THE STRUCTURE OF HOPF ALGEBRAS WITH A WEAK PROJECTION

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1. Introduction

In [12] Radford describes completely the structure of Hopf algebras with a projection: Assume H is a bialgebra and $H' \subset H$ is a Hopf subalgebra such that there is a bialgebra map $\pi\colon H \to H'$ whose restriction to H' is the identity. Then H decomposes as a tensor product $H' \otimes Q$. The vector space Q has the structures of an algebra, a coalgebra, an H'-module, and an H'-comodule that satisfy a list of compatibility conditions which, in more recent terminology, say precisely that Q is a bialgebra in the braided monoidal category of Yetter-Drinfeld modules. In particular, one should note that Q is not a bialgebra in the usual sense. In the literature, there are various other ways of constructing bialgebra or Hopf algebra structures on a tensor product $H' \otimes Q$: Bismash or bicrossed products, more generally bicrossed products with cocycles, and double crossproducts [8, 6] building on previous results [16, 18]. In the present paper we study Hopf algebras with a weak projection (to be defined below). This setup is sufficiently general to include bicrossproducts and double crossproducts in the construction. Moreover, all quantized universal enveloping algebras are examples of Hopf algebras with weak projections (onto the analogs of their Cartan subalgebras). By a recent result of Ştefan and Van Oystaeyen [17], any finite dimensional pointed Hopf algebra over a field of characteristic zero has a weak projection onto its coradical.

Let H be a bialgebra and H' a Hopf subalgebra. We define a weak projection to be a retraction $\pi\colon H \to H'$ for the inclusion map which is a coalgebra map and left H'-linear. Given a weak projection, one can decompose $H = H' \otimes Q$ with $Q \cong H/H'^{+}H$. The relevant coalgebra structure on the tensor product $H' \otimes Q$ is that of a cosmash product $H' \ltimes Q$ with respect to a suitable coaction $Q \ni q \mapsto q_{[0]} \otimes q_{[1]} \in Q \otimes H'$. In [13] we have studied Q-cleft H-comodule algebras of a certain type and characterized them as a generalized type of crossed products. As a special case, we have seen that the multiplication on H itself can be written as

$$(x \ltimes p)(y \ltimes q) = x(p_{(1)} \rightharpoonup y_{(1)})\tau(p_{(2)} \leftharpoonup y_{(2)}|q_{(1)[0]}) \ltimes (p_{(3)} \leftharpoonup y_{(3)}q_{(1)[1]}) \cdot q_{(2)}$$

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in terms of maps

$$\begin{aligned} Q\otimes H' \ni q\otimes x &\mapsto q \leftharpoonup x \in Q\\ Q\otimes H' \ni q\otimes x &\mapsto q \rightharpoonup x \in H'\\ Q\otimes Q \ni p\otimes q &\mapsto p\cdot q \in Q\\ Q\otimes Q \ni p\otimes q &\mapsto \tau(p|q) \in H'. \end{aligned}$$

We have conjectured a list of axioms for these operation that ensure that $H' \ltimes Q$ becomes a bialgebra in this way.

In the present paper we derive both the above general formula for multiplication in a cosmash product and a list of axioms for the operations listed. More generally we consider, for H' a bialgebra and Q an H'-comodule coalgebra, all possible structures making the cosmash product $H' \ltimes Q$ a bialgebra with H' as a subbialgebra with the canonical projection of $H' \ltimes Q$ onto H' a left linear map.

We organize the investigation of such structures in several steps according to the following principle: A bialgebra H containing H' as a subbialgebra is a bialgebra in the monoidal category of H'-bimodule coalgebras. An H'-bimodule coalgebra is a right H'-module in the category of left H'-module coalgebras. We are looking for such a bialgebra H which is a cosmash product. In section 2 we start by describing maps $F\colon H' \ltimes P \to H' \ltimes Q$ in terms of their "constituents" $F^{(1)}\colon P \to H'$ and $F^{(2)}\colon P \to Q$, which satisfy certain axioms. We derive simple formulas that describe the constituents of a composition. In section 3 we then describe H'-bimodule coalgebra structures on $H' \ltimes Q$. Since these are given by maps $H' \ltimes Q \otimes H' \to H' \ltimes Q$, and since $H' \ltimes Q \otimes H'$ is itself a cosmash product, all we have to do is use the characterization of maps between cosmash products obtained in Section 2, and write out the constituents of the module axioms. In section 4 we prepare for bialgebra structures by observing that for two bimodule coalgebras $H' \ltimes P$ and $H' \ltimes Q$, the tensor product $(H' \ltimes P) \otimes_{H'} (H' \ltimes Q)$ is again a cosmash product $H' \ltimes (P \otimes Q)$. We compute the necessary structures on $P \otimes Q$, and we provide formulas for the constituents of a tensor product of morphisms. In section 5 we write out in terms of constituents the axioms for a multiplication $\nabla\colon (H' \ltimes Q) \otimes_{H'} (H' \ltimes Q) \to H' \ltimes Q$, using the formulas describing tensor products. We also write out the associativity axioms in terms of constituents; the complete description of bialgebra structures on $H' \ltimes Q$ thus obtained is listed in Theorem 5.1. As a striking feature we only note here that Q has a multiplication that is not necessarily associative, but whose non-associativity is controlled by a two-cocycle in equation (5.13)

In Theorem 5.3 we characterize when the bialgebra structures we have described are Hopf algebra structures. Basically, this is the case when H' is a Hopf algebra, and Q fulfills a natural generalization of the requirement to be a Hopf algebra (still suited in the present situation where Q is not a bialgebra due to its nonassociative multiplication).

In section 6 we show in what way Radford's biproducts as well as Majid's bicrossproducts and double crossproducts are examples of our construction. More generally, the special case of trivial cocycle τ in our construction gives the trivalent products previously studied in [2], see also [5]. (The more recent [3] also allows cocycles, but the construction of the present paper does not seem to be a special case.) We also recall from [13] why Hopf algebras with a triangular decomposition, like quantized enveloping algebras, have weak projections in our sense.

One of the special cases of our construction discussed in section 6 is also a special case of a bicrossproduct $A\,{}^{\alpha}{\bowtie}_{\tau}B$ of two Hopf algebras with a cocycle $\tau\colon B \otimes B \to A$ and a dual cocycle $\alpha\colon B \to A \otimes A$; namely, the special case in which $\alpha = \varepsilon$ is trivial. When applied in this situation, Theorem 5.3 yields that $A\,{}^{\alpha}{\bowtie}_{\tau}B$ has an antipode, where no condition on invertibility of τ has to be imposed (in fact, invertibility of τ is then a consequence). This leads us to consider, in an appendix, antipodes for bicrossproduct bialgebras $A\,{}^{\alpha}{\bowtie}_{\tau}B$ with both a cocycle and a dual cocycle. Results of Hofstetter [6] and Andruskiewitsch and Devoto [1] say that when A and B are Hopf algebras and both α and τ are invertible, then $A\,{}^{\alpha}{\bowtie}_{\tau}B$ is a Hopf algebra. We prove that it is in fact enough to assume that α *or* τ is invertible.

Throughout the paper we work over a fixed base field k. Unadorned tensor products are over k, algebras are k-algebras etc. For the comultiplication of a coalgebra C we use the variant $\Delta(c) = c_{(1)} \otimes c_{(2)}$ of Sweedler's notation; we write $\varepsilon\colon C \to k$ for the counit. If confusion seems likely, we will write Δ_C and ε_C. We write S for the antipode of a Hopf algebra. Let H' be a bialgebra. We will use the following notation for the comodule structure map of a right H'-comodule Q:

$$\rho'\colon Q \ni q \mapsto q_{[0]} \otimes q_{[1]} \in Q \otimes H'$$

Recall that an H'-comodule coalgebra is a coalgebra Q which is a right H'-comodule satisfying

$$\Delta(q_{[0]}) \otimes q_{[1]} = q_{(1)[0]} \otimes q_{(2)[0]} \otimes q_{(1)[1]}q_{(2)[1]}$$

and $\varepsilon(q_{[0]})q_{[1]} = \varepsilon(q)1$.

2. The category of cosmash products

Let H' be a bialgebra. We recall the following well known construction: given a right H'-comodule coalgebra Q one can form the cosmash or cosemidirect product $H' \ltimes Q$ which is $H' \otimes Q$ as a k-space and has the following structure of left H'-module coalgebra:

$$\Delta(x \ltimes q) = x_{(1)} \ltimes q_{(1)[0]} \otimes x_{(2)}q_{(1)[1]} \ltimes q_{(2)} \tag{2.1}$$

$\varepsilon(x \ltimes q) = \varepsilon(x)\varepsilon(q)$ and $x(y \ltimes q) = xy \ltimes q$. The following Lemma characterizes cosmash products as special module coalgebras:

Lemma 2.1. *The following are equivalent for an H'-module coalgebra C:*

1. *C is isomorphic to a cosmash product $C \cong H' \ltimes Q$ for an H'-comodule coalgebra Q.*
2. *There is an isomorphism $\psi\colon C \cong H' \otimes \overline{C}$ of left H'-modules and right $\overline{C}$-comodules such that $(H' \otimes \varepsilon)\psi\colon C \to H'$ is a coalgebra map, where $\overline{C} = C/H'^{+}C$.*
3. *There is a left H'-linear coalgebra map $\pi\colon C \to H'$, a k-space Q and a map $\nu\colon C \to Q$, left H'-linear with respect to the trivial module structure on Q, such that $\psi = (\pi \otimes \nu)\Delta\colon C \to H' \otimes Q$ is an isomorphism.*

We will only sketch the proof. The equivalence of (1) and (2) is well known at least for the dual case of smash products. Given a cosmash product $C = H' \ltimes Q$ we can define maps as in (3) by $\pi(x \ltimes q) = x\varepsilon(q)$ and $\nu(x \ltimes q) = \varepsilon(x)q$. Given maps as in (3) let $\varphi\colon Q \to C$ be such that $\varphi(q) = \psi^{-1}(1 \otimes q)$. Then the comodule coalgebra structure on Q such that ψ is an isomorphism $C \cong H' \ltimes Q$ can be found as $\Delta_Q = (\nu \otimes \nu)\Delta_C\varphi\colon Q \to Q \otimes Q$, $\varepsilon_Q = \varepsilon_C\varphi$ and $\rho' = (\nu \otimes \pi)\Delta_C\varphi\colon Q \to Q \otimes H'$.

The following two Lemmas record couniversal properties of cosmash products that should be considered (at least dual to) standard material.

