THE MONOIDAL CENTER CONSTRUCTION AND BIMODULES

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ABSTRACT. Let $\mathcal{C}$ be a cocomplete monoidal category such that the tensor product in $\mathcal{C}$ preserves colimits in each argument. Let A be an algebra in $\mathcal{C}$. We show (under some assumptions including "faithful flatness" of A) that the center of the monoidal category $({}_A\mathcal{C}_A, \underset{A}{\otimes})$ of A-A-bimodules is equivalent to the center of $\mathcal{C}$ (hence in a sense trivial): $\mathcal{Z}({}_A\mathcal{C}_A) \cong \mathcal{Z}(\mathcal{C})$.

Assuming A to be a commutative algebra in the center $\mathcal{Z}(\mathcal{C})$, we compute the center $\mathcal{Z}(\mathcal{C}_A)$ of the category of right A-modules (considered as a subcategory of ${}_A\mathcal{C}_A$ using the structure of $A \in \mathcal{Z}(\mathcal{C})$. We find $\mathcal{Z}(\mathcal{C}_A) \cong \operatorname{dys} \mathcal{Z}(\mathcal{C})_A$, the category of dyslectic right A-modules in the braided category $\mathcal{Z}(\mathcal{C})$.

1. INTRODUCTION

Braided monoidal categories are situated at the intersection of quantum group theory and low-dimensional topology (more precisely invariants of knots and three-manifolds, see [11, 3] for general references).

They also provide a framework for generalizations of commutative algebra based on a flip of tensor factors more general than the ordinary flip (as proposed by Manin [5] for symmetric monoidal categories and well-established for the special case of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces) or generalizations of Hopf algebra theory [4].

The center construction [2] is a machinery providing fundamental examples of braided categories. It assigns to any monoidal category $\mathcal{C}$ a braided monoidal category $\mathcal{Z}(\mathcal{C})$ whose objects are objects of $\mathcal{C}$ together with a specified way of permuting them past other objects of $\mathcal{C}$ in a tensor product. All possible braidings on $\mathcal{C}$ are parametrized by sections $\mathcal{C} \to \mathcal{Z}(\mathcal{C})$ of the underlying functor. The center is a categorical version of the Hopf algebraic construction of the Drinfeld double: Given a k-Hopf algebra H one can consider the monoidal category ${}_H\mathcal{M}$ of H-modules; taking the center of this category one obtains the category ${}_H^H\mathcal{YD}$ of Yetter-Drinfeld modules, which is equivalent to the category of modules over the Drinfeld double of H if H is finite dimensional.

The topic of the present paper is the computation of the centers of bimodule categories.

More precisely, assume given an algebra A within a monoidal category $\mathcal{C}$. Then one can form the category ${}_A\mathcal{C}_A$ of bimodules over A in $\mathcal{C}$, which is again a monoidal category (assuming $\mathcal{C}$ has coequalizers and they are preserved under tensor products). To ${}_A\mathcal{C}_A$ one can then apply the center construction to obtain another braided monoidal category. Note that the process is superficially parallel to the route from the category of vector spaces via the category of modules over a Hopf algebra to its center, the category of modules over the double. Thus one might expect that $\mathcal{Z}({}_A\mathcal{C}_A)$ is a new and quite complicated example of a braided monoidal category. However, there are results in special cases that indicate the contrary: Assume H is a Hopf algebra with bijective antipode over a field k, and consider $\mathcal{C} = \mathcal{M}^H$. The center of $\mathcal{C}$ is the category $\mathcal{YD}_H^H$ of Yetter-Drinfeld modules. An algebra in $\mathcal{C}$ is just an H-comodule algebra A. In [9] we have shown that, in the (very special) case that A is a faithfully flat H-Galois extension of its coinvariants, the category $\mathcal{Z}({}_A\mathcal{M}_A^H)$ does not depend on the structure of A at all: We have an equivalence of braided monoidal categories $\mathcal{YD}_H^H \to \mathcal{Z}({}_A\mathcal{M}_A^H)$. In [10] we have established the same result, an equivalence $\mathcal{YD}_H^H \cong \mathcal{Z}({}_A\mathcal{M}_A^H)$, under the assumption that H is finite-dimensional, but dropping all assumptions on A. The proof of the first result makes extensive use of the Hopf-Galois situation. The proof of the second result reduces the problem to the first result by dualizing H and passing from A to a smash product. However, the restrictions on H and A in the two cases are somewhat disjoint, so one expects a more general result to hold, with a more natural proof avoiding Hopf-Galois theory. In fact, we shall show in section 3 that for any monoidal category $\mathcal{C}$ and algebra A in $\mathcal{C}$ one has a category equivalence $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}({}_A\mathcal{C}_A)$ (under suitable restrictions on $\mathcal{C}$ and A which we shall not discuss until section 3; in particular, the result covers comodule algebras over any Hopf algebra over a field).

In section 4, we modify the problem by restricting our attention to special bimodules. Let $\mathcal{B}$ be a braided monoidal category, and A a commutative (with respect to the braiding) algebra in $\mathcal{B}$. Then one can consider the category $\mathcal{B}_A$ of right A-modules as a monoidal subcategory of ${}_A\mathcal{B}_A$: Every right A-module becomes naturally a left A-module by switching over the module structure using the braiding. In fact, one can do this in two ways: using the braiding or its inverse. This leads to the definition of dyslectic modules by Pareigis [8] as those modules that do not see the difference between the two ways of making them bimodules. Pareigis shows that the category $\operatorname{dys}\mathcal{B}_A$ of dyslectic right A-modules is a braided monoidal category, with the braiding obtained by factoring the braiding in $\mathcal{B}$ to the appropriate tensor products over A. Now assume given a monoidal category $\mathcal{C}$, and a commutative algebra A in the center $\mathcal{Z}(\mathcal{C})$. Again, one can consider $\mathcal{C}_A$ as a full monoidal subcategory of the monoidal category ${}_A\mathcal{C}_A$. By a slight generalization of the reasoning in [8], one obtains a braided monoidal functor $\operatorname{dys}\mathcal{Z}(\mathcal{C})_A \to \mathcal{Z}(\mathcal{C}_A)$. The main result of section 4 is that this functor is indeed an equivalence. On the categorical level, this does not (as opposed to the case of general bimodules treated in section 3) mean that the center of $\mathcal{C}_A$ thus computed is trivial; in particular, some of the representation theory of A is still present in the structure of dys $\mathcal{Z}(\mathcal{C})_A$. However, the result means that there are severe restrictions on an object $X \in \mathcal{Z}(\mathcal{C}_A)$: It has to be an object of $\mathcal{Z}(\mathcal{C})$ (i.e. permutable with all objects of $\mathcal{C}$, not just A-modules); the way it permutes with objects of $\mathcal{C}_A$ does not "see" their A-module structure; the way that X is an object in the center is strongly linked with the structure of $A \in \mathcal{Z}(\mathcal{C})$ by the requirement that the module structure $X \otimes A \to X$ is a morphism in $\mathcal{Z}(\mathcal{C})$; finally, there is the dyslexia restriction on the module structure.

As an application of the results in sections 3 and 4 we will study an intermediate case in section 5: We assume given an algebra A in a monoidal category $\mathcal{C}$, and a subalgebra R in A which is also a commutative algebra in $\mathcal{Z}(\mathcal{C})$ and "braided central" in A in a natural sense (this is a generalization to a braided situation of the condition that A be an R-algebra). We then study the category of those A-bimodules on which R acts "braided centrally". Under suitable assumptions including faithful flatness of A over R we can compute the center of this category to be equivalent to dys $\mathcal{Z}(\mathcal{C})_R$. As a special case we cover results obtained in [9] for the case of an H-Galois extension A/B and its subalgebra A^B.

2. Preliminaries

In this section we fix notations and conventions on a number of subjects that we will be concerned with in the rest of the paper.

Monoidal categories. A monoidal category (also called a tensor category) is a category $\mathcal{C}$ with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object $I \in \mathcal{C}$ such that $\otimes$ is associative up to a coherent natural isomorphism $\alpha\colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, and I is a unit for $\otimes$ up to coherent natural isomorphisms $I \otimes X \cong X \cong X \otimes I$. We refer to Kassel's book [3] for precise definitions. We shall freely assume (backed by MacLanes coherence theorem, see [3, XI.5]) that all our monoidal categories are strict, i. e. the tensor product is associative on the nose and I is a unit for $\otimes$.

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$. The structure of a monoidal functor for $\mathcal{F}$ is an isomorphism $\xi = \xi_{XY}\colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ which is coherent in the sense that the two possibilities of composing a morphism $\mathcal{F}(X) \otimes \mathcal{F}(Y) \otimes \mathcal{F}(Z) \to \mathcal{F}(X \otimes Y \otimes Z)$ from instances of ξ coincide, and $\xi_{IX} = \xi_{XI} = 1_{\mathcal{F}(X)}$ holds. A monoidal functor is a pair $(\mathcal{F}, \xi)$ in which $\mathcal{F}$ is a functor and ξ a monoidal functor structure.

A prebraiding for a monoidal category $\mathcal{C}$ is a morphism $\sigma_{XY}\colon X \otimes Y \to Y \otimes X$, natural in $X, Y \in \mathcal{C}$, satisfying, for all $X, Y, Z \in \mathcal{C}$ the equations

$$\sigma_{X\otimes Y,Z} = (\sigma_{XZ} \otimes Y)(X \otimes \sigma_{YZ})$$
$$\sigma_{X,Y\otimes Z} = (Y \otimes \sigma_{XZ})(\sigma_{XY} \otimes Z)$$
$$\sigma_{XI} = \sigma_{IX} = 1_X$$

A braiding is a prebraiding which is an isomorphism (in which case the last axiom in the list is superfluous). A (pre)braided monoidal category is a monoidal category with a specified (pre)braiding.

