MORITA BASE CHANGE IN QUANTUM GROUPOIDS

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Abstract. Let L be a quantum semigroupoid, more precisely a $\times_R$-bialgebra in the sense of Takeuchi. We describe a procedure replacing the algebra R by any Morita equivalent, or in fact more generally any $\sqrt{\text{Morita}}$ equivalent (in the sense of Takeuchi) algebra S to obtain a $\times_S$-bialgebra $\widetilde{H}$ with the same monoidal representation category.

1. Introduction

Quantum groupoids (or Hopf algebroids) are algebraic structures designed to be the analogs of (the function algebras of) groupoids in the realm of noncommutative geometry. A groupoid consists of a set G of arrows, and a set V of vertices. Thus a quantum groupoid consists of an algebra L (the function algebra on the noncommutative space of arrows) and an algebra R (the function algebra on the noncommutative space of vertices. The assignment to an arrow of its source and target vertices defines two maps $G \rightrightarrows V$. Thus the definition of a quantum groupoid involves two maps $R \rightrightarrows L$; it turns out to be the right choice to assume one of these to be an algebra, the other an anti-algebra map, and to assume that the images of the two commute. Since multiplication in G is an only partially defined map, comultiplication in L maps from L to some tensor product $L \otimes_R L$; one has to make the right choice of module structures to define the tensor product, and one needs to assume that comultiplication actually maps to a certain subspace of $L \otimes_R L$ to be able to state that comultiplication is assumed to be an algebra map.

The first version of a quantum (semi)groupoid or bialgebroid or Hopf algebroid was considered by Takeuchi [25], following work of Sweedler [24] in which R is by assumption commutative. Actually Takeuchi invents his $\times_R$-bialgebras from different motivations, involving generalizations of Brauer groups, and does not seem to be thinking of groupoids at all. Lu [14] and Xu [28] reinvent his notion, now with the motivation by noncommutative-geometric groupoids in mind. (Actually most of Lu's or Xu's definition is the very same as Takeuchi's up to changes in notation, at least as far as comultiplication is concerned. For a detailed translation, and the removal of any doubt about the notion of counit, consult Brzeziński and Militaru [3].)

Of the other possible definitions of a quantum groupoid we should mention the weak Hopf algebras of Böhm and Szlachányi [2], see also the recent survey [17] by Nikshych and Vainerman and the literature cited there, and the notion of a face algebra due to Hayashi [8, 10]. Face algebras were shown to be precisely the $\times_R$-bialgebras in which R is commutative and separable in [20]. Also, face algebras are precisely the weak bialgebras whose target counital subalgebras are commutative. Etingof and Nikshych [5] have shown that weak Hopf algebras are $\times_R$-bialgebras.

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In fact weak bialgebras are precisely those $\times_R$-bialgebras in which R is Frobenius-separable (for example semisimple over the complex numbers) [23].

In the present paper we will discuss a construction that allows us to replace the algebra R in any $\times_R$-bialgebra L by a Morita-equivalent algebra S to obtain a $\times_S$-bialgebra that has the same representation theory, more precisely a monoidal category of representations equivalent to that of L. In fact we can, more generally, replace R by any $\sqrt{\text{Morita}}$ equivalent algebra S. The notion of $\sqrt{\text{Morita}}$ equivalence is due to Takeuchi [26]. Two algebras R, S are by definition $\sqrt{\text{Morita}}$ equivalent if we have an equivalence of k-linear monoidal categories ${}_R\mathcal{M}_R \cong {}_S\mathcal{M}_S$. The definition is already at the heart of our application: A $\times_R$-bialgebra can be characterized as having a monoidal category of representations with tensor product based on the tensor product in ${}_R\mathcal{M}_R$. However, for some purposes it does seem that Morita base change (replacing R by a Morita equivalent algebra S) is more well-behaved than the more general $\sqrt{\text{Morita}}$ base change (replacing R by a $\sqrt{\text{Morita}}$ equivalent algebra S): We will show that Morita base change respects duality.

Morita (or $\sqrt{\text{Morita}}$) base change can serve two immediate purposes: One is to produce new examples of quantum groupoids. The other, and perhaps more useful one, is to on the contrary reduce the supply of essentially different examples — we can consider two $\times_R$-bialgebras to be not very essentially different if they are obtained from each other by $\sqrt{\text{Morita}}$ base change. Note that the equivalence relation thus imposed on $\times_R$-bialgebras is weaker than the natural relation that would consider two $\times_R$-bialgebras to be equivalent if their monoidal categories of representations are equivalent. In fact this latter equivalence relation is known to be important and nontrivial also in the realm of ordinary bialgebras, where Morita base change is meaningless. Thus Morita base change presents a possibility of relating different $\times_R$-bialgebras very closely, in a way that cannot occur between ordinary bialgebras.

Let us state very briefly two ways in which $\sqrt{\text{Morita}}$ base change reduces the supply of examples: If R is an Azumaya k-algebra, then any $\times_R$-bialgebra is, up to $\sqrt{\text{Morita}}$ base change, an ordinary bialgebra. Over the field of complex numbers, every weak bialgebra is, up to Morita base change, a face algebra. Of course, in neither case our results show that certain $\times_R$-bialgebras are entirely superfluous, since examples may occur in natural situations that come with a specific choice of R.

The plan of the paper is as follows: After recalling some definitions in Section 2 and Section 3 we present the general $\sqrt{\text{Morita}}$ base change procedure in Section 4. More detailed information on Morita base change will be given in Section 5. In Section 6 we discuss the canonical Tannaka duality of Hayashi [11, 10]; this construction assigns a face algebra F to any finite split semisimple k-linear monoidal category. For example, it assigns such a face algebra to the category of representations of a split semisimple (quasi)Hopf algebra H. At first sight, there is no apparent relation between the original H and Hayashi's F (beyond, of course, the fact that their monoidal representation categories are equivalent). We show that F can be obtained from H in two steps: First, one applies a kind of smash product construction that builds from H a $\times_H$-bialgebra isomorphic to $H \otimes H \otimes H^*$ as a vector space. Next, applying Morita base change to replace the base H by the Morita equivalent product of copies of the field, one obtains a face algebra — which turns out to be Hayashi's face algebra F. In Section 7 we compute the dimension of the face algebra obtained by Morita base change from a certain weak Hopf algebra constructed by Nikshych and Vainerman from a subfactor of a type II_1 factor. It turns out that Morita base change reduces the dimension from 122 to 24 without affecting the monoidal category of representations.

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2. Hopf algebroids

In this section we will very briefly recall the necessary definitions and notations on $\times_R$-bialgebras. For more details we refer to [24, 25, 19].

Throughout the paper, k denotes a commutative base ring, and all modules, algebras, unadorned tensor products etc. are understood to be over k

Let R be a k-algebra. We denote the opposite algebra by $\overline{R}$, we let $R \ni r \mapsto \overline{r} \in \overline{R}$ denote the obvious k-algebra antiisomorphism, and abbreviate the enveloping algebra $R^e := R \otimes \overline{R}$. We write $r\overline{s} := r \otimes \overline{s} \in R \otimes \overline{R}$ for $r, s \in R$.

For our purposes, a handy characterization of $\times_R$-bialgebras is the following [19, Thm.5.1]: A $\times_R$-bialgebra L is an R^e-ring (that is, a k-algebra equipped with a k-algebra map $R^e \to L$, which we write $r \otimes \overline{s} \mapsto r\overline{s}$) for which the category ${}_L\mathcal{M}$ is equipped with a monoidal category structure such that the "underlying" functor ${}_L\mathcal{M} \to {}_{R^e}\mathcal{M}$ is a strict monoidal functor. Here, the monoidal category structure on ${}_{R^e}\mathcal{M}$ is induced via the identification with the category ${}_R\mathcal{M}_R$ of bimodules; we denote tensor product in ${}_{R^e}\mathcal{M}$ by $\diamond_R$, or $\diamond$ if no confusion is likely.

Thus, for two L-modules M, N, there is an L-module structure on $M \diamond_R N$, and this tensor product of L-modules defines a monoidal category structure on ${}_L\mathcal{M}$. The connection with the original definition in [25] is that the module structure on $M \diamond N$ can be described in terms of a certain comultiplication on L, which, however, has a more intricate definition than in the ordinary bialgebra case. First of all, the comultiplication is an algebra map $L \to L \times_R L$ into a certain subset $L \times_R L \subset L \diamond_R L$ which has an algebra structure induced by that of $L \otimes L$, and whose definition we shall now recall.

The notations $\int_r := \int_{r\in R}$ and $\int^r := \int^{r \in R}$, which we will introduce only by example, are due to MacLane, see [24, 25]. For $M, N \in {}_{R^e}\mathcal{M}_{R^e}$ we let

$$\int_r {}_{\overline{r}}M \otimes {}_rN := M \otimes N / \langle \overline{r}m \otimes n - m \otimes rn | r \in R, m \in M, n \in N\rangle$$

and we let $\int^r M_{\overline{r}} \otimes N_r \subset M \otimes N$ denote the k-submodule consisting of all elements $\sum m_i \otimes n_i \in M \otimes N$ satisfying $\sum m_i\overline{r} \otimes n_i = \sum m_i \otimes n_i r$ for all $r \in R$. Note $\int_r {}_{\overline{r}}M \otimes {}_rN = M \diamond_R N$ for $M, N \in {}_{R^e}\mathcal{M}$.