Lemma 2.2. *Let Q be an H'-comodule coalgebra and C a coalgebra. There is a bijection between*

1. *Coalgebra maps $F\colon C \to H' \ltimes Q$*
2. *Pairs $(F^{[1]}, F^{[2]}) \in \mathrm{Coalg}(C, H') \times \mathrm{Coalg}(C, Q)$ satisfying*

$$(F^{[2]} \otimes F^{[1]})\Delta(c) = F^{[2]}(c_{(2)})_{[0]} \otimes F^{[1]}(c_{(1)})F^{[2]}(c_{(2)})_{[1]} \tag{2.2}$$

given by $F = (F^{[1]} \otimes F^{[2]})\Delta$ and $F^{[1]} = \pi F$, $F^{[2]} = \nu F$.

Lemma 2.3. *With notations as in the preceding Lemma assume further that B is an H'-module coalgebra. Then F is H'-linear if and only if both $F^{[1]}$ and $F^{[2]}$ are H'-linear, where H' carries the regular, and Q the trivial H'-module structure.*

It is easy to see that, as a module coalgebra, a cosmash product has a universal property in addition to the couniversal ones listed above. This is based on the standard bijection between H'-linear maps from $H' \otimes P$ to an H'-module M and k-linear maps from P to M:

Lemma 2.4. *Let P be an H'-comodule coalgebra and D an H'-module coalgebra. There is a bijection between*

1. *H'-module coalgebra maps $G\colon H' \ltimes P \to D$*
2. *k-linear maps $G_0\colon P \to D$ satisfying $\varepsilon G_0 = \varepsilon$ and*

$$\Delta G_0(p) = G_0(p_{(1)[0]}) \otimes p_{(1)[1]}G_0(p_{(2)}) \tag{2.3}$$

given by $G(x \ltimes p) = xG_0(p)$ and $G_0(p) = G(1 \ltimes p)$.

It is now easy to combine the universal and couniversal properties of cosmash products to obtain a description of module coalgebra maps between two cosmash products in terms of two constituents:

Proposition 2.5. *Let P and Q be two H'-comodule coalgebras. A bijection between*

1. *H'-module coalgebra maps $F\colon H' \ltimes P \to H' \ltimes Q$ and*
2. *Pairs $(F^{(1)}, F^{(2)}) \in \operatorname{Hom}(P, H') \times \operatorname{Coalg}(P, Q)$ satisfying*

$$\Delta F^{(1)}(p) = F^{(1)}(p_{(1)[0]}) \otimes p_{(1)[1]} F^{(1)}(p_{(2)}) \tag{2.4}$$

$$F^{(2)}(p_{(2)})_{[0]} \otimes F^{(1)}(p_{(1)}) F^{(2)}(p_{(2)})_{[1]} = F^{(2)}(p_{(1)[0]}) \otimes p_{(1)[1]} F^{(1)}(p_{(2)}) \tag{2.5}$$

$$\varepsilon F^{(1)}(p) = \varepsilon(p) \tag{2.6}$$

for all $p \in P$

is given by $F(x \ltimes p) = xF^{(1)}(p_{(1)}) \ltimes F^{(2)}(p_{(2)})$, $F^{(1)} = \pi F \varphi$ and $F^{(2)} = \nu F \varphi$.

Proof. By Lemmas 2.2 and 2.3, giving an H'-module coalgebra map $F\colon H' \ltimes P \to H' \ltimes Q$ is the same as giving two H'-module coalgebra maps $F^{[1]}\colon H' \ltimes P \to H'$ and $F^{[2]}\colon H' \ltimes P \to Q$ satisfying (2.2) for $b \in H' \ltimes P$. By Lemma 2.4 the H'-module coalgebra maps $F^{[1]}$ and $F^{[2]}$ are described equivalently by maps $F^{(1)}\colon P \to H'$ and $F^{(2)}\colon P \to Q$ such that $F^{(2)}$ is a coalgebra map (note that Q has the trivial H'-module structure when writing out (2.3) for $F^{(2)}$), $\varepsilon F^{(1)} = \varepsilon$, and (2.4) holds. Written out for $b = x \ltimes p \in H' \ltimes P$, equation (2.2) reads

$$F^{(2)}(p_{(1)[0]}) \otimes x p_{(1)[1]} F^{(1)}(p_{(2)}) = F^{(2)}(p_{(2)})_{[0]} \otimes x F^{(1)}(p_{(1)}) F^{(2)}(p_{(2)})_{[1]}$$

which is equivalent to (2.5) for all $p \in P$. □

In the following, we shall refer to the maps $F^{(1)}$ and $F^{(2)}$ as the first and second *constituents* of F. The key to our subsequent calculations will be to do them in terms of constituents. The following Lemma records the relevant formulas for the constituents of a composition.

Lemma 2.6. *Let P, Q, R be H'-comodule coalgebras, and let $F\colon H' \ltimes P \to H' \ltimes Q$ and $G\colon H' \ltimes Q \to H' \ltimes R$ be two H'-module coalgebra maps. Then we have*

$$(GF)^{(1)} = F^{(1)} * G^{(1)} F^{(2)} \tag{2.7}$$

$$(GF)^{(2)} = G^{(2)} F^{(2)} \tag{2.8}$$

Proof. We compute

$$\begin{aligned}(GF)^{(1)}(p) = \pi GF(1 \ltimes p) &= \pi G(F^{(1)}(p_{(1)}) \ltimes F^{(2)}(p_{(2)})) \\ &= F^{(1)}(p_{(1)}) \pi G(1 \ltimes F^{(2)}(p_{(2)})) = F^{(1)}(p_{(1)}) G^{(1)}(F^{(2)}(p_{(2)}))\end{aligned}$$

for $p \in P$, which proves the first equation, while

$$\begin{aligned}(GF)^{(2)}(p) = \nu GF(1 \ltimes p) = \nu G(F^{(1)}(p_{(1)}) \ltimes F^{(2)}(p_{(2)}))\\ = \varepsilon F^{(1)}(p_{(1)})\nu G(1 \ltimes F^{(2)}(p_{(2)})) = G^{(2)}F^{(2)}(p)\end{aligned}$$

proves the second. □

3. Cosmash bimodule coalgebras

If C is an H'-module coalgebra and D is a coalgebra, then $C \otimes D$ with the module structure induced by that of the left tensorand is an H'-module coalgebra.

Now assume that $C = H' \ltimes Q$ is a cosmash product. It is straightforward to check that the identity on $H' \otimes Q \otimes C$ is an isomorphism $C \otimes D \cong H' \ltimes (Q \otimes C)$ of H'-module coalgebras, where $Q \otimes C$ denotes the tensor product coalgebra with the H'-comodule structure induced by the left tensorand. If $F\colon H' \ltimes P \to H' \ltimes Q$ is an H'-module coalgebra morphism and $f\colon D \to D'$ is a coalgebra homomorphism, then

$$(F \otimes f)^{(1)} = F^{(1)} \otimes \varepsilon \tag{3.1}$$

$$(F \otimes f)^{(2)} = F^{(2)} \otimes f. \tag{3.2}$$

Assume that D is a bialgebra and Q is an H'-comodule coalgebra. We shall now describe how to endow the cosmash product $C = H' \ltimes Q$ with an additional right D-module structure making it an H'-D-bimodule coalgebra. This is achieved precisely by giving a map $\mu\colon C \otimes D \to C$ of H'-module coalgebras satisfying the conditions for a right D-module structure:

$$\mu(\mu \otimes D) = \mu(C \otimes \nabla) \tag{3.3}$$

$$\mu(C \otimes \eta) = \mathrm{id}_C\,. \tag{3.4}$$

It only remains to write out the axioms for such a map μ in terms of its constituents $\mu^{(1)}\colon Q \otimes D \to H'$ and $\mu^{(2)}\colon Q \otimes D \to Q$. We write them as $\mu^{(1)}(q \otimes d) = q \rightharpoonup d$ and $\mu^{(2)}(q \otimes d) = q \leftharpoonup d$. We claim that the result is the following:

Proposition 3.1. *Let Q be an H'-comodule coalgebra, and D a bialgebra. A bijection between*

1. *Right D-module structures on $H' \ltimes Q$ making it an H'-D-bimodule coalgebra, and*

2. *Pairs* $(\leftharpoonup, \rightharpoonup)$, *where* $\leftharpoonup$ *is a* D*-module coalgebra structure on* Q *and* $\rightharpoonup\colon Q \otimes D \to H'$ *is a map satisfying*

$$q \rightharpoonup (xy) = (q_{(1)} \rightharpoonup x_{(1)})((q_{(2)} \leftharpoonup x_{(2)}) \rightharpoonup y) \tag{3.5}$$

$$q \rightharpoonup 1 = \varepsilon(q)1 \tag{3.6}$$

$$\Delta(q \rightharpoonup x) = q_{(1)[0]} \rightharpoonup x_{(1)} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x_{(2)}) \tag{3.7}$$

$$\varepsilon(q \rightharpoonup x) = \varepsilon(q)\varepsilon(x) \tag{3.8}$$

$$\begin{aligned}(q_{(2)} \leftharpoonup x_{(2)})_{[0]} \otimes (q_{(1)} \rightharpoonup x_{(1)})(q_{(2)} \leftharpoonup x_{(2)})_{[1]}\\ = q_{(1)[0]} \leftharpoonup x_{(1)} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x_{(2)})\end{aligned} \tag{3.9}$$

for all $q \in Q$ *and* $x, y \in D$

is given by

$$(x \ltimes q)y = x(q_{(1)} \rightharpoonup y_{(1)}) \ltimes q_{(2)} \leftharpoonup y_{(2)}$$
$$q \leftharpoonup y = \nu((1 \ltimes q)y) \qquad q \rightharpoonup y = \pi((1 \ltimes q)y)$$

for $x \in H'$, $q \in Q$ *and* $y \in D$.

Proof. The conditions that $\mu^{(1)}$ and $\mu^{(2)}$ are the constituents of a morphism $\mu\colon H' \ltimes Q \otimes D \to H' \ltimes Q$ are listed in Lemma 2.5: $\mu^{(2)}$ has to be a coalgebra map, which is one half of the condition above that $\leftharpoonup$ make Q a D-module coalgebra. Condition (2.4) for $\mu^{(1)}$ is precisely (3.7) above, and (2.6) is precisely (3.8). Condition (2.5) from Lemma 2.5 is precisely (3.9) above.