Let $\mathcal{C}$ be a (strict) monoidal category and $\mathcal{D}$ a monoidal subcategory. The left weak centralizer $\mathcal{Z}_0^\ell(\mathcal{D};\mathcal{C})$ of $\mathcal{D}$ in $\mathcal{C}$ has as its objects pairs $(X,\sigma_{X,-})$ where $X\in\mathcal{C}$, and $\sigma_{X,Y}\colon X\otimes Y\to Y\otimes X$ is a morphism in $\mathcal{C}$, natural in $Y\in\mathcal{D}$, satisfying

$$\sigma_{X,Y\otimes Z}=(Y\otimes\sigma_{XZ})(\sigma_{XY}\otimes Z)$$
$$\sigma_{XI}=\mathrm{id}_X$$

for all $Y,Z\in\mathcal{D}$. The centralizer $\mathcal{Z}(\mathcal{D};\mathcal{C})$ of $\mathcal{D}$ in $\mathcal{C}$ is the full subcategory of $\mathcal{Z}_0^\ell(\mathcal{D};\mathcal{C})$ consisting of those objects $(X,\sigma_{X,-})$ in which σ is an isomorphism. The left weak centralizer $\mathcal{Z}_0^\ell(\mathcal{C};\mathcal{C})$ is a monoidal category with $(X,\sigma_{X,-})\otimes(Y,\sigma_{Y,-}):=(X\otimes Y,\sigma_{X\otimes Y,-})$, where $\sigma_{X\otimes Y,-}$ is defined by $\sigma_{X\otimes Y,Z}=(\sigma_{XZ}\otimes Y)(X\otimes\sigma_{YZ})$ for $Z\in\mathcal{D}$. The centralizer is a full monoidal subcategory $\mathcal{Z}(\mathcal{D};\mathcal{C})\subset\mathcal{Z}_0^\ell(\mathcal{D};\mathcal{C})$. The left weak center of $\mathcal{C}$ is $\mathcal{Z}_0^\ell(\mathcal{C}):=\mathcal{Z}_0^\ell(\mathcal{C};\mathcal{C})$. The center of $\mathcal{C}$ is $\mathcal{Z}(\mathcal{C}):=\mathcal{Z}(\mathcal{C};\mathcal{C})$. The (left weak) center is a (pre)braided category with (pre)braiding

$$\sigma_{XY}\colon(X,\sigma_{X,-})\otimes(Y,\sigma_{Y,-})\to(Y,\sigma_{Y,-})\otimes(X,\sigma_{X,-}).$$

A reference for braided monoidal categories and the center construction is again Kassel's book [3]. Prebraided monoidal categories and (weak) centralizers are obtained by rather obvious weakenings of the axioms.

Algebras and modules. (Extensive references for module theory in monoidal categories are [6, 7].)

Assume given a (strict) monoidal category $\mathcal{C}$. An algebra in $\mathcal{C}$ is an object $A\in\mathcal{C}$ with a multiplication, which we shall denote $\nabla_A\colon A\otimes A\to A$ (or ∇ if no confusion is likely), and a unit, denoted $\eta_A\colon I\to A$ (or just η) fulfilling

$$\nabla(A\otimes\nabla)=\nabla(\nabla\otimes A)\colon A\otimes A\otimes A\to A$$

and $\nabla(A\otimes\eta)=\nabla(\eta\otimes A)=1_A$. We shall sometimes use the abbreviated notation

$$\eta_m\colon X_1\otimes\ldots\otimes X_n\to X_1\otimes\ldots X_{m-1}\otimes A\otimes X_m\otimes\ldots X_n$$

for the morphism "inserting η" at the indicated position, for objects $X_1,\ldots,X_n\in\mathcal{C}$. Let A be an algebra in $\mathcal{C}$. A left A-module in $\mathcal{C}$ is an object $M\in\mathcal{C}$ with a left module structure, which we shall denote $\lambda_M\colon A\otimes M\to M$ (or just λ), satisfying

$$\lambda(A\otimes\lambda)=\lambda(\nabla\otimes M)\colon A\otimes A\otimes M\to M$$

and $\lambda(\eta\otimes M)=1_M$. A right A-module M has a right module structure $\rho=\rho_M\colon M\otimes A\to M$ satisfying $\rho(\rho\otimes A)=\rho(M\otimes\nabla)$ and $\rho(M\otimes\eta)=1_M$, and a bimodule is a left as well as right A-module satisfying in addition $\rho(\lambda\otimes A)=\lambda(A\otimes\rho)\colon A\otimes M\otimes A\to M$. We denote the categories of right (left, bi-) modules in $\mathcal{C}$ by $\mathcal{C}_A$ (${}_A\mathcal{C}$, ${}_A\mathcal{C}_A$).

Assume that $\mathcal{C}$ has coequalizers. Let $M\in\mathcal{C}_A$ and $N\in{}_A\mathcal{C}$. The tensor product $M\underset{A}{\otimes}N$ and the epimorphism $\nu=\nu_{MN}\colon M\otimes N\to M\underset{A}{\otimes}N$ are defined by a coequalizer

$$M \otimes A \otimes N \overset{\lambda \otimes 1}{\underset{1 \otimes \rho}{\rightrightarrows}} M \otimes N \xrightarrow{\nu} M \underset{A}{\otimes} N.$$

If $M, N \in {}_A\mathcal{C}_A$, then $M \underset{A}{\otimes} N \in {}_A\mathcal{C}_A$. If we assume that all functors $- \otimes V \colon \mathcal{C} \to \mathcal{C}$ as well as $V \otimes -$ for $V \in \mathcal{C}$ preserve coequalizers, then ${}_A\mathcal{C}_A$ becomes in this way a monoidal category, which we shall (contrary to the facts) assume to be strict by identifying $(M \underset{A}{\otimes} N) \underset{A}{\otimes} P = M \underset{A}{\otimes} (N \underset{A}{\otimes} P) =: M \underset{A}{\otimes} N \underset{A}{\otimes} P$ for $M, N, P \in {}_A\mathcal{C}_A$. Note that $\underset{A}{\otimes}$ preserves coequalizers in each argument, and preserves epimorphisms in each argument provided $\otimes$ does.

The underlying functor $\mathcal{C}_A \to \mathcal{C}$ is right adjoint, with the left adjoint mapping $V \in \mathcal{C}$ to $V \otimes A \in \mathcal{C}_A$, with adjunction morphism $\eta_2 \colon V \to V \otimes A$. For each $M \in {}_A\mathcal{C}$ we have a canonical isomorphism $(V \otimes A) \underset{A}{\otimes} M \cong V \otimes M$, which is a morphism in $\mathcal{C}_A$ if $M \in {}_A\mathcal{C}_A$. Of course the underlying functor ${}_A\mathcal{C} \to \mathcal{C}$ also has a left adjoint $V \mapsto A \otimes V$, with canonical isomorphisms $M \underset{A}{\otimes} (A \otimes V) \cong M \otimes V$ for $M \in \mathcal{C}_A$.

Faithfully flat descent. The technique of faithfully flat descent relates, for a faithfully flat ring extension $B \subset A$, B-modules to A-modules endowed with an additional structure. As explained in [1, 4.2], faithfully flat descent is a special case of Beck's theorem.The latter result is of a purely categorical nature and allows to prove analogs of faithfully flat descent in monoidal categories. We only list the relevant definitions and results for "faithfully flat" extensions A of the neutral object I.

Definition 2.1. Let $\mathcal{C}$ be a monoidal category and A an algebra in $\mathcal{C}$. Let $X \in {}_A\mathcal{C}$ with module structure $\lambda \colon A \otimes X \to X$. A descent datum on X is a morphism $\theta \colon X \to A \otimes X$ of A-modules such that

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "\theta"'] & A \otimes X \arrow[d, "A \otimes \theta"] \\ A \otimes X \arrow[r, "A \otimes \eta \otimes X"'] & A \otimes A \otimes X \end{tikzcd}$$

commutes and $\lambda\theta = \mathrm{id}_X$ holds.

Left A-modules equipped with a descent datum form a category in the obvious way: A morphism of descent data from (X, θ_X) to (Y, θ_Y) is a module morphism $f \colon X \to Y$ with $\theta_Y f = (A \otimes f)\theta_X$.

For $V \in \mathcal{C}$ we call $A \otimes \eta \otimes V \colon A \otimes V \to A \otimes A \otimes V$ the canonical descent datum on the A-module $A \otimes V$.

Theorem 2.2. *Let $\mathcal{C}$ be a monoidal category with equalizers and $A \in \mathcal{C}$ an algebra such that $A \otimes - \colon \mathcal{C} \to \mathcal{C}$ preserves equalizers. Then the category of left*

A-modules equipped with a descent datum is equivalent to the category $\mathcal{C}$. The equivalence maps $V \in \mathcal{C}$ to $A \otimes V$ equipped with the canonical descent datum, and its inverse maps $X \in {}_A\mathcal{C}$ to X_0 defined by an equalizer

$$X_0 \xrightarrow{\iota} X \overset{\theta}{\underset{\eta \otimes X}{\rightrightarrows}} A \otimes X$$

In particular,

$$A \otimes X_0 \xrightarrow{A \otimes \iota} A \otimes X \xrightarrow{\lambda} X$$

is an isomorphism.

As a special case we have an equalizer

$$V \xrightarrow{\eta_1} A \otimes V \overset{\eta_1}{\underset{\eta_2}{\rightrightarrows}} A \otimes A \otimes V$$

for all $V \in \mathcal{C}$. Also note that if (X, θ) is an A-module with descent datum, and $W \in \mathcal{C}$, then $(X \otimes W, \theta \otimes W)$ is also an A-module with descent datum, and $(X \otimes W)_0 = X_0 \otimes W$. This follows since the inverse of the functor $(-)_0$ maps the right hand side to $A \otimes X_0 \otimes W = X \otimes W$.

We mention one ingredient of the proof of Theorem 2.2 for later use:

$$A \xrightarrow{\eta_2} A \otimes A \overset{\eta_2}{\underset{\eta_3}{\rightrightarrows}} A \otimes A \otimes A \tag{2.1}$$

is a split equalizer (the maps $\nabla\colon A \otimes A$ and $\nabla \otimes A\colon A \otimes A \otimes A \to A \otimes A$ satisfy the equations $\nabla \eta_2 = 1_A$, $(\nabla \otimes A)\eta_2 = 1_{A \otimes A}$ and $(\nabla \otimes A)\eta_3 = \eta_2 \nabla\colon A \otimes A \to A \otimes A$). Split equalizers are preserved under arbitrary functors. In particular

$$V \otimes A \otimes W \xrightarrow{\eta_3} V \otimes A \otimes A \otimes W \overset{\eta_3}{\underset{\eta_4}{\rightrightarrows}} V \otimes A \otimes A \otimes A \otimes W$$

is an equalizer for all $V, W \in \mathcal{C}$.