For two R^e-bimodules M and N we abbreviate

$$M \times_R N := \int^s \int_r {}_{\overline{r}}M_{\overline{s}} \otimes {}_rN_s.$$

If M, N are R^e-rings, then so is $M \times_R N$, with multiplication given by $(\sum m_i \otimes n_i)(\sum m'_j \otimes n'_j) = \sum m_i m'_j \otimes n_i n'_j$, and R^e-ring structure

$$R^e \ni r \otimes \overline{s} \mapsto r \otimes \overline{s} \in M \times_R N.$$

For $M,N,P \in {}_{R^e}\mathcal{M}_{R^e}$ one defines

$$M \times_R P \times_R N := \int^{s,u} \int_{r,t} {}_{\overline{r}}M_{\overline{s}} \otimes {}_{r,\overline{t}}P_{s,\overline{u}} \otimes {}_t N_u$$

(where $\int^{s,u} := \int^s \int^u = \int^u \int^s$). There are associativity maps

$$(M \times_R P) \times_R N \xrightarrow{\alpha} M \times_R P \times_R N$$

$$M \times_R (P \times_R N) \xrightarrow{\alpha'} M \times_R P \times_R N$$

given on elements by the obvious formulas (doing nothing), but which need not be isomorphisms. If M, N and P are R^e-rings, so is $M \times_R N \times_R P$, and α, α' are R^e-ring maps.

An R^e-ring structure on the algebra $\mathrm{E} = \mathrm{End}(R)$ is given by $r \otimes \overline{s} \mapsto (t \mapsto rts)$. We have, for any $M \in {}_{R^e}\mathcal{M}_{R^e}$, two R^e-bimodule maps

$$\theta : M \times_R \mathrm{End}(R) \to M; \qquad m \otimes f \mapsto \overline{f(1)}m$$

$$\theta' : \mathrm{End}(R) \times_R M \to M; \qquad f \otimes m \mapsto f(1)m.$$

which are R^e-ring homomorphisms if M is an R^e-ring.

Now we are prepared to write down the definition of a $\times_R$-bialgebra L. This is by definition an R^e-ring equipped with a comultiplication, a map $\Delta\colon L \to L \times_R L$ of R^e-rings over R^e, and a counit, a map $\varepsilon : L \to \mathrm{E}$ of R^e-rings, such that

$$\alpha(\Delta \times_R L)\Delta = \alpha'(L \times_R \Delta)\Delta\colon L \to L \times_R L \times_R L \tag{2.1}$$

$$\theta(L \times_R \varepsilon)\Delta = \mathrm{id}_L = \theta(\varepsilon \times_R L)\Delta. \tag{2.2}$$

For $\times_R$-bialgebras we will make use of the usual Sweedler notation, writing $\Delta(\ell) =: \ell_{(1)} \otimes \ell_{(2)} \in L \times_R L$.

If L is a $\times_R$-bialgebra, then the module structure on the tensor product $M \diamond_R N$ of $M, N \in {}_L\mathcal{M}$ can be described in terms of the comultiplication of L by the usual formula $\ell(m \otimes n) = \ell_{(1)}m \otimes \ell_{(2)}n$.

The suitable definition of comodules over a $\times_R$-bialgebra L is as follows: A left L-comodule is an R-bimodule M together with a map $\lambda\colon M \to L \times_R M$ of R-bimodules such that

$$\alpha'(L \times_R \lambda)\lambda = \alpha(\Delta \times_R M)\lambda\colon M \to L \times_R L \times_R M$$

and $\theta'(\varepsilon \times_R M)\lambda = \mathrm{id}_M$ hold. We will denote by ${}^L\mathcal{M}$ the category of left L-comodules. We will use Sweedler notation in the form $\lambda(m) = m_{(-1)} \otimes m_{(0)}$ and $\alpha(\Delta \times_R M)(m) = m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)}$ for L-comodules.

The category ${}^L\mathcal{M}$ of left L-comodules over a $\times_R$-bialgebra is monoidal. The tensor product of $M, N \in {}^L\mathcal{M}$ is their tensor product $M \otimes_R N$ over R, equipped with the comodule structure

$$M \underset{R}{\otimes} N \to L \times_R (M \underset{R}{\otimes} N)$$

$$m \otimes n \mapsto m_{(-1)}n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}$$

In [21, Thm. and Def.3.5] we have introduced a notion of $\times_R$-Hopf algebra. It is rather different from that of a Hopf algebroid given by Lu [14], although Lu's bialgebroids are the same as $\times_R$-bialgebras. By definition, a $\times_R$-bialgebra is a $\times_R$-Hopf algebra if and only if the map

$$\beta\colon L \underset{\overline{R}}{\otimes} L \ni \ell \otimes m \mapsto \ell_{(1)} \otimes \ell_{(2)}m \in L \diamond L$$

is a bijection. Note that this is equivalent to a well-known characterization of Hopf algebras among ordinary bialgebra. By [21] it is equivalent to saying that the underlying functor ${}_L\mathcal{M} \to {}_{R^e}\mathcal{M}$ preserves inner hom-functors. More precisely, for each $M \in {}_L\mathcal{M}$ the functor ${}_L\mathcal{M} \ni N \mapsto N \diamond M \in {}_L\mathcal{M}$ has a right adjoint $\underline{\hom}(M, -)$. The $\times_R$-bialgebra is a $\times_R$-Hopf algebra if and only if a canonically defined map $\underline{\hom}(M, N) \to \mathrm{Hom}_{\overline{R}-}(M, N)$ is a bijection for all $M, N \in {}_L\mathcal{M}$.

Let us finally recall two special cases of the notion of a $\times_R$-bialgebra. Weak bialgebras and weak Hopf algebras were introduced by Böhm and Szlachányi [2]. We refer to the survey [17] by Nikshych and Vainerman and the literature cited there. It was shown in [5] that weak Hopf algebras are $\times_R$-bialgebras. More details and a converse are in [23]. By definition, a weak bialgebra H is a k-coalgebra and k-algebra such that comultiplication is multiplicative, but not necessarily unit-preserving (and neither is multiplication assumed to be comultiplicative). There are specific axioms replacing the "missing" compatibility axioms for a bialgebra, namely, for $f, g, h \in H$:

$$\varepsilon(fgh) = \varepsilon(fg_{(1)})\varepsilon(g_{(2)}h) = \varepsilon(fg_{(2)})\varepsilon(g_{(1)}h),$$
$$1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1'_{(1)} \otimes 1'_{(2)} = 1_{(1)} \otimes 1'_{(1)}1_{(2)} \otimes 1'_{(2)}.$$

If H is a weak bialgebra, then the target counital subalgebra H_t consists by definition of all elements of the form $\varepsilon(1_{(1)}h)1_{(2)}$ with $h \in H$. The source counital subalgebra H_s is the target counital subalgebra in the coopposite of H. It turns out that H_t is a subalgebra which is Frobenius-separable (i.e. a multi-matrix algebra when $k = \mathbb{C}$ is the field of complex numbers), anti-isomorphic to H_s, and that H has the structure of a $\times_R$-bialgebra for $R = H_t$, in which H_s is the image of $\overline{R}$ in H. Moreover, any $\times_R$-bialgebra in which R is Frobenius-separable can be obtained in this way from a weak bialgebra. A weak Hopf algebra is by definition a weak bialgebra H with an antipode, which in turn is an anti-automorphism of H whose axioms we shall not recall. The antipode maps H_t isomorphically onto H_s and vice versa. We have shown in [23] that a weak bialgebra has an antipode if and only if the associated $\times_R$-bialgebra is a $\times_R$-Hopf algebra.

The face algebras introduced earlier by Hayashi [8, 10] are recovered as a yet more special case of weak bialgebras, namely that where the target (and source) counital subalgebra is commutative. In particular, as shown in [20], a face algebra H the same thing as a $\times_R$-bialgebra in which R is commutative and separable. We will only be using the case where the base field is the field of complex numbers, so that R is a direct product of copies of the field. In particular, the images of the minimal idempotents of R in H form a distinguished family of idempotents in H, which feature prominently in Hayashi's original definition (along, of course, with the images of the corresponding idempotents in $\overline{R}$). We shall refer to them as the face idempotents of H; their number, or the dimension of R, is an important structure element of H.

3. Morita- and $\sqrt{\text{Morita}}$-equivalence

Suppose R, S are Morita equivalent k-algebras (we shall write $R \overset{\mathrm{M}}{\sim} S$ for short). Then by definition the categories ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ of left modules are equivalent as k-linear abelian categories. In this situation, one also gets an equivalence $({}_R\mathcal{M}_R, \otimes_R) \cong ({}_S\mathcal{M}_S, \otimes_S)$ of k-linear monoidal categories. This can be seen by applying Watts' theorem [27], which says that the monoidal category ${}_R\mathcal{M}_R$ can be viewed as the category of right exact k-linear endofunctors of ${}_R\mathcal{M}$. Somewhat more useful is the following explicit description of the monoidal equivalence: When $R \overset{\mathrm{M}}{\sim} S$, fix a strict Morita context (R, S, P, Q, f, g). In particular, we have $P \in {}_S\mathcal{M}_R$, $Q \in {}_R\mathcal{M}_S$, $f\colon P \otimes_R Q \to S$ an isomorphism of S-bimodules and $g\colon Q \otimes_S P \to R$ an isomorphism of R-bimodules. An equivalence is given by

$$\mathcal{F}\colon {}_R\mathcal{M} \ni M \mapsto P \underset{R}{\otimes} M \in {}_S\mathcal{M},$$

and we can describe a matching equivalence of bimodule categories

$$(\hat{\mathcal{F}}, \xi)\colon ({}_R\mathcal{M}_R, \underset{R}{\otimes}) \to ({}_S\mathcal{M}_S, \underset{S}{\otimes})$$

as follows: We set $\hat{\mathcal{F}}(M) = P \otimes_R M \otimes_R Q$ as S-bimodules, and we define the monoidal functor structure

$$\xi\colon \mathcal{F}(M) \underset{S}{\otimes} \mathcal{F}(N) \to \mathcal{F}(M \underset{R}{\otimes} N)$$

as the composition

$$\begin{aligned} P \underset{R}{\otimes} M \underset{R}{\otimes} Q \underset{S}{\otimes} P \underset{R}{\otimes} N \underset{R}{\otimes} Q \xrightarrow{P \otimes M \otimes g \otimes N \otimes Q} & P \underset{R}{\otimes} M \underset{R}{\otimes} R \underset{R}{\otimes} N \underset{R}{\otimes} Q \\ & \cong P \underset{R}{\otimes} M \underset{R}{\otimes} N \underset{R}{\otimes} Q \end{aligned}$$

