recall that a quasibialgebra H is an associative algebra with an algebra map $\Delta\colon H \to H \otimes H$ called comultiplication, an algebra map $\varepsilon\colon H \to k$ that is a counit for Δ, and an invertible element $\phi \in H \otimes H \otimes H$ such that

$$(\varepsilon \otimes H)\Delta(h) = h = (H \otimes \varepsilon)\Delta(h), \tag{3.1}$$

$$(H \otimes \Delta)\Delta(h) \cdot \phi = \phi \cdot (\Delta \otimes H)\Delta(h), \tag{3.2}$$

$$(H \otimes H \otimes \Delta)(\phi) \cdot (\Delta \otimes H \otimes H)(\phi) = (1 \otimes \phi) \cdot (H \otimes \Delta \otimes H)(\phi) \cdot (\phi \otimes 1), \tag{3.3}$$

$$(H \otimes \varepsilon \otimes H)(\phi) = 1 \tag{3.4}$$

hold for all $h \in H$. We will write $\Delta(h) =: h_{(1)} \otimes h_{(2)}$, $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$, and $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)}$.

A finite-dimensional quasibialgebra is the same as the dual of a finite-dimensional coquasibialgebra (historically, vice versa would be more to the point).

A quasiantipode (S,α,β) for a quasibialgebra H consists of an anti-algebra endomorphism S of H, and elements $\alpha,\beta\in H$, such that

$$S(h_{(1)})\alpha h_{(2)}=\varepsilon(h)\alpha, \qquad h_{(1)}\beta S(h_{(2)})=\varepsilon(h)\beta,$$
$$\phi^{(1)}\beta S(\phi^{(2)})\alpha\phi^{(3)}=1, \qquad S(\phi^{(-1)})\alpha\phi^{(-2)}\beta\phi^{(-3)}=1$$

hold in H, for $h\in H$. A quasi-Hopf algebra is a quasibialgebra with a quasiantipode. Note that we disagree in this definition with Drinfeld who requires S to be a bijection. We will return to this in Section 4 where we give another proof for a recent result of Bulacu and Caenepeel, which says that the antipode of a finite-dimensional quasi-Hopf algebra is automatically bijective.

The main result of this section characterizes finite quasi-Hopf algebras via rigidity of their module category, or the structure theorem for Hopf modules. The functor $\mathcal{R}$ in the theorem is the formal dual to (and older than) the functor $\mathcal{L}$ in the preceding section; it is due to Hausser and Nill [4]. We shall recall some details in the proof.

Theorem 3.1. *Let H be a finite-dimensional quasibialgebra. The following are equivalent:*

1. *H is a quasi-Hopf algebra.*
2. *The monoidal category ${}_H\mathcal{M}_f$ is rigid.*
3. *The functor $\mathcal{R}\colon {}_H\mathcal{M}\to {}_H\mathcal{M}_H^H$ is a category equivalence.*

Proof. The equivalence of (2) and (3) follows by duality from Theorem 2.6. The implication (1) $\Longrightarrow$ (2) is the motivation for the definition of a quasiantipode in Drinfeld's paper [3]; the question whether one should require bijectivity of the antipode is actually irrelevant here. To recall some details: If V is a finite-dimensional H-module, then a dual object for V in ${}_H\mathcal{M}$ is the dual vector space V^* with the H-module structure given by the transpose of the action via S, and the evaluation and coevaluation maps

$$\mathrm{ev}\colon V^*\otimes V\ni\varphi\otimes v\mapsto\varphi(\alpha v)\in k$$
$$\mathrm{db}\colon k\ni 1\mapsto\beta v_i\otimes v^i.$$

We shall now prove (3) $\Longrightarrow$ (1). We start by recalling the form of $\mathcal{R}$: it is the composition of the functor ${}_H\mathcal{M}\to {}_H\mathcal{M}_H$ which is given by restriction of the right module structure along $\varepsilon\colon H\to k$, and the cofree right comodule functor ${}_H\mathcal{M}_H\to {}_H\mathcal{M}_H^H$, where comodule now means comodule over the coassociative coalgebra H in the monoidal category ${}_H\mathcal{M}_H$. We observe that $\mathcal{R}$ is right adjoint, being the composition of two rather standard right adjoint functors: The left adjoint to the cofree comodule functor is just the underlying functor ${}_H\mathcal{M}_H^H\to {}_H\mathcal{M}_H$, while the left adjoint to the restriction functor ${}_H\mathcal{M}\to {}_H\mathcal{M}_H$ is the induction functor, in this case mapping $M\in {}_H\mathcal{M}_H$ to $M/MH^+\in {}_H\mathcal{M}$. Thus we have the overall left adjoint $\mathcal{F}$ to $\mathcal{R}$ mapping $M\in {}_H\mathcal{M}_H^H$ to $\overline{M}:=M/MH^+\in {}_H\mathcal{M}$, and it is easy to find the unit of adjunction to be