Assuming now that μ is a morphism, we compute

$$\begin{aligned}(\mu(\mu \otimes D))^{(1)}(q \otimes x \otimes y) &= ((\mu \otimes D)^{(1)} * \mu^{(1)}(\mu \otimes D)^{(2)})(q \otimes x \otimes y)\\ &= ((\mu^{(1)} \otimes \varepsilon) * \mu^{(1)}(\mu^{(2)} \otimes D))(q \otimes x \otimes y) = (q_{(1)} \rightharpoonup x_{(1)})((q_{(2)} \leftharpoonup x_{(2)}) \rightharpoonup y)\end{aligned}$$

and

$$\begin{aligned}(\mu(C \otimes \nabla))^{(1)}(q \otimes x \otimes y) &= ((C \otimes \nabla)^{(1)} * \mu^{(1)}(C \otimes \nabla)^{(2)})(q \otimes x \otimes y)\\ &= (\varepsilon * \mu^{(1)}(Q \otimes \nabla))(q \otimes x \otimes y) = q \rightharpoonup (xy),\end{aligned}$$

so that the first constituent of equation (3.3) is precisely (3.5). The first constituent of equation (3.4) is precisely (3.6). It is easy to check that the second constituents of (3.3) and (3.4) combine to be the axioms for Q to be a right D-module. □

Proposition 3.2. *Let* D *be a bialgebra, and assume that the cosmash products* $H' \ltimes P$ *and* $H' \ltimes Q$ *are endowed with right* D*-module structures making them* H'*-*D*-bimodule coalgebras.*

An H'*-module coalgebra map* $F\colon H' \ltimes P \to H' \ltimes Q$ *is right* D*-linear if and only if* $F^{(2)}$ *is a* D*-module map and*

$$F^{(1)}(p_{(1)})(F^{(2)}(p_{(2)}) \rightharpoonup x) = (p_{(1)} \rightharpoonup x_{(1)})F^{(1)}(p_{(2)} \leftharpoonup x_{(2)}) \tag{3.10}$$

holds for all $p \in P$ *and* $x \in D$.

Proof. Since

$$(\mu(F\otimes D))^{(1)}(p\otimes x)=(F\otimes D)^{(1)}(p_{(1)}\otimes x_{(1)})\mu^{(1)}(F\otimes D)^{(2)}(p_{(2)}\otimes x_{(2)})$$
$$=(F^{(1)}\otimes\varepsilon)(p_{(1)}\otimes x_{(1)})\mu^{(1)}(F^{(2)}\otimes D)(p_{(2)}\otimes x_{(2)})=F^{(1)}(p_{(1)})(F^{(2)}(p_{(2)})\rightharpoonup x)$$

and

$$(F\mu)^{(1)}(p\otimes x)=\mu^{(1)}(p_{(1)}\otimes x_{(1)})F^{(1)}\mu^{(2)}(p_{(2)}\otimes x_{(2)})$$
$$=(p_{(1)}\rightharpoonup x_{(1)})F^{(1)}(p_{(2)}\leftharpoonup x_{(2)}),$$

the first constituent of the defining equation $\mu(F\otimes D)=F\mu$ for a D-module map is precisely equation (3.10). It is easy to check that the second constituent is precisely the condition that $F^{(2)}$ be a module map. □

4. Tensor products of bimodule coalgebras

If C,D are H'-bimodule coalgebras, then their tensor product $C\otimes_{H'}D$ over H' is in a natural way an H'-bimodule coalgebra. This describes a structure of monoidal category on the category of all H'-bimodule coalgebras, with H' as the neutral object.

If $\pi_C\colon C\to H'$ and $\pi_D\colon D\to H'$ are left H'-module coalgebra maps, then so is

$$\pi\colon C\otimes_{H'}D\ni c\otimes d\mapsto\pi_C(c\pi_D(d))\in H'.$$

Assume $C=H'\ltimes P$ and $D=H'\ltimes Q$ are cosmash products H'-bimodule coalgebras. Then $\pi\colon C\otimes_{H'}D\to H'$ fulfills $\pi(x\ltimes p\otimes y\ltimes q)=x(p\rightharpoonup y)\varepsilon(q)$. The next Lemma states that $C\otimes_{H'}D$ is again a cosmash product in this situation, and computes the relevant structures

Lemma 4.1. *Let $C=H'\ltimes P$ and $D=H'\ltimes Q$ be two cosmash products, endowed with right H'-module structures making them H'-H'-bimodule coalgebras. Then the canonical isomorphism*

$$\psi\colon(H'\otimes P)\otimes_{H'}(H'\otimes Q)\to H'\otimes P\otimes Q$$

is an isomorphism of H'-bimodule coalgebras

$$\psi\colon(H\ltimes P)\otimes_{H'}(H'\ltimes Q)\to H'\ltimes(P\otimes Q)$$

when we endow $P\otimes Q$ with the following structures:

$$\Delta(p\otimes q)=p_{(1)}\otimes q_{(1)[0]}\otimes p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)}\tag{4.1}$$
$$\varepsilon(p\otimes q)=\varepsilon(p)\varepsilon(q)\tag{4.2}$$
$$\rho'(p\otimes q)=p_{(1)[0]}\otimes q_{[0]}\otimes p_{(1)[1]}(p_{(2)}\rightharpoonup q_{[1]})\tag{4.3}$$
$$(p\otimes q)\leftharpoonup x=p\leftharpoonup(q_{(1)}\rightharpoonup x_{(1)})\otimes q_{(2)}\leftharpoonup x_{(2)}\tag{4.4}$$
$$(p\otimes q)\rightharpoonup x=p\rightharpoonup(q\rightharpoonup x)\tag{4.5}$$

Proof. Define

$$\nu\colon (H'\ltimes P)\otimes_{H'}(H'\ltimes Q)\to P\otimes Q$$

to be the unique map, left H'-linear with respect to the trivial module structure on $P\otimes Q$, satisfying $\nu(1\ltimes p\otimes 1\ltimes q)=p\otimes q$. One finds

$$\nu(x\ltimes p\otimes y\ltimes q)=\nu(x(p_{(1)}\rightharpoonup y_{(1)})\ltimes p_{(2)}\leftharpoonup y_{(2)}\otimes 1\ltimes q)=\varepsilon(x)p\rightharpoonup y\otimes q.$$

Since the map

$$(\pi\otimes\nu)\Delta\colon (H'\ltimes P)\otimes_{H'}(H'\ltimes Q)\to H'\otimes P\otimes Q$$

coincides with the canonical isomorphism ψ by the calculation

$$\begin{aligned}(\pi\otimes\nu)\Delta(x\ltimes p\otimes 1\ltimes q)&=\pi(x_{(1)}\ltimes p_{(1)[0]}\otimes 1\ltimes q_{(1)[0]})\otimes\nu(x_{(2)}p_{(1)[1]}\ltimes p_{(2)}\otimes q_{(1)[1]}\otimes q_{(2)})\\&=x_{(1)}\otimes\nu(x_{(2)}\ltimes p\otimes 1\ltimes q)=x\otimes p\otimes q,\end{aligned}$$

$(H'\ltimes P)\otimes_{H'}(H'\ltimes Q)$ is a cosmash product as claimed. We compute

$$\begin{aligned}\Delta(p\otimes q)&=(\nu\otimes\nu)\Delta(1\ltimes p\otimes 1\ltimes q)\\&=\nu(1\ltimes p_{(1)[0]}\otimes 1\otimes q_{(1)[0]})\otimes\nu(p_{(1)[1]}\otimes p_{(2)}\otimes q_{(1)[1]}\otimes q_{(2)})\\&=p_{(1)}\otimes q_{(1)[0]}\otimes p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)}\end{aligned}$$

$$\rho'(p\otimes q)=(\nu\otimes\pi)\Delta(1\ltimes p\otimes 1\ltimes q)=p_{(1)[0]}\otimes q_{[0]}\otimes p_{(1)[1]}(p_{(2)}\rightharpoonup q_{[1]}),$$

and, from $(1\ltimes p\otimes 1\ltimes q)x=1\ltimes p\otimes q_{(1)}\rightharpoonup x_{(1)}\otimes q_{(2)}\leftharpoonup x_{(2)}$,

$$(p\otimes q)\rightharpoonup x=\nu((1\ltimes p\otimes 1\ltimes q)x)=p\rightharpoonup(q\rightharpoonup x)$$

and

$$(p\otimes q)\leftharpoonup x=\pi((1\ltimes p\otimes 1\ltimes q)x)=p\leftharpoonup(q_{(1)}\rightharpoonup x_{(1)})\otimes q_{(2)}\leftharpoonup x_{(2)}\qquad\square$$

Proposition 4.2. *Let $H'\ltimes P$, $H'\ltimes P'$, $H'\ltimes Q$ and $H'\ltimes Q'$ be cosmash products endowed with right H'-module structures making them H'-bimodule coalgebras. Let $F\colon H'\ltimes P\to H'\ltimes P'$ and $G\colon H'\ltimes Q\to H'\ltimes Q'$ be two H'-bimodule coalgebra maps. Then*

$$(F\otimes_{H'}G)^{(1)}(p\otimes q)=F^{(1)}(p_{(1)})(F^{(2)}(p_{(2)})\rightharpoonup G^{(1)}(q))\tag{4.6}$$

$$(F\otimes_{H'}G)^{(2)}(p\otimes q)=F^{(2)}(p)\leftharpoonup G^{(1)}(q_{(1)})\otimes G^{(2)}(q_{(2)})\tag{4.7}$$

in particular

$$(F\otimes_{H'}\mathrm{id})^{(1)}=F^{(1)}\otimes\varepsilon\tag{4.8}$$

$$(id\otimes_{H'}G)^{(1)}(p\otimes q)=p\rightharpoonup G^{(1)}(q)\tag{4.9}$$

$$(F\otimes_{H'}\mathrm{id})^{(2)}=F^{(2)}\otimes\mathrm{id}\,.\tag{4.10}$$

for $p\in P$ and $q\in Q$.

Proof. It suffices to prove the first two equations, of which the remaining three are special cases. We compute

$$\begin{aligned}(F\otimes_{H'}G)^{(1)}(p\otimes q)&=\pi(F\otimes_{H'}G)(1\ltimes p\otimes 1\ltimes q)\\&=\pi(F^{(1)}(p_{(1)})\ltimes F^{(2)}(p_{(2)})\otimes G^{(1)}(q_{(1)})\ltimes G^{(2)}(q_{(2)}))\\&=F^{(1)}(p_{(1)})(F^{(2)}(p_{(2)})\rightharpoonup G^{(1)}(q))\end{aligned}$$

and

$$\begin{aligned}(F\otimes_{H'}G)^{(2)}(p\otimes q)&=\nu(F^{(1)}(p_{(1)})\ltimes F^{(2)}(p_{(2)})\otimes G^{(1)}(q_{(1)})\ltimes G^{(2)}(q_{(2)}))\\&=F^{(2)}(p)\leftharpoonup G^{(1)}(q_{(1)})\otimes G^{(2)}(q_{(2)})\qquad\square\end{aligned}$$

5. Bialgebras with a left projection

Let H be a bialgebra and $H'\subset H$ a Hopf subalgebra (i.e. a subbialgebra that is a Hopf algebra). We say that a map $\pi\colon H\to H'$ is a weak (left) projection if π is a left H'-linear coalgebra map and $\pi|_{H'}=\mathrm{id}_{H'}$. Since π is convolution invertible (with convolution inverse $S\pi$) the left H'-module coalgebra structure of H is that of a cosmash product: For $Q=H/H'^{+}H$ endowed with the H'-comodule coalgebra structure $\rho'(\overline{a})=\overline{a_{(2)}}\otimes S(\pi(a_{(1)}))\pi(a_{(3)})$ we have the isomorphism $\psi=(\pi\otimes\nu)\Delta\colon H\to H'\ltimes Q$ where $\nu\colon H\to Q$ is the canonical epimorphism.

