QUOTIENTS OF FINITE QUASI-HOPF ALGEBRAS

PETER SCHAUENBURG

Abstract. Let H be a finite-dimensional quasi-Hopf algebra. We show for each quotient quasibialgebra Q of H that Q is a quasi-Hopf algebra whose dimension divides the dimension of H.

1. Introduction

In [7] Nichols and Zoeller prove what is now known as the Nichols-Zoeller Theorem: A finite dimensional Hopf algebra H over a field k is a free module over every Hopf subalgebra $K \subset H$. This answers affirmatively one of Kaplansky's conjectures on Hopf algebras in the finite-dimensional case. The Nichols-Zoeller Theorem and some related results are an important tool in the study of finite-dimensional Hopf algebras.

Quasi-Hopf algebras, introduced by Drinfeld [2], are a generalization of ordinary Hopf algebras that can be motivated most easily by looking at representation categories: The category of modules over a Hopf algebra is a monoidal category, with the module structure on the tensor product over the base field of two modules given by the diagonal action via comultiplication. The same thing is still true for quasibialgebras; the difference is that now the tensor product of representations is associative with an associativity isomorphism that differs from the ordinary one for vector spaces.

In [11] we have proved the direct generalization of the Nichols-Zoeller Theorem to quasi-Hopf algebras: A finite-dimensional quasi-Hopf algebra H over a field k is a free module over every quasi-Hopf subalgebra $K \subset H$. The generalization is made possible by the introduction of Hopf modules over quasibialgebras by Hausser and Nill [4]. Hopf modules and the structure theorem for Hopf modules are a key ingredient in the proof of the Nichols-Zoeller Theorem (and the subject of a generalization of the statement of the theorem). Although quasi-Hopf algebras are not coassociative coalgebras, and thus comodules over them are not immediately defined, one can still define Hopf bimodules, and prove a structure theorem for them.

One of the standard applications of the Nichols-Zoeller theorem is to investigate the structure of (semisimple) Hopf algebras by dimension counting: The dimension of a Hopf subalgebra has to divide the dimension of the large Hopf algebra, a version of the classical Lagrange theorem for finite groups. This can serve to narrow down the possible examples in classification attempts. To be yet more specific, one standard argument is that the number of one-dimensional representations of a Hopf algebra H has to divide the dimenision of H. Unfortunately, the version of the Nichols-Zoeller Theorem for quasi-Hopf algebras provided in [11] does not help at

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all in this situation: The one-dimensional representations of H are not related to a quasi-Hopf subalgebra, but rather to a quotient quasi-Hopf algebra. For a quotient Hopf algebra Q of an ordinary Hopf algebra H, the Nichols-Zoeller Theorem implies by duality that H is a cofree Q-comodule. In the quasi-Hopf case, this does not even make sense to ask, since Q is not a coassociative coalgebra, hence H is not a comodule in the usual sense.

We shall nevertheless prove that the dimension of a finite-dimensional quasi-Hopf algebra H is divisible by the dimension of any quotient quasi-Hopf algebra Q. The key to this is the construction of an inclusion of quasi-Hopf algebras with the same ratio of dimensions as that between H and Q. In the ordinary Hopf case this is found by simply dualizing. In the quasi-Hopf case, we will find that H and Q^* generate a quasi-Hopf subalgebra $D(Q;H)$ in the Drinfeld double $D(H)$ of H, and $\dim D(Q;H) = \dim H \dim Q$, while $\dim D(H) = (\dim H)^2$.

Without doubt, one could verify the claims just made on $D(Q;H)$ by direct calculations with the rather complicated quasi-Hopf algebra structure of the Drinfeld double given by Hausser and Nill [3] (while the earlier description by Majid [5] is perhaps to indirect for this purpose). Instead, we will do a closer analysis of $D(Q;H)$ in Section 3, giving parallel interpretations for its modules to those of the double $D(H)$. This will allow us to show our claims without much calculations. Moreover, we will be able in Section 4 to show more than the "Lagrange" statement that $\dim Q$ divides $\dim H$: We will show that $\dim Q$ divides the dimension of any Hopf module in ${}^{Q}_{H}\mathcal{M}_{H}$, parallel to the results of Nichols and Zoeller who also show a freeness result for Hopf modules, not only for the Hopf algebras themselves.

In addition to the results summarized so far, we will show in Section 2 that any quotient quasibialgebra of a finite-dimensional quasi-Hopf algebra is a quotient quasi-Hopf algebra itself. In the case of ordinary Hopf algebras, this is due to Nichols [6], whose arguments we will vary in the necessary manner, replacing a canonical map $H \otimes H \to H \otimes H$ by its quasi-Hopf version due to Drinfeld. Nichols' result applies (by duality) to subbialgebras of finite-dimensional Hopf algebras, whereas it is not true for subquasibialgebras of finite-dimensional quasi-Hopf algebras, although we will give a positive result under additional hypotheses.

