HOPF BIMODULES OVER HOPF-GALOIS EXTENSIONS, MIYASHITA-ULBRICH ACTIONS, AND MONOIDAL CENTER CONSTRUCTIONS

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1 Introduction

The main result, theorem 4.3, of this paper is the calculation of the center of the monoidal category ${}_A\mathcal{M}_A^H$ of relative Hopf bimodules over the flat H-Galois extension A.

Let H be a Hopf algebra with bijective antipode over the field k. In [10] we have studied the structure of two-sided two-cosided Hopf modules over H and found a category equivalence ${}_H^H\mathcal{M}_H^H \cong \mathcal{YD}_H^H$ with the category of Yetter-Drinfel'd modules [17, 8]. Knowing the latter is the center of $\mathcal{M}_H$, the category of right H-modules, and ${}_H\mathcal{M}_H^H \cong \mathcal{M}_H$, we conclude that $\mathcal{Z}({}_H\mathcal{M}_H^H) \cong \mathcal{YD}_H^H \cong {}_H^H\mathcal{M}_H^H$.

In [11] we have constructed a monoidal functor ${}_H^H\mathcal{M}_H^H \to \mathcal{Z}({}_A\mathcal{M}_A^H)$ for any H-comodule algebra A, generalizing the functor cited above for $A = H$. This of course can be described equivalently as a functor $\mathcal{YD}_H^H \to \mathcal{Z}({}_A\mathcal{M}_A^H)$, which we know to be an equivalence if $A = H$.

In section 4 we will establish the same result for an important class of H-comodule algebras, namely for flat Hopf-Galois extensions. The crucial part of the proof, namely the reconstruction of a Yetter-Drinfel'd module from an object of $\mathcal{Z}({}_A\mathcal{M}_A^H)$, uses (a minor generalization of) the Miyashita-Ulbrich action considered in [15, 2]. (The author thanks Alexander Milinski for drawing his attention to the occurrence of Yetter-Drinfel'd modules in this context.) An adjunction between ${}_A\mathcal{M}_A^H$ and the category of modules over the (commutative) algebra A^B in $\mathcal{YD}_H^H$ is described in section 3.

In section 5 we come back to that adjunction. For a flat H-Galois extension we exhibit an adjunction between ${}_{A^B}\left(\mathcal{YD}_H^H\right)$ and the centralizer of a subcategory (described by a certain commutation relation) in ${}_A\mathcal{M}_A^H$. The adjunction is an equivalence under more restrictive hypotheses on A. As a corollary, we exhibit a close relation of the center of a subcategory of ${}_A\mathcal{M}_A^H$ with the concept of dyslectic modules of [7].

Our interest in the centers and centralizers considered in this paper has two sources of motivation.

First, braided monoidal categories are a basic tool in the construction and understanding of knot invariants [3, 9]. The center construction [4] and its extension [5] to ribbon categories is a machinery providing fundamental examples of braided categories. From this viewpoint, the result of theorem 4.3 is a negative answer to a special case of the following question: Does application of the center construction to the monoidal category ${}_A\mathcal{M}_A^H$ yield essentially new braided categories with potential applications to knot theory? The center computed in corollary 5.4 also only yields a special case of the type of braided category considered in [7].

The second motivation of this paper stems from the theory of quantum groups and quantum spaces, in particular their differential calculi. The well known examples show that the noncommutativity of the algebra of a quantum space or a bimodule of differential forms is more often than not controlled by some kind of commutation relation, often connected with a solution of the Yang-Baxter equation. For that reason, commutative algebras in braided categories (like A^B in $\mathcal{YD}_H^H$ in this paper) have been studied as a nice framework for noncommutative algebraic geometry. One may hope that the condition that a bimodule be contained in the center of a category or at least the centralizer of a specified subcategory, can serve as a condition excluding too noncommutative behavior of a differential calculus. A look at [7] shows that such a condition can be different from that of imposing the "obvious" commutation relation. Also, maps $M \underset{A}{\otimes} N \to N \underset{A}{\otimes} M$ "switching" tensorands in a nonstandard way, and subject to the braid equation, play a key role in many constructions of quantum group theory. We cite only the construction [16] of the exterior algebra over a bicovariant differential calculus over a quantum group. Its intimate relation to the braiding of ${}_H^H\mathcal{M}_H^H$, as explained in [12], brings this particular example close to the braided categories considered in this paper.

2 Preliminaries

The following remarks on (weak) centers and centralizers of monoidal categories are an insignificant generalization of the treatment in [4].