Examples from Hopf algebra theory. Let k be a commutative ring and H a k-flat k-bialgebra. Then the category $\mathcal{M}^H$ of all right H-comodules is a monoidal category; the tensor product of two right H-comodules is their tensor product over k, with the comodule structure induced by means of multiplication in H. We will denote comultiplication in H by $\Delta\colon H \to H \otimes H$ and write $\Delta(h) = h_{(1)} \otimes h_{(2)}$; a right H-comodule structure for the comodule V will be denoted $V \ni v \mapsto v_{(0)} \otimes v_{(1)} \in V \otimes H$. The category $\mathcal{M}^H$ has colimits and limits, and tensor product preserves colimits in each variable. An algebra in $\mathcal{M}^H$ is a right H-comodule algebra, that is, an algebra A endowed with a right H-comodule structure satisfying $(xy)_{(0)} \otimes (xy)_{(1)} = x_{(0)}y_{(0)} \otimes x_{(1)}y_{(1)}$ and $1_{(0)} \otimes 1_{(1)} = 1 \otimes 1 \in A \otimes H$. A right A-module in $\mathcal{M}^H$ is called a Hopf module: It is a right H-comodule as well as right A-module M satisfying $(ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$ for all $m \in M$ and $a \in A$. We write $(\mathcal{M}^H)_A = \mathcal{M}^H_A$, ${}_A(\mathcal{M}^H) = {}_A\mathcal{M}^H$ and ${}_A(\mathcal{M}^H)_A = {}_A\mathcal{M}^H_A$.

The left weak centralizer of $\mathcal{M}^H$ is the category $\mathcal{YD}^H_H$ of right-right Yetter-Drinfeld modules over H. An object of $\mathcal{YD}^H_H$ is a right H-comodule as well as right H-module V (with module structure denoted $\leftharpoonup$) satisfying

$$v_{(0)} \leftharpoonup h_{(1)} \otimes v_{(1)}h_{(2)} = (v \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(v \leftharpoonup h_{(2)})_{(1)}$$

for all $v \in V$ and $h \in H$. The equivalence

$$\mathcal{YD}^H_H \ni V \mapsto (V, \sigma_{V,-}) \in \mathcal{Z}^\ell_0(\mathcal{M}^H)$$

is given by defining

$$\sigma_{VW} \colon V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes v \leftharpoonup w_{(1)} \in W \otimes V$$

for $W \in \mathcal{M}^H$. If H^{op} is a Hopf algebra with antipode $\overline{S}$ (e. g. if H is a Hopf algebra with bijective antipode S and $\overline{S} = S^{-1}$) then $\mathcal{YD}^H_H \cong \mathcal{Z}(\mathcal{M}^H)$, with $\sigma^{-1}_{VW}(w \otimes v) = v \leftharpoonup \overline{S}(w_{(1)}) \otimes w_{(0)}$.

3. The center construction on general bimodules

The main result of this section is that under suitable conditions no interesting new examples of braided monoidal categories can ever arise from applying the center construction to the monoidal category of A-bimodules over an algebra A in a monoidal category $\mathcal{C}$. Instead, we will find that $\mathcal{Z}({}_A\mathcal{C}_A) \cong \mathcal{Z}(\mathcal{C})$ under these conditions.

To even have a monoidal category of bimodules to begin with, we need some general hypotheses:

General Hypotheses 3.1. Throughout the section, we assume that $\mathcal{C}$ is a (strict) monoidal category with coequalizers, and that $\mathcal{D} \subset \mathcal{C}$ is a monoidal subcategory closed under coequalizers. We assume that for each object $V \in \mathcal{C}$ the functors $V \otimes -, - \otimes V \colon \mathcal{C} \to \mathcal{C}$ preserve coequalizers and epimorphisms.

Under no further assumptions on an algebra A in $\mathcal{C}$, one can construct a functor $\mathcal{F} \colon \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}({}_A\mathcal{C}_A)$; we will treat, more generally, weak centralizers of subcategories:

Proposition 3.2. *Let $A \in \mathcal{D}$ be an algebra. Then a monoidal functor*

$$\mathcal{F}_0 \colon \mathcal{Z}^\ell_0(\mathcal{D}; \mathcal{C}) \to \mathcal{Z}^\ell_0({}_A\mathcal{D}_A; {}_A\mathcal{C}_A)$$

is defined as follows: $\mathcal{F}_0(V, \sigma_{V,-}) = (A \otimes V, \tilde{\sigma}_{A \otimes V,-})$, where the left A-module structure of $A \otimes V$ is the obvious one, the right A-module structure is

$$A \otimes V \otimes A \xrightarrow{A \otimes \sigma_{VA}} A \otimes A \otimes V \xrightarrow{\nabla \otimes V} A \otimes V,$$

and $\tilde{\sigma}$ is the unique left module morphism making the diagrams

$$\begin{tikzcd}
V \otimes M \arrow[r, "\sigma_{VM}"] \arrow[d, "\nu_{A\otimes V,M}(\eta\otimes V\otimes M)"'] & M \otimes V \arrow[d, "\cong"] \\
(A\otimes V)\underset{A}{\otimes} M \arrow[r, "\tilde\sigma_{A\otimes V,M}"'] & M \underset{A}{\otimes} (A\otimes V)
\end{tikzcd}$$

commute for all $M \in {}_A\mathcal{D}_A$. The monoidal functor structure of $\mathcal{F}_0$ is given by the canonical isomorphisms $A\otimes V\otimes W \cong (A\otimes V)\underset{A}{\otimes}(A\otimes W)$ for $V, W \in \mathcal{Z}_0^\ell(\mathcal{D};\mathcal{C})$.

The functor $\mathcal{F}_0$ restricts to a functor

$$\mathcal{F}\colon \mathcal{Z}(\mathcal{D};\mathcal{C}) \to \mathcal{Z}({}_A\mathcal{D}_A;{}_A\mathcal{C}_A).$$

The functors $\mathcal{F}_0\colon \mathcal{Z}_0^\ell(\mathcal{C}) \to \mathcal{Z}_0^\ell({}_A\mathcal{C}_A)$ and $\mathcal{F}\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}({}_A\mathcal{C}_A)$ are prebraided (resp. braided).

We shall skip the proof of the Proposition altogether: one has to show, among other things, that the bimodule structure is well-defined, that the morphism $\tilde\sigma$ is a well-defined bimodule morphism, and that it fulfills the axioms for an object of the (weak left) center (or centralizer).

One should view $\mathcal{F}_0(V,\sigma_{V,-})$ as a trivial object in $\mathcal{Z}_0^\ell({}_A\mathcal{D}_A;{}_A\mathcal{C}_A)$; let us discuss this a little for the case of $\mathcal{F}\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}({}_A\mathcal{C}_A)$: As a left A-module, $A\otimes V$ is obviously free over $V \in \mathcal{C}$ with respect to the "embedding" $\iota = \eta_2\colon V \to A$. But by definition of the right module structure, $\sigma_{VA}\colon V\otimes A \to A\otimes V$ is a right A-module isomorphism: this just means that

$$\begin{tikzcd}
V\otimes A\otimes A \arrow[rr, "V\otimes\nabla"] \arrow[d, "\sigma\otimes A"'] & & V\otimes A \arrow[d, "\sigma"] \\
A\otimes V\otimes A \arrow[r, "A\otimes\sigma"'] & A\otimes A\otimes V \arrow[r, "\nabla\otimes V"'] & A\otimes V
\end{tikzcd}$$

commutes, which it does by naturality of σ with respect to ∇. so $\mathcal{F}(V)$ is also free as a right A-module with respect to the same $\iota\colon V \to \mathcal{F}(V)$. As to the switching morphisms $\tilde\sigma_{A\otimes V,-}$, they are, on the "basis" V, just given by the original $\sigma_{V,-}$. Stated another way: Since $\mathcal{F}(V)$ is free as both a left and a right A-module over V, one can identify $\mathcal{F}(V)\underset{A}{\otimes} M \cong V\otimes M$ and $M\underset{A}{\otimes}\mathcal{F}(V) \cong M\otimes V$, and with this identification $\tilde\sigma_{V\otimes A,M} = \sigma_{VM}$. This (almost) means that $\tilde\sigma_{A\otimes V,M}$ is completely indifferent to the module structures of M (except these are present, of course, in the identifications).

Now our main theorems in this section say that under some additional assumptions in fact every object in the center of ${}_A\mathcal{C}_A$ is trivial in the above sense.

Theorem 3.3. *Assume that $\mathcal{C}$ has equalizers. Let A be an algebra in $\mathcal{D}$ such that the functor $A\otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves and reflects equalizers, and the functor $-\otimes A\colon \mathcal{C} \to \mathcal{C}$ reflects equalizers. Assume that for all $V \in \mathcal{D}$ and $W \in \mathcal{C}$ there* *exists an epimorphism $V_1 \to V$ in $\mathcal{D}$ such that $\eta_2\colon V_1 \otimes W \to V_1 \otimes A \otimes W$ is a monomorphism.*

Then the functor $\mathcal{F}\colon \mathcal{Z}(\mathcal{D};\mathcal{C}) \to \mathcal{Z}({}_A\mathcal{D}_A;{}_A\mathcal{C}_A)$ from Proposition 3.2 is a category equivalence.

Theorem 3.4. *Let $A \in \mathcal{D}$ be an algebra. Assume that $\mathcal{C}$ has equalizers, and that $A \otimes -\colon \mathcal{C} \to \mathcal{C}$ preserves and reflects equalizers.*

Assume that we have a functor $\mathcal{R}_1\colon \mathcal{D} \to \mathcal{D}$, for each $V \in \mathcal{D}$ an object $\mathcal{R}_2(V) \in \mathcal{D}$, and a coequalizer

$$\mathcal{R}_2(V) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathcal{R}_1(V) \xrightarrow{\pi_0} V$$

in $\mathcal{D}$, in which π_0 is a natural transformation, such that for all $V \in \mathcal{D}$, $W \in \mathcal{C}$ and $i = 1, 2$

$$\mathcal{R}_i(V) \otimes W \xrightarrow{\eta_2} \mathcal{R}_i(V) \otimes A \otimes W \underset{\eta_3}{\overset{\eta_2}{\rightrightarrows}} \mathcal{R}_i(V) \otimes A \otimes A \otimes W$$

is an equalizer.

Assume that for all $U \in \mathcal{C}$ and $V, W \in \mathcal{D}$ there are epimorphisms $\hat{V} \to V$ and $\hat{W} \to W$ in $\mathcal{D}$ such that $\eta_3\colon \hat{V} \otimes \hat{W} \otimes U \to \hat{V} \otimes \hat{W} \otimes A \otimes U$ is a monomorphism.

Then the functor $\mathcal{F}_0\colon \mathcal{Z}_0^\ell(\mathcal{D};\mathcal{C}) \to \mathcal{Z}_0^\ell({}_A\mathcal{D}_A;{}_A\mathcal{C}_A)$ from Proposition 3.2 is a category equivalence. If moreover $-\otimes A\colon \mathcal{C} \to \mathcal{C}$ reflects isomorphisms, then $\mathcal{F}\colon \mathcal{Z}(\mathcal{D};\mathcal{C}) \to \mathcal{Z}({}_A\mathcal{D}_A, {}_A\mathcal{C}_A)$ is also an equivalence.