$$M\ni m\mapsto\overline{m_{(0)}}\otimes m_{(1)}\in M/MH^+\otimes H.$$

In particular,

$$\widetilde{\vartheta}\colon H_{\bullet}\otimes{}_{\bullet}H^{\bullet}_{\bullet}\ni g\otimes h\mapsto\overline{g\phi^{(1)}\otimes h_{(1)}\phi^{(2)}}\otimes h_{(2)}\phi^{(3)}\in\overline{H_{\bullet}\otimes{}_{\bullet}H^{\bullet}_{\bullet}}\otimes H$$

is an isomorphism. While $\widetilde{\vartheta}$ is a morphism in ${}_H\mathcal{M}_H^H$ by construction, where the structures are as indicated by the dots, we may observe that it is also a left H-module map with respect to another set of module structures: the action of H on the left tensor factor of $H \otimes H$ gives another H-bimodule ${}_\bullet H_\bullet \otimes H_\bullet$, and thus a left module $\overline{{}_\bullet H \otimes H}$; it is obvious that $\widetilde{\vartheta}$ is a module map with respect to these structures as well. In particular $\overline{{}_\bullet H \otimes H}^{\dim H} \cong H^{\dim H}$, so that we have $\overline{{}_\bullet H \otimes H} \cong H$ as left modules by Krull-Schmidt. Pick an isomorphism $\widetilde{\gamma}\colon \overline{H \otimes H} \to H$ of left H-modules. Setting $\gamma(h) = \widetilde{\gamma}(\overline{1 \otimes h})$ we find $\widetilde{\gamma}(\overline{g \otimes h}) = g\gamma(h)$. Next, we use the regular left $H \otimes H$-module structure on $H \otimes H$, which induces an $H \otimes H$-module structure on $\overline{H \otimes H}$; via $\widetilde{\gamma}$, we get an $H \otimes H$-module structure on H, such that the action of the left tensor factor is the regular module structure of H. In any such $H \otimes H$-module structure, the action of the right tensor factor has the form $h \circ g = gS(h)$ for some algebra antiendomorphism S of H. Thus

$$g\gamma(\ell h) = \widetilde{\gamma}(\overline{g \otimes \ell h}) = \widetilde{\gamma}(\overline{g \otimes h})S(\ell) = g\gamma(h)S(\ell)$$

for all $g, h, \ell \in H$. In particular $\gamma(h) = \beta S(h)$ for $\beta := \gamma(1)$. We define $\vartheta := (\widetilde{\gamma} \otimes H)\widetilde{\vartheta}\colon H \otimes H \to H \otimes H$ and find $\vartheta(g \otimes h) = g\phi^{(1)}\beta S(h_{(1)}\phi^{(2)}) \otimes h_{(2)}\phi^{(3)}$. Note that

$$\vartheta\colon H_\bullet \otimes {}_\bullet H_\bullet^\bullet \to {}_\bullet({}_S H) \otimes {}_\bullet H_\bullet^\bullet$$

is a morphism in ${}_H\mathcal{M}_H^H$ with the indicated structures, where ${}_S H$ denotes the left H-module structure on H given by S. In addition, ϑ is an H-module map with respect to the left H-module structures given by the regular action of H on the left tensor factors. We may summarize the three variants of H-linearity in the formula

$$\vartheta((g \otimes h)\xi(j_{(1)} \otimes j_{(2)})) = (g \otimes h_{(2)})\vartheta(\xi)(S(h_{(1)}) \otimes j)$$

for $g, h, j \in H$ and $\xi \in H \otimes H$, in which all multiplications are now in the algebra $H \otimes H$.