In an appendix, we will give a categorical proof, rather free of computations, of the canonical isomorphism $H \otimes H \to H \otimes H$ given by Drinfeld in [2] and used crucially in Section 2.

2. Antipodes for quotients and subobjects

Recall that a quasibialgebra $H = (H, \Delta, \varepsilon, \phi)$ consists of an algebra H, algebra maps $\Delta\colon H \to H \otimes H$ and $\varepsilon\colon H \to k$, and an invertible element $\phi \in H^{\otimes 3}$, the associator, such that

$$(\varepsilon \otimes H)\Delta(h) = h = (H \otimes \varepsilon)\Delta(h), \tag{2.1}$$

$$(H \otimes \Delta)\Delta(h) \cdot \phi = \phi \cdot (\Delta \otimes H)\Delta(h), \tag{2.2}$$

$$(H \otimes H \otimes \Delta)(\phi) \cdot (\Delta \otimes H \otimes H)(\phi) = (1 \otimes \phi) \cdot (H \otimes \Delta \otimes H)(\phi) \cdot (\phi \otimes 1), \tag{2.3}$$

$$(H \otimes \varepsilon \otimes H)(\phi) = 1 \tag{2.4}$$

hold for all $h \in H$. We will write $\Delta(h) =: h_{(1)} \otimes h_{(2)}$, $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$, and $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)}$.

We define a morphism or map of quasibialgebras from (H,ϕ) to (L,ψ) to be an algebra map $f\colon H\to L$ compatible with comultiplication $\Delta f=(f\otimes f)\Delta$ and with the counit $\varepsilon f=\varepsilon$, and finally satisfying $(f\otimes f\otimes f)(\phi)=\psi$.

A subquasibialgebra is a subalgebra L of H that has a quasibialgebra structure for which the inclusion is a quasibialgebra map. Equivalently L is a subalgebra of H satisfying $\Delta(H)\subset H\otimes H$ and $\phi\in H\otimes H\otimes H$. A quotient quasibialgebra of H is a quasibialgebra Q for which there is a surjective quasibialgebra map $\nu\colon H\to Q$. Equivalently, Q is isomorphic to H/I for an ideal $I\subset H$ satisfying $\Delta(I)\subset I\otimes H+H\otimes I$ and $\varepsilon(I)=0$; a coassociator for $Q=H/I$ is then the canonical image of ϕ in $Q\otimes Q\otimes Q$.

Recall further that a quasi-antipode for a quasibialgebra (H,ϕ) is a triple (S,α,β) in which $\alpha,\beta\in H$, and S is an anti-algebra endomorphism of H satisfying

$$S(h_{(1)})\alpha h_{(2)}=\varepsilon(h)\alpha, \qquad h_{(1)}\beta S(h_{(2)})=\varepsilon(h)\beta,$$
$$\phi^{(1)}\beta S(\phi^{(2)})\alpha\phi^{(3)}=1, \qquad S(\phi^{(-1)})\alpha\phi^{(-2)}\beta\phi^{(-3)}=1$$

for $h\in H$. A quasi-Hopf algebra is a quasibialgebra with a quasi-antipode. Note that our definition differs from Drinfeld's in that we do not require the antipode to be bijective. For finite-dimensional quasibialgebras it was recently shown by Bulacu and Caenepeel [1] that bijectivity of the antipode is automatic.

We define a quasibialgebra map $f\colon H\to H'$ between quasi-Hopf algebras (H,ϕ,S,α,β) and $(H',\phi',S',\alpha',\beta')$ to be a quasi-Hopf algebra map if $S'f=fS$, $f(\alpha)=\alpha'$, and $f(\beta)=\beta'$. We define a quasi-Hopf subalgebra to be a subquasibialgebra $L\subset H$ that has a quasi-antipode so that the inclusion is a quasi-Hopf algebra map. Equivalently, L is a subquasibialgebra such that $S(L)\subset L$, and $\alpha,\beta\in L$. Further we define a quotient quasi-Hopf algebra of H to be a quasi-Hopf algebra Q with a surjective quasi-Hopf algebra map $\nu\colon H\to Q$. Equivalently, a quotient quasi-Hopf algebra is a quotient quasibialgebra $Q\cong H/I$ such that $S(I)=I$. Then S induces an antiautomorphism on the quotient, which is a quasiantipode together with the canonical images of α and β in Q.