Before we start the proof, let us discuss the hypotheses of (and differences between) Theorem 3.3 and Theorem 3.4. We will not discuss the assumption that equalizers exist.

Clearly the theorems are false when we drop all assumptions on A: as an example consider $\mathcal{C}$ the category ${}_{\mathbb{Z}}\mathcal{M}$ of abelian groups, $A = \mathbb{Q}$. Then $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \not\cong {}_{\mathbb{Q}}\mathcal{M} = \mathcal{Z}({}_{\mathbb{Q}}\mathcal{M})$.

This example is ruled out by the "faithful flatness" assumption that $A \otimes -$ preserve equalizers. It may seem harmless to also assume that $-\otimes A$ preserve equalizers. Since then both $\eta_1\colon W \to A \otimes W$ and $\eta_2\colon V \to V \otimes A$ are monic for all $V, W \in \mathcal{C}$, the assumption that $\eta_2\colon V \otimes W \to V \otimes A \otimes W$ is also monic seems not overly restrictive, either.

These remarks show that Theorem 3.3 is a reasonable result, so it may be hard to understand why one would put up with the rather unwieldy technical hypotheses of Theorem 3.4. However, Theorem 3.3 makes no statement about the *weak* center, and, maybe more importantly, we will see an application in section 5 where $A \otimes -$ is assumed to preserve and reflect equalizers, while this does not imply anything about $-\otimes A$. Assume that $A \otimes -$ preserves and reflects equalizers. By faithfully flat descent, this implies that

$$W \xrightarrow{\eta_1} A \otimes W \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} A \otimes A \otimes W \tag{3.1}$$

is an equalizer for any $W \in \mathcal{C}$. However, in the course of the proof of Theorem 3.4 we shall be needing more. In a first approximation, it would be good if (3.1) remained an equalizer after tensoring on the left with any $V \in \mathcal{C}$. Note that this is stronger than the remaining conditions in Theorem 3.4, and trivially satisfied if the tensor product in $\mathcal{C}$ preserves equalizers in its right argument. The condition is also satisfied if $\mathcal{C}$ is braided, for then tensoring with V on the left is the same as tensoring on the right. To use the same argument it is even enough to have some braided category $\mathcal{B}$ with equalizers and a monoidal functor $\mathcal{C} \to \mathcal{B}$ which is faithful and preserves equalizers. These observations cover the following corollary:

Corollary 3.5. *Let k be a commutative ring, and H a k-flat k-bialgebra. Let A be a k-faithfully flat right H-comodule algebra. Then*

$$\mathcal{Z}_0^\ell({}_A\mathcal{M}_A^H) \cong \mathcal{Z}_0^\ell(\mathcal{M}^H) \cong \mathcal{YD}_H^H \text{ and } \mathcal{Z}({}_A\mathcal{M}_A^H) \cong \mathcal{Z}(\mathcal{M}^H).$$

If H^{op} is a Hopf algebra, then

$$\mathcal{Z}_0^\ell({}_A\mathcal{M}_A^H) = \mathcal{Z}({}_A\mathcal{M}_A^H) \cong \mathcal{YD}_H^H.$$

(Note that the equivalence $\mathcal{Z}({}_A\mathcal{M}_A^H) \cong \mathcal{YD}_H^H$ is also an immediate application of Theorem 3.3.) Special cases of this result are [9, Thm. 4.3], where k is a field, H is a bialgebra with skew antipode, and A is a flat H-Galois extension of its coinvariant subalgebra, and [10, Cor. 6.6], where H is a finite Hopf algebra.

We will encounter an example in section 5 where Theorem 3.4 applies while we have no information whether the equalizer (3.1) is preserved under tensoring with every $V \in \mathcal{C}$. The idea is then that every V has a (functorial) 'resolution' by objects $\mathcal{R}_i(V)$ such that (3.1) is preserved under tensoring with them. The hypotheses of Theorem 3.4 are still potentially a little weaker than the conditions encountered in section 5 that inspired them.

Proof of Theorem 3.3 and Theorem 3.4. We have to construct an inverse functor $\mathcal{G}_0$ for $\mathcal{F}_0$ (resp. $\mathcal{G}$ for $\mathcal{F}$). Let $(X, \tilde{\sigma}_{X\,-})$ be an object in $\mathcal{Z}_0^\ell({}_A\mathcal{D}_A; {}_A\mathcal{C}_A)$.

For $V \in \mathcal{D}$ abbreviate $\overline{V} := A \otimes V \otimes A \in {}_A\mathcal{D}_A$. Consider the following morphisms:

$$\phi_V \colon X \otimes V \otimes A \cong X \underset{A}{\otimes} \overline{V} \xrightarrow{\tilde{\sigma}} \overline{V} \underset{A}{\otimes} X \cong A \otimes V \otimes X$$

$$\phi := \phi_I \colon X \otimes A \to A \otimes X$$

$$\theta_V \colon X \otimes V \xrightarrow{X \otimes V \otimes \eta} X \otimes V \otimes A \xrightarrow{\phi}_V A \otimes V \otimes X$$

$$\theta := \theta_I \colon X \to A \otimes X$$

$$t = t_{VW} \colon \overline{V \otimes W} \xrightarrow{A \otimes V \otimes \eta \otimes W \otimes A} A \otimes V \otimes A \otimes W \otimes A \cong \overline{V} \underset{A}{\otimes} \overline{W}.$$

Since t is a morphism in ${}_A\mathcal{D}_A$, and $\tilde{\sigma}$ makes X an object of the left weak centralizer, the diagram

$$\begin{tikzcd}
X \underset{A}{\otimes} \overline{V \otimes W} \arrow[r, "X \underset{A}{\otimes} t"] \arrow[dd, "\tilde{\sigma}"'] & X \underset{A}{\otimes} \overline{V} \underset{A}{\otimes} \overline{W} \arrow[d, "\tilde{\sigma} \underset{A}{\otimes} \overline{W}"] \\
 & \overline{V} \underset{A}{\otimes} X \underset{A}{\otimes} \overline{W} \arrow[d, "\overline{V} \underset{A}{\otimes} \tilde{\sigma}"] \\
\overline{V \otimes W} \underset{A}{\otimes} X \arrow[r, "t \underset{A}{\otimes} X"'] & \overline{V} \underset{A}{\otimes} \overline{W} \underset{A}{\otimes} X
\end{tikzcd}$$

commutes. Up to the canonical isomorphisms $X \underset{A}{\otimes} \overline{V} \cong X \otimes V \otimes A$, $\overline{V} \underset{A}{\otimes} X \cong A \otimes V \otimes X$ etc. we can rewrite this as

$$\begin{tikzcd}
X \otimes V \otimes W \otimes A \arrow[r, "\eta_3"] \arrow[dd, "\phi_{V \otimes W}"'] & X \otimes V \otimes A \otimes W \otimes A \arrow[d, "\phi_V \otimes W \otimes A"] \\
 & A \otimes V \otimes X \otimes W \otimes A \arrow[d, "A \otimes V \otimes \phi_W"] \\
A \otimes V \otimes W \otimes X \arrow[r, "\eta_3"'] & A \otimes V \otimes A \otimes W \otimes X
\end{tikzcd}$$

which is the same as the lower pentagon in

$$\begin{tikzcd}
X \otimes V \otimes W \arrow[r, "\eta_3"] \arrow[d, "\eta_4"'] & X \otimes V \otimes A \otimes W \arrow[r, "\phi_V \otimes W"] \arrow[d, "\eta_5"] & A \otimes V \otimes X \otimes W \arrow[d, "\eta_5"] \\
X \otimes V \otimes W \otimes A \arrow[r, "\eta_3"'] \arrow[d, "\phi_{V \otimes W}"'] & X \otimes V \otimes A \otimes W \otimes A \arrow[r, "\phi_V \otimes W \otimes A"'] & A \otimes V \otimes X \otimes W \otimes A \arrow[d, "A \otimes V \otimes \phi_W"] \\
A \otimes V \otimes W \otimes X \arrow[rr, "\eta_3"'] & & A \otimes V \otimes A \otimes W \otimes X
\end{tikzcd}$$

Since the upper two rectangles commute trivially, we have found that

$$\begin{tikzcd}
X \otimes V \otimes W \arrow[r, "\theta_V \otimes W"] \arrow[d, "\theta_{V \otimes W}"'] & A \otimes V \otimes X \otimes W \arrow[d, "A \otimes V \otimes \theta_W"] \\
A \otimes V \otimes W \otimes X \arrow[r, "\eta_3"'] & A \otimes V \otimes A \otimes W \otimes X
\end{tikzcd} \tag{3.2}$$

commutes.

Upon specializing $V = W = I$ in (3.2), we obtain the first axiom for θ being a descent datum on the left A-module X. The other axiom is easily obtained by applying naturality of $\tilde{\sigma}$ to the multiplication $\overline{I} = A \otimes A \to A$, and it follows that multiplication induces an isomorphism $A \otimes X_0 \to X$ when we define X_0 by an equalizer

$$X_0 \xrightarrow{\iota} X \underset{\eta \otimes X}{\overset{\theta}{\rightrightarrows}} A \otimes X.$$

For the rest of the proof, we shall identify $A \otimes X_0 = X$ whenever convenient.

Next, we specialize $V = I$ in (3.2). After renaming $W = V$, we find a commutative diagram

$$\begin{tikzcd}
X \otimes V \arrow[r, "\theta \otimes V"] \arrow[d, "\theta_V"'] & A \otimes X \otimes V \arrow[d, "A \otimes \theta_V"] \\
A \otimes V \otimes X \arrow[r, "\eta_2"'] & A \otimes A \otimes V \otimes X
\end{tikzcd}$$

whence $\theta_V \colon X \otimes V \to A \otimes V \otimes X$ is a morphism of descent data from the descent datum $\theta \otimes V$ on $X \otimes V$ to the canonical descent datum on $A \otimes V \otimes X$. Hence, there is a unique morphism $\theta'_V \colon X_0 \otimes V \to V \otimes X$ such that

$$\theta_V = \left(X \otimes V \cong A \otimes X_0 \otimes V \xrightarrow{A \otimes \theta'_V} A \otimes V \otimes X \right). \tag{3.3}$$

Now we specialize $W = I$ in (3.2) and find that the diagram

$$\begin{tikzcd}
X \otimes V \arrow[r, "\theta_V"] \arrow[d, "\theta_V"'] & A \otimes V \otimes X \arrow[d, "A \otimes V \otimes \theta"] \\
A \otimes V \otimes X \arrow[r, "\eta_3"'] & A \otimes V \otimes A \otimes X
\end{tikzcd}$$

hence also

$$\begin{tikzcd}
X_0 \otimes V \arrow[r, "\theta'_V"] \arrow[d, "\theta'_V"'] & V \otimes X \arrow[d, "V \otimes \theta"] \\
V \otimes X \arrow[r, "\eta_2"'] & V \otimes A \otimes X
\end{tikzcd}$$

commutes, so θ'_V factors over the equalizer of $\eta_2, V \otimes \theta \colon V \otimes X \to V \otimes A \otimes X$.