Let $\mathcal{D}$ be a monoidal category and $\mathcal{C} \subset \mathcal{D}$ a subcategory with $I \in \mathcal{C}$. The left weak centralizer of $\mathcal{C}$ in $\mathcal{D}$, denoted by $\mathcal{W}_\ell(\mathcal{C};\mathcal{D})$, is a category whose objects are pairs (A,a) where $A \in \mathcal{D}$, and $a_X : A \otimes X \to X \otimes A$ is a morphism, natural in $X \in \mathcal{C}$, such that the diagrams

$$\begin{tikzcd}
A \otimes X \otimes Y \arrow[rr, "a_{X\otimes Y}"] \arrow[dr, "a_X \otimes Y"'] & & X \otimes Y \otimes A \\
& X \otimes A \otimes Y \arrow[ur, "X \otimes a_Y"'] &
\end{tikzcd} \tag{2.1}$$

commute for all $X, Y \in \mathcal{C}$, and $a_I : A \otimes I \to I \otimes A$ is the canonical isomorphism. A morphism $f : (A,a) \to (B,b)$ in $\mathcal{W}_\ell(\mathcal{C};\mathcal{D})$ is a morphism $f : A \to B$ in $\mathcal{D}$ such that the diagrams

$$\begin{tikzcd}
A \otimes X \arrow[r, "a"] \arrow[d, "f \otimes X"'] & X \otimes A \arrow[d, "X \otimes f"] \\
B \otimes X \arrow[r, "b"'] & X \otimes B
\end{tikzcd}$$

commute for all $X \in \mathcal{C}$. The centralizer $\mathcal{Z}(\mathcal{C};\mathcal{D})$ of $\mathcal{C}$ in $\mathcal{D}$ is the full subcategory in the weak left centralizer whose objects are those (A,a) in which a is an isomorphism. The left weak centralizer and the centralizer of $\mathcal{C}$ in $\mathcal{D}$ are monoidal categories with tensor product $(A,a)\otimes(B,b) = (A\otimes B, (a\otimes id)(id\otimes b))$.

The left weak center of $\mathcal{D}$ is $\mathcal{W}_\ell(\mathcal{D}) := \mathcal{W}_\ell(\mathcal{D};\mathcal{D})$. It is a prebraided monoidal category with prebraiding $\sigma_{(A,a),(B,b)} = a_B : (A,a)\otimes(B,b) \to (B,b)\otimes(A,a)$.

The center of $\mathcal{D}$ is $\mathcal{Z}(\mathcal{D}) := \mathcal{Z}(\mathcal{D};\mathcal{D})$. It is a braided monoidal category.

We need to fix some conventions on algebras and modules in (prebraided) monoidal categories. Let A be an algebra in the monoidal category $\mathcal{C}$. We denote by $\mathcal{C}_A$ the category of right A-modules, by ${}_A\mathcal{C}$ the category of left A-modules and by ${}_A\mathcal{C}_A$ the category of A-A-bimodules. As a special case let H be a bialgebra and consider the monoidal category $\mathcal{M}^H$ of right H-comodules. An algebra A in $\mathcal{M}^H$ is just a right H-comodule algebra. We write $\mathcal{M}^H_A := (\mathcal{M}^H)_A$; this is just the category of (relative, as opposed to the case $A = H$) Hopf modules, that is, right H-comodules and right A-modules

M with $(ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$ for $a \in A$ and $m \in M$. (For coalgebra and comodule structures we use Sweedler's notation without the summation symbol.) Hopf modules in ${}_A\mathcal{M}^H$ are defined similarly. We denote by ${}_A\mathcal{M}_A^H = {}_A(\mathcal{M}^H)_A$ the category of Hopf bimodules.

As another example, consider the monoidal category $\mathcal{M}_H$ of right H-modules. An algebra in $\mathcal{M}_H$ is a right H-module algebra, that is, an algebra and an H-module with $(xy) \leftharpoonup h = (x \leftharpoonup h_{(1)})(y \leftharpoonup h_{(2)})$ for $x, y \in A$ and $h \in H$, where $\leftharpoonup$ denotes the H-module structure. An object $M \in (\mathcal{M}_H)_A$ is an H-module and A-module with $(mx) \leftharpoonup h = (m \leftharpoonup h_{(1)})(x \leftharpoonup h_{(2)})$ for $m \in M$, $x \in A$ and $h \in H$.

Let H be a bialgebra. A (right-right) Yetter-Drinfel'd module is a right H-module (with the action denoted by $\leftharpoonup$) and right H-comodule V with the compatibility condition $v_{(0)} \leftharpoonup h_{(1)} \otimes v_{(1)}h_{(2)} = (v \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(v \leftharpoonup h_{(2)})_{(1)}$ for all $h \in H$ and $v \in V$. We denote by $\mathcal{YD}_H^H$ the category of right-right Yetter-Drinfel'd modules. Yetter-Drinfel'd modules were introduced in [17].