Theorem 2.1. *Let H be a quasi-Hopf algebra over a field k, and let Q be a finite dimensional quotient quasibialgebra of H. Then Q is a quotient quasi-Hopf algebra.*

Proof. By [2, Prop. 1.5] we have an isomorphism

$$\varphi\colon H\otimes H\ni g\otimes h\mapsto gS(\phi^{(-1)})\alpha\phi^{(-2)}h_{(1)}\otimes\phi^{(-3)}h_{(2)}\in H\otimes H \tag{2.5}$$

with inverse given by

$$\varphi^{-1}(g\otimes h)=g\phi^{(1)}\beta S(h_{(1)}\phi^{(2)})\otimes h_{(2)}\phi^{(3)}$$

Let $\nu\colon H\ni h\mapsto\overline{h}\in Q$ denote the canonical epi. Define

$$\overline{\varphi}\colon Q\otimes Q\ni p\otimes q\mapsto p\overline{S(\phi^{(-1)})\alpha\phi^{(-2)}}q_{(1)}\otimes q_{(2)}\in Q\otimes Q.$$

Then the diagram

$$\begin{tikzcd} H\otimes H \arrow[r, "\varphi"] \arrow[d, "\nu\otimes\nu"] & H\otimes H \arrow[d, "\nu\otimes\nu"] \\ Q\otimes Q \arrow[r, "\overline{\varphi}"] & Q\otimes Q \end{tikzcd}$$

commutes, and thus $\overline{\varphi}$ is onto. Since Q is finite dimensional, $\overline{\varphi}$ is an isomorphism. From the diagram we conclude that

$$\overline{\varphi}^{-1}(\overline{g}\otimes\overline{h})=(\nu\otimes\nu)\varphi^{-1}(g\otimes h),$$

hence

$$\theta\colon Q\ni q\mapsto(Q\otimes\varepsilon)\overline{\beta}^{-1}(1\otimes q)\in Q$$

satisfies $\theta(\overline{h})=\overline{\beta S(h)}$ for all $h\in H$. Now define

$$\overline{S}\colon Q\ni q\mapsto\overline{S(\phi^{(-1)})\alpha\phi^{(-2)}}\theta(q\overline{\phi^{(-3)}})\in Q.$$

Then for $h\in H$ we have

$$\overline{S}(\overline{h})=\overline{S(\phi^{(-1)})\alpha\phi^{(-2)}}\cdot\overline{\beta S(h\phi^{(-3)})}=\overline{S(\phi^{(-1)})\alpha\phi^{(-2)}\beta S(\phi^{(-3)})}\cdot\overline{S(h)}=\overline{S(h)},$$

showing that S maps the kernel of ν into itself. □

Remark 2.2. The direct analog of Theorem 2.1 for subquasibialgebras instead of quotients is false for quite trivial reasons. To see this consider a quasi-Hopf algebra (H,ϕ,S,α,β) and a quasi-Hopf subalgebra $K\subset H$. By the remark following the definition of a quasi-Hopf algebra in [2], we can obtain another quasi-Hopf structure $(H,\phi,S',\alpha',\beta')$ for any unit $u\in H$ by setting $S'(h)=uS(h)u^{-1}$, $\alpha'=u\alpha$, $\beta'=\beta u^{-1}$, while leaving ϕ unchanged. Of course, it may happen that K is not a quasi-Hopf subalgebra for this new quasi-Hopf structure (for example, if α is a unit, and $u\notin K$).

However, we can provide quasiantipodes for subquasibialgebras under some additional assumptions:

Proposition 2.3. *Let H be a quasi-Hopf algebra with coassociator ϕ and quasi-antipode (S,α,β).*

Let $K\subset H$ be a finite-dimensional subquasibialgebra such that

$$S(\phi^{(-1)})\alpha\phi^{(-2)}\otimes\phi^{(-3)}\in K\otimes K.$$

Then K is a quasi-Hopf subalgebra.

Proof. Again we consider the canonical map φ from (2.5). By our extra assumptions, we see that $\varphi(K\otimes K)\subset K\otimes K$, and consider the map $\varphi'\colon K\otimes K\to K\otimes K$ given by restricting φ. It is injective since φ is, hence bijective by finite dimensionality. The inverse of φ' is given by the restriction of φ^{-1}, so we see that for $x\in K$

$$K\ni(K\otimes\varepsilon)\varphi^{-1}(1\otimes x)=\beta S(x),$$

hence in particular $\beta\in K$, and for all $x\in K$

$$K\ni S(\phi^{(-1)})\alpha\phi^{(-2)}\beta S(x\phi^{(-3)})=S(x).$$

Finally $\alpha=(K\otimes\varepsilon)\varphi(1\otimes 1)\in K$, so K is a quasi-Hopf subalgebra. □

Remark 2.4. As a special case of Proposition 2.3, a finite-dimensional subquasibialgebra $K\subset H$ of a quasi-Hopf algebra H is a quasi-Hopf subalgebra provided that it contains a subquasibialgebra $L\subset K$ which is a quasi-Hopf subalgebra of H.