At this point, we have to argue differently in the situations of Theorem 3.3 and Theorem 3.4 to arrive at the following

Claim: There exists a unique natural (in $V \in \mathcal{D}$) transformation $\sigma_{X_0,V}\colon X_0 \otimes V \to V \otimes X_0$ with

$$\theta'_V = \left(X_0 \otimes V \xrightarrow{\sigma_{X_0 V}} V \otimes X_0 \xrightarrow{V \otimes \iota} V \otimes X\right).$$

In the situation of Theorem 3.3 this follows immediately from the fact that

$$V \otimes X_0 \xrightarrow{V \otimes \iota} V \otimes X \underset{\theta}{\overset{\eta_2}{\rightrightarrows}} V \otimes A \otimes X \tag{3.4}$$

is an equalizer. To see this, note first that ϕ is a right A-module morphism, hence

$$\phi = \left(A \otimes X_0 \otimes A \xrightarrow{\theta} A \otimes A \otimes X_0 \otimes A \xrightarrow{A \otimes \rho} A \otimes A \otimes X_0\right).$$

Using (3.3) we find that $\phi = A \otimes \phi'$ for $\phi' = \rho(A \otimes \theta')\colon A \otimes X_0 \to X_0 \otimes A$. Since $A \otimes -$ reflects equalizers, we know that ϕ' is a right A-module isomorphism. Now to see that (3.4) is an equalizer, it suffices to show that it is after tensoring with A on the right, that is, it suffices to show

$$V \otimes X_0 \otimes A \xrightarrow{\eta_2} V \otimes A \otimes X_0 \otimes A \underset{\eta_3}{\overset{\eta_2}{\rightrightarrows}} V \otimes A \otimes A \otimes X_0 \otimes A$$

is an equalizer. But for any object $T \in \mathcal{C}$ we have an isomorphism $T \otimes \phi'\colon T \otimes X_0 \otimes A \to T \otimes A \otimes X_0$, so we really have to check that

$$V \otimes A \otimes X_0 \xrightarrow{\eta_2} V \otimes A \otimes A \otimes X_0 \underset{\eta_3}{\overset{\eta_2}{\rightrightarrows}} V \otimes A \otimes A \otimes A \otimes X_0$$

is an equalizer, which it is since

$$A \xrightarrow{\eta_1} A \otimes A \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} A \otimes A \otimes A$$

is a split equalizer, hence preserved under the functor $V \otimes (-) \otimes X_0$.

Let us now turn to proving the claim in the situation of Theorem 3.4. This time we know that at least

$$\mathcal{R}_i(V) \otimes X_0 \xrightarrow{1 \otimes \iota} \mathcal{R}_i(V) \otimes X \underset{\eta_2}{\overset{1 \otimes \theta}{\rightrightarrows}} \mathcal{R}_i(V) \otimes A \otimes X$$

is an equalizer for $i = 1, 2$. It follows that we have unique morphisms

$$\sigma^{(i)}_V\colon X_0 \otimes \mathcal{R}_i(V) \to \mathcal{R}_i(V) \otimes X_0$$

with

$$\theta'_{\mathcal{R}_i(V)} = \left(X_0 \otimes \mathcal{R}_i(V) \xrightarrow{\sigma^{(i)}} \mathcal{R}_i(V) \otimes X_0 \xrightarrow{1 \otimes \iota} \mathcal{R}_i(V) \otimes X\right).$$

Let $j \in \{1,2\}$. In the diagram

$$\begin{tikzcd} X_0 \otimes \mathcal{R}_2(V) \arrow[r, "\sigma^{(2)}"] \arrow[d, "1\otimes\pi_j"'] & \mathcal{R}_2(V)\otimes X_0 \arrow[r, "\eta_2"] \arrow[d, "\pi_j\otimes 1"] & \mathcal{R}_2(V)\otimes A\otimes X_0 \arrow[d, "\pi_j\otimes 1\otimes 1"] \\ X_0\otimes\mathcal{R}_1(V) \arrow[r, "\sigma^{(1)}"'] & \mathcal{R}_1(V)\otimes X_0 \arrow[r, "\eta_2"'] & \mathcal{R}_1(V)\otimes A\otimes X_0 \end{tikzcd}$$

the outside commutes by naturality of θ', and the right square commutes trivially. Since the lower right arrow η_2 is monic, the left square commutes also. Thus we can define

$$\sigma_{X_0,V}\colon X_0\otimes V\to V\otimes X_0$$

by passing to the coequalizer of π_1,π_2. We note that

$$\theta'_V = (X_0\otimes V \xrightarrow{\sigma} V\otimes X_0 \xrightarrow{V\otimes\iota} V\otimes X) \tag{3.5}$$

since this holds for $\mathcal{R}_i(V)$ instead of V. To show that $\sigma_{X_0,V}$ is natural in $V\in\mathcal{D}$, it is enough to note that $\sigma_V^{(1)}$ is natural in V, so is $\pi_0\colon\mathcal{R}_1(V)\to V$, and π_0 is epic.

Our next claim is that (in the situation of either Theorem 3.3 or Theorem 3.4) $(X_0,\sigma_{X_0,-})\in\mathcal{Z}_0^\ell(\mathcal{D};\mathcal{C})$. We can rewrite (3.2) as

$$\begin{tikzcd} X_0\otimes V\otimes W \arrow[r, "\theta'_V\otimes W"] \arrow[d, "\theta'_{V\otimes W}"'] & V\otimes A\otimes X_0\otimes W \arrow[d, "V\otimes A\otimes\theta'_W"] \\ V\otimes W\otimes A\otimes X_0 \arrow[r, "\eta_2"'] & V\otimes A\otimes W\otimes A\otimes X_0 \end{tikzcd}$$

Now $\theta'_V = (V\otimes\eta\otimes X_0)\sigma_{X_0,V}$, and hence we have proved that

$$\sigma_{X_0,V\otimes W} = (V\otimes\sigma_{X_0,W})(\sigma_{X_0,V}\otimes W) \tag{3.6}$$

holds after composing on the left with

$$V\otimes\eta\otimes W\otimes\eta\otimes X_0\colon V\otimes W\otimes X_0\to V\otimes A\otimes W\otimes A\otimes X_0.$$

In the situation of Theorem 3.3 we can, by hypothesis, choose an epimorphism $V_1\to V$ such that

$$V_1\otimes\eta\otimes W\otimes X_0\colon V_1\otimes W\otimes X_0\to V_1\otimes A\otimes W\otimes X_0$$

is a monomorphism; in the situation of Theorem 3.4 $V_1 := \mathcal{R}_1(V)$ has this property. We conclude that $\sigma_{X_0,V_1\otimes W} = (V_1\otimes\sigma_{X_0,W})(\sigma_{X_0,V_1}\otimes W)$ holds after composing on the left with

$$V_1\otimes W\otimes\eta\otimes X_0\colon V_1\otimes W\otimes X_0\to V_1\otimes W\otimes A\otimes X_0.$$

By naturality of σ with respect to the epimorphisms $V_1 \to V$ we find that (3.6) holds after composing on the left with

$$V \otimes W \otimes \eta \otimes X_0 \colon V \otimes W \otimes X_0 \to V \otimes W \otimes A \otimes X_0.$$

In the situation of Theorem 3.3 we can argue as above that the latter morphism is in fact monic, so that (3.6) holds: It is enough to show that it is monic after tensoring on the right with A, but then tensoring on the right with $X_0 \otimes A$ is the same as tensoring with $A \otimes X_0$, and $\eta_1 \colon A \to A \otimes A$ is split monic.

In the situation of Theorem 3.4 we can at least choose epimorphisms $\hat{V} \to V$ and $\hat{W} \to W$ such that $\hat{V} \otimes \hat{W} \otimes \eta \otimes X_0$ is a monomorphism. Then (3.6) holds with V, W replaced by $\hat{V}, \hat{W}$. Again by naturality we can pass from $\hat{V}, \hat{W}$ to V, W and have proved (3.6).

This proves, in the situation of either of the theorems, the first axiom for $(X_0, \sigma_{X_0,-}) \in \mathcal{Z}_0^\ell(\mathcal{C})$. We skip the second axiom.

We have constructed a functor $\mathcal{G}_0 \colon \mathcal{Z}_0^\ell({}_A\mathcal{D}_A; {}_A\mathcal{C}_A) \to \mathcal{Z}_0^\ell(\mathcal{D}; \mathcal{C})$ in the situation of Theorem 3.4. To find that we also have functors $\mathcal{G} \colon \mathcal{Z}({}_A\mathcal{D}_A; {}_A\mathcal{C}_A) \to \mathcal{Z}(\mathcal{D}; \mathcal{C})$ under the relevant additional hypothesis in Theorem 3.4, respectively in the situation of Theorem 3.3, it remains to check that $\sigma_{X_0,V}$ is an isomorphism provided $X \in \mathcal{Z}({}_A\mathcal{D}_A; {}_A\mathcal{C}_A)$ and $- \otimes A$ reflects isomorphisms: Specializing $V = I$ in (3.9), we see that $A \otimes \sigma_{X_0,A}$, hence also $\sigma_{X_0,A}$ is an isomorphism. Using (3.9) again, we find that $A \otimes \sigma_{X_0,V \otimes A} = (A \otimes V \otimes \sigma_{X_0,A})(A \otimes \sigma_{X_0,V} \otimes A)$ and hence $A \otimes \sigma_{X_0,V} \otimes A$ is an isomorphism. It follows that $(X_0, \sigma_{X_0,-}) \in \mathcal{Z}(\mathcal{D}; \mathcal{C})$.