As a first application

$$h_{(1)}\beta S(h_{(2)}) = (H \otimes \varepsilon)\vartheta(h_{(1)} \otimes h_{(2)}) = (H \otimes \varepsilon)(\vartheta(1 \otimes 1)(1 \otimes h)) = \beta\varepsilon(h)$$

for $h \in H$. Next, we set $\alpha := (H \otimes \varepsilon)\vartheta^{-1}(1 \otimes 1)$, and find

$$(H \otimes \varepsilon)\vartheta^{-1}(g \otimes h) = (H \otimes \varepsilon)((g \otimes 1)\vartheta^{-1}(1 \otimes 1)(h_{(1)} \otimes h_{(2)})) = g\alpha h$$

for all $g, h \in H$. This implies further

$$S(h_{(1)})\alpha h_{(2)} = (H \otimes \varepsilon)\vartheta^{-1}(S(h_{(1)}) \otimes h_{(2)}) = (H \otimes \varepsilon)((1 \otimes h)\vartheta^{-1}(1 \otimes 1)) = \varepsilon(h)\alpha,$$

for $h \in H$, and

$$1 = (H \otimes \varepsilon)\vartheta^{-1}\vartheta(1 \otimes 1) = (H \otimes \varepsilon)\vartheta^{-1}(\phi^{(1)}\beta S(\phi^{(2)}) \otimes \phi^{(3)}) = \phi^{(1)}\beta S(\phi^{(2)})\alpha\phi^{(3)}.$$

We can determine the inverse of ϑ using that $H_\bullet \otimes {}_\bullet H_\bullet^\bullet$ is the cofree right H-comodule within ${}_H\mathcal{M}_H$ over H, so that we have

$$\begin{aligned}\vartheta^{-1}(g \otimes h) &= (H \otimes \varepsilon)\vartheta^{-1}((g \otimes h)_{(0)}) \otimes (g \otimes h)_{(1)} \\ &= (H \otimes \varepsilon)\vartheta^{-1}(gS(\phi^{(-1)}) \otimes \phi^{(-2)}h_{(1)}) \otimes \phi^{(-3)}h_{(2)} \\ &= gS(\phi^{(-1)})\alpha\phi^{(-2)}h_{(1)} \otimes \phi^{(-3)}h_{(2)}.\end{aligned}$$

We find that

$$\begin{aligned}1 = (H \otimes \varepsilon)\vartheta\vartheta^{-1}(1 \otimes 1) &= (H \otimes \varepsilon)\vartheta(S(\phi^{(-1)})\alpha\phi^{(-2)} \otimes \phi^{(-3)}) \\ &= S(\phi^{(-1)})\alpha\phi^{(-2)}\beta S(\phi^{(-3)}),\end{aligned}$$

which was the last axiom missing to show that (S, α, β) is a quasiantipode. $\square$

Remark 3.2.
1. The map ϑ occurs already in Drinfeld's paper [3]. Its inverse is the proper quasi-Hopf analog of the canonical map $\kappa\colon H \otimes H \ni g \otimes h \mapsto gh_{(1)} \otimes h_{(2)} \in H \otimes H$ for an ordinary bialgebra H. The canonical map κ is well-known to be a bijection if and only if H has an antipode. Note, however, that both ϑ and ϑ^{-1} involve the quasiantipode, while bijectivity of the naively copied map κ does not seem to have a relation to the question when a quasibialgebra H is a quasi-Hopf algebra.
2. As in Corollary 2.7 we see that $V_{\bullet} \otimes {}_{\bullet}H_{\bullet}^{\bullet} \cong {}_{S}V \otimes {}_{\bullet}H_{\bullet}^{\bullet} \in {}_{H}\mathcal{M}_{H}^{H}$ for every $V \in \mathcal{M}_H$, when H is a finite-dimensional quasi-Hopf algebra. The same can be shown explicitly if H is not finite-dimensional, see [4, 10].

4. Bijectivity of the antipode

Bulacu and Caenepeel have proved that the antipode (mapping) of a finite-dimensional quasi-Hopf algebra H is always a bijection. As in the ordinary Hopf case, they show this along with the existence of integrals, and the fact that H is a Frobenius algebra. The standard textbook [11, 1, 5] proof does not work in the quasi-Hopf situation: It is based on finding a Hopf module structure on the dual H^*. A structure of Hopf module in ${}_{H}\mathcal{M}_{H}^{H}$ is indeed given in [4] to do integral theory, but only *using the assumption* that the antipode is bijective.