3. The partial double

Throughout the section, we let H denote a quasi-Hopf algebra. The key property of a quasibialgebra is that its modules form a monoidal category: The tensor product of $V, W \in {}_H\mathcal{M}$ is their tensor product $V \otimes W$ over k, endowed with the diagonal module structure $h(v \otimes w) = h_{(1)}v \otimes h_{(2)}w$; the neutral object is k with the trivial module structure given by ε. The associativity isomorphism in the category is

$$(U \otimes V) \otimes W \ni u \otimes v \otimes w \mapsto \phi^{(1)}u \otimes \phi^{(2)}v \otimes \phi^{(3)}w \in U \otimes (V \otimes W)$$

for $U, V, W \in {}_H\mathcal{M}$.

The opposite of a quasibialgebra and the tensor product of two quasibialgebras are naturally quasibialgebras. Thus ${}_H\mathcal{M}_H$ is also a monoidal category, with associativity isomorphism

$$(U \otimes V) \otimes W \ni u \otimes v \otimes w \mapsto \phi^{(1)}u\phi^{(-1)} \otimes \phi^{(2)}v\phi^{(-2)} \otimes \phi^{(3)}w\phi^{(-3)} \in U \otimes (V \otimes W).$$

We will make free use of the formalism of (co)algebra and (co)module theory within monoidal categories. When C, D are coalgebras in ${}_H\mathcal{M}_H$, we will use the abbreviations ${}^C_H\mathcal{M}_H, {}_H\mathcal{M}^D_H, {}^C_H\mathcal{M}^D_H$ for the categories of left C-comodules, right D-comodules, and C-D-bicomodules within the monoidal category ${}_H\mathcal{M}_H$.

We see that H itself is a coassociative coalgebra within the monoidal category ${}_H\mathcal{M}_H$. Thus we can define a Hopf module $M \in {}_H\mathcal{M}^H_H$ to be a right H-comodule within the category ${}_H\mathcal{M}_H$. Written out explicitly, this definition is the same as that of Hausser and Nill [4, Def.3.1]. Hausser and Nill have also proved a structure theorem for such Hopf modules, which says that the functor

$$\mathcal{R}\colon {}_H\mathcal{M} \ni V \mapsto .V \otimes .H^{\cdot}_{\cdot} \in {}_H\mathcal{M}^H_H$$

is an equivalence of categories. We have used this equivalence as a basis for a description of the Drinfeld double of H in [12]. In [10, Expl.4.10] we have repeated this description with a general $\mathcal{C}$-categorical technique, which we shall follow once more now to obtain a relative double $D(L; H)$ for any quasibialgebra map $\nu\colon H \to L$.

For any right H comodule M in ${}_H\mathcal{M}_H$ and any $P \in {}_H\mathcal{M}_H$ we can form the right H-comodule $P \otimes M$ in ${}_H\mathcal{M}_H$, which gives us a functor ${}_H\mathcal{M}_H \times {}_H\mathcal{M}^H_H \to {}_H\mathcal{M}^H_H$ that makes ${}_H\mathcal{M}^H_H$ into a left ${}_H\mathcal{M}_H$-category in the sense of Pareigis [8]. Being equivalent to ${}_H\mathcal{M}^H_H$, the category ${}_H\mathcal{M}$ is then also a left ${}_H\mathcal{M}_H$-category, which means that we have a functor $\diamondsuit\colon {}_H\mathcal{M}_H \times {}_H\mathcal{M} \to {}_H\mathcal{M}$ and a coherent natural isomorphism $\Omega\colon (P \otimes Q)\diamondsuit V \to P\diamondsuit(Q\diamondsuit V)$ for $P, Q \in {}_H\mathcal{M}_H$ and $V \in {}_H\mathcal{M}$.

Now let C be a coalgebra in ${}_H\mathcal{M}_H$. Since ${}_H\mathcal{M}$ is a left ${}_H\mathcal{M}_H$-category, it makes sense (see [8]) to talk about C-comodules within ${}_H\mathcal{M}$, which form a category ${}^C({}_H\mathcal{M})$, which in our situation is naturally equivalent to ${}^C_H\mathcal{M}^H_H$, with the equivalence ${}^C({}_H\mathcal{M}) \cong {}^C_H\mathcal{M}^H_H$ induced by $\mathcal{R}$.