We will not check that $\mathcal{G}_0\mathcal{F}_0$ (resp. $\mathcal{G}\mathcal{F}$) is isomorphic to the identity. We claim that the isomorphism $A \otimes X_0 \cong X$ induced by multiplication is a component of an isomorphism $\mathcal{F}_0\mathcal{G}_0 \cong \mathcal{I}d$ (resp. $\mathcal{F}\mathcal{G} \cong \mathcal{I}d$). In detail, we have to show that the right A-module structure ρ of X is

$$\rho = (A \otimes X_0 \otimes A \xrightarrow{A \otimes \sigma} A \otimes A \otimes X_0 \xrightarrow{\nabla \otimes X_0} A \otimes X_0) \tag{3.7}$$

and that

$$\begin{tikzcd}
X_0 \otimes M \arrow[r, "\sigma_{X_0,M}"] \arrow[d, "\nu(\eta \otimes X_0 \otimes M)"'] & M \otimes X_0 \arrow[d, "\cong"] \\
(A \otimes X_0) \underset{A}{\otimes} M \arrow[r, "\tilde{\sigma}_{XM}"'] & M \underset{A}{\otimes} (A \otimes X_0)
\end{tikzcd} \tag{3.8}$$

commutes for all $M \in {}_A\mathcal{D}_A$. For the first claim, we consider the morphism $t' = \nabla \otimes A \colon \overline{A} \to \overline{I}$ to get a commutative diagram

$$
\begin{tikzcd}
X \underset{A}{\otimes} \overline{A} \arrow[r, "X \underset{A}{\otimes} t'"] \arrow[d, "\tilde{\sigma}"'] & X \underset{A}{\otimes} \overline{I} \arrow[d, "\tilde{\sigma}"] \\
\overline{A} \underset{A}{\otimes} X \arrow[r, "t' \underset{A}{\otimes} X"'] & \overline{I} \underset{A}{\otimes} X
\end{tikzcd}
$$

which is equivalent to

$$
\begin{tikzcd}
X \otimes A \otimes A \arrow[r, "\rho \otimes A"] \arrow[d, "\phi_A"'] & X \otimes A \arrow[d, "\phi"] \\
A \otimes A \otimes X \arrow[r, "\nabla \otimes X"'] & A \otimes X
\end{tikzcd}
$$

which yields, by composition with $\eta_3 \colon X \otimes A \to X \otimes A \otimes A$,

$$
\begin{tikzcd}
X \otimes A \arrow[r, "\rho"] \arrow[d, "\theta_A"'] & X \arrow[d, "\theta"] \\
A \otimes A \otimes X \arrow[r, "\nabla \otimes X"'] & A \otimes X
\end{tikzcd}
$$

This now is, by (3.5), the outside of the diagram

$$
\begin{tikzcd}
A \otimes X_0 \otimes A \arrow[r, "\rho"] \arrow[d, "A \otimes \sigma"'] & A \otimes X_0 \arrow[dd, "\eta_2"] \\
A \otimes A \otimes X_0 \arrow[ur, "\nabla \otimes X_0"'] \arrow[d, "\eta_3"'] & \\
A \otimes A \otimes A \otimes X_0 \arrow[r, "\nabla \otimes A \otimes X_0"'] & A \otimes A \otimes X_0
\end{tikzcd}
$$

whose lower part commutes trivially. We conclude that the desired equality holds after composing on the left with $\eta_2 \colon A \otimes X_0 \to A \otimes A \otimes X_0$. Since this is a monomorphism, we are done with our first claim, (3.7).

Since for all $M \in {}_A\mathcal{D}_A$ we have a canonical epimorphism $\overline{M} \to M$, it is enough to verify the second claim that (3.8) commutes for $M = \overline{V}$, $V \in \mathcal{D}$. By construction, ϕ_V is a right A-module map, and thus, by construction of θ_V, we have

$$\phi_V = \left(A \otimes X_0 \otimes V \otimes A \xrightarrow{\theta_V \otimes A} A \otimes V \otimes A \otimes X_0 \otimes A \xrightarrow{A \otimes V \otimes \rho} A \otimes V \otimes A \otimes X_0\right).$$

Thus the diagram

$$\begin{tikzcd}
A \otimes X_0 \otimes V \otimes A \arrow[r, "\theta_V \otimes A"] \arrow[dr, "A \otimes \sigma \otimes A"'] & A \otimes V \otimes A \otimes X_0 \otimes A \arrow[r, "A \otimes V \otimes \rho"] & A \otimes V \otimes A \otimes X_0 \\
 & A \otimes V \otimes X_0 \otimes A \arrow[u, "\eta_3"] \arrow[ur, "A \otimes V \otimes \sigma"'] &
\end{tikzcd}$$

in which the left triangle is (3.5), and the right triangle uses (3.7), shows that

$$\phi_V = A \otimes \sigma_{X_0, V \otimes A} \colon A \otimes X_0 \otimes V \otimes A \to A \otimes V \otimes A \otimes X_0. \tag{3.9}$$

Consider the following diagram:

$$\begin{tikzcd}
X_0 \otimes A \otimes V \otimes A \arrow[r, "\sigma_{X_0, A \otimes V \otimes A}"] \arrow[d, "\nu\eta_1"'] & A \otimes V \otimes A \otimes X_0 \arrow[d, "\cong"] \\
X \underset{A}{\otimes} \overline{V} \arrow[r, "\tilde{\sigma}_{X, \overline{V}}"'] \arrow[d, "\cong"'] & \overline{V} \underset{A}{\otimes} X \arrow[d, "\cong"] \\
A \otimes X_0 \otimes V \otimes A \arrow[r, "\phi_V"'] & A \otimes V \otimes A \otimes X_0
\end{tikzcd}$$

We want to show that the top square commutes. The bottom square commutes by definition of ϕ_V. The composition of the right vertical isomorphisms is the identity. The composition of the left vertical morphisms equals $\sigma_{X_0, A} \otimes V \otimes A$, by (3.7). Hence, the claim follows from (3.9). □

4. The center construction on commutative bimodules

In the preceding section we have seen that under favorable conditions the center of the full bimodule category ${}_A\mathcal{C}_A$ is "trivial" in the sense that it is isomorphic to the center of $\mathcal{C}$ and completely indifferent to the structure of A. In this section we will consider *commutative* algebras (in a braided sense) and compute the center of the category of commutative bimodules over a commutative algebra.

Recall that an algebra A in a prebraided monoidal category $(\mathcal{B}, \sigma)$ is said to be commutative (or braided commutative, or σ-commutative if there is need to emphasize this) if its multiplication satisfies $\nabla = \nabla\sigma$.

Over a commutative algebra A in a prebraided monoidal category there are two obvious notions of "commutative" bimodules. These are the bimodules in the images of the functors Σ and Γ below; one of the two can be formulated if A is a commutative algebra in the weak left center of a monoidal category.

Lemma 4.1. *Let $\mathcal{C}$ be a strict monoidal category and $(\mathcal{B},\sigma)$ a prebraided monoidal category. Assume $\mathcal{C}$ and $\mathcal{B}$ have coequalizers and tensor products preserve coequalizers in each argument.*

(1) *Let A be a commutative algebra in $\mathcal{Z}_0^\ell(\mathcal{C})$. Then every $M \in \mathcal{C}_A$ is a left A-module via*
$$\lambda = \left(A \otimes M \xrightarrow{\sigma_{AM}} M \otimes A \xrightarrow{\rho} M\right).$$
This defines a section $\Sigma\colon \mathcal{C}_A \to {}_A\mathcal{C}_A$ for the underlying functor. The image of Σ is a monoidal subcategory of ${}_A\mathcal{C}_A$ canonically equivalent to $\mathcal{C}_A$. We shall consider $\mathcal{C}_A$ as a subcategory of ${}_A\mathcal{C}_A$ in this way.

(2) *As a special case one has a functor $\Sigma\colon \mathcal{B}_A \to {}_A\mathcal{B}_A$ if A is a commutative algebra in $\mathcal{B}$.*

(3) *Similarly, for a commutative algebra A in $\mathcal{B}$ one has a section $\Sigma\colon {}_A\mathcal{B} \to {}_A\mathcal{B}_A$ of the underlying functor. We shall consider ${}_A\mathcal{B}$ as a subcategory of ${}_A\mathcal{B}_A$ in this way.*

(4) *Assume $\mathcal{B}$ is braided and $A \in \mathcal{B}$ is a commutative algebra. Then as another special case of (1) we can consider the functor $\Sigma^-\colon \mathcal{B}_A \to {}_A\mathcal{B}_A$ given by endowing $M \in \mathcal{B}_A$ with the left module structure $A\otimes M \xrightarrow{\sigma^{-1}} M\otimes A \xrightarrow{\rho} M$. Note that the images of Σ^- and Γ coincide.*

The following definition is due to Pareigis [8].

Definition 4.2. Let $(\mathcal{B},\sigma)$ be a prebraided monoidal category and A a commutative algebra in $\mathcal{B}$. A right A-module M is said to be dyslectic if
$$\begin{tikzcd}
M \otimes A \arrow[r, "\sigma"] \arrow[dr, "\rho"'] & A \otimes M \arrow[r, "\sigma"] & M \otimes A \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}$$
commutes. We denote by $\operatorname{dys}\mathcal{B}_A \subset \mathcal{B}_A$ the category of dyslectic right A-modules.

Remark 4.3. If $\mathcal{B}$ is braided, then $\operatorname{dys}\mathcal{B}_A$ is the equalizer of Σ and Σ^- and can be identified with the intersection of the images of Σ and Σ^-. If $\mathcal{B}$ is only prebraided, then $\operatorname{dys}\mathcal{B}_A$ can be identified with the intersection of the images of Σ and Γ, and equals the equalizer of Σ and $\Gamma\mathcal{U}_\ell\Sigma$, where $\mathcal{U}_\ell\colon {}_A\mathcal{B}_A \to {}_A\mathcal{B}$ denotes the underlying functor.

Assume $\mathcal{B}$ is braided. Pareigis [8] has shown that $\operatorname{dys}\mathcal{B}_A$ is a monoidal subcategory of $\mathcal{B}_A$, and that $\operatorname{dys}\mathcal{B}_A$ is braided. The braiding of $\tilde{\sigma}$ of $\operatorname{dys}\mathcal{B}_A$ is obtained by factoring the braiding in $\mathcal{B}$, that is, the diagrams

$$\begin{tikzcd} M \otimes N \arrow[r, "\sigma"] \arrow[d, "\nu"'] & N \otimes M \arrow[d, "\nu"] \\ M \underset{A}{\otimes} N \arrow[r, "\tilde{\sigma}"'] & N \underset{A}{\otimes} M \end{tikzcd}$$

commute for $M, N \in \operatorname{dys} \mathcal{B}_A$.