It is useful to know that the equivalence $\mathcal{F}$ and the monoidal equivalence $\hat{\mathcal{F}}$ are compatible in the following sense: The category ${}_R\mathcal{M}$ is in a natural way a left ${}_R\mathcal{M}_R$-category in the sense of Pareigis [18], that is, a category on which ${}_R\mathcal{M}_R$ acts (by tensor product). The compatibility says that the following diagram commutes up to coherent natural isomorphisms:

$$\begin{tikzcd} {}_R\mathcal{M}_R \times {}_R\mathcal{M} \arrow[r, "\otimes_R"] \arrow[d, "\hat{\mathcal{F}} \times \mathcal{F}"] & {}_R\mathcal{M} \arrow[d, "\mathcal{F}"] \\ {}_S\mathcal{M}_S \times {}_S\mathcal{M} \arrow[r, "\otimes_S"] & {}_S\mathcal{M} \end{tikzcd}$$

Takeuchi [26] has introduced and investigated the notion of $\sqrt{\text{Morita}}$-equivalence of k-algebras; by his definition, two k-algebras R, S are $\sqrt{\text{Morita}}$-equivalent, written $R \overset{\sqrt{\mathrm{M}}}{\sim} S$, if there is an equivalence of k-linear monoidal categories ${}_R\mathcal{M}_R \cong {}_S\mathcal{M}_S$. By the above, Morita equivalence clearly implies $\sqrt{\text{Morita}}$-equivalence. On the other hand, since ${}_R\mathcal{M}_R \cong {}_{R^e}\mathcal{M}$, the enveloping algebras of $\sqrt{\text{Morita}}$-equvialent algebras are Morita equivalent, so that

$$R \overset{\mathrm{M}}{\sim} S \quad \Rightarrow \quad R \overset{\sqrt{\mathrm{M}}}{\sim} S \quad \Rightarrow \quad R^e \overset{\mathrm{M}}{\sim} S^e.$$

Neither of the reverse implications holds.

Note that a bimodule $M \in {}_R\mathcal{M}_R$ has a left dual object in the monoidal category $({}_R\mathcal{M}_R, \otimes_R)$ if and only if it is finitely generated projective as a right R-module. It follows that any equivalence of monoidal categories ${}_R\mathcal{M}_R \cong {}_S\mathcal{M}_S$ maps left (or right) finitely generated projective modules to left (or right) finitely generated projective modules. This reduces to a standard fact on projective modules, if the equivalence comes from a Morita equivalence $R \overset{\mathrm{M}}{\sim} S$, for then it maps $M \in {}_R\mathcal{M}_R$ to $P \otimes_R M \otimes_R Q$, which is finitely generated projective as left S-module if ${}_RM$ is finitely generated projective, for ${}_SP$ and ${}_RQ$ are finitely generated projective.

4. A $\sqrt{\text{Morita}}$-base change principle

An A-ring L for a k-algebra A is an algebra in the monoidal category of A-bimodules. As an immediate consequence, if $A \overset{\mathrm{M}}{\sim} B$, then an A-ring is essentially the same as a B-ring. Moreover, if $\widetilde{L}$ is the B-ring corresponding to the A-ring L, then $\widetilde{L}$-modules are essentially the same as L-modules, since the actions of ${}_B\mathcal{M}_B$ on ${}_B\mathcal{M}$ and of ${}_A\mathcal{M}_A$ on ${}_A\mathcal{M}$ are compatible with the equivalences. Thus we have:

Lemma 4.1. *Let L be an A-ring over the k-algebra A. Assume given a k-algebra B and a strict Morita context (A, B, C, D, ϕ, ψ). Then there is a B-ring $\widetilde{L}$, and a category equivalence $\mathcal{G}\colon {}_L\mathcal{M} \to {}_{\widetilde{L}}\mathcal{M}$ lifting the equivalence $\mathcal{F}\colon {}_A\mathcal{M} \to {}_B\mathcal{M}$ given by tensoring with C, as in the following diagram:*

$$\begin{array}{ccc} {}_L\mathcal{M} & \xrightarrow{\mathcal{G}} & {}_{\widetilde{L}}\mathcal{M} \\ \downarrow & & \downarrow \\ {}_A\mathcal{M} & \xrightarrow{\mathcal{F}} & {}_B\mathcal{M} \end{array}$$

in which the vertical arrows are the underlying functors induced by the A-ring structure of L, and the B-ring structure of $\widetilde{L}$, respectively.

Explicitly, the B-ring $\widetilde{L}$ is given by $\widetilde{L} := C \otimes_A L \otimes_A D$ with unit map

$$B \cong C \underset{A}{\otimes} D \xrightarrow{C \otimes_A \eta \otimes_A D} C \underset{A}{\otimes} L \underset{A}{\otimes} D$$

in which η is the unit map of the A-ring L, and multiplication map

$$\widetilde{L} \underset{B}{\otimes} \widetilde{L} = C \underset{A}{\otimes} L \underset{A}{\otimes} D \underset{B}{\otimes} C \underset{A}{\otimes} L \underset{A}{\otimes} D \cong C \underset{A}{\otimes} L \underset{A}{\otimes} L \underset{A}{\otimes} D \xrightarrow{C \otimes_A \nabla \otimes_A D} C \underset{A}{\otimes} L \underset{A}{\otimes} D.$$

When M is a L-module, then the $\widetilde{L}$-module $\mathcal{F}(M)$ is $C \otimes_A M$ equipped with the $\widetilde{L}$-module structure

$$\widetilde{L} \underset{B}{\otimes} \mathcal{F}(M) = C \underset{A}{\otimes} L \underset{A}{\otimes} D \underset{B}{\otimes} C \underset{A}{\otimes} M = C \underset{A}{\otimes} L \underset{A}{\otimes} M \xrightarrow{C \otimes_A \mu_M} C \underset{A}{\otimes} M = \mathcal{F}(M).$$

Remark 4.2. Assume that the algebras involved in the situation above are multi-matrix algebras over a field k.

Then the Bratteli diagram of the inclusion $B \subset \widetilde{L}$ is the same as that of the inclusion $A \subset L$, except that the ranks of the components of A have to be replaced by the ranks of the components of B, and the top floor representing $\widetilde{L}$ has to be adjusted accordingly.

In fact the number of edges between a vertex on the bottom floor of the Bratelli diagram and a vertex on the top floor is the multiplicity of an irreducible representation of A in a certain irreducible representation of L. Evidently these numbers do not change in the construction in the Lemma.

Theorem 4.3. *Let L be a $\times_R$-bialgebra for a k-algebra R. Let S be a k-algebra which is $\sqrt{\text{Morita}}$-equivalent to R. Then there is a $\times_S$-bialgebra $\widetilde{L}$ whose module category is equivalent to that of L, as a monoidal category.*

More precisely, assume given a monoidal category equivalence $\mathcal{F}\colon {}_{R^e}\mathcal{M} \to {}_{S^e}\mathcal{M}$. Then there is a $\times_S$-bialgebra $\widetilde{L}$ and a monoidal category equivalence $\mathcal{G}\colon {}_L\mathcal{M} \to {}_{\widetilde{L}}\mathcal{M}$

making

$$
\begin{tikzcd}
{}_L\mathcal{M} \arrow[r, "\mathcal{G}"] \arrow[d, "\mathcal{U}"] & {}_{\widetilde{L}}\mathcal{M} \arrow[d, "\widetilde{\mathcal{U}}"] \\
{}_{R^e}\mathcal{M} \arrow[r, "\mathcal{F}"] & {}_{S^e}\mathcal{M}
\end{tikzcd}
\tag{4.1}
$$

a commutative diagram of monoidal functors (in which the vertical arrows are underlying functors.

Proof. We know already that there is an S^e-ring $\widetilde{L}$ and a category equivalence $\mathcal{G}$ making the square in the Theorem a commutative diagram of k-linear functors. We can endow ${}_{\widetilde{L}}\mathcal{M}$ with a monoidal category structure such that $\mathcal{G}$ is a monoidal functor. Then the underlying functor $\widetilde{\mathcal{U}}$ is monoidal as well, since it can be written as the composition $\widetilde{\mathcal{U}} = \mathcal{F}\mathcal{U}\mathcal{G}^{-1}$. Now [19, Thm.5.1] implies that there exists a $\times_S$-bialgebra structure on $\widetilde{L}$ inducing the given monoidal category structure on ${}_{\widetilde{L}}\mathcal{M}$. $\square$

Definition 4.4. Let L be a $\times_R$-bialgebra, and S a k-algebra $\sqrt{\text{Morita}}$-equivalent to R. We will say that the $\times_S$-bialgebra $\widetilde{L}$ obtained from L as in the proof of Theorem 4.3 is obtained from L by a $\sqrt{\text{Morita}}$ base change.