By a reasoning similar to that in [5], the monoidal category $\mathcal{YD}_H^H$ is isomorphic to the left weak center of $\mathcal{M}^H$ (to the center if H is a Hopf algebra with bijective antipode), where, for $W \in \mathcal{M}^H$ and $V \in \mathcal{YD}_H^H$ we have

$$\sigma_{VW} : V \otimes W \ni v \otimes w \mapsto w_{(0)} \otimes v \leftharpoonup w_{(1)} \in W \otimes V \tag{2.2}$$

In particular, this equation also defines a (pre-)braiding for $\mathcal{YD}_H^H$. The same equation, now for $W \in \mathcal{YD}_H^H$ and $V \in \mathcal{M}_H$, also exhibits $\mathcal{YD}_H^H$ as the right weak center of $\mathcal{M}_H$ (defined in the obvious way), which is again isomorphic to the center if H is a Hopf algebra with bijective antipode.

Let $(\mathcal{C}, \sigma)$ be a prebraided monoidal category and A a commutative algebra in $\mathcal{C}$ (that is, the multiplication $\nabla : A \otimes A \to A$ is assumed to satisfy $\nabla = \nabla\sigma$). We say that A acts left symmetrically on $M \in {}_A\mathcal{C}_A$, if the right and left module structures ρ and λ of M satisfy $\lambda = \rho\,\sigma_{AM}$. We say that A acts right symmetrically on M if $\rho = \lambda\,\sigma_{MA}$. The category ${}_A\mathcal{C}$ is naturally isomorphic to the full subcategory of those $M \in {}_A\mathcal{C}_A$ on which A acts right symmetrically, see [7] (and an analogous statement holds for left symmetric actions). A module $M \in {}_A\mathcal{C}$ is dyslectic, see [7], if the module structure λ satisfies $\lambda = \lambda\,\sigma_{MA}\,\sigma_{AM}$. We denote by ${}_A\tilde{\mathcal{C}}$ the category of dyslectic left A-modules. ${}_A\tilde{\mathcal{C}}$ is naturally isomorphic to the category of A-A-bimodules on which A acts both left and right symmetrically. By [7], the category of dyslectic left A-modules is prebraided (braided if $\mathcal{C}$ is).

As an example, consider an algebra $A \in \mathcal{YD}_H^H$. It is commutative if

$$xy = y_{(0)}(x \leftharpoonup y_{(1)}) \tag{2.3}$$

holds for all $x, y \in H$. Let M be an A-A-bimodule in $\mathcal{YD}_H^H$ (and A commutative). A acts left symmetrically on M iff the formula (2.3) holds for all $x \in A$ and $y \in M$, and right symmetrically iff it holds for all $x \in M$ and $y \in A$.

Let H be a Hopf algebra. In [10] it was shown that the well known, [14, p.84], category equivalence

$$\begin{array}{rcl} {}^H_H\mathcal{M} & \cong & \underline{Vec} \\ \Gamma & \mapsto & {}^{\mathrm{co}\, H}\Gamma \\ H \otimes V & \leftarrow\!\shortmid & V \end{array}$$

extends to a monoidal equivalence $({}^H_H\mathcal{M}^H_H, \underset{H}{\otimes}) \cong \mathcal{YD}^H_H$. If the antipode of H is a bijection, we also have a category equivalence ${}_H\mathcal{M}^H_H \cong \mathcal{M}_H$. We conclude that, if H is a Hopf algebra with bijective antipode, then we have a category equivalence

$$\mathcal{Z}({}_H\mathcal{M}^H_H) \cong \mathcal{Z}(\mathcal{M}_H) \cong \mathcal{YD}^H_H.$$

We will generalize this category equivalence in Theorem 4.3.

Throughout the paper, we fix a field k; all vector spaces, algebras etc. are over the ground field k.

For an algebra B and a B-B-bimodule N we denote the centralizer of B in N by $N^B := \{n \in N | \forall b \in B: \; bn = nb\}$.