Theorem 4.4. *Let $\mathcal{C}$ be a strictly monoidal category with coequalizers; assume the tensor product preserves coequalizers in each argument. Let $(A, \sigma_{A,-})$ be a commutative algebra in $\mathcal{Z}_0^\ell(\mathcal{C})$.*

An equivalence $\mathcal{F}_0\colon {}_A\mathcal{Z}_0^\ell(\mathcal{C}) \to \mathcal{Z}_0^\ell(\mathcal{C}_A; {}_A\mathcal{C}_A)$ of monoidal categories is defined by $\mathcal{F}_0(X, \sigma_{X,-}) = (X, \tilde{\sigma}_{X,-})$ where $\tilde{\sigma}_{X,-}$ is obtained by factoring $\sigma_{X,-}$ to the tensor product over A, that is, where

$$\begin{tikzcd} X \otimes M \arrow[r, "\sigma_{XM}"] \arrow[d, "\nu"'] & M \otimes X \arrow[d, "\nu"] \\ X \underset{A}{\otimes} M \arrow[r, "\tilde{\sigma}_{XM}"'] & M \underset{A}{\otimes} X \end{tikzcd}$$

commutes for all $M \in \mathcal{C}_A$; the bimodule structure on X in $\mathcal{F}_0(X, \sigma_{X,-})$ is the underlying A-bimodule structure of the $(A, \sigma_{A,-})$-bimodule structure of $(X, \sigma_{X,-})$.

If $(A, \sigma_{A,-}) \in \mathcal{Z}(\mathcal{C})$, then $\mathcal{F}_0$ restricts to an equivalence $\mathcal{F}\colon {}_A\mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C}_A; {}_A\mathcal{C}_A)$.

Proof. We skip showing that $\mathcal{F}_0$ and $\mathcal{F}$ are well-defined: This can be done by essentially the same reasoning as in the proof that $\operatorname{dys} \mathcal{B}_A$ is braided for commutative A in a braided category $\mathcal{B}$ in [8].

For any $V \in \mathcal{C}$ consider $\overline{V} = V \otimes A \in \mathcal{C}_A$ and $\iota = \eta_2\colon V \to \overline{V}$. Note that

$$\lambda_{\overline{V}} = \rho_{\overline{V}} \sigma_{A,\overline{V}} = (V \otimes \nabla)(V \otimes \sigma_{AA})(\sigma_{AV} \otimes A) = (V \otimes \nabla)(\sigma_{AV} \otimes A)$$

since A is commutative. In particular $\sigma_{AV}\colon A \otimes V \to \overline{V}$ is a left A-module morphism (compare the remarks following Proposition 3.2).

Let $(X, \tilde{\sigma}_{X,-}) \in \mathcal{Z}_0^\ell(\mathcal{C}_A; {}_A\mathcal{C}_A)$. For $V \in \mathcal{C}$ define $\sigma_{XV}\colon X \otimes V \to V \otimes X$ by

$$\sigma_{XV} = \left(X \otimes V \xrightarrow{\nu\eta_3} X \underset{A}{\otimes} \overline{V} \xrightarrow{\tilde{\sigma}} \overline{V} \underset{A}{\otimes} X \cong V \otimes X \right).$$

It is not hard to check that $(X, \sigma_{X,-}) \in \mathcal{Z}_0^\ell(\mathcal{C})$. If $(A, \sigma_{A,-}) \in \mathcal{Z}(\mathcal{C})$ and $(X, \tilde{\sigma}_{X,-}) \in \mathcal{Z}(\mathcal{C}_A; {}_A\mathcal{C}_A)$, then $(X, \sigma_{X,-}) \in \mathcal{Z}(\mathcal{C})$ since $\nu\eta_3\colon X \otimes V \to X \underset{A}{\otimes} \overline{V}$ is an isomorphism.

To complete the construction of a functor $\mathcal{G}_0\colon \mathcal{Z}_0^\ell(\mathcal{C}_A; {}_A\mathcal{C}_A) \to {}_A\mathcal{Z}_0^\ell(\mathcal{C})$, we still have to show that X is a left A-module in $\mathcal{Z}_0^\ell(\mathcal{C})$, that is, that $\lambda\colon A \otimes X \to X$

is a morphism in the left weak centralizer, and that X as a bimodule is in the image of $\Gamma\colon {}_A\mathcal{Z}_0^\ell(\mathcal{C}) \to {}_A\mathcal{Z}_0^\ell(\mathcal{C})_A$, that is, that

$$\rho = \lambda\sigma_{XA}\colon X \otimes A \to X.$$

We prove the second claim first by considering the diagram

$$\begin{tikzcd}
X \otimes A \arrow[r, "\nu\eta_3"] \arrow[d, "\sigma"'] & X \underset{A}{\otimes} \overline{A} \arrow[r, "X \underset{A}{\otimes} \nabla"] \arrow[d, "\tilde\sigma"] & X \underset{A}{\otimes} A \arrow[d, "\tilde\sigma"] \\
A \otimes X \arrow[r, "\cong"'] & \overline{A} \underset{A}{\otimes} X \arrow[r, "\nabla \underset{A}{\otimes} X"'] & A \underset{A}{\otimes} X
\end{tikzcd}$$

Up to the canonical identifications $X \underset{A}{\otimes} A \cong X \cong A \underset{A}{\otimes} X$ the composition of the top arrows is the right module structure ρ, the composition of the bottom arrows is the left module structure λ of X, and the right vertical arrow is the identity. The left square commutes by definition of $\sigma_{X,-}$ and the right square commutes by naturality of $\tilde\sigma$ with respect to $\nabla\colon \overline{A} \to A$.

To check that $\lambda\colon A \otimes X \to X$ is a morphism in $\mathcal{Z}_0^\ell(\mathcal{C})$ we consider $V \in \mathcal{C}$ and have to show commutativity of the outside of the following diagram:

$$\begin{tikzcd}
A \otimes X \otimes V \arrow[rr, "A \otimes \sigma"] \arrow[dr, "\nu\eta_4"] \arrow[ddd, "\lambda \otimes V"'] & & A \otimes V \otimes X \arrow[r, "\sigma \otimes X"] \arrow[d, "\cong"] & V \otimes A \otimes X \arrow[ddl, "\nu"'] \arrow[ddd, "V \otimes \lambda"] \\
& A \otimes X \underset{A}{\otimes} \overline{V} \arrow[r, "A \otimes \tilde\sigma"] \arrow[d, "\lambda \underset{A}{\otimes} \overline{V}"'] & A \otimes \overline{V} \underset{A}{\otimes} X \arrow[d, "\lambda \underset{A}{\otimes} X"'] & \\
& X \underset{A}{\otimes} \overline{V} \arrow[r, "\tilde\sigma"'] & \overline{V} \underset{A}{\otimes} X \arrow[dr, "\cong"] & \\
X \otimes V \arrow[ur, "\nu\eta_3"] \arrow[rrr, "\sigma"'] & & & V \otimes X
\end{tikzcd}$$

The quadrangle adjacent to the left of the top arrows commutes by definition of σ_{XV}. To see that the triangle-shaped quadrangle to the right of it commutes, note that the composition of the two vertical arrows can be rewritten as

$$A \otimes V \otimes X \xrightarrow{\eta_3} A \otimes \overline{V} \otimes X \xrightarrow{\lambda \otimes X} \overline{V} \otimes X \xrightarrow{\nu} \overline{V} \underset{A}{\otimes} X.$$

and by definition of the left module structure of $\overline{V}$ we have

$$\sigma_{AV} = (A \otimes V \xrightarrow{\eta_3} A \otimes \overline{V} \xrightarrow{\lambda} \overline{V}).$$

The triangle on the right commutes trivially. The quadrangle on the bottom is again the definition of σ_{XV}. The quadrangle on the left commutes trivially, and finally the central rectangle is left A-linearity of $\tilde{\sigma}$.

We are done constructing a potential quasiinverse $\mathcal{G}_0$ for $\mathcal{F}_0$ (and $\mathcal{G}$ for $\mathcal{F}$). It remains to check that $\mathcal{G}_0\mathcal{F}_0$ and $\mathcal{F}_0\mathcal{G}_0$ are identity functors; since $\mathcal{G}_0$ as well as $\mathcal{F}_0$ leave the underlying bimodule structure of all objects unchanged, we only have to pay attention to the switching morphisms.

Let $(X, \sigma_{X,-}) \in {}_A\mathcal{Z}_0^\ell(\mathcal{C})$ and write $\mathcal{G}_0\mathcal{F}_0(X, \sigma_{X,-}) =: (X, \sigma'_{X,-})$. We have to show $\sigma = \sigma'$. For $V \in \mathcal{C}$ consider the diagram

$$\begin{tikzcd}
X \otimes V \arrow[rr, "f"] \arrow[d, "\eta_3"] & & V \otimes X \arrow[dl, "\eta_3"'] \arrow[d, "\eta_2"] \\
X \otimes V \otimes A \arrow[r, "\sigma \otimes A"'] \arrow[d, "\nu"] & V \otimes X \otimes A \arrow[r, "V \otimes \sigma"'] & V \otimes A \otimes X \arrow[d, "\nu"] \\
X \otimes_A \overline{V} \arrow[rr, "\tilde{\sigma}_{X\overline{V}}"'] & & \overline{V} \otimes_A X
\end{tikzcd}$$

in which the lower pentagon commutes by definition of $\tilde{\sigma}$, and the triangle commutes trivially. The upper quadrangle commutes trivially if $f = \sigma$, and the outside of the diagram commutes by definition of $\mathcal{G}_0$ if and only if $f = \sigma'$.