By [10, Thm.3.3] and the remarks preceding it, $\diamondsuit$ induces a functor ${}_H\mathcal{M}_H \ni P \mapsto P\diamondsuit H \in {}_H\mathcal{M}_H$, and we have an isomorphism $(P\diamondsuit H) \otimes_H V \cong P\diamondsuit V$, natural in $P \in {}_H\mathcal{M}_H$ and $V \in {}_H\mathcal{M}$.

By [10, Cor.3.8], $C\diamondsuit H$ has an H-coring structure in such a way that one has a category equivalence ${}^{C\diamondsuit H}\mathcal{M} \cong {}^C({}_H\mathcal{M})$ that commutes with the underlying functors to ${}_H\mathcal{M}$. (Here ${}^{C\diamondsuit H}\mathcal{M}$ denotes the category of left comodules over the H-coring $C\diamondsuit H$.) For any coalgebra morphism $f\colon C \to D$ in ${}_H\mathcal{M}_H$ we obtain a commutative diagram of functors

$$\begin{tikzcd}
{}^{C\Diamond H}\mathcal{M} \arrow[r] \arrow[d, "{}^{f\Diamond H}\mathcal{M}"] & {}^{C}({}_H\mathcal{M}) \arrow[r, "\mathcal{R}"] \arrow[d, "{}^{f}({}_H\mathcal{M})"] & {}^{C}_{H}\mathcal{M}^{H}_{H} \arrow[d, "{}^{f}_{H}\mathcal{M}^{H}_{H}"] \\
{}^{D\Diamond H}\mathcal{M} \arrow[r] & {}^{D}({}_H\mathcal{M}) \arrow[r, "\mathcal{R}"] & {}^{D}_{H}\mathcal{M}^{H}_{H}
\end{tikzcd}$$

for the H-coring map $f\Diamond H\colon C\Diamond H \to D\Diamond H$.

From [12, Sec.4] we know that $P\Diamond V \cong P\otimes V$ as vector spaces, functorially in $P \in {}_H\mathcal{M}_H$ and $V \in {}_H\mathcal{M}$. In particular we have $C\Diamond H \cong C\otimes H$ as right H-modules, functorially in the coalgebra C in ${}_H\mathcal{M}_H$.

Now assume that C is finite dimensional. Then $C\Diamond H$ is a finitely generated free right H-module, so the dual algebra $\operatorname{Hom}_{-H}(C\Diamond H, H)$ of the H-coring $C\Diamond H$ fulfills ${}_{(C\Diamond H)^\vee}\mathcal{M} \cong {}^{C\Diamond H}\mathcal{M}$. Note that $(C\Diamond H)^\vee \cong H\otimes C^*$ as vector spaces, functorially in C.

Finally, we specialize to a class of examples of coalgebras in ${}_H\mathcal{M}_H$. Whenever $\nu\colon H \to L$ is a morphism of quasibialgebras, we can consider L as a coalgebra in ${}_H\mathcal{M}_H$ with respect to its usual comultiplication, and the H-bimodule structure induced along ν. We write $D(L;H) := (L\Diamond H)^\vee$ for the dual algebra of the coring $L\Diamond H$. Note that $D(L;H) \cong H\otimes L^*$ as vector spaces. The isomorphism is natural in L, meaning that for any morphism $f\colon L \to M$ of quasibialgebras, the induced morphism $D(f;H)\colon D(M;H) \to D(L;H)$ corresponds to $H\otimes f^*\colon H\otimes M^* \to H\otimes L^*$. In particular it is surjective (resp. injective) if f is injective (resp. surjective).

The same calculations as those made to prove [12, Lem.3.2] prove more generally that ${}^L_H\mathcal{M}^H_H$ is a monoidal category, the tensor product of $M, N \in {}^L_H\mathcal{M}^H_H$ being $M\otimes_H N$ with the left and right comodule structures

$$M \underset{H}{\otimes} N \ni m\otimes n \mapsto m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)} \in L\otimes (M \underset{H}{\otimes} N)$$
$$M \underset{H}{\otimes} N \ni m\otimes n \mapsto m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)} \in (M \underset{H}{\otimes} N)\otimes H.$$

Note that the underlying functor ${}^L_H\mathcal{M}^H_H \to {}_H\mathcal{M}^H_H$ is monoidal.