Our aim is to describe the structures on Q responsible for the bialgebra structure on the cosmash product $H'\ltimes Q$.

Theorem 5.1. *Let H' be a bialgebra and Q an H'-comodule coalgebra. Assume we are given maps*

$$\begin{aligned}Q\otimes H'\ni q\otimes x&\mapsto q\leftharpoonup x\in Q\\Q\otimes H'\ni q\otimes x&\mapsto q\rightharpoonup x\in H'\\\nabla\colon Q\otimes Q\ni p\otimes q&\mapsto p\cdot q\in Q\\\tau\colon Q\otimes Q\ni p\otimes q&\mapsto \tau(p|q)\in H'\end{aligned}$$

and an element $1\in Q$ such that $\leftharpoonup$ makes Q an H'-module coalgebra, $1\in Q$ is grouplike, invariant under the action of H' and coinvariant under the coaction of H', and the equations

$$q\rightharpoonup(xy)=(q_{(1)}\rightharpoonup x_{(1)})((q_{(2)}\leftharpoonup x_{(2)})\rightharpoonup y)\tag{5.1}$$

$$q\rightharpoonup 1=\varepsilon(q)1\tag{5.2}$$

$$\Delta(q\rightharpoonup x)=q_{(1)[0]}\rightharpoonup x_{(1)}\otimes q_{(1)[1]}(q_{(2)}\rightharpoonup x_{(2)})\tag{5.3}$$

$$\varepsilon(q\rightharpoonup x)=\varepsilon(q)\varepsilon(x)\tag{5.4}$$

$$\begin{aligned}&(q_{(2)}\leftharpoonup x_{(2)})_{[0]}\otimes(q_{(1)}\rightharpoonup x_{(1)})(q_{(2)}\leftharpoonup x_{(2)})_{[1]}\\&\qquad=q_{(1)[0]}\leftharpoonup x_{(1)}\otimes q_{(1)[1]}(q_{(2)}\rightharpoonup x_{(2)})\end{aligned}\tag{5.5}$$

$$\begin{aligned}((p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)})_{[0]} \otimes \tau(p_{(1)}|q_{(1)[0]})((p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)})_{[1]} \\ = p_{(1)[0]} \cdot q_{(1)[0]} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)} \leftharpoonup q_{(1)[2]}|q_{(2)})\end{aligned} \tag{5.6}$$

$$\Delta(p \cdot q) = p_{(1)} \cdot q_{(1)[0]} \otimes (p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)} \tag{5.7}$$

$$\varepsilon(p \cdot q) = \varepsilon(p)\varepsilon(q) \tag{5.8}$$

$$\Delta(\tau(p|q)) = \tau(p_{(1)[0]}|q_{(1)[0]}) \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)} \leftharpoonup q_{(1)[2]}|q_{(2)}) \tag{5.9}$$

$$\varepsilon(\tau(p|q)) = \varepsilon(p)\varepsilon(q) \tag{5.10}$$

$$\begin{aligned}(p_{(1)} \rightharpoonup (q_{(1)} \rightharpoonup x_{(1)})_{(1)})\tau(p_{(2)} \leftharpoonup (q_{(1)} \rightharpoonup x_{(1)})_{(2)}|q_{(2)} \leftharpoonup x_{(2)}) \\ = \tau(p_{(1)}|q_{(1)[0]})((p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)} \rightharpoonup x)\end{aligned} \tag{5.11}$$

$$(p \cdot q) \leftharpoonup x = (p \leftharpoonup (q_{(1)} \rightharpoonup x_{(1)})) \cdot (q_{(2)} \leftharpoonup x_{(2)}) \tag{5.12}$$

$$(p \cdot q) \cdot r = (p \leftharpoonup \tau(q_{(1)}|r_{(1)[0]})) \cdot ((q_{(2)} \leftharpoonup r_{(1)[1]}) \cdot r_{(2)}) \tag{5.13}$$

$$\begin{aligned}(p_{(1)} \rightharpoonup \tau(q_{(1)}|r_{(1)[0]})_{(1)})\tau(p_{(2)} \leftharpoonup \tau(q_{(1)}|r_{(1)[0]})_{(2)}|(q_{(2)} \leftharpoonup r_{(1)[1]}) \cdot r_{(2)}) \\ = \tau(p_{(1)}|q_{(1)[0]})\tau((p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)}|r)\end{aligned} \tag{5.14}$$

$$1 \rightharpoonup x = x \tag{5.15}$$

$$p \cdot 1 = 1 \cdot p = p \tag{5.16}$$

$$\tau(1|p) = \tau(p|1) = \varepsilon(p) \tag{5.17}$$

are satisfied for all $x, y \in H'$ and $p, q, r \in Q$.

Then the cosmash product $H' \ltimes Q$ is a bialgebra with unit $1 \ltimes 1$ and multiplication

$$(x \ltimes p)(y \ltimes q) = x(p_{(1)} \rightharpoonup y_{(1)})\tau(p_{(2)} \leftharpoonup y_{(2)}|q_{(1)[0]}) \ltimes (p_{(3)} \leftharpoonup y_{(3)}q_{(1)[1]}) \cdot q_{(2)}.$$

The map $j\colon H' \ni x \mapsto x \ltimes 1 \in H' \ltimes Q$ is a bialgebra map satisfying $\pi j = \mathrm{id}_{H'}$.

Every bialgebra structure on $H' \ltimes Q$ such that H' is a subbialgebra of $H' \ltimes Q$ and $\pi\colon H' \ltimes Q \to Q$ is a left H'-linear coalgebra map with $\pi|_{H'} = \mathrm{id}_{H'}$ is of the form described above.

Proof. We begin by restating the problem of defining suitable bialgebra structures on $H' \ltimes Q$:

Remark 5.2. Let H be a left H'-module coalgebra and $\pi\colon H \to H'$ a left H'-module coalgebra map. Giving H a bialgebra structure with H' as a subbialgebra and π a coalgebra map is equivalent to defining the following structures:

1. A right H'-module coalgebra structure on H making it an H'-bimodule coalgebra.
2. An algebra structure on the object H of the monoidal category of H'-H'-bimodule coalgebras, that is, maps $\nabla\colon H \otimes_{H'} H \to H$ and $j\colon H' \to H$ of H'-H'-bimodule coalgebras satisfying $\nabla(\nabla \otimes_{H'} \mathrm{id}) = \nabla(\mathrm{id} \otimes_{H'} \nabla)$ and $\nabla(j \otimes_{H'} \mathrm{id}) = \nabla(\mathrm{id} \otimes_{H'} j) = \mathrm{id}_H$.

Now assume that $H = H' \ltimes Q$. We have seen in Proposition 3.1 that defining a right H'-module structure on H making it an H'-bimodule coalgebra is the same as giving a right H'-module structure $\rightharpoonup$ on Q making it an H'-module coalgebra, and a map $\rightharpoonup$ as above satisfying equations (5.1) through (5.5).

Next, we wish to describe, in terms of constituents, a map ∇ as in Remark 5.2. We shall be using freely the description of tensor products and their morphisms obtained in section 4. A left H'-module coalgebra map $\nabla\colon H\otimes_{H'}H\to H$ is described equivalently by its constituents $\nabla^{(1)}\colon Q\otimes Q\to H'$ and $\nabla^{(2)}\colon Q\otimes Q\to Q$, subject to the conditions listed in Proposition 2.5. We write $\nabla^{(1)}=:\tau$ and $\nabla^{(2)}(p\otimes q)=:p\cdot q$. Then the condition that $\nabla^{(2)}$ be a coalgebra map is precisely (5.7) and (5.8). Because of

$$\begin{aligned}
&\nabla^{(2)}((p\otimes q)_{(1)[0]})\otimes(p\otimes q)_{(1)[1]}\nabla^{(1)}((p\otimes q)_{(2)})\\
&\quad=\nabla^{(2)}((p_{(1)}\otimes q_{(1)[0]})_{[0]})\otimes(p_{(1)}\otimes q_{(1)[0]})_{[1]}\nabla^{(1)}(p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)})\\
&\qquad=p_{(1)[0]}\cdot q_{(1)[0]}\otimes p_{(1)[1]}(p_{(2)}\rightharpoonup q_{(1)[1]})\tau(p_{(3)}\leftharpoonup q_{(1)[2]}|q_{(2)})
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^{(2)}((p\otimes q)_{(2)})_{[0]}\otimes\nabla^{(1)}((p\otimes q)_{(1)})\nabla^{(2)}((p\otimes q)_{(2)})_{[1]}\\
&\quad=((p_{(2)}\leftharpoonup q_{(1)[1]})\cdot q_{(2)})_{[0]}\otimes\tau(p_{(1)}|q_{(1)[0]})((p_{(2)}\leftharpoonup q_{(1)[1]})\cdot q_{(2)})_{[1]}
\end{aligned}$$

equation (2.5) for ∇ is precisely (5.6).

The calculation

$$\begin{aligned}
&\nabla^{(1)}((p\otimes q)_{(1)[0]})\otimes(p\otimes q)_{(1)[1]}\nabla^{(1)}((p\otimes q)_{(2)})\\
&\quad=\nabla^{(1)}((p_{(1)}\otimes q_{(1)[0]})_{[0]})\otimes(p_{(1)}\otimes q_{(1)[0]})_{[1]}\nabla^{(1)}(p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)})\\
&\qquad=\tau(p_{(1)[0]}|q_{(1)[0]})\otimes p_{(1)[1]}(p_{(2)}\rightharpoonup q_{(1)[1]})\tau(p_{(3)}\leftharpoonup q_{(1)[2]}|q_{(2)})
\end{aligned}$$

shows (2.4) for ∇ to be equivalent to (5.9). Clearly (2.6) becomes (5.10) in the present situation.