Now let $(X, \tilde{\sigma}_{X,-}) \in \mathcal{Z}_0^\ell(\mathcal{C}_A; {}_A\mathcal{C}_A)$ and write $\mathcal{F}_0\mathcal{G}_0(X, \tilde{\sigma}_{X,-}) =: (X, \tilde{\tilde{\sigma}}_{X,-})$. We have to show $\tilde{\tilde{\sigma}} = \tilde{\sigma}$. For $M \in \mathcal{C}_A$ let $\pi\colon \overline{M} \to M$ denote the canonical epimorphism. Consider the diagram

$$\begin{tikzcd}
X \otimes M \arrow[r, "\sigma_{XM}"] \arrow[d, "\eta_3"] & M \otimes X \arrow[d, "\eta_2"] \\
X \otimes \overline{M} \arrow[d, "\nu"] & \overline{M} \otimes X \arrow[d, "\nu"] \\
X \otimes_A \overline{M} \arrow[r, "\tilde{\sigma}_{X\overline{M}}"] \arrow[d, "X \otimes_A \pi"'] & \overline{M} \otimes_A X \arrow[d, "\pi \otimes_A X"] \\
X \otimes_A M \arrow[r, "f"'] & M \otimes_A X
\end{tikzcd}$$

The top hexagon commutes by definition of σ. The lower rectangle commutes for $f = \tilde{\sigma}_{X\overline{M}}$ by naturality of $\tilde{\sigma}$ with respect to π. The composition of the vertical arrows is checked to be $\nu\colon X \otimes M \to X \underset{A}{\otimes} M$ and $\nu\colon M \otimes X \to M \underset{A}{\otimes} X$, respectively (since ν is natural with respect to π, and $\pi\eta_2 = 1_M$). Hence commutativity of the outside of the diagram is the definition of $f = \tilde{\tilde{\sigma}}_{X,M}$. $\square$

Corollary 4.5. *Let $\mathcal{C}$ be a strictly monoidal category with coequalizers; assume the tensor product preserves coequalizers in each argument. Let $(A, \sigma_{A,-})$ be a commutative algebra in $\mathcal{Z}_0^\ell(\mathcal{C})$.*

Then the functor $\mathcal{F}_0$ from Theorem 4.4 induces an equivalence

$$\mathcal{F}_0\colon \operatorname{dys} \mathcal{Z}_0^\ell(\mathcal{C})_A \cong \mathcal{Z}_0^\ell(\mathcal{C}_A)$$

of prebraided monoidal categories. If $(A, \sigma_{A,-}) \in \mathcal{Z}(\mathcal{C})$, then $\mathcal{F}_0$ restricts to an equivalence

$$\mathcal{F}\colon \operatorname{dys} \mathcal{Z}(\mathcal{C})_A \cong \mathcal{Z}(\mathcal{C}_A)$$

of braided monoidal categories.

5. Algebras over braided commutative algebras

Let $\mathcal{B}$ be a braided monoidal category, and R a commutative algebra in $\mathcal{B}$. There is a natural braided analog of the notion of an R-algebra in $\mathcal{B}$: This should be an algebra in $\mathcal{B}$ with an algebra morphism $\iota\colon R \to A$ such that

(5.1)
$$\begin{tikzcd} R \otimes A \arrow[rr, "\sigma_{RA}"] \arrow[dr, "\nabla(\iota \otimes A)"'] & & A \otimes R \arrow[dl, "\nabla(A \otimes \iota)"] \\ & A & \end{tikzcd}$$

commutes. More generally, when $\mathcal{C}$ is a monoidal category and $(R, \sigma_{R,-})$ a commutative algebra in $\mathcal{Z}_0^\ell(\mathcal{C})$, we can define an R-algebra to be an algebra A in $\mathcal{C}$ such that the diagram (5.1) commutes.

Just as in the case of ordinary R-algebras, there is a natural restriction on A-bimodules: We say that R acts braided centrally on an A-bimodule M if the diagram

$$\begin{tikzcd} R \otimes M \arrow[rr, "\sigma"] \arrow[dr, "\lambda(\iota \otimes M)"'] & & M \otimes R \arrow[dl, "\rho(M \otimes \iota)"] \\ & M & \end{tikzcd}$$

commutes.

In this section we will put together the results from sections 3 and 4 to study the center of the category of those A-bimodules for an R-bialgebra on which R acts centrally.

Note that an $(R, \sigma_{R,-})$-algebra A is the same as an algebra in the monoidal category $\mathcal{C}_R$, and the category of those A-bimodules on which R acts centrally is naturally identified with ${}_A(\mathcal{C}_R)_A$.

Theorem 5.1. *Let $\mathcal{C}$ be a monoidal category with coequalizers and equalizers; assume that the tensor product preserves coequalizers and epimorphisms in each argument.*

Let $(R, \sigma_{R,-})$ be a commutative algebra in $\mathcal{Z}_0^\ell(\mathcal{C})$ and A an R-algebra in $\mathcal{C}$. Assume that $A \underset{R}{\otimes} -\colon {}_R\mathcal{C} \to \mathcal{C}$ preserves and reflects equalizers.

(1) *If $-\underset{R}{\otimes} A\colon \mathcal{C}_R \to \mathcal{C}$ reflects equalizers, and $\iota_2\colon V \otimes W \to V \otimes A \underset{R}{\otimes} W$ is monic for all $W \in {}_R\mathcal{C}_R$ and $V \in \mathcal{C}$, and $R \in \mathcal{Z}(\mathcal{C})$, then we have category equivalences*

$$\mathcal{Z}({}_A(\mathcal{C}_R)\,;{}_A\mathcal{C}_A) \cong {}_R\mathcal{Z}(\mathcal{C}) \quad \text{and} \quad \mathcal{Z}({}_A(\mathcal{C}_R)_A) \cong \operatorname{dys}\mathcal{Z}(\mathcal{C})_R.$$

(2) *If*

$$V \otimes W \xrightarrow{\iota_2} V \otimes A \underset{R}{\otimes} W \overset{\iota_2}{\underset{\iota_3}{\rightrightarrows}} V \otimes A \underset{R}{\otimes} A \underset{R}{\otimes} W$$

is an equalizer for all $V \in \mathcal{C}$ and $W \in {}_R\mathcal{C}_R$, then we have category equivalences

$$\mathcal{Z}_0^\ell({}_A(\mathcal{C}_R)\,;{}_A\mathcal{C}_A) \cong {}_R\mathcal{Z}_0^\ell(\mathcal{C}) \quad \text{and} \quad \mathcal{Z}_0^\ell({}_A(\mathcal{C}_R)_A) \cong \operatorname{dys}\mathcal{Z}_0^\ell(\mathcal{C})_R.$$

Proof. In each part we only show the first of the equivalences. As to (1), we have a chain of equivalences

$$\mathcal{Z}({}_A(\mathcal{C}_R)_A) \cong \mathcal{Z}(\mathcal{C}_R) \cong \operatorname{dys}\mathcal{Z}(\mathcal{C})_R$$

the right hand one of which is a direct application of Theorem 4.4. To deduce the left hand one from Theorem 3.3, we consider $V \in \mathcal{C}_R$ and put $V_1 = V \otimes R$. Then since the left R-module monomorphism $\iota \underset{R}{\otimes} W\colon A \underset{R}{\otimes} W$ is, by assumption, preserved under $V \otimes -$, it is preserved under $V_1 \underset{R}{\otimes} -$.

To prove (2), we use a similar chain of equivalences

$$\mathcal{Z}_0^\ell({}_A(\mathcal{C}_R)_A) \cong \mathcal{Z}_0^\ell(\mathcal{C}_R) \cong \operatorname{dys}\mathcal{Z}_0^\ell(\mathcal{C})_R,$$

the second one of which is again a direct application of Theorem 4.4. For the first one, we construct $\mathcal{R}_1, \mathcal{R}_2\colon \mathcal{C}_R \to \mathcal{C}_R$ by $\mathcal{R}_1(V) = V \otimes R$ and $\mathcal{R}_2 = \mathcal{R}_1^2$. We let $\pi_0\colon \mathcal{R}_1(V) \to V$ be the canonical epimorphism, $\pi_1 = \pi_0\colon \mathcal{R}_1\mathcal{R}_1(V) \to \mathcal{R}_1(V)$, and $\pi_2 = \mathcal{R}_1(\pi_0)\colon \mathcal{R}_1\mathcal{R}_1(V) \to \mathcal{R}_1(V)$. Then we have a coequalizer

$$\mathcal{R}_2(V) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} \mathcal{R}_1(V) \xrightarrow{\pi_0} V$$

and for every $W \in {}_R\mathcal{C}_R$ and $i = 1, 2$

$$\mathcal{R}_i(V) \underset{R}{\otimes} W \xrightarrow{\iota_2} \mathcal{R}_i(V) \underset{R}{\otimes} A \underset{R}{\otimes} W \overset{\iota_2}{\underset{\iota_3}{\rightrightarrows}} \mathcal{R}_i(V) \underset{R}{\otimes} A \underset{R}{\otimes} A \underset{R}{\otimes} W$$

is an equalizer by our hypotheses (since $\mathcal{R}_1(V) \underset{R}{\otimes} T \cong V \otimes T$ for all $V \in {}_R\mathcal{C}$). In the notations of Theorem 4.4 we can choose $\hat{V} = \mathcal{R}_1(V)$ and $\hat{W} = \mathcal{R}_1(V)$ with the canonical epimorphisms to V, resp. W to satisfy the last of the hypotheses, since $\mathcal{R}_1(V) \underset{R}{\otimes} \mathcal{R}_1(W) \cong \mathcal{R}_1(V \otimes W)$. □

Reasoning similarly as for Corollary 3.5, we have the following application of (2):

Corollary 5.2. *Let k be a commutative ring, H a k-flat k-bialgebra and A a right H-comodule algebra. Let $R \subset A$ be a subcomodule algebra endowed with a right H-module structure (denoted $\leftharpoondown$) such that $R \in \mathcal{YD}^H_H$, and $ra = a_{(0)}(r \leftharpoondown a_{(1)})$ holds for all $r \in R$ and $a \in A$. Denote by $\mathfrak{C}$ the full subcategory of ${}_A\mathcal{M}^H_A$ of those Hopf bimodules M for which $rm = m_{(0)}(r \leftharpoondown m_{(1)})$ holds for all $m \in M$ and $r \in R$. If A is faithfully flat as a right R-module, then we have category equivalences*

$$\mathcal{Z}^\ell_0(\mathfrak{C}; {}_A\mathcal{M}^H_A) \cong {}_R\left(\mathcal{YD}^H_H\right) \text{ and } \mathcal{Z}^\ell_0(\mathfrak{C}) \cong \operatorname{dys}\left(\mathcal{YD}^H_H\right)_R.$$

If H^{op} is a Hopf algebra, then also

$$\mathcal{Z}(\mathfrak{C}; {}_A\mathcal{M}^H_A) \cong {}_R\left(\mathcal{YD}^H_H\right) \text{ and } \mathcal{Z}(\mathfrak{C}) \cong \operatorname{dys}\left(\mathcal{YD}^H_H\right)_R.$$

Example 5.3. Let H be a k-flat k-Hopf algebra and A a right H-Galois extension of $B = A^{\operatorname{co} H}$. Denote by A^B the centralizer of B in A. The Miyashita-Ulbrich action of H on A^B is the unique right H-module structure such that $ra = a_{(0)}(r \leftharpoondown a_{(1)})$ holds for all $r \in A^B$ and $a \in A$. In this situation the preceding corollary covers Theorem 5.3 and the part when A is a faithfully flat right A^B-module of Corollary 5.4 of [9].

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