The equivalence $\mathcal{R}$ is a monoidal equivalence. We make ${}_{D(L;H)}\mathcal{M}$ a monoidal category in such a way that the equivalence ${}_{D(L;H)}\mathcal{M} \cong {}^L_H\mathcal{M}^H_H$ is a monoidal functor. Thus, for any quasibialgebra maps $H \xrightarrow{\nu} L \xrightarrow{f} M$ we obtain a commutative diagram of monoidal functors

$$\begin{tikzcd}
{}_{(M\Diamond H)^\vee}\mathcal{M} \arrow[r, "{}_{(f\Diamond H)^\vee}\mathcal{M}"] \arrow[d] & {}_{(L\Diamond H)^\vee}\mathcal{M} \arrow[r, "{}_{(\varepsilon\Diamond H)^\vee}\mathcal{M}"] \arrow[d] & {}_H\mathcal{M} \arrow[d] \\
{}^{M}_{H}\mathcal{M}^{H}_{H} \arrow[r, "{}^{f}_{H}\mathcal{M}^{H}_{H}"] & {}^{L}_{H}\mathcal{M}^{H}_{H} \arrow[r] & {}_{H}\mathcal{M}^{H}_{H}
\end{tikzcd}$$

By a trivial modification of the monoidal category structures (cf. [9, Rem.5.3] for an analogous trick) we can make sure that the functors in the top row are strict monoidal functors. This implies that $D(L;H)$ is a quasibialgebra, and that $D(f;H)\colon D(M;H) \to D(L;H)$ is a quasibialgebra map.

4. Lagrange's theorem

Consider a finite-dimensional quasi-Hopf algebra H, and a quotient quasi-Hopf algebra Q. As a particular case of the constructions in the preceding section, we obtain injective quasibialgebra maps $H \to D(Q;H) \to D(H;H)$. Note that $D(H;H) = D(H)$ is the Drinfeld double of H, which is a quasi-Hopf algebra. It follows from Proposition 2.3 that $D(Q;H)$ is a quasi-Hopf algebra as well, and a quasi-Hopf subalgebra of $D(H)$. By [11], $D(H)$ is a free $D(Q;H)$-module, so in particular $\dim D(Q;H) = \dim Q \dim H$ divides $\dim D(H) = (\dim H)^2$. Cancelling $\dim H$ we get:

Corollary 4.1. *Let H be a finite-dimensional quasi-Hopf algebra, and Q a quotient quasibialgebra of H. Then $\dim Q$ divides $\dim H$.*

As an immediate application we have:

Corollary 4.2. *Let H be a finite-dimensional quasi-Hopf algebra. Then the number of one-dimensional representations of H divides $\dim H$.*

Proof. We pass to the dual picture: Representations of H are comodules over the dual coalgebra H^*. One-dimensional comodules correspond to grouplike elements of H^*. These grouplikes span a sub-coquasibialgebra of H^*, which corresponds to a quotient quasibialgebra of H, whose dimension divides the dimension of H by the preceding corollary. □

Nichols and Zoeller [7] do not only prove a freeness theorem for Hopf algebra inclusions $K \subset H$, but also a freeness theorem for Hopf modules in ${}^H_K\mathcal{M}$. In particular, for any finite-dimensional $M \in {}^H_K\mathcal{M}$, they show that $\dim K \mid \dim M$. Suppose that H is a finite-dimensional quasi-Hopf algebra and Q a quotient quasibialgebra. Then Q is a coalgebra in the monoidal category ${}_H\mathcal{M}_H$, so that we have a well-defined notion of Hopf module in ${}^Q_H\mathcal{M}_H$ (while ${}^Q_H\mathcal{M}$ is not defined).

Proposition 4.3. *Let H be a finite-dimensional quasi-Hopf algebra, Q a quotient quasibialgebra of H, and $M \in {}^Q_H\mathcal{M}_H$. Then $\dim Q$ divides $\dim M$.*

Proof. Consider the commutative diagram of functors (first without the dotted arrows)

$$\begin{tikzcd}
{}_{D(H)}\mathcal{M} \arrow[r, "\mathcal{R}"] \arrow[d, shift right, "\mathcal{V}"'] & {}^H_H\mathcal{M}^H_H \arrow[d, shift right, "\mathcal{U}"'] \\
{}_{D(Q;H)}\mathcal{M} \arrow[r, "\mathcal{R}"] \arrow[u, shift right, dotted] & {}^Q_H\mathcal{M}^H_H \arrow[u, shift right, dotted]
\end{tikzcd}$$

in which the functor $\mathcal{V}$ is the underlying functor induced by the inclusion $D(Q;H) \to D(H)$, and $\mathcal{U}$ is the underlying functor induced by the projection $H \to Q$. The horizontal arrows are induced by the category equivalence $\mathcal{R}\colon {}_H\mathcal{M} \to {}_H\mathcal{M}^H_H$. Since they are equivalences, the diagram also commutes for the dotted arrows, if these denote the right adjoint functors to $\mathcal{V}$ and $\mathcal{U}$. Now the right adjoint to the underlying functor $\mathcal{U}$ is given by cotensor product with H, taken within the monoidal category ${}_H\mathcal{M}_H$ (dually to the induction functor for an algebra inclusion), whereas the right adjoint to $\mathcal{V}$ is the usual coinduction functor for the algebra inclusion $D(Q;H) \subset D(H)$. This means that for $W \in {}_{D(Q;H)}\mathcal{M}$ we have

$$H \underset{Q}{\square} \mathcal{R}(W) \cong \mathcal{R}(\mathrm{Hom}_{D(Q;H)}(D(H), W)).$$