Having dealt with ∇ as a left module coalgebra map, we next have to consider the conditions, listed in Proposition 3.2, for ∇ to be a right H'-module map. Now the condition that $\nabla^{(2)}$ be an H'-module map is precisely (5.12) above, and the calculations

$$\begin{aligned}
&\nabla^{(1)}((p\otimes q)_{(1)})(\nabla^{(2)}((p\otimes q)_{(2)})\rightharpoonup x)\\
&\qquad=\tau(p_{(1)}|q_{(1)[0]})(((p_{(2)}\leftharpoonup q_{(1)[1]})\cdot q_{(2)})\rightharpoonup x)
\end{aligned}$$

and

$$\begin{aligned}
&((p\otimes q)_{(1)}\rightharpoonup x_{(1)})\nabla^{(1)}((p\otimes q)_{(2)}\leftharpoonup x_{(2)})\\
&\quad=(p_{(1)}\otimes q_{(1)[0]})\rightharpoonup x_{(1)})\tau((p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)})\leftharpoonup x_{(2)})\\
&\quad=(p_{(1)}\rightharpoonup q_{(1)[0]}\rightharpoonup x_{(1)})\tau(p_{(2)}\leftharpoonup q_{(1)[1]}(q_{(2)}\rightharpoonup x_{(2)})|q_{(3)}\leftharpoonup x_{(3)})\\
&\quad=(p_{(1)}\rightharpoonup(q_{(1)}\rightharpoonup x_{(1)})_{(1)})\tau(p_{(2)}\leftharpoonup(q_{(1)}\rightharpoonup x_{(1)})_{(2)}|q_{(2)}\leftharpoonup x_{(2)})
\end{aligned}$$

show that (3.10) reads, for ∇, precisely (5.11).

Finally, it remains to deal with the equations that make ∇ an algebra structure for H. Since

$$(\nabla(\nabla\otimes_{H'}\mathrm{id}))^{(2)}=\nabla^{(2)}(\nabla\otimes_{H'}\mathrm{id})^{(2)}=\nabla^{(2)}(\nabla^{(2)}\otimes\mathrm{id})$$

and

$$\begin{aligned}(\nabla(\mathrm{id}\otimes_{H'}\nabla))^{(2)}(p\otimes q\otimes r)&=\nabla^{(2)}(\mathrm{id}\otimes_{H'}\nabla)^{(2)}(p\otimes q\otimes r)\\&=\nabla^{(2)}(p\leftharpoonup\nabla^{(1)}((q\otimes r)_{(1)})\otimes\nabla^{(2)}((q\otimes r)_{(2)})\\&=(p\leftharpoonup\tau(q_{(1)}\otimes r_{(1)[0]}))\cdot((q_{(2)}\leftharpoonup r_{(1)[1]})\cdot r_{(2)}),\end{aligned}$$

the second constituent of the equation $\nabla(\nabla\otimes_{H'}\mathrm{id})=\nabla(\mathrm{id}\otimes_{H'}\nabla)$ is precisely (5.13), while the calculations

$$\begin{aligned}(\nabla(\nabla\otimes_{H'}\mathrm{id}))^{(1)}(p\otimes q\otimes r)&=((\nabla\otimes_{H'}\mathrm{id})^{(1)}*\nabla^{(1)}(\nabla\otimes_{H'}\mathrm{id})^{(2)})(p\otimes q\otimes r)\\&=((\nabla^{(1)}\otimes\varepsilon)*\nabla^{(1)}(\nabla^{(2)}\otimes\mathrm{id}))(p\otimes q\otimes r)\\&=\nabla^{(1)}((p\otimes q)_{(1)})\varepsilon(r_{(1)[0]})\nabla^{(1)}(\nabla^{(2)}\otimes\mathrm{id})((p\otimes q)_{(2)}\leftharpoonup r_{(1)[1]}\otimes r_{(2)})\\&=\nabla^{(1)}((p\otimes q)_{(1)})\nabla^{(1)}(\nabla^{(2)}\otimes\mathrm{id})((p\otimes q)_{(2)}\otimes r)\\&=\tau(p_{(1)}|q_{(1)[0]})\tau((p_{(2)}\leftharpoonup q_{(1)[1]})\cdot q_{(2)}|r)\end{aligned}$$

and

$$\begin{aligned}&(\nabla(\mathrm{id}\otimes_{H'}\nabla))^{(1)}(p\otimes q\otimes r)\\&=(\mathrm{id}\otimes_{H'}\nabla)^{(1)}(p_{(1)}\otimes\xi_{(1)[0]})\nabla^{(1)}(\mathrm{id}\otimes_{H'}\nabla)^{(2)}(p_{(2)}\leftharpoonup\xi_{(1)[1]}\otimes\xi_{(2)})\\&=(p_{(1)}\rightharpoonup\nabla^{(1)}(\xi_{(1)[0]}))\nabla^{(1)}(p_{(2)}\leftharpoonup\xi_{(1)[1]}\nabla^{(1)}(\xi_{(2)})\otimes\nabla^{(2)}(\xi_{(3)}))\\&=(p_{(1)}\rightharpoonup\nabla^{(1)}(\xi_{(1)})_{(1)})\nabla^{(1)}(p_{(2)}\leftharpoonup\nabla^{(1)}(\xi_{(1)})_{(2)}\otimes\nabla^{(2)}(\xi_{(2)}))\\&=(p_{(1)}\rightharpoonup\tau(q_{(1)}|r_{(1)[0]})_{(1)})\tau(p_{(2)}\leftharpoonup\tau(q_{(1)}|r_{(1)[0]})_{(2)}|(q_{(2)}\leftharpoonup r_{(1)[1]})\cdot r_{(2)}),\end{aligned}$$

where we have abbreviated $\xi=q\otimes r$, show the first constituent to be equivalent to (5.14).

A left H'-module coalgebra map $j\colon H'=H'\ltimes k\to H'\ltimes Q$ is described by a suitable element $1\in Q$. We omit the calculations showing that axioms (5.15) through (5.17) and the properties that 1 is grouplike, invariant and coinvariant under H', are the conditions for such an element to describe a map j as in Remark 5.2 satisfying $\pi j=\mathrm{id}$.

To end the proof it remains to note that

$$\begin{aligned}\nabla(x\ltimes p\otimes y\ltimes q)&=\nabla((x\ltimes p)y\otimes 1\ltimes q)=\nabla(x(p_{(1)}\rightharpoonup y_{(1)})\ltimes p_{(2)}\leftharpoonup y_{(2)}\otimes 1\ltimes q)\\&=x(p_{(1)}\rightharpoonup y_{(1)})\tau((p_{(2)}\leftharpoonup y_{(2)})_{(1)}|q_{(1)[0]})\ltimes((p_{(2)}\leftharpoonup y_{(2)})_{(2)}\leftharpoonup q_{(1)[1]})\cdot q_{(2)}\\&=x(p_{(1)}\rightharpoonup y_{(1)})\tau(p_{(2)}\leftharpoonup y_{(2)}|q_{(1)[1]})\ltimes(p_{(3)}\leftharpoonup y_{(3)}q_{(1)[1]})\cdot q_{(2)},\end{aligned}$$

using the identification $(H'\ltimes Q)\otimes_{H'}(H'\ltimes Q)\cong H'\ltimes(Q\otimes Q)$. □

The following Theorem characterizes when a cosmash product bialgebra is a Hopf algebra. Note that if Q were a bialgebra, condition (2)(b) below would be equivalent to Q being a Hopf algebra.

Theorem 5.3. *Let H' be a bialgebra and Q an H'-comodule coalgebra equipped with the structures described in Theorem 5.1, so that $H := H' \ltimes Q$ is a bialgebra. Then the following are equivalent:*

1. *H is a Hopf algebra.*
2. (a) *H' is a Hopf algebra, and*
   (b) *The map $\beta_Q\colon Q \otimes Q \ni p \otimes q \mapsto p \cdot q_{(1)} \otimes q_{(2)} \in Q \otimes Q$ is a bijection.*

Proof. Assume that H is a Hopf algebra. Since $H' \subset H$ is a subbialgebra, and H is a free, hence faithfully flat left H'-module, it follows from [14, Cor.2.4] that H' is a Hopf subalgebra. We have shown in [13, Rem.5.14] that β_Q is bijective if H is a Hopf algebra.

Now assume (2) and consider the map $\beta_H\colon H \otimes H \ni g \otimes h \mapsto gh_{(1)} \otimes h_{(1)} \in H \otimes H$. It is well known that H is a Hopf algebra iff β_H is a bijection. Obviously β_H is left H-linear with H acting on the left tensor factors. β_H is also left H-colinear when we use the codiagonal coaction for the domain, and the coaction on the left tensor factor for the codomain. Note that $H \otimes H$ is a left H-Hopf module in each of these ways. Hence, by restricting the module strucures and using π for the comodule strucutures, $H \otimes H$ is also a left H'-Hopf module, and β_H is a map of left H'-Hopf modules. It follows by the structure theorem for Hopf modules that it is sufficient to show that $k \otimes_{H'} \beta_H$ is a bijection. Since $k \otimes_{H'} H \cong Q$ via $k \otimes_{H'} \nu$, it suffices to show that the map

$$\theta\colon Q \otimes H \ni p \otimes h \mapsto \nu((1 \ltimes p)h_{(1)}) \otimes h_{(2)} \in Q \otimes H$$

is bijective. Now

$$\begin{aligned}\theta(q \otimes y \ltimes q) &= \nu((1 \ltimes q)(y_{(1)} \ltimes q_{(1)[0]})) \otimes y_{(2)} q_{(1)[1]} \ltimes q_{(2)} \\ &= (q \leftharpoonup y_{(1)}) \cdot q_{(1)[0]} \otimes y_{(2)} q_{(1)[1]} \ltimes q_{(2)}.\end{aligned}$$

We can write $\theta = \theta' \theta''$ where

$$\theta'\colon Q \otimes H' \otimes Q \ni p \otimes y \otimes q \mapsto p \cdot q_{(1)[0]} \otimes y q_{(1)[1]} \otimes q_{(2)} \in Q \otimes H' \otimes Q$$
$$\theta''\colon Q \otimes H' \otimes Q \ni p \otimes y \otimes q \mapsto p \leftharpoonup y_{(1)} \otimes y_{(2)} \otimes q \in Q \otimes H' \otimes Q.$$

Note that θ'' is a bijection since H' is a Hopf algebra. Since β_Q is right Q-colinear with Q coacting on the right tensor factors, the inverse of β_Q is right Q-colinear as well, hence has the form $\beta_Q^{-1}(p \otimes q) = \tilde{S}(p \otimes q_{(1)}) \otimes q_{(2)}$ for some map $\tilde{S}\colon Q \otimes Q \to Q$. Note that $\tilde{S}(p \otimes q_{(1)}) \cdot q_{(2)} = \varepsilon(q)p = \tilde{S}(p \cdot q_{(1)} \otimes q_{(2)})$ for all $p, q \in Q$. We claim that the map

$$\overline{\theta}'\colon Q \otimes H' \otimes Q \ni p \otimes y \otimes q \mapsto \tilde{S}(p \otimes q_{(1)[0]}) \otimes y q_{(1)[1]} \otimes q_{(2)} \in Q \otimes H' \otimes Q$$

is the inverse of θ'. In fact

$$\overline{\theta}'\theta'(p\otimes y\otimes q) = \overline{\theta}'(p\cdot q_{(1)[0]}\otimes yq_{(1)[1]}\otimes q_{(2)}) = \tilde{S}(p\cdot q_{(1)[0]}\otimes q_{(2)[0]})\otimes yq_{(1)[1]}q_{(2)[1]}\otimes q_{(3)}$$
$$= \tilde{S}(p\cdot q_{(1)[0](1)}\otimes q_{(1)[0](2)})\otimes yq_{(1)[1]}\otimes q_{(2)} = p\varepsilon(q_{(1)[0]})\otimes yq_{(1)[1]}\otimes q_{(2)} = p\otimes y\otimes q$$

and similarly for composition in the reverse order. □

Lemma 5.4. *Assume that condition (2) in the preceding Theorem is satisfied. Let $S\colon Q\to Q$ be the map defined by $\beta_Q^{-1}(1\otimes q) = S(q_{(1)})\otimes q_{(2)}$. Then we have*

$$S(h) = \pi^{-1}(\varphi(S(\nu(h_{(1)}))\leftharpoonup \pi^{-1}(h_{(2)}))_{(1)}h_{(3)})\varphi(S(\nu(h_{(1)}))\leftharpoonup \pi^{-1}(h_{(2)}))_{(2)}$$

for $h\in H'\ltimes Q$, and

$$S(1\ltimes q) = \pi^{-1}((1\ltimes S(q_{(1)[0]})\leftharpoonup S(q_{(1)[1]}))_{(1)}(q_{(1)[2]}\ltimes q_{(2)}))(1\ltimes S(q_{(1)[0]})\leftharpoonup S(q_{(1)[1]}))_{(2)}$$

for $q\in Q$, where $\pi^{-1} = S\pi$ is the convolution inverse for π.