Thus a $\times_S$-bialgebra obtained from a $\times_R$-bialgebra L by a $\sqrt{\text{Morita}}$-base change has the same monoidal representation category as L itself. The difference is "merely" in a change of the base algebra.

We will be somewhat sloppy in our terminology: Given a $\times_R$-bialgebra L and $S \overset{\sqrt{\mathrm{M}}}{\sim} R$, we will speak of *the* $\times_S$-bialgebra $\widetilde{L}$ obtained from L by $\sqrt{\text{Morita}}$ base change. This suppresses the choice of a monoidal category equivalence ${}_R\mathcal{M}_R \cong {}_S\mathcal{M}_S$, by which (and not by S alone) $\widetilde{L}$ is determined.

Corollary 4.5. *Let L be a $\times_R$-bialgebra, where R is an Azumaya k-algebra. Then L can be obtained by $\sqrt{\text{Morita}}$-base change from an ordinary bialgebra H.*

Proof. Takeuchi [26, Ex.2.4] has shown that the algebra R is Azumaya if and only if $R \overset{\sqrt{\mathrm{M}}}{\sim} k$. $\square$

Takeuchi also gives the following elegant description of the monoidal category equivalence ${}_R\mathcal{M}_R \to \mathcal{M}_k$: It maps M to the centralizer M^R of R in M, whereas its inverse maps $V \in \mathcal{M}_k$ to $V \otimes R$ with the obvious R-bimodule structure. Thus, the ordinary k-bialgebra associated to a $\times_R$-bialgebra L is

$$H = \{\ell \in L | \forall r \in R\colon r\ell = \overline{r}\ell \wedge \ell r = \ell\overline{r}\}$$

whereas L can be obtained from H by merely tensoring with two copies of R, one of which gives the left R^e-module structure, and the other one the right R^e-module structure of $R \otimes L \otimes R$.

In a sense our corollary says that examples of $\times_R$-bialgebras in which R is Azumaya are irrelevant; they are just versions of ordinary bialgebras in which the base ring is enlarged, without affecting the representation theory. That would also apply to examples like that considered by Kadison in [12, Thm.5.2]. In fact in the example of a $\times_R$-bialgebra T there, R is Azumaya over its center Z. Moreover, the two algebra maps $Z \to T$ coming from the maps $R \to T$ and $\overline{R} \to H$ coincide by construction and have central image. Thus T can be considered as a $\times_R$-bialgebra for the Z-algebra R; since this is Azumaya, Corollary 4.5 applies, so that T can be obtained by $\sqrt{\text{Morita}}$ base change from a Z-bialgebra. However, we should rush to concede that Corollary 4.5 does of course not rule out that interesting examples of $\times_R$-bialgebras over Azumaya k-algebras arise naturally. In fact Kadison's example is constructed from a natural situation that comes with a natural choice of base R. Moreover, the example gives us the opportunity to point out a certain subtlety about $\sqrt{\text{Morita}}$ base change: The $\times_R$-bialgebra T in [12, Sec.4] occurs in duality with another $\times_R$-bialgebra S, which can also be considered as a $\times_R$-bialgebra for the Z-algebra R. Thus, if R is Azumaya over Z, then S can be reduced by $\sqrt{\text{Morita}}$ base change to a Z-bialgebra S', while T can be replaced by a Z-bialgebra T'. However, we do not have any indication that S' and T' are still dual to each other. It is conceivable that the duality only shows over the ring R. We will show below that Morita base change is compatible with duality.

Closing the section, let us show that $\sqrt{\text{Morita}}$ base change preserves the property of a $\times_R$-bialgebra of being a $\times_R$-Hopf algebra in the sense of [21]:

Proposition 4.6. *Let L be a $\times_R$-bialgebra, $S \overset{\sqrt{M}}{\sim} R$, and let $\widetilde{L}$ be the $\times_S$-bialgebra obtained from L by $\sqrt{\text{Morita}}$ base change.*

$\widetilde{L}$ is a $\times_S$-Hopf algebra if and only if L is a $\times_R$-Hopf algebra.

Proof. In the diagram (4.1), the horizontal functors are monoidal equivalences, hence preserve inner hom-functors. Thus the left hand vertical functor preserves inner hom-functors if and only the right hand one does. □

5. Morita base change

Morita equivalence implies $\sqrt{\text{Morita}}$ equivalence. Thus, given a $\times_R$-bialgebra L and a k-algebra S Morita equivalent to R, we can apply $\sqrt{\text{Morita}}$ base change (which, of course, we shall call Morita base change in this case) to L to obtain a $\times_S$-bialgebra $\widetilde{L}$ with equivalent monoidal module category.

5.1. **Morita base change — explicitly.** To find out what the result looks like more explicitly, fix a Morita context (R, S, P, Q, f, g). We will write $f(p \otimes q) = pq$, $g(q \otimes p) = qp$, $f^{-1}(1_S) = p_i \otimes q^i \in P \otimes_R Q$ and $g^{-1}(1_R) = q_i \otimes p^i \in Q \otimes_S P$ (with a summation over upper and lower indices understood). Write $\overline{P} \in {}_{\overline{R}}\mathcal{M}_{\overline{S}}$ for the bimodule opposite to P, and $\overline{p}$ with $p \in P$ for a typical element; similarly for $\overline{Q} \in {}_{\overline{S}}\mathcal{M}_{\overline{R}}$. Somewhat dangerously we write $P^e := P \otimes \overline{Q} \in {}_{S^e}\mathcal{M}_{R^e}$ and $Q^e := Q \otimes \overline{P} \in {}_{R^e}\mathcal{M}_{S^e}$, so that the bimodules P^e and Q^e induce the equivalence ${}_{R^e}\mathcal{M} \cong {}_{S^e}\mathcal{M}$ underlying the $\sqrt{\text{Morita}}$ equivalence between R and S induced by the Morita equivalence between R and S.

To keep our formulas a manageable size, we will write $p\overline{q} := p \otimes \overline{q} \in P \otimes \overline{Q} = P^e$, and similarly for the typical elements of Q^e.

Now let L be a $\times_R$-bialgebra. The $\times_S$-bialgebra $\widetilde{L}$ obtained from L by Morita base change has underlying S^e-bimodule $P^e \otimes_{R^e} L \otimes_{R^e} Q^e$. The equivalence ${}_L\mathcal{M} \cong {}_{\widetilde{L}}\mathcal{M}$ sends $M \in {}_L\mathcal{M}$ to $P^e \otimes_{R^e} M$, with the $\widetilde{L}$-module structure given by

$$(p_1\overline{q_1} \otimes \ell \otimes q_2\overline{p_2})(p_3\overline{q_3} \otimes m) = p_1\overline{q_1} \otimes \ell(q_2p_3)(\overline{q_3p_2})m$$

for $p_1,p_2,p_3\in P$ and $q_1,q_2,q_3\in Q$. The monoidal functor structure of the equivalence is given by

$$\begin{aligned}\xi\colon (P^e\underset{R^e}{\otimes}M)\underset{S}{\diamond}(P^e\underset{R^e}{\otimes}N)&\xrightarrow{\sim}P^e\underset{R^e}{\otimes}(M\underset{R}{\diamond}N)\\ p_1\overline{q_1}\otimes m\otimes p_2\overline{q_2}\otimes n&\mapsto p_1\overline{q_2}\otimes m\otimes(q_1p_2)n\\ &=p_1\overline{q_{(2)}}\otimes\overline{q_1p_2}m\otimes n\\ p\overline{q_i}\otimes m\otimes p^i\overline{q}&\leftarrow p\overline{q}\otimes m\otimes n\end{aligned}$$

for $M,N\in{}_L\mathcal{M}$, $m\in M$, $n\in N$, $p_1,p_2\in P$, $q_1,q_2\in Q$.

It follows that the $\widetilde{L}$-module structure of the tensor product of two $\widetilde{L}$-modules coming via the equivalence from L-modules M,N can be computed as the composition

$$\begin{aligned}&(P^e\underset{R^e}{\otimes}L\underset{R^e}{\otimes}Q^e)\underset{S^e}{\otimes}((P^e\underset{R^e}{\otimes}M)\underset{S}{\diamond}(P^e\underset{R}{\otimes^e}N))\\ &\qquad\xrightarrow{\mathrm{id}\otimes\xi}(P^e\underset{R^e}{\otimes}L\underset{R^e}{\otimes}Q^e)\underset{S^e}{\otimes}(P^e\underset{R^e}{\otimes}(M\underset{R}{\diamond}N))\\ &\qquad\qquad\xrightarrow{\mu}P^e\underset{R^e}{\otimes}(M\underset{R}{\diamond}N)\xrightarrow{\xi^{-1}}(P^e\underset{R^e}{\otimes}M)\underset{S}{\diamond}(P^e\underset{R^e}{\otimes}N)\end{aligned}$$