3 Miyashita-Ulbrich actions

Throughout this section we assume that H is a bialgebra, and the H-comodule algebra A is a Hopf-Galois extension. Recall that A is an H-Galois extension of B if $B = A^{\mathrm{co}\, H}$ and the canonical mapping $\kappa : A \underset{B}{\otimes} A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$ is a bijection. For calculations with the inverse of κ we adopt a notation similar to that in [13]. We define a map $\gamma : H \to A \underset{B}{\otimes} A$ by $\gamma(h) = \kappa^{-1}(1 \otimes h)$ and write $\gamma(h) =: \sum \ell_i(h) \otimes r_i(h)$ (Caution: the finite index set needed might a priori depend on $h \in H$). From [13, (3.4)], we take, for the case that H is a

Hopf algebra, the following identities for γ, where $g, h \in H$, $x \in A$ and $b \in B$:

$$\nabla\gamma(h) = \sum \ell_i(h) r_i(h) = \varepsilon(h) \tag{3.1}$$

$$\sum \ell_i(h) \underset{B}{\otimes} r_i(h)_{(0)} \otimes r_i(h)_{(1)} = \sum \ell_i(h_{(1)}) \underset{B}{\otimes} r_i(h_{(1)}) \otimes h_{(2)} \tag{3.2}$$

$$\sum \ell_i(h)_{(0)} \underset{B}{\otimes} r_i(h) \otimes \ell_i(h)_{(1)} = \sum \ell_i(h_{(2)}) \underset{B}{\otimes} r_i(h_{(2)}) \otimes S(h_{(1)}) \tag{3.3}$$

$$\gamma(gh) = \sum \ell_i(h)\ell_j(g) \underset{B}{\otimes} r_j(g) r_i(h) \tag{3.4}$$

$$\sum \ell_i(h) \underset{B}{\otimes} 1 \underset{B}{\otimes} r_i(h) = \sum \ell_i(h_{(1)}) \underset{B}{\otimes} r_i(h_{(1)}) \ell_j(h_{(2)}) \underset{B}{\otimes} r_j(h_{(2)}) \tag{3.5}$$

$$\sum x_{(0)} \ell_i(x_{(1)}) \underset{B}{\otimes} r_i(x_{(1)}) = 1 \underset{B}{\otimes} x \tag{3.6}$$

Finally, for $h \in H$:

$$\gamma(h) \in (A \underset{B}{\otimes} A)^B \tag{3.7}$$

The only formula of those cited that actually uses the antipode of H is (3.3); the proof of the remaining ones in [13] does not use the antipode either. From (3.3) it follows that for $h \in H$ we have

$$\sum \ell_i(h_{(2)})_{(0)} \underset{B}{\otimes} r_i(h_{(2)}) \otimes h_{(1)} \ell_i(h_{(2)})_{(1)} = \sum \ell_i(h) \underset{B}{\otimes} r_i(h) \otimes 1 \tag{3.8}$$

This formula does not involve the antipode, and we can prove it for an arbitrary bialgebra H using the π-method of [2]: Applying the functor $\mathrm{Hom}_{A-}(—, A \underset{B}{\otimes} A \otimes H)$ to the left A-linear isomorphism $\kappa : A \underset{B}{\otimes} A \to A \otimes H$ yields an isomorphism

$$\pi : \mathrm{Hom}(H, A \underset{B}{\otimes} A \otimes H) \to \mathrm{Hom}_{B-}(A, A \underset{B}{\otimes} A \otimes H)$$

with $\pi(f)(x) = x_{(0)} f(x_{(1)})$. We apply this to the maps $f_i : H \to A \underset{B}{\otimes} A \otimes H$ given by

$$\begin{aligned} f_1(h) &= \sum \ell_i(h_{(2)})_{(0)} \underset{B}{\otimes} r_i(h_{(2)}) \otimes h_{(1)} \ell_i(h_{(2)})_{(1)} \\ f_2(h) &= \sum \ell_i(h) \underset{B}{\otimes} r_i(h) \otimes 1 \end{aligned}$$

to get

$$\begin{aligned}
\pi(f_1)(x) &= x_{(0)} \sum \ell_i(x_{(2)})_{(0)} \underset{B}{\otimes} r_i(x_{(2)}) \otimes x_{(1)} \ell_i(x_{(2)})_{(1)} \\
&= L\left(x_{(0)} \ell_i(x_{(1)}) \underset{B}{\otimes} r_i(x_{(1)})\right) \\
&= L(1 \underset{B}{\otimes} x) \\
&= 1 \underset{B}{\otimes} x \otimes 1 \\
&= x_{(0)} \sum \ell_i(x_{(1)}) \underset{B}{\otimes} r_i(x_{(1)}) \otimes 1 \\
&= \pi(f_2)(x),
\end{aligned}$$

where we define $L : A \underset{B}{\otimes} A \to A \underset{B}{\otimes} A \otimes H$ by $L(x \otimes y) = x_{(0)} \otimes y \otimes x_{(1)}$. This shows $f_1 = f_2$, which is (3.8).