Proof. Define $S\colon H\to H$ by the above formula. Since we know that H has an antipode, it suffices to show $S(h_{(1)})h_{(2)} = \varepsilon(h)1$. Note $\pi^{-1}(h_{(1)})h_{(2)} = \varphi\nu(h)$ by the calculation $\pi^{-1}(x_{(1)}\ltimes q_{(1)[0]})(x_{(2)}q_{(1)[1]}\ltimes q_{(2)}) = S(x_{(1)})x_{(2)}\ltimes q = \varepsilon(x)1\ltimes q$. Hence

$$S(h_{(1)})h_{(2)} = \varphi\nu(\varphi(S(\nu(h_{(1)}))\leftharpoonup \pi^{-1}(h_{(2)}))h_{(3)}).$$

For $h = x\ltimes q$, we calculate

$$\nu(h_{(1)})\otimes\pi(h_{(2)})\otimes h_{(3)} = \nu(x_{(1)}\ltimes q_{(1)[0]})\otimes\pi((x_{(2)}q_{(1)[1]}\ltimes q_{(2)})_{(1)})\otimes(x_{(2)}q_{(1)[1]}\ltimes q_{(2)})_{(2)}$$
$$= q_{(1)[0]}\otimes\pi(x_{(1)}q_{(1)[1]}\ltimes q_{(2)[0]})\otimes x_{(2)}q_{(2)[1]}q_{(2)[1]}\ltimes q_{(3)}$$
$$= q_{(1)[0]}\otimes x_{(1)}q_{(1)[1]}\otimes x_{(2)}q_{(1)[2]}\ltimes q_{(2)}$$

and hence

$$S(h_{(1)})h_{(2)} = \varphi\nu(\varphi(S(q_{(1)[0]})\leftharpoonup S(x_{(1)}q_{(1)[1]}))(x_{(2)}q_{(1)[2]}\ltimes q_{(2)}))$$
$$= \varphi((S(q_{(1)[0]})\leftharpoonup S(x_{(1)}q_{(1)[1]})x_{(2)}q_{(1)[2]})\cdot q_{(2)}) = \varphi(S(q_{(1)})\cdot q_{(2)}) = \varepsilon(q)1$$

using $\nu((1\ltimes r)(y\ltimes t)) = (r\leftharpoonup y)\cdot t$ and $S(q_{(1)})\cdot q_{(2)} = \varepsilon(q)1$. The second formula in the lemma is a special case of the first. □

Note that to have an antipode on H it was not necessary to assume that the cocycle τ is convolution invertible, as one might expect from the known results on bicrossproducts with a cocycle and dual cocycle like [6].

Corollary 5.5. *Let $H = H'\ltimes Q$ be a bialgebra as in Theorem 5.1, and assume that the equivalent conditions of Theorem 5.3 are satisfied. Then the cocycle $\tau\colon Q\otimes Q\to H'$ is convolution invertible when we use the coalgebra structure*

$$\Delta(p\otimes q) = p_{(1)}\otimes q_{(1)[0]}\otimes p_{(2)}\leftharpoonup q_{(1)[1]}\otimes q_{(2)}$$

on $Q\otimes Q$.

The antipode of H can be written as

$$S(x\ltimes q) = (\tau^{-1}(S(q_{(1)[0]})_{(1)}|q_{(2)[0]})\ltimes S(q_{(1)[0]})_{(2)}\leftharpoonup q_{(2)[1]})S(xq_{(1)[1]}q_{(2)[2]})$$

Proof. We can invoke [13, Thm.5.13] since H is clearly a Q-cleft H-comodule algebra with cleft map $\varphi\colon Q \to H$, to show that τ is convolution invertible. Since $\mathrm{id}_H = (j\pi) * (\varphi\nu)$, we have $S = (\varphi^{-1}\nu) * (j\pi^{-1})$, hence

$$\begin{aligned} S(x \ltimes q) &= \varphi^{-1}(q_{[0]})S(xq_{[1]}) \\ &= (\tau^{-1}(S(q_{[0](1)})_{(1)}|q_{[0](2)[0]}) \ltimes S(q_{[0](1)})_{(2)} \leftharpoonup q_{[0](2)[1]})S(xq_{[1]}) \\ &= (\tau^{-1}(S(q_{(1)[0]})_{(1)}|q_{(2)[0]}) \ltimes S(q_{(1)[0]})_{(2)} \leftharpoonup q_{(2)[1]})S(xq_{(1)[1]}q_{(2)[2]}) \end{aligned}$$

using the explicit formula for φ^{-1} obtained in [13, Cor.5.15]. □

6. Special cases

The setup in section 5 is sufficiently general to include as special cases of the construction three previously known important sources of examples for Hopf algebras: Radford's biproducts [12], Majid's bicrossproducts [8], previously studied in special cases by various other authors [16, 18], and double crossproducts, again by Majid [8], which include Drinfeld's quantum double construction.

Consider a bialgebra structure on a cosmash product $H' \ltimes Q$ as in Theorem 5.1. We have a diagram of k-linear maps

$$H' \underset{j}{\overset{\pi}{\leftrightarrows}} H' \ltimes Q \underset{\varphi}{\overset{\nu}{\rightleftarrows}} Q$$

in which j is a bialgebra map, π is a left H'-module coalgebra map satisfying $\pi(1) = 1$, ν is a coalgebra map satisfying $\nu(1) = 1$, and $\varphi(1) = 1$. Various special cases arise according to additional properties that we can require of the maps π, ν and/or φ.

6.1. **The case of a bimodule coalgebra projection.** Since for $q \in Q$ and $x, y \in H'$ we have $\pi((x \ltimes q)y) = x(q \rightharpoonup y)$ and $\pi(x \ltimes q)y = x\varepsilon(q)y$, the map π is a right H'-module map if and only if $\rightharpoonup$ is trivial, that is if and only if $q \rightharpoonup x = \varepsilon(q)x$ for all $q \in Q$ and $x \in H'$.

This case is of interest for the following reason: As in [13, Prop.5.2] we can conclude that whenever we have subspaces $\mathfrak{a}, \mathfrak{b}$ of a bialgebra H and a Hopf subalgebra H' such that multiplication induces an isomorphism $\mathfrak{a} \otimes H' \otimes \mathfrak{b} \cong H$ (a triangular decomposition of H) and both $\mathfrak{a}\, H'$ and $H'\, \mathfrak{b}$ are subbialgebras of H that admit projections in the sense of Radford onto H' (or even only H'-bimodule coalgebra projections) then H admits an H'-bimodule coalgebra projection onto H'. A triangular decomposition of the type just described is well known to exist for all quantized enveloping algebras, with H' the analog of the Cartan subalgebra.

Let us inspect for the special case when π is an H'-bimodule map the equations in Theorem 5.1. Equations (5.1) through (5.4) just become trivial, as well as (5.15). Of course, those of the remaining equations that do not contain $\rightharpoonup$ at all just remain unchanged. We record the changes in the others: Equation (5.5) says, for trivial $\rightharpoonup$, precisely that Q is a Yetter-Drinfeld H'-module; since we also have the conditions that Q is both a module and a comodule coalgebra, we can say that Q has to be a coalgebra in the category of Yetter-Drinfeld H'-modules. Equation (5.9) receives the minor simplification to

$$\Delta(\tau(p|q)) = \tau(p_{(1)[0]}|q_{(1)[0]}) \otimes p_{(1)[1]}q_{(1)[1]}\tau(p_{(2)} \leftharpoonup q_{(1)[2]}|q_{(2)}), \tag{6.1}$$

equation (5.11) simplifies to

$$x_{(1)}\tau(p \leftharpoonup x_{(2)}|q \leftharpoonup x_{(3)}) = \tau(p|q)x \tag{6.2}$$

which, if H' is a Hopf algebra, is equivalent to saying that $\tau\colon Q \otimes Q \to H'$ is an H'-module map with respect to the right adjoint action of H' on itself. Equation (5.12) now reads

$$(p \cdot q) \leftharpoonup x = (p \leftharpoonup x_{(1)}) \cdot (q \leftharpoonup x_{(2)}) \tag{6.3}$$

and says that $\cdot\colon Q \otimes Q \to Q$ is an H'-module map. Equation (5.14) receives the very minor simplification

$$\begin{aligned} \tau(q_{(1)}|r_{(1)[0]})_{(1)}\tau(p \leftharpoonup \tau(q_{(1)}|r_{(1)[0]})_{(2)}|(q_{(2)} \leftharpoonup r_{(1)[1]}) \cdot r_{(2)} \\ = \tau(p_{(1)}|q_{(1)[0]})\tau((p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)}|r). \end{aligned} \tag{6.4}$$

6.2. **Bicrossproducts with (half) a cocycle.** The map $\nu\colon H' \ltimes Q \to Q$ is multiplicative (which, of course, entails that Q is a bialgebra) if and only if $\leftharpoonup$ is trivial, that is, $q \leftharpoonup x = \varepsilon(x)q$ for all $q \in Q$ and $x \in H'$.