hence is given by

$$\begin{aligned}&(p_1\overline{q_1}\otimes\ell\otimes q_2\overline{p_2})(p_3\overline{q_3}\otimes m\otimes p_4\overline{q_4}\otimes n)\\ &\quad=\xi^{-1}((p_1\overline{q_1}\otimes\ell\otimes q_2\overline{p_2})(p_3\overline{q_4}\otimes m\otimes(q_3p_4)n)\\ &\quad=\xi^{-1}(p_1\overline{q_1}\otimes\ell(q_2p_3)(\overline{q_4p_2})(m\otimes(q_3p_4)n))\\ &\quad=\xi^{-1}(p_1\overline{q_1}\otimes\ell_{(1)}(q_2p_3)m\otimes\ell_2(\overline{q_4p_2})(q_3p_4)n)\\ &\quad=p_1\overline{q_i}\otimes\ell_{(1)}(q_2p_3)m\otimes p^i\overline{q_1}\otimes\ell_2(\overline{q_4p_2})(q_3p_4)n\\ &\quad=p_1\overline{q_i}\otimes\ell_{(1)}(q_2p_3)m\otimes p^i\overline{q_1}\otimes\ell_2(q_3p_j)(q^jp_4)(\overline{q_4p_2})n\\ &\quad=p_1\overline{q_i}\otimes\ell_{(1)}(q_2p_3)(\overline{q_3p_j})m\otimes p^i\overline{q_1}\otimes\ell_{(2)}(q^jp_4)(\overline{q_4p_2})n\\ &\quad=(p_1\overline{q_i}\otimes\ell_{(1)}\otimes q_2\overline{p_j})(p_3\overline{q_3}\otimes m)\otimes(p^i\overline{q_1}\otimes\ell_{(2)}\otimes q^j\overline{p_2})(p_4\overline{q_4}\otimes n)\end{aligned}$$

for $p_1,\dots,p_4\in P$, $q_1,\dots,q_4\in Q$, $\ell\in L$, $m\in M$, and $n\in N$. this proves that the comultiplication in $\widetilde{L}$ is given by the formula

$$\Delta(p_1\overline{q_1}\otimes\ell\otimes q_2\overline{p_2})=(p_1\overline{q_i}\otimes\ell_{(1)}\otimes q_2\overline{p_j})\otimes(p^i\overline{q_1}\otimes\ell_{(2)}\otimes q^j\overline{p_2})$$

for $p_1,p_2\in P$ and $q_1,q_2\in Q$.

5.2. **Weak bialgebras versus face algebras.** Let us be yet more concrete for the case that R is a multi-matrix algebra $R=\bigoplus_{\alpha=1}^n M_{d_\alpha}(k)$, and $S=k^n$. A Morita context (R,S,P,Q,f,g) can be given as follows: P is generated as a right R-module by one element p which is a sum $p=\sum_{\alpha=1}^n E_{11}^{(\alpha)}$ of minimal idempotents (where we have denoted the matrix units in the α-th component by $E_{ij}^{(\alpha)}$). Q is generated as left R-module by the same element p. both maps f,g are given by matrix multiplication. We have $f^{-1}(1_S)=p\otimes p$, and $g^{-1}(1_R)=\sum_{\alpha=1}^n\sum_{i=1}^{d_\alpha}E_{i1}^{\alpha}\otimes E_{1i}^{(\alpha)}$. Let L be a $\times_R$-bialgebra, and $\widetilde{L}$ the $\times_S$-bialgebra obtained from it by Morita base change. Then $\widetilde{L}=p\overline{p}Lp\overline{p}\subset L$, with multiplication given by multiplication in L, unit $p\overline{p}$, and comultiplication

$$\Delta(p\overline{p}\ell p\overline{p}) = p\overline{E_{i1}^{(\alpha)}}\ell_{(1)}p\overline{p} \otimes E_{1i}^{(\alpha)}\overline{p}\ell_{(2)}p\overline{p}.$$

Now let $k = \mathbb{C}$ be the field of complex numbers. Then a $\times_R$-bialgebra for a multi-matrix algebra R is the same as a weak bialgebra in the sense of Böhm and Szlachányi [2, 1]. If R is commutative, then this is in turn the same thing as a face algebra in the sense of Hayashi [10]. Thus Morita base change says that Hayashi's face algebras are a sufficiently general case of weak bialgebras, at least as long as we are interested in the respective module categories:

Corollary 5.1. *Let H be a weak bialgebra over the field of complex numbers.*

Then H can be obtained by Morita base change from a weak bialgebra whose source counital subalgebra is commutative.

In particular, there is a face algebra F and a monoidal category equivalence ${}_H\mathcal{M} \cong {}_F\mathcal{M}$.

H is a weak Hopf algebra if and only if F is a face Hopf algebra.

Remark 5.2. For the case of semisimple H, it follows in fact from Hayashi's canonical Tannaka duality [10, 11] that there is a face algebra F and a monoidal category equivalence ${}_H\mathcal{M} \cong {}_F\mathcal{M}$. The corollary above shows the same for non-semisimple H, but it is also a different result in the semisimple case. A peculiar feature of Hayashi's canonical Tannaka duality (on which we will give more details in the next section) is that it yields semisimple face algebras with the same number of face idempotents and irreducible representations. This clearly needs not be the case for the face algebras obtained by Morita base change. A trivial example is the trivial Morita base change applied to an ordinary Hopf algebra, which leaves us with the same Hopf algebra, or only one face idempotent. More examples will appear below.

5.3. **Duality.** There is a well-behaved notion of duality for $\times_R$-bialgebras, developed in [21], and shown to be compatible with the duality for weak bialgebras in [23]. The main difficulty in the definitions is to sort out how the four module structures in a $\times_R$-bialgebra should be translated through the duality, and to check that the formulas defining the dual structures are well defined with respect to the various tensor products over R. A specialty is that one can define the $\times_R$-bialgebra analog of the opposite or coopposite of the dual of an ordinary bialgebra, but not the direct analog of the dual (unless one wants to allow two versions of "left" and "right" bialgebroids like Kadison and Szlachányi [13]). More generally, one defines [21, Def.5.1] a skew pairing between two $\times_R$-bialgebras Λ and L to be a k-linear map $\tau\colon \Lambda \otimes L \to R$ satisfying

$$((r \otimes \overline{s})\xi(t \otimes \overline{u})|\ell)v = r\tau(\xi|(t \otimes \overline{v})\ell(u \otimes \overline{s})), \tag{5.1}$$

$$\tau(\xi|\ell m) = \tau(\overline{\tau(\xi_{(2)}|m)}\xi_{(1)}|\ell), \quad \tau(\xi|1) = \varepsilon(\xi)(1), \tag{5.2}$$

$$\tau(\xi\zeta|\ell) = \tau(\xi|\tau(\zeta|\ell_{(1)})\ell_{(2)}), \qquad \tau(1|\ell) = \varepsilon(\ell)(1) \tag{5.3}$$

for all $\xi, \zeta \in \Lambda$, $\ell, m \in L$, $r, s, t, u, v \in R$.

Proposition 5.3. *Let $\tau\colon \Lambda \otimes L \to R$ be a skew pairing between $\times_R$-bialgebras Λ and L. Let $S \overset{M}{\sim} R$, and let $\widetilde{\Lambda}, \widetilde{L}$ be the $\times_S$-bialgebras obtained from Λ and L by Morita base change.*

Then a skew pairing $\widetilde{\tau}\colon \widetilde{\Lambda}\otimes\widetilde{L}\to S$ *can be defined by*

$$\begin{aligned}\widetilde{\tau}(p_1\otimes\overline{q_1}\otimes\xi\otimes q_2\otimes\overline{p_2}|p_3\otimes\overline{q_3}\otimes\ell\otimes q_4\otimes\overline{p_4}) &= p_1\tau(\overline{q_1p_4}\xi(q_2p_3)(\overline{q_4p_2})|\ell)q_3\\ &= p_1\tau(\xi|(q_2p_3)\ell(q_4p_2)(\overline{q_1p_4}))q_3\end{aligned}$$

for $p_1,\dots,p_4\in P$, $q_1,\dots,q_4\in Q$, $\xi\in\Lambda$, *and* $\ell\in L$.

Proof. To make the formulas a manageable size, we abbreviate $p\overline{q}:=p\otimes\overline{q}\in P^e=P\otimes\overline{Q}$ for $p\in P$ and $q\in Q$, and similarly for elements of Q^e.

We omit checking (5.1) for $\widetilde{\tau}$. Now let $p_1,\dots,p_6\in P$, $q_1,\dots,q_6\in Q$, $\xi\in\Lambda$, and $\ell,m\in L$. Then