In particular, since $D(H)$ is a free $D(Q;H)$-module of rank $\dim H/\dim Q$, and since $\mathcal{R}$ multiplies dimensions by $\dim H$, we have

$$\dim(H \underset{Q}{\square} M) = \frac{\dim H}{\dim Q} \dim M$$

whenever $M \in {}^Q_H\mathcal{M}^H_H$ is finite dimensional.

Now consider a finite-dimensional $V \in {}^Q_H\mathcal{M}_H$. Then we can tensor V with $H \in {}_H\mathcal{M}^H_H$ to obtain $M = V \otimes H \in {}^Q_H\mathcal{M}^H_H$. Calculating within the monoidal category ${}_H\mathcal{M}_H$ we have

$$(H \underset{Q}{\square} V) \otimes H \cong H \underset{Q}{\square} (V \otimes H) = H \underset{Q}{\square} M.$$

It follows that

$$\dim H \dim(H \underset{Q}{\square} V) = \frac{\dim H}{\dim Q} \dim M$$

or $\dim(H \,\square_Q\, V) = \frac{\dim H}{\dim Q} \dim V$.

But on the other hand $H \,\square_Q\, V \in {}^H_H\mathcal{M}_H$. By the left-right switched version of the structure theorem of Hausser and Nill, any Hopf module in ${}^H_H\mathcal{M}_H$ is a free left H-module, so that $\dim H$ divides $\dim(H \,\square_Q\, V)$. Thus $\dim Q$ divides $\dim V$. $\square$

Appendix A. A dogmatic proof of a formula of Drinfeld

In this section we return to the isomorphism φ from equation (2.5). Its proof in Drinfeld's paper [2] is not particularly hard, but does involve a calculation with the coassociator element of H, the pentagon identity (2.3) and the identities defining a quasi-antipode. The "dogma" alluded to in this section's title says that such calculations should be banned. After all, the pentagon identity precisely ensures that ${}_H\mathcal{M}$ is a monoidal category, and the quasi-antipode axioms precisely ensure that the dual vector space of a finite-dimensional module can be made into a dual object inside that monoidal category. Since the axioms are more or less equivalent to the categorical properties, no further reference to the former should be necessary, and using the latter should lead to easier and more conceptual proofs.

So now we let H be a finite-dimensional quasi-Hopf algebra and $V \in {}_H\mathcal{M}$ a finite-dimensional H-module. Then V has the dual object $V^* \in {}_H\mathcal{M}$, with module structure given by the transpose of the right module structure V_S induced via the quasi-antipode, evaluation $\mathrm{ev}\colon V^* \otimes V \ni \varphi \otimes v \mapsto \varphi(\alpha v) \in k$ and dual basis $\mathrm{db}\colon k \to V \otimes V^*$, $\mathrm{db}(1) = \beta v_i \otimes v^i$, where v_i and v^i are a pair of dual bases in V and V^*, and summation is suppressed. It follows that the functor

$$\mathcal{F}\colon {}_H\mathcal{M} \ni W \mapsto .W \otimes .V \in {}_H\mathcal{M}$$

has the right adjoint

$$\mathcal{G}\colon {}_H\mathcal{M} \ni X \mapsto X \otimes V^* \in {}_H\mathcal{M}$$

with unit u and counit c of the adjunction given by

$$u = \left(X \xrightarrow{X\otimes \mathrm{db}} X \otimes (V \otimes V^*) \xrightarrow{\Phi^{-1}} (X \otimes V) \otimes V^*\right)$$

$$c = \left((X \otimes V^*) \otimes V \xrightarrow{\Phi} X \otimes (V^* \otimes V) \xrightarrow{X\otimes \mathrm{ev}} X\right).$$

We have the standard isomorphism

$$\Gamma = \left(X \otimes V^* \xrightarrow{\cong} \mathrm{Hom}(V, X) \xrightarrow{\cong} \mathrm{Hom}_{H-}(.H \otimes V, X)\right)$$

with $\Gamma(x\otimes\varphi)(h\otimes v)=h\varphi(v)x$ and $\Gamma^{-1}(f)=f(1\otimes v_i)\otimes v^i$. Via the isomorphism Γ we have another right adjoint

$$\mathcal{G}'\colon {}_H\mathcal{M}\ni X\mapsto \mathrm{Hom}_{H-}(.H\otimes V,X)\in {}_H\mathcal{M}.$$