The multiplication in $H' \ltimes Q$ is now of the form familiar from Hopf crossed products:

$$(x \ltimes p)(y \ltimes q) = x(p_{(1)} \rightharpoonup y)\tau(p_{(2)}|q_{(1)}) \ltimes p_{(3)} \cdot q_{(2)}.$$

Equations (5.1) and (5.2) turn into the condition that $\rightharpoonup$ is a measuring. Equation (5.13) now expresses associativity of multiplication in Q, and (5.7) and (5.8) turn into the condition that Q is a bialgebra. Equations (5.11) and (5.14) turn out to be the familiar condition for $\rightharpoonup$ and τ to be a cocycle and weak action in the sense of the theory of Hopf crossed products as reviewed in [10]:

$$\begin{gathered} (p_{(1)} \rightharpoonup (q_{(1)} \rightharpoonup x))\tau(p_{(2)}|q_{(2)}) = \tau(p_{(1)}|q_{(1)})(p_{(2)} \cdot q_{(2)} \rightharpoonup x) \\ \tau(p_{(1)}|q_{(1)})\tau(p_{(2)} \cdot q_{(2)}|r) = (p_{(1)} \rightharpoonup \tau(q_{(1)}|r_{(1)}))\tau(p_{(2)}|q_{(2)} \cdot r_{(2)}) \\ \tau(q|1) = \tau(1|q) = \varepsilon(q) \\ 1 \rightharpoonup q = q. \end{gathered}$$

All that remains of the other equations in the axiom system of Theorem 5.1 is

$$\Delta(q \rightharpoonup x) = q_{(1)[0]} \rightharpoonup x_{(1)} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x_{(2)})$$
$$\varepsilon(q \rightharpoonup x) = \varepsilon(q)\varepsilon(x)$$
$$q_{(2)[0]} \otimes (q_{(1)} \rightharpoonup x)q_{(2)[1]} = q_{(1)[0]} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x)$$
$$(p_{(2)} \cdot q_{(2)})_{[0]} \otimes \tau(p_{(1)}|q_{(1)})(p_{(2)} \cdot q_{(2)})_{[1]}$$
$$= p_{(1)[0]} \cdot q_{(1)[0]} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)}|q_{(2)})$$
$$\rho'(1) = 1 \otimes 1$$
$$\Delta\tau(p|q) = \tau(p_{(1)[0]}|q_{(1)[0]}) \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})\tau(p_{(3)}|q_{(2)})$$
$$\varepsilon\tau(p|q) = \varepsilon(p)\varepsilon(q)$$

These equations are a special case of those listed in [6], see also [9, Thm.6.2.2]. One should not that we have not recovered the full generality of the constructions in [6, 9] since there the coalgebra structure, which is a cosmash product in our case, also contains a (dual) cocycle.

6.3. **The case of a trivial cocycle.** Since $\varphi(p)\varphi(q) = \tau(p_{(1)}|q_{(1)[0]}) \ltimes (p_{(2)} \leftharpoonup q_{(1)[1]}) \cdot q_{(2)}$, the map φ is multiplicative if and only if $\tau(p|q) = \varepsilon(p)\varepsilon(q)1$ holds for $p, q \in Q$. Note that in this case Q has to be associative as a subalgebra of $H' \ltimes Q$. Multiplication in $H' \ltimes Q$ is now given by

$$(x \ltimes p)(y \ltimes q) = x(p_{(1)} \rightharpoonup y_{(1)}) \ltimes (p_{(2)} \leftharpoonup y_{(2)})q$$

and we have the following simplified list of axioms

$$q \rightharpoonup xy = (q_{(1)} \rightharpoonup x_{(1)})((q_{(2)} \leftharpoonup x_{(2)}) \rightharpoonup y)$$
$$q \rightharpoonup 1 = \varepsilon(q)1$$
$$\Delta(q \rightharpoonup x) = q_{(1)[0]} \rightharpoonup x_{(1)} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x_{(2)})$$
$$\varepsilon(q \rightharpoonup x) = \varepsilon(q)\varepsilon(x)$$
$$(q_{(2)} \leftharpoonup x_{(2)})_{[0]} \otimes (q_{(1)} \rightharpoonup x_{(1)})(q_{(2)} \leftharpoonup x_{(2)})_{[1]}$$
$$= q_{(1)[0]} \leftharpoonup x_{(1)} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x_{(2)})$$
$$\rho'(p \cdot q) = p_{(1)[0]} \cdot q_{(1)[0]} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]})$$
$$\Delta(p \cdot q) = p_{(1)} \cdot q_{(1)[0]} \otimes (p_{(2)} \leftharpoonup q_{(1)[1]})q_{(2)}$$
$$\varepsilon(p \cdot q) = \varepsilon(p)\varepsilon(q)$$
$$(p \cdot q) \leftharpoonup x = (p \leftharpoonup (q_{(1)} \rightharpoonup x_{(1)}))(q_{(2)} \leftharpoonup x_{(2)})$$

plus the requirements that $\cdot$ makes Q an associative algebra, and $\rightharpoonup$ makes H' a Q-module.

The products with trivial cocycle thus described are a special case of the trivalent products previously introduced in [2] to unify bicrossproducts, double crossproducts and biproducts, see also [5].

6.4. **Bicrossproducts.** Both maps φ and ν are multiplicative if and only if both $\leftharpoonup$ and τ are trivial. Then multiplication in $H' \ltimes Q$ is given by a smash product construction

$$(x \ltimes p)(y \ltimes q) = x(p_{(1)} \rightharpoonup y) \ltimes p_{(2)} \cdot q.$$

In this case all that remains of the axiom system in the preceding paragraph is the following: Q is a bialgebra, H' is a Q-module algebra via $\rightharpoonup$, Q is an H'-comodule coalgebra, and, leaving out conditions involving counits or units, the remaining conditions are

$$\begin{aligned}
\Delta(q \rightharpoonup x) &= q_{(1)[0]} \rightharpoonup x_{(1)} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x_{(2)}) \\
\rho'(p \cdot q) &= p_{(1)[0]} \cdot q_{(1)[0]} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup q_{(1)[1]}) \\
q_{(2)[0]} \otimes (q_{(1)} \rightharpoonup x)q_{(2)[1]} &= q_{(1)[0]} \otimes q_{(1)[1]}(q_{(2)} \rightharpoonup x).
\end{aligned}$$

These conditions are known as those for a matched pair of bialgebras [18]. Majid [8] has shown that with these conditions the cosmash product and smash product is a bialgebra, called a bismash or bicrossed product.

6.5. **Double crossproducts.** We have seen that φ is multiplicative if and only if τ is trivial. Let us now write out the conditions for φ to be a coalgebra map:

$$1 \ltimes q_{(1)} \otimes 1 \ltimes q_{(2)} = \Delta(1 \ltimes q) = 1 \ltimes q_{(1)[0]} \otimes q_{(1)[1]} \ltimes q_{(2)}$$

for all $q \in Q$ is equivalent to $\rho'(q) = q \otimes 1$ for all $q \in Q$, that is, to the coaction ρ' of H' on Q being trivial. Assume that φ is multiplicative and comultiplicative. (Then Q is a subbialgebra of $H' \ltimes Q$ hence itself a bialgebra.) In this case the multiplication in $H' \ltimes Q$ takes the form

$$(x \ltimes p)(y \ltimes q) = x(p_{(1)} \rightharpoonup y_{(1)}) \ltimes (p_{(2)} \leftharpoonup y_{(2)})q$$

as in Majid's construction of a double crossproduct [8]. The conditions for $\rightharpoonup$ and $\leftharpoonup$ that we obtain by specializing the list in Theorem 5.1 are precisely those obtained by Majid: Q is an H' module coalgebra; H' is a Q-module coalgebra by the specializations of (3.7), (3.8), (5.11) and (5.15). Furthermore, we have the conditions

$$\begin{gathered}
q \rightharpoonup (xy) = (q_{(1)} \rightharpoonup x_{(1)})((q_{(2)} \leftharpoonup x_{(2)}) \rightharpoonup y) \\
q \rightharpoonup 1 = \varepsilon(q)1 \\
(p \cdot q) \leftharpoonup x = (p \leftharpoonup (q_{(1)} \rightharpoonup x_{(1)})) \cdot (q_{(2)} \leftharpoonup x_{(2)}) \\
1 \leftharpoonup x = \varepsilon(x)1 \\
q_{(2)} \leftharpoonup x_{(2)} \otimes q_{(1)} \rightharpoonup x_{(1)} = q_{(1)} \leftharpoonup x_{(1)} \otimes q_{(2)} \rightharpoonup x_{(2)}
\end{gathered}$$

where the last one comes from (5.5).

6.6. **Biproducts.** The map π is a bialgebra map if and only if $\rightharpoonup$ and τ are trivial, that is, $q \rightharpoonup x = \varepsilon(q)x$ and $\tau(p|q) = \varepsilon(p)\varepsilon(q)$ for $p, q \in Q$ and $x \in H'$. One easily verifies that all that remains of the axioms in Theorem 5.1 in this case is that Q is a bialgebra in the braided monoidal category of right Yetter-Drinfel'd modules over H', and the structure of $H' \ltimes Q$ as an algebra is that of a semidirect product with respect to the action $\leftharpoonup$ of H' on Q. This construction and axioms were treated in [12].

Appendix . A remark on bicrossproduct Hopf algebras

In this appendix we shall be dealing with a different construction of bialgebra and Hopf algebra structures on tensor products from the one that has been the topic of the bulk of the paper, namely with bicrossproducts $A\ {}^{\alpha}{\bowtie}_{\tau} B$ involving a cocycle τ and a dual cocycle α.

Throughout the section (with the exception of the more general Proposition A.1) we assume the following situation:

Let A and B be Hopf algebras. Let

$$\begin{aligned}
\rightharpoonup &: B \otimes A \to A\\
\tau &: B \otimes B \to A\\
\rho &: B \to B \otimes A\\
\alpha &: B \to A \otimes A
\end{aligned}$$

be k-linear maps (a weak action of B on A, a cocycle, a weak coaction of A on B and a dual cocylce, respectively), such that $H := A\ {}^{\alpha}{\bowtie}_{\tau} B := A \otimes B$ is a bialgebra with the multiplication

$$(a \bowtie b)(a' \bowtie b') = a(b_{(1)} \rightharpoonup a')\tau(b_{(2)}|b'_{(1)}) \bowtie b_{(3)}b'_{(2)} \tag{A.1}$$

and comultiplication

$$\Delta_H(a \bowtie b) = a_{(1)}\alpha^{(1)}(b_{(1)}) \bowtie b_{(2)[0]} \otimes a_{(2)}\alpha^{(2)}(b_{(1)})b_{(2)[1]} \bowtie b_{(3)} \tag{A.2}$$

is a bialgebra with unit $1 \bowtie 1$ and counit $\varepsilon \otimes \varepsilon$. We have used the notation $\rho(b) = b_{[0]} \otimes b_{[1]}$ and $\alpha(b) = \alpha^{(1)}(b) \otimes \alpha^{(2)}(b)$. The precise conditions that have to be imposed on the weak (co-)actions and (dual) cocycles to make $A\ {}^{\alpha}{\bowtie}_{\tau} B$ a bialgebra were discussed in detail in [6] and [9]. In the "abelian" case where A is commutative and B is cocommutative, the conditions can be interpreted in terms of Hopf algebra cohomology [6, 7]. In [6] question when H has an antipode is also answered; Andruskiewitsch and Devoto [1] independently (note that only the abelian case was treated in the more accessible version [7] of Hofstetters thesis) proved that H has an antipode if and only if both τ and α are convolution invertible.