$$\begin{aligned}
&\widetilde{\tau}((p_1\overline{q_1}\otimes\xi\otimes q_2\overline{p_2})_{(1)}|p_3\overline{q_3}\otimes\ell\otimes q_4\overline{p_4}\overline{\widetilde{\tau}((p_1\overline{q_1}\otimes\xi\otimes q_2\overline{p_2})_{(2)}|p_5\overline{q_5}\otimes m\otimes q_6\overline{p_6})})\\
&\quad=\widetilde{\tau}(p_1\overline{q_i}\otimes\xi_{(1)}\otimes q_2\overline{p_j}|p_3\overline{q_3}\otimes\ell\otimes q_4\overline{p_4}\overline{\widetilde{\tau}(p^i\overline{q_1}\otimes\xi_{(2)}\otimes q^j\overline{p_2}|p_5\overline{q_5}\otimes m\otimes q_6\overline{p_6})})\\
&\quad=\widetilde{\tau}(p_1\overline{q_i}\otimes\xi_{(1)}\otimes q_2\overline{p_j}|p_3\overline{q_3}\otimes\ell\otimes q_4\overline{p_4}\overline{p^i\tau(\overline{q_1p_6}\xi_{(2)}(q^jp_5)(\overline{q_6p_2})|m)q_5})\\
&\quad=\widetilde{\tau}(p_1\overline{q_i}\xi_{(1)}q_2\overline{p_5}|p_3\overline{q_3}\ell(\overline{q_5p_4})\overline{\tau(\overline{q_1p_6}\xi_{(2)}(\overline{q_6p_2})|m)}q_4\overline{p^i})\\
&\quad=p_1\tau(\overline{q_ip^i}\xi_{(1)}(q_2p_3)(\overline{q_4p_5})|\ell(\overline{q_5p_4})\overline{\tau(\overline{q_1p_6}\xi_{(2)}\overline{q_6p_2}|m)})q_3\\
&\quad=p_1\tau(\xi_{(1)}(q_2p_3)|\ell(q_4p_5)\overline{\tau(\overline{q_1p_6}\xi_{(2)}\overline{q_6p_2}|\overline{q_5p_4}m)})q_3\\
&\quad=p_1\tau(\overline{q_1p_6}\xi(q_2p_3)(\overline{q_6p_2})|\ell(q_4p_5)(\overline{q_5p_4})m)q_3\\
&\quad=\widetilde{\tau}(p_1\overline{q_1}\otimes\xi\otimes q_2\overline{p_2}|p_3\overline{q_3}\otimes\ell(q_4p_5)(\overline{q_5p_4})m\otimes q_6\overline{p_6})\\
&\quad=\widetilde{\tau}(p_1\overline{q_1}\otimes\xi\otimes q_2\overline{p_2}|(p_3\overline{q_3}\otimes\ell\otimes q_3\overline{p_4})(p_5\overline{q_5}\otimes m\otimes q_6\overline{p_6}))
\end{aligned}$$

proves (5.2) for $\widetilde{\tau}$ (we omit treating the second part).

The proof for (5.3) is similar. □

By definition, a skew pairing $\tau\colon\Lambda\otimes L\to R$ induces a map $\phi\colon\Lambda\to\mathrm{Hom}_{\overline{R}-}(L,R)$. There is an R^e-ring structure [21, Lem.5.5] on $L^\vee:=\mathrm{Hom}_{\overline{R}-}(L,R)$ for which ϕ is a morphism of R^e-rings. In particular, the induced R^e-bimodule structure [21, Def.5.4] satisfies $(r\overline{s}\xi t\overline{u})(\ell)=r\xi(t\ell u\overline{s})$ for $r,s,t,u\in R$, $\xi\in L^\vee$, and $\ell\in L$.

If L is finitely generated projective as left $\overline{R}$-module, then $L^\vee$ has a $\times_R$-bialgebra structure [21, Thm.5.12] such that evaluation defined a skew pairing between $L^\vee$ and L. We call this $\times_R$-bialgebra the left dual of L.

Proposition 5.4. *Let* L *be a* $\times_R$*-bialgebra that is finitely generated projective as left* $\overline{R}$*-module.*

Let $S\overset{M}{\sim}R$, *and let* $\widetilde{L}$ *be the* $\times_S$*-bialgebra obtained from* L *by Morita base change. Then* $\widetilde{L}$ *is finitely generated projective as left* $\overline{S}$*-module, and its left dual* $\times_S$*-bialgebra* $\widetilde{L}^\vee$ *is isomorphic to the* $\times_S$*-bialgebra* $\widetilde{L^\vee}$ *that is obtained from the left dual* $L^\vee$ *of* L *by Morita base change.*

Proof. $\widetilde{L}=P^e\otimes_{R^e}L\otimes_{R^e}Q^e$ is finitely generated projective as left $\overline{S}$-module since the modules P_R, ${}_{\overline{S}}\overline{Q}$, ${}_{\overline{R}}L$, and ${}_{R^e}Q^e$ are finitely generated projective.

Since the R-modules P and Q are finitely generated projective and each other's dual, we have $\mathrm{Hom}(P\otimes_R M,V)\cong\mathrm{Hom}(M,V)\otimes_R Q$ for any $M\in{}_R\mathcal{M}$ and any k-module V, and similarly $\mathrm{Hom}(N\otimes_R Q,V)\cong P\otimes_R\mathrm{Hom}(N,V)$. We use this three times in the second isomorphism in the following calculation. The first isomorphism uses the category equivalence ${}_{\overline{R}}\mathcal{M}\cong{}_{\overline{S}}\mathcal{M}$ given by tensoring with $\overline{Q}$. The fourth isomorphism is an instance of the general isomorphism $\mathrm{Hom}_{R-}(M,P)\cong\mathrm{Hom}_{R-}(M,R)\otimes_R P$, in the third we have used that $\int^r$ commutes with tensor products by flat (in particular by projective) modules. The last step merely replaces the rightmost P by $\overline{P}$.

$$\begin{aligned}\operatorname{Hom}_{\overline{S}-}(\widetilde{L},S) = \operatorname{Hom}_{\overline{S}-}(P^e \underset{R^e}{\otimes} L \underset{R^e}{\otimes} Q^e, P\underset{R}{\otimes} Q) &\cong \int^r \operatorname{Hom}(P \underset{R}{\otimes} {}_{\overline{r}}L \underset{R^e}{\otimes} Q^e, P_r)\\ &\cong \int^r \int_{s,t,u} P_s \otimes \overline{Q}_{\overline{t}} \otimes \operatorname{Hom}({}_{u\overline{r}}L_{s\overline{t}}, P_r) \otimes {}_uQ\\ &\cong \int_{s,t,u} P_s \otimes \overline{Q}_{\overline{t}} \otimes \int^r \operatorname{Hom}({}_{u\overline{r}}L_{s\overline{t}}, P_r) \otimes {}_uQ\\ \cong \int_{s,t,u,v} P_s \otimes \overline{Q}_{\overline{t}} \otimes \operatorname{Hom}_{\overline{R}}({}_uL_{s\overline{t}}, {}_vR) \otimes {}_uQ \otimes {}_vP &\cong P^e \underset{R^e}{\otimes} L^\vee \underset{R^e}{\otimes} Q^e.\end{aligned}$$

Let $F\colon P^e \otimes_{R^e} L^\vee \otimes_{R^e} Q^e \to \widetilde{L}^\vee$ denote the resulting isomorphism. It is easy to check that

$$F(p_1\overline{q_1} \otimes \xi \otimes q_2\overline{p_2})(p_3\overline{q_3} \otimes \ell \otimes q_4\overline{p_4}) = p_1\xi((q_2p_3)\ell(q_4p_2)(\overline{q_1p_4}))q_3$$

for $p_1,\dots,p_4 \in P$, $q_1,\dots,q_4 \in Q$, $\xi \in L^\vee$ and $\ell \in L$. In other words, the evaluation of $\widetilde{L}^\vee$ on $\widetilde{L}$ can be written as the composition

$$\widetilde{L}^\vee \otimes \widetilde{L} \xrightarrow{F\otimes\widetilde{L}} \widetilde{L^\vee} \otimes \widetilde{L} \xrightarrow{\widetilde{\tau}} S,$$

where $\widetilde{\tau}$ is the skew pairing of $\times_S$-bialgebras induced by the evaluation $\tau\colon L^\vee \otimes L \to R$. It follows that F is an isomorphism of $\times_S$-bialgebras. □

Remark 5.5. Let R, S, L be as above. Since ${}^L\mathcal{M} \cong {}_{L^\vee}\mathcal{M}$ as monoidal categories by [21, Cor.5.15], we can conclude that ${}^L\mathcal{M} \cong {}^{\widetilde{L}}\mathcal{M}$ as monoidal categories. Moreover, the equivalence is induced by the monoidal category equivalence ${}_R\mathcal{M}_R \cong {}_S\mathcal{M}_S$. Explicitly, it asssigns to $M \in {}^L\mathcal{M}$ the S-bimodule $P \otimes_R M \otimes_R Q$ endowed with the left $\widetilde{L}$-comodule structure

$$\begin{aligned}P \underset{R}{\otimes} M \underset{R}{\otimes} Q &\to (P^e \underset{R^e}{\otimes} L \underset{R^e}{\otimes} Q^e) \times_S (P \underset{R}{\otimes} M \underset{R}{\otimes} Q)\\ p \otimes m \otimes q &\mapsto (p \otimes \overline{q_i} \otimes m_{(-1)} \otimes q \otimes \overline{p_j}) \otimes (p^i \otimes \otimes m_{(0)} \otimes q^j)\end{aligned}$$

(where $M \ni m \mapsto m_{(-1)} \otimes m_{(0)} \in L \times_R M$ denotes the comodule structure on M).

We conjecture that the same formula can be used to define a category equivalence ${}^L\mathcal{M} \cong {}^{\widetilde{L}}\mathcal{M}$ when L is not finitely generated projective as left $\overline{R}$-module.

Remark 5.6. Let L be a $\times_R$-bialgebra which is finitely generated projective as left $\overline{R}$-module, and let $R \overset{\sqrt{\mathrm{M}}}{\sim} S$. Let the monoidal equivalence ${}_{R^e}\mathcal{M} \cong {}_{S^e}\mathcal{M}$ be induced by a Morita context involving the modules $C \in {}_{S^e}\mathcal{M}_{R^e}$ and $D \in {}_{R^e}\mathcal{M}_{S^e}$. By the remarks closing Section 3, we know that $C \otimes_{R^e} L$ is a finitely generated projective left $\overline{S}$-module. Since D is a finitely generated projective left R^e-module, we can conclude that $\widetilde{L} = C \otimes_{R^e} L \otimes_{R^e} D$ is a finitely generated projective left $\overline{S}$-module.

However, we do not know in this situation whether $\widetilde{L}^\vee$ and $\widetilde{L^\vee}$ are isomorphic $\times_S$-bialgebras.