In this section we will give a rather short proof for:

Theorem 4.1 (Bulacu–Caenepeel [2]). *Let H be a finite-dimensional quasi-Hopf algebra with antipode (S, α, β). Then H is a Frobenius algebra, and S is bijective.*

Proof. Since $H \otimes H$ is an $H \otimes H$-$H \otimes H$-bimodule, we get an $H \otimes H$-$H \otimes H$-bimodule structure on

$$\mathrm{Hom}_{H\otimes k-}(H\otimes H, H) := \{F\colon H\otimes H \to H | \forall h \in H, \xi \in H\otimes H\colon F((h\otimes 1)\xi) = hF(\xi)\}$$

by setting $((g \otimes h)F(j \otimes \ell))(\xi) = F((1 \otimes \ell)\xi(g \otimes h))j$ for all $g, h, j, \ell \in H$, $F \in \mathrm{Hom}_{H\otimes k-}(H \otimes H, H)$ and $\xi \in H \otimes H$. Thus ϑ induces an automorphism ϑ^t of $\mathrm{Hom}_{H\otimes k-}(H \otimes H, H)$ satisfying

$$\vartheta^t((S(g_{(1)}) \otimes h)F(j \otimes g_{(2)})) = (h_{(1)} \otimes h_{(2)})\vartheta^t(F)(j \otimes g).$$

We have a bijection $T\colon H \otimes H^* \to \mathrm{Hom}_{H\otimes k-}(H \otimes H, H)$ given by $T(g \otimes \varphi)(h \otimes j) = h\varphi(j)g$. It is straightforward to check that T is an $H \otimes H$-$H \otimes H$-bimodule map.

Thus ϑ^t induces an automorphism of $H \otimes H^*$ that is an isomorphism between the left H-action on the right tensor factor, and the diagonal left H-action. The latter has the structure of a Hopf module ${}^{\bullet}_{\bullet}H_{\bullet} \otimes {}_{\bullet}H^* \in {}^{H}_{H}\mathcal{M}_{H}$, so by the structure theorem for Hopf modules (which applies in the left-right switched version since H^{bop} is a quasi-Hopf algebra), it is a free H-module, isomorphic to $H^{\dim H}$. By the Krull-Schmidt Theorem, $H^* \cong H$ as left H-modules. Thus H is a Frobenius algebra. Now we consider once more the isomorphism ϑ^t, and identify $H \otimes H^*$ with $H \otimes H$ as left modules. We see that $H_{\bullet} \otimes {}_{\bullet}H \cong {}_{\bullet}H_{\bullet} \otimes {}_{\bullet}H$ as H-H-bimodules. But ${}_{\bullet}H_{\bullet} \otimes {}_{\bullet}H \cong {}_{\bullet}H_{\bullet} \otimes H_S$ by the left-right switched version of part (2) of Remark 3.2, and it follows that we have an isomorphism of right H-modules

$$H \cong k \underset{H}{\otimes} (H_{\bullet} \otimes {}_{\bullet}H) \cong k \underset{H}{\otimes} ({}_{\bullet}H_{\bullet} \otimes H_S) \cong H_S.$$

Thus S is a bijection. $\square$

Our short proof took advantage of the structure theorem for Hopf modules as well as the isomorphism ϑ. It may be worthwhile to note that one does not really need the full generality of the structure theorem, but can use more directly the information contained in the map ϑ:

For $V \in \mathcal{M}_H$, we can define

$$\vartheta_V : V \otimes H \ni v \otimes h \mapsto v\phi^{(1)}\beta S(h_{(1)}\phi^{(2)}) \otimes h_{(2)}\phi^{(3)} \in V \otimes H$$

and

$$\vartheta_V^{-1} : V \otimes H \ni v \otimes h \mapsto vS(\phi^{(-1)})\alpha\phi^{(-2)}h_{(1)} \otimes \phi^{(-3)}h_{(2)} \in V \otimes H.$$

Since ϑ_V and ϑ_V^{-1} are natural in $V \in \mathcal{M}_H$ and mutually inverse isomorphisms for $V = H$, we see that they are mutually inverse isomorphisms for any $V \in \mathcal{M}_H$. In particular, we see that $V_{\bullet} \otimes {}_{\bullet}H_{\bullet} \cong {}_SV \otimes {}_{\bullet}H_{\bullet}$ as H-bimodules, so that every right H-module of the form $V_{\bullet} \otimes H_{\bullet}$ is free. Since H^{bop} is a quasi-Hopf algebra, every H-H-bimodule of the form ${}_{\bullet}H_{\bullet} \otimes {}_{\bullet}V$ with $V \in {}_H\mathcal{M}$ is is isomorphic to ${}_{\bullet}H_{\bullet} \otimes V_S$, hence free as a left H-module. No other cases of the structure theorem for Hopf modules were used in our proof.

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