The unit u' and counit c' of the adjunction are given by

$$u'=\left(X\xrightarrow{u}(X\otimes V)\otimes V^*\xrightarrow{\Gamma}\mathrm{Hom}_{H-}(Q,X\otimes V)\right)$$
$$c'=\left(\mathrm{Hom}_{H-}(Q,X)\otimes V\xrightarrow{\Gamma^{-1}\otimes V}(X\otimes V^*)\otimes V\xrightarrow{c}X\right)$$

so that

$$\begin{aligned}u'(x)(h\otimes v)&=\Gamma(u(x))(h\otimes v)=\Gamma(\phi^{(-1)}x\otimes\phi^{(-2)}\beta v_i\otimes\phi^{(-3)}v^i)(h\otimes v)\\&=h(\phi^{(-3)}v^i)(v)\cdot(\phi^{(-1)}x\otimes\phi^{(-2)}\beta v_i)=hv^i(v)\cdot(\phi^{(-1)}x\otimes\phi^{(-2)}\beta S(\phi^{(-3)})v_i)\\&\qquad h_{(1)}\phi^{(-1)}x\otimes h_{(2)}\phi^{(-2)}\beta S(\phi^{(-3)})v\end{aligned}$$

and

$$\begin{aligned}c'(f\otimes v)&=c(f(1\otimes v_i)\otimes v^i\otimes v)=\phi^{(1)}f(1\otimes v_i)(\phi^{(2)}v^i)(\alpha\phi^{(3)}v)\\&=f(\phi^{(1)}\otimes v_i)v^i(S(\phi^{(2)})\alpha\phi^{(3)}v)=f(\phi^{(1)}\otimes S(\phi^{(2)})\alpha\phi^{(3)}v).\end{aligned}$$

One checks that the relevant H-module structure on $\mathrm{Hom}_{H-}(H\otimes V,X)$ (making Γ an H-module map) is given by $(hf)(g\otimes v)=f(gh_{(1)}\otimes S(h_{(2)})v)$. In other words, we have $\mathcal{G}'(X)=\mathrm{Hom}_{H-}(Q,X)$ for the H-bimodule $Q=.H.\otimes.(V_S)$. In particular $\mathcal{G}'$ is the right adjoint in the standard hom-tensor adjunction with left adjoint $\mathcal{F}'=Q\otimes_H(-)$. We denote the counit and unit of that standard adjunction by c'',u''. Now left adjoints are unique, so that we get mutually inverse isomorphisms

$$\Lambda=\left(Q\underset{H}{\otimes}V\xrightarrow{Q\otimes_H u'}Q\underset{H}{\otimes}\mathrm{Hom}_{H-}(Q,X\otimes V)\xrightarrow{c''}X\otimes V\right)$$
$$\Lambda^{-1}=\left(X\otimes V\xrightarrow{u''\otimes V}\mathrm{Hom}_{H-}(Q,Q\underset{H}{\otimes}X)\otimes V\xrightarrow{c'}Q\underset{H}{\otimes}X\right).$$

We compute

$$\Lambda(h\otimes v\otimes x)=u'(x)(h\otimes v)=h_{(1)}\phi^{(-1)}x\otimes h_{(2)}\phi^{(-2)}\beta S(\phi^{(-3)})v$$

and

$$\Lambda^{-1}(x\otimes v)=c'(u''(x)\otimes v)=u''(x)(\phi^{(1)}\otimes S(\phi^{(2)})\alpha\phi^{(3)}v)=\phi^{(1)}\otimes S(\phi^{(2)})\alpha\phi^{(3)}v\otimes x$$

Finally, we specialize $V=X=H$, and identify $Q\otimes_H H\cong Q=H\otimes H$ to find $\Lambda(h\otimes g)=h_{(1)}\phi^{(-1)}\otimes h_{(2)}\phi^{(-2)}\beta S(\phi^{(-3)})g$ and $\Lambda^{-1}(h\otimes g)=\phi^{(1)}h_{(1)}\otimes S(h_{(2)})S(\phi^{(2)})\alpha\phi^{(3)}g$.

We have obtained the version of (2.5) for the opposite and coopposite quasi-Hopf algebra to H.

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Mathematisches Institut der Universität München, Theresienstr. 39, 80333 München, Germany

E-mail address: `schauen@rz.mathematik.uni-muenchen.de`