Recall that the left dual $L^\vee$ is finitely generated projective as left R-module. For $\times_R$-bialgebras H such that ${}_RH$ is finitely generated projective, one can define a right dual $\times_R$-bialgebra ${}^\vee H$ in such a way that ${}^\vee(L^\vee) \cong L$.

Now let $\hat{L} := {}^\vee\widetilde{L^\vee}$ be the right dual $\times_S$-bialgebra of the $\times_S$-bialgebra obtained by $\sqrt{\text{Morita}}$ base change from the right dual of L (note that $\widetilde{L^\vee}$ is a finitely generated projective left S-module by reasoning similar to that used for ${}_{\overline{S}}\widetilde{L}$). Then we have equivalences of monoidal categories

$$^{L}\mathcal{M} \cong {}_{L^\vee}\mathcal{M} \cong {}_{\widetilde{L^\vee}}\mathcal{M} \cong {}^{\hat{L}}\mathcal{M}.$$

If our $\sqrt{\text{Morita}}$ equivalence comes from a Morita equivalence, then $\hat{L} \cong \widetilde{L}$. Otherwise, it seems that we have another version of $\sqrt{\text{Morita}}$ base change, suitable for comodules instead of modules.

More conjecturally, such a dual version of $\sqrt{\text{Morita}}$ base change should also be possible if L is not assumed to be finitely generated projective as left $\overline{R}$-module.

6. Canonical Tannaka duality

In this section we let k be a field. Let $\mathcal{C}$ be a semisimple k-linear tensor category with a finite number of simple objects whose endomorphism rings are isomorphic to k. Hayashi [11, 10] has proved that $\mathcal{C}$ is equivalent to the category of modules over a finite dimensional face algebra F. The construction can of course be applied to ${}_H\mathcal{M}$ where H is a split semisimple quasi-Hopf algebra, though it is not so clear how F is related to H.

In this section we will describe a connection between the "given" H and the "canonical" F. This proceeds in two steps. First, one uses a generalized smash product construction that produces a $\times_H$-bialgebra L isomorphic to $H \otimes H \otimes H^*$ as a vector space, and with ${}_L\mathcal{M} \cong {}_H\mathcal{M}$ as monoidal categories. In a second step, we use Morita base change to replace H by the Morita equivalent product of copies of the base field. The result is a face algebra $\widetilde{L}$, and we shall show that $\widetilde{L} \cong F$.

Let us first recall some elements of Hayashi's construction. An important step is the construction of a monoidal functor $\Omega_0 \colon \mathcal{C} \to {}_R\mathcal{M}_R$, where $R = k^n$ and n is the number of isomorphism classes of simple objects in $\mathcal{C}$. We will not go into details on the second important step, which is the construction of a unique face algebra F with n face idempotents such that Ω_0 factors over a monoidal equivalence $\Omega \colon \mathcal{C} \to {}_F\mathcal{M}$. (In fact Hayashi uses modules rather than comodules, which is of no importance since the face algebra he constructs is finite dimensional.)

Let Λ be the set of isomorphism classes of simple objects in $\mathcal{C}$. For simplicity we let $R \cong k^n$ have the set Λ as its canonical basis of idempotents.

Hayashi's canonical functor Ω_0 sends $X \in \mathcal{C}$ to the R-bimodule $\Omega_0(X)$ with $\mu\Omega_0(X)\lambda = \mathrm{Hom}_{H-}(L_\mu, X \otimes L_\lambda)$, where, compared with Hayashi's comvention, we have switched the sides in R-bimodules and replaced tensor product in $\mathcal{C}$ by its opposite. The monoidal functor structure ω of Ω_0 is the map

$$\begin{aligned}
\Omega_0(X) \underset{R}{\otimes} \Omega_0(Y) &\xrightarrow{\omega} \Omega_0(X \otimes Y) \\
\mu\Omega_0(X)\rho \otimes \rho\Omega_0(Y)\lambda &\to \Omega_0(X \otimes Y) \\
f \otimes g &\mapsto (L_\mu \xrightarrow{f} X \otimes L_\rho \xrightarrow{X \otimes g} X \otimes (Y \otimes L_\lambda) \xrightarrow{\Phi^{-1}} (X \otimes Y) \otimes L_\lambda),
\end{aligned}$$

where $\mu, \rho, \lambda \in \Lambda$, and Φ denotes the associator isomorphism in the category $\mathcal{C}$.

Now let H be a split semisimple quasi-Hopf algebra. We will apply Hayashi's constructions to $\mathcal{C} = {}_H\mathcal{M}$, and investigate the relation of F to H.

The first step is a construction suggested by Hausser and Nill (see [7], Proposition 3.11 and the remarks following the proof): They have defined a category ${}_H\mathcal{M}_H^H$ of Hopf modules over H, which is monoidal in such a way that the underlying functor

$\mathcal{U}\colon {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$ is a strict monoidal functor. They show ${}_H\mathcal{M}_H^H$ to be equivalent as a monoidal category to ${}_H\mathcal{M}$ via a monoidal functor

$$(\mathcal{R},\xi)\colon {}_H\mathcal{M} \ni V \to V\otimes H \in {}_H\mathcal{M}_H^H.$$

Now by translating the coaction of H on a Hopf module into an action of the dual H^*, one can describe Hopf modules in ${}_H\mathcal{M}_H^H$ equivalently as modules over a certain generalized smash product $L := (H\otimes H^{\mathrm{op}})\# H^*$. This kind of classification of Hopf modules by modules over an algebra which is a product of several copies of H and its dual goes back to Cibils and Rosso [4]. We refer the reader to [22, Ex.4.12] for details on the construction of L. Since the underlying functor

$${}_{(H\otimes H^{\mathrm{op}})\# H^*}\mathcal{M} \cong {}_H\mathcal{M}_H^H \to {}_H\mathcal{M}_H$$

is strictly monoidal, it follows from [19, Thm.5.1] that L has the structure of a $\times_H$-bialgebra such that ${}_L\mathcal{M} \cong {}_H\mathcal{M}_H^H \cong {}_H\mathcal{M}$ are equivalences of monoidal categories.

H being split semisimple, it is Morita equivalent to a direct product of copies of k. We claim that Hayashi's F results from applying the appropriate Morita base change to L.

To see this, we have to verify that the diagram of monoidal functors

$$\begin{tikzcd} {}_H\mathcal{M} \arrow[r, "\mathcal{R}"] \arrow[d, "\Omega_0"] & {}_H\mathcal{M}_H^H \arrow[d, "\mathcal{U}"] \\ {}_R\mathcal{M}_R & {}_H\mathcal{M}_H \arrow[l, "\mathcal{F}"] \end{tikzcd}$$

commutes up to isomorphism of monoidal functors. Here $\mathcal{F}$ denotes the monoidal functor given by the Morita equivalence between R and H.

For $\lambda\in\Lambda$, now the set of simple modules in ${}_H\mathcal{M} = \mathcal{C}$, fix a minimal idempotent $e_\lambda \in H$ such that $L_\lambda := He_\lambda \in \Lambda$.

The functor $\mathcal{UR}$ maps $X\in {}_H\mathcal{M}$ to $\mathcal{UR}(X) = .X\otimes .H. \in {}_H\mathcal{M}_H$; here the dots indicate that $X\otimes H$ is equipped with the diagonal left H-module structure and the right module structure induced by that of the right tensor factor. The functor $\mathcal{F}$ maps $M\in {}_H\mathcal{M}_H$ to the R-bimodule defined by $\mu\mathcal{F}(M)\lambda = e_\mu M e_\lambda$, so that $\mathcal{FUR}(X)$ satisfies

$$\begin{aligned}\mu\mathcal{FUR}(X)\lambda = e_\mu\mathcal{UR}(X)e_\lambda = e_\mu(X\otimes H)e_\lambda &\cong \mathrm{Hom}_{H-}(He_\mu, X\otimes He_\lambda)\\ &= \mathrm{Hom}_{H-}(L_\mu, X\otimes L_\lambda) = \mu\Omega_0(X)\lambda.\end{aligned}$$

Note that the isomorphism $\psi\colon \Omega_0(X) \cong \mathcal{F}(X\otimes H)$ we have found maps $f\in \mathrm{Hom}_{H-}(L_\mu, X\otimes L_\lambda)$ to $\psi(f) = f(e_\mu) \in e_\mu(X\otimes H)e_\lambda$.

We still have to show that $\Omega_0 \cong \mathcal{FUR}$ as *monoidal* functors.

The monoidal functor structure ξ of $\mathcal{R}$ is the isomorphism

$$(X\otimes H)\underset{H}{\otimes}(Y\otimes H) = X\otimes(Y\otimes H) \xrightarrow{\Phi^{-1}} (X\otimes Y)\otimes H$$

in which the first equality is the canonical identification.

For $M,N\in {}_H\mathcal{M}_H$ we can identify $\mathcal{F}(M\otimes_H N)\subset M\otimes_H N$ with $\mathcal{F}(M)\otimes_R \mathcal{F}(N)$, which makes $\mathcal{F}$ a strict monoidal functor. In particular, the monoidal functor structure of $\mathcal{FUR}$ is the restriction of that of $\mathcal{R}$; we shall denote it by ξ again.