We will show that indeed one half of the invertibility assumptions on the cocycle and dual cocycle are superfluous: If α *or* τ is invertible then so is the other one and H has an antipode, which is bijective if so are the antipodes of A and B (as

Akira Masuoka pointed out to me, the question of bijectivity of the antipode of a bicrossed product with cocycles is not settled in the literature.)

One special case of this claim has already been proved in section 5 (and section 6). If the dual cocycle α is trivial, the double crossproduct is a special case of the product discussed in section 5 and hence has an antipode by Theorem 5.3. It was this special case that motivated the appendix.

Bicrossproducts $A\,{}^{\alpha}\!\bowtie_{\tau} B$ can be characterized as the possible middle terms of a split short exact sequence of bialgebras starting in A and ending in B. (We shall avoid using the terminology of short exact sequences in our precise statements below, so that we can also avoid discussing the definitions involved.) Thus we ask when the middle term of such a short exact sequence has an antipode. In fact, under some faithful flatness assumptions, this can be done even for the case when the sequence is not split:

Proposition A.1. *Let H be a bialgebra and $I \subset H$ a biideal such that $B = H/I$ is a Hopf algebra and $A := H^{\operatorname{co} B} \subset H$ is a subbialgebra which is a Hopf algebra.*

If H is a left faithfully flat B-Galois extension of A, then H is a Hopf algebra. If the antipodes of A and B are bijective, then so is the antipode of H.

Proof. To see that H is a Hopf algebra, it is sufficient to show that

$$\beta_H\colon H \otimes H \ni g \otimes h \mapsto gh_{(1)} \otimes h_{(2)} \in H \otimes H$$

is a bijection. We make $H \otimes H$ a Hopf module in the category $\mathcal{M}_H^B$ of relative Hopf modules for the B-comodule algebra H in two ways: $M = H \otimes H$ is a Hopf module in $\mathcal{M}_H^H$ with the module and comodule structure coming from the regular module and comodule structure on the right tensor factor; $N = H \otimes H$ is a Hopf module in $\mathcal{M}_H^H$ with the same comodule structure as M, but the diagonal module structure. $\beta_H\colon M \to N$ is a morphism of Hopf modules in $\mathcal{M}_H^H$, hence also in $\mathcal{M}_H^B$. By Schneider's structure theorem for Hopf modules [15], it is sufficient to show that

$$\beta_H{}^{\operatorname{co} B}\colon M^{\operatorname{co} B} \to N^{\operatorname{co} B}$$

is a bijection. Now $M^{\operatorname{co} B} = N^{\operatorname{co} B} = H \otimes A$ and

$$\beta_H{}^{\operatorname{co} B}\colon H \otimes A \ni h \otimes a \mapsto ha_{(1)} \otimes a_{(2)} \in H \otimes A$$

is a bijection since A is a Hopf algebra.

If B has bijective antipode, then the opposite algebra H^{op} is a left faithfully flat B^{op}-Galois extension of A^{op}, which is a Hopf algebra since A has bijective antipode. By the first part of the Theorem H^{op} is a Hopf algebra, hence H has bijective antipode. □

From the Proposition it is easy to deduce when a bicrossed product with cocycles has an antipode. We need to fix some more notation: We define a system of mappings

$$A \underset{\pi}{\overset{j}{\rightleftarrows}} H \underset{\varphi}{\overset{\nu}{\rightleftarrows}} B$$

by $j(a) = a \bowtie 1$, $\pi(a \bowtie b) = a\varepsilon(b)$, $\nu(a \bowtie b) = \varepsilon(a)b$ and $\varphi(b) = 1 \bowtie b$. Note that j and ν are bialgebra maps. We shall consider A as a subbialgebra of H via j and B as a quotient bialgebra of H via ν; consequently we shall use the notation $\nu(h) =: \overline{h}$. Note also that φ is a right B-comodule map and π is a left A-module map. One has $h = \pi(h_{(1)})\varphi(\overline{h_{(2)}})$ for all $h \in H$.

We define an operation $\wedge\colon B \otimes A \to A$ by $b \wedge a = \varphi(b_{(1)})a\varphi(S(b_{(2)}))$. The operation $\wedge$ was used in [11]; the calculation

$$\begin{aligned}\varphi(b_{(1)})a\varphi(S(b_{(2)})) &= (1 \bowtie b_{(1)})(a \bowtie S(b_{(2)}))\\ &= (b_{(1)} \rightharpoonup a)\tau(b_{(2)}|S(b_{(5)})) \bowtie b_{(3)}S(b_{(4)}) = (b_{(1)} \rightharpoonup a)\tau(b_{(2)}|S(b_{(3)})) \bowtie 1\end{aligned}$$

shows that $\wedge$ is well defined.

Theorem A.2. *Assume given a bicrossproduct $H = A\ {}^{\alpha}{\bowtie}_{\tau} B$ of two Hopf algebras A and B with cocycle τ and dual cocycle α. Assume that the cocycle τ is convolution invertible. Then H is a Hopf algebra and the dual cocycle α is also convolution invertible. We have*

$$S_H(h) = \varphi^{-1}(\overline{h_{(2)}})_{(1)}S_A(h_{(1)}\varphi^{-1}(\overline{h_{(2)}})_{(2)}) \tag{A.3}$$

$$S_H(h) = \varphi^{-1}(\overline{h_{(3)}})_{(1)}S_A(\pi(h_{(1)})(\overline{h_{(2)}} \wedge \pi(\varphi^{-1}(\overline{h_{(3)}})_{(2)})) \tag{A.4}$$

$$S_H(1 \bowtie b) = \varphi^{-1}(b_{(2)})_{(1)}S_A(b_{(1)} \wedge \pi(\varphi^{-1}(b_{(2)})_{(2)})) \tag{A.5}$$

$$\alpha^{-1}(b) = \varphi(b_{(1)[0]})\varphi^{-1}(b_{(3)})_{(1)} \otimes b_{(1)[1]}\varphi(b_{(2)})\varphi^{-1}(b_{(3)})_{(2)} \tag{A.6}$$

where

$$\varphi^{-1}(b) = \tau^{-1}(S(b_{(2)})|b_{(3)}) \bowtie S_B(b_{(1)}).$$

If the antipodes of A and B are bijective, then so is the antipode of H.

Proof. Since τ is convolution invertible, H is a B-cleft extension of A, that is, φ is convolution invertible with the inverse as stated in the Theorem by [4]. Thus we are in the situation of Proposition A.1 with $I = A^+H = A^+ \bowtie B$ and H has an antipode, which is bijective if so are the antipodes of A and B.

We define a k-linear map $S_H\colon H \to H$ by the right hand side of (A.4). Then (A.5) is an immediate consequence of the formula $\pi((1 \bowtie b)_{(1)}) \otimes \overline{(1 \bowtie b)_{(2)}} = 1 \otimes b$. We have to show show that S_H is the antipode and satisfies (A.3).

By looking at the explicit form of φ^{-1} we have $\varphi^{-1}(b)_{(1)} \otimes \overline{\varphi^{-1}(b)_{(2)}} = \varphi^{-1}(b_{(2)}) \otimes S_B(b_{(1)})$ for all $b \in B$. For $h \in H$ we calculate, in $H \otimes H$:

$$\begin{aligned}\varphi^{-1}(\overline{h_{(2)}})_{(1)} \otimes h_{(1)}\varphi^{-1}(\overline{h_{(2)}})_{(2)} &= \varphi^{-1}(\overline{h_{(2)}})_{(1)} \otimes h_{(1)}\pi(\varphi^{-1}(\overline{h_{(2)}})_{(2)})\varphi(\overline{\varphi^{-1}(\overline{h_{(2)}})_{(3)}})\\ &= \varphi^{-1}(\overline{h_{(3)}})_{(1)} \otimes h_{(1)}\pi(\varphi^{-1}(\overline{h_{(3)}})_{(2)})\varphi(S_B(\overline{h_{(2)}})\\ &= \varphi^{-1}(\overline{h_{(4)}})_{(1)} \otimes \pi(h_{(1)})\varphi(\overline{h_{(2)}})\pi(\varphi^{-1}(\overline{h_{(4)}})_{(2)})\varphi(S_B(\overline{h_{(3)}})\\ &= \varphi^{-1}(\overline{h_{(3)}})_{(1)} \otimes \pi(h_{(1)})(\overline{h_{(2)}} \wedge \pi(\varphi^{-1}(\overline{h_{(3)}})_{(2)})).\end{aligned}$$

which shows (A.3). The calculation

$$\begin{aligned} h_{(1)}S_H(h_{(2)}) &= h_{(1)}\varphi^{-1}(\overline{h_{(3)}})_{(1)}S_A(h_{(2)}\varphi^{-1}(\overline{h_{(3)}})_{(2)}) \\ &= (h_{(1)}\varphi^{-1}(\overline{h_{(2)}}))_{(1)}S_A((h_{(1)}\varphi^{-1}(\overline{h_{(2)}}))_{(2)}) = \varepsilon(h_{(1)}\varphi^{-1}(\overline{h_{(2)}}))\cdot 1 = \varepsilon(h)\cdot 1 \end{aligned}$$

shows S_H to be the antipode of H.

The formula (A.2) for comultiplication can be rewritten as

$$\Delta\varphi(b) = \alpha^{(1)}(b_{(1)})\varphi(b_{(2)[0]}) \otimes \alpha^{(2)}(b_{(1)})b_{(2)[1]}\varphi(b_{(3)}).$$

Knowing that α and φ are convolution invertible we can solve this to give (A.6). □

Remark A.3. Of course, there is a dual version of Proposition A.1, resulting in a dual version of Theorem A.2, which we state without giving details: If In Theorem A.2 we assume α instead of τ to be invertible, then H is again a Hopf algebra, and τ is invertible. We have

$$\begin{aligned} S_H(h) &= \pi^{-1}(\varphi(S_B(\overline{h_{(1)}}))_{(1)}h_{(2)})\varphi(S_B(\overline{h_{(1)}}))_{(2)} \\ \tau^{-1}(\overline{g}|\overline{h}) &= \pi^{-1}(g_{(1)}h_{(1)})\pi(g_{(2)})(\overline{g_{(2)}} \rightharpoonup \pi(h_{(2)}), \end{aligned}$$

where $\pi^{-1}(a \ltimes b) = \overline{\alpha}^{(1)}(b)S_A(\overline{\alpha}^{(2)}(b))S_A(a)$. Here $\overline{\alpha}^{(1)}(b) \otimes \overline{\alpha}^{(2)}(b) := \alpha^{-1}(b)$.

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