We need to show that the diagrams

$$\begin{tikzcd}
\Omega_0(X) \otimes_R \Omega_0(Y) \arrow[r, "\omega"] \arrow[d, "\psi\otimes\psi"] & \Omega_0(X \otimes Y) \arrow[d, "\psi"] \\
\mathcal{FUR}(X) \otimes_R \mathcal{FUR}(Y) \arrow[r, "\xi"] & \mathcal{FUR}(X \otimes Y)
\end{tikzcd}$$

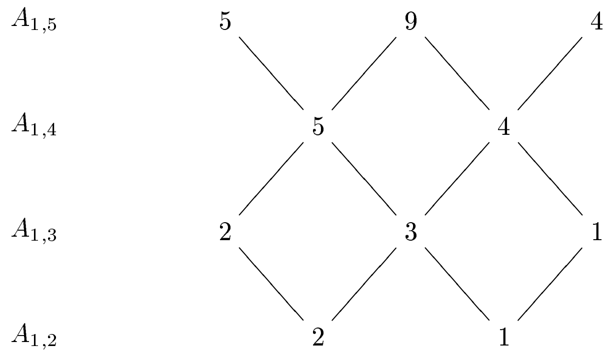
commute for $X, Y \in {}_H\mathcal{M}$. Let $f \in \mathrm{Hom}_H(L_\mu, X \otimes L_\rho)$ and $g \in \mathrm{Hom}_H(L_\rho, Y \otimes L_\lambda)$. Then $\psi\omega(f \otimes g) = \omega(f \otimes g)(e_\mu) = \Phi^{-1}(X \otimes g)f(e_\mu)$. On the other hand, write $f(e_\mu) = \sum x_i \otimes h_i$ with $x_i \in X$ and $h_i \in L_\rho$. Then we have

$$\begin{aligned}
\xi(\psi \otimes \psi)(f \otimes g) &= \xi(f(e_\mu) \otimes g(e_\rho)) = \Phi^{-1}(\sum x_i \otimes h_i g(e_\rho)) \\
&= \Phi^{-1}(\sum x_i \otimes g(h_i e_\rho)) = \Phi^{-1}(\sum x_i \otimes g(h_i)) = \Phi^{-1}(X \otimes g)f(e_\mu).
\end{aligned}$$

7. An example from Subfactor theory

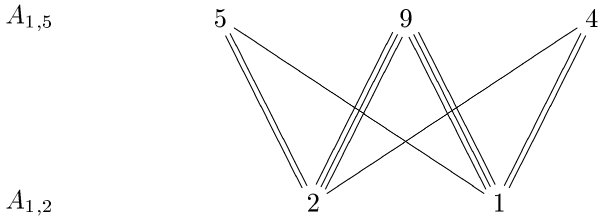
Nikshych and Vaĭnerman have shown how to associate a weak Hopf algebra to a subfactor of finite depth of a von Neumann algebra factor [15, 16]. The case of a subfactor $N \subset M$ of a type II_1 factor of index $\beta = 4\cos^2\frac{\pi}{n+3}$ with $n \geq 2$ is treated in more detail in [15, 17]. The associated weak Hopf algebra can be described as follows (we summarize the beginning of [17, sec.2.7]): Let $\mathcal{A}_{\beta,k}$ be the Temperley-Lieb algebra as in [6], that is, the unital algebra freely generated by idempotents $e_1, \dots, e_{k-1}$ subject to the relations $\beta e_i e_j e_i = e_i$ for $|i-j| = 1$ and $e_i e_j = e_j e_i$ for $|i-j| \geq 2$. Then $\mathcal{A}_{\beta,k}$ is semisimple for $k \leq n+1$ by the choice of β (cf. [6, §2.8]). Define $A_{1,k}$ by $A_{1,k} = \mathcal{A}_{\beta,k+1}$ if $k \leq n+1$, and let $A_{1,k+1}$ be obtained by applying the Jones basic construction to the inclusion $A_{1,k-1} \subset A_{1,k}$ for $k \geq n+1$. Thus $H := A_{1,2n-1}$ is generated by idempotents $e_1, \dots, e_{2n-1}$, and contains $A_{1,n-1}$, generated by $e_1, \dots, e_{n-1}$, and $A_{n+1,2n-1}$, generated by $e_{n+1}, \dots, e_{2n-1}$, as subalgebras. Nikshych and Vaĭnerman describe a weak Hopf algebra structure of H with target counital subalgebra $H_t = A_{1,n-1}$ and $H_s = A_{n+1,2n-1}$. For $n = 2$, $H_t \cong \mathbb{C}\oplus\mathbb{C}$ is commutative, and $H \cong M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ is a face algebra of dimension 13.

We shall examine the case $n = 3$. Here the Bratteli diagram of the inclusions $A_{1,2} \subset \cdots \subset A_{1,5}$ is obtained from that of the inclusion $A_{1,2} \subset A_{1,3}$ (which is found in [6]) by applying Jones' basic construction twice:

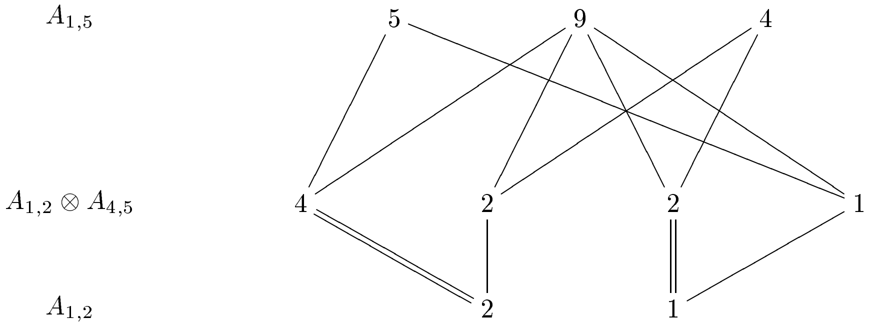


We see that $H = A_{1,5}$ has dimension $5^2 + 9^2 + 4^2 = 122$, and its counital subalgebras are isomorphic to $\mathbb{C} \oplus M_2(\mathbb{C})$, of dimension 5. We will apply Morita base change to this example, reducing the counital subalgebra to $\mathbb{C} \oplus \mathbb{C}$. We will not derive an explicit description of the resulting face algebra, but will be content with determining its algebra structure, hence its dimension.

The Bratteli diagram for the inclusion $H_t \subset H$ is obtained by composing the stories of the Bratteli diagram above:

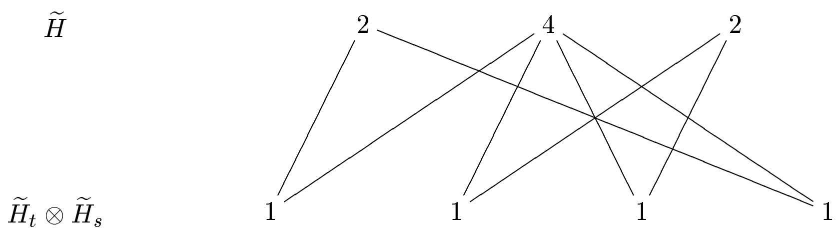


To apply Morita base change, we need the Bratteli diagram for the map $H_s \otimes H_t \to H$. Of course $H_s \otimes H_t \cong M_4(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$, and the antipode of H, which exchanges H_s and H_t, will switch the two copies of $M_2(\mathbb{C})$. The lower story of the following tower is the inclusion $H_t \subset H_t \otimes H_s$:



We claim that the upper story is the Bratteli diagram for the map $H_t \otimes H_s \to H$. It will help to know that, since the antipode switches the two vertices labelled 2 on the middle floor, any top floor vertex has the same number of edges to each of these vertices. In particular, there can be no edge from the 5 on the top floor to a 2 on the middle floor, since there is only one 1 on the bottom floor linked to 5. Also, there can be no more than one link from 5 to the 1 on the middle floor, since there should be only one link to the 1 on the bottom floor. This makes the two links from 5 to the middle floor inevitable as shown. There should be three links from the top 9 to the bottom 2. These cannot be accounted for by three links to the left 2 on the middle floor, since that would also entail three links to the right 2, and hence six links to the bottom 1. So there has to be one link to the 4 and one to the left 2, hence also the right 2 on the middle floor, which makes the one link to the 1 on the middle floor also inevitable. Finally the one link from the top 4 to the bottom 2 can only be accounted for by a single link from the top 4 to the left 2 on the middle floor, hence there also has to be one to the right 2, and there is no room for more.

Now we apply Morita base change to pass from H with counital subalgebra $H_t \cong \mathbb{C} \oplus M_2(\mathbb{C})$ to $\widetilde{H}$ with counital subalgebra $\widetilde{H}_t \cong \mathbb{C} \oplus \mathbb{C}$. The Bratteli diagram for the inclusion $\widetilde{H}_t \oplus \widetilde{H}_s \subset \widetilde{H}$ is the same as for $H_t \oplus H_s \subset H$, but with all ranks on the lower floor replaced by 1:



The resulting ranks on the upper floor show that $\dim \widetilde{H} = 2^2 + 4^2 + 2^2 = 24$. (Remember that $\dim H = 122$.)

Remark 7.1. By Hayashi's canonical Tannaka duality, there is a face algebra F with ${}_F\mathcal{M} \cong {}_H\mathcal{M}$ as monoidal categories, where F has three face idempotents (since H has three isomorphism classes of irreducible modules) whereas our $\widetilde{H}$ has two. Hayashi has also described another procedure to associate a face algebra to any subfactor of a II_1 factor [9], which will, however, also yield a face algebra that has as many faces as isomorphism classes of irreducible modules. By contrast, applying Morita base change to the weak Hopf algebra $A_{1,2n-1}$ of Nikshych and Vaĭnerman will yield a face algebra with one face less than isomorphism classes of irreducible modules whenever n is odd.

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