Using (3.8) instead of (3.3) below makes our main results valid in the case of an H-Galois extension with H only a bialgebra. However, it seems to be an open question if such extensions can exist if H does not have an antipode. By [1, Prop. 5, (3)] H is a Hopf algebra if A is cleft and augmented.

The Miyashita-Ulbrich action of H on E^B, where E is an algebra and $\alpha : B \to E$ is an algebra homomorphism, was introduced in [2], generalizing the actions on A^B and on an algebra E with an algebra homomorphism $\alpha : B \to Z(E)$ introduced in [15]. We need a slightly more general version: If M is any A-A-bimodule, then we define the Miyashita-Ulbrich action of H on M^B by

$$M^B \otimes H \ni x \otimes h \mapsto x \leftharpoonup h := \sum \ell_i(h) x r_i(h) \in M^B. \tag{3.9}$$

$x \leftharpoonup h$ is a well defined element of M if $x \in M^B$, and $x \leftharpoonup h \in M^B$ then follows from (3.7). The Miyashita-Ulbrich action makes M^B a right H-module by (3.4):

$$(x \leftharpoonup g) \leftharpoonup h = \sum \ell_j(h) \ell_i(g)\, x\, r_i(g) r_j(h) = \sum \ell_i(gh)\, x\, r_i(gh) = x \leftharpoonup gh$$

It should be noted that the formulas proved for the Miyashita-Ulbrich action in [2] show that it is intimately connected with the theory of Yetter-Drinfel'd modules. In fact [2, (3.11)] shows A^B to be a Yetter-Drinfel'd module; by [2, (3.4)], A^B is in fact a commutative algebra in the braided category of Yetter-Drinfel'd modules. We redo the calculations for our slightly more general setting.

First, let $M \in {}_A\mathcal{M}_A^H$ be a Hopf module. Then M^B is a subcomodule of M, and an H-module with the Miyashita-Ulbrich action. We claim that M^B is in fact a Yetter-Drinfel'd module:

$$
\begin{aligned}
&(x \leftharpoonup h_{(2)})_{(0)} \otimes h_{(1)}(x \leftharpoonup h_{(2)})_{(1)} \\
&= \left(\sum \ell_i(h_{(2)})\, x\, r_i(h_{(2)})\right)_{(0)} \otimes h_{(1)} \left(\sum \ell_i(h_{(2)})\, x\, r_i(h_{(2)})\right)_{(1)} \\
&= \sum \ell_i(h_{(2)})_{(0)}\, x_{(0)}\, r_i(h_{(2)}) \otimes h_{(1)}\ell_i(h_{(2)})_{(1)} x_{(1)} h_{(3)} \\
&= \sum \ell_i(h_{(1)})\, x_{(0)} r_i(h_{(1)}) \otimes x_{(1)} h_{(2)} \\
&= x_{(0)} \leftharpoonup h_{(1)} \otimes x_{(1)} h_{(2)}
\end{aligned}
$$

where the second and third equality use (3.2) and (3.8), respectively. From (3.5) it follows that the formula

$$
\begin{aligned}
(xy) \leftharpoonup h &= \sum \ell_i(h)\, x\, y\, r_i(h) = \sum \ell_i(h_{(1)})\, x\, r_i(h_{(1)}) \ell_j(h_{(2)})\, y\, r_j(h_{(2)}) \\
&= (x \leftharpoonup h_{(1)})(y \leftharpoonup h_{(2)})
\end{aligned} \tag{3.10}
$$

holds for $h \in H$ and either $x \in A^B$ and $y \in M^B$ or $x \in M^B$ and $y \in A^B$.

For elements $x \in A$ and $m \in M^B$ the formula

$$
mx = x_{(0)}(m \leftharpoonup x_{(1)}) \tag{3.11}
$$

follows from (3.6): $x_{(0)}(m \leftharpoonup x_{(1)}) = \sum x_{(0)} \ell_i(x_{(1)}) m r_i(x_{(1)}) = mx$.

Since the braiding of $\mathcal{YD}_H^H$ maps $a \otimes x \in A^B \otimes A^B$ to $x_{(0)} \otimes a \leftharpoonup x_{(1)}$, we conclude from (3.10) and (3.11):

3.1. Corollary: *Let H be a bialgebra and A a right H-Galois extension of B. Then A^B is a commutative algebra in the prebraided category $\mathcal{YD}_H^H$.*

This allows us to formulate the following result:

3.2. Theorem: *Let H be a bialgebra and A a right H-Galois extension of B. Then there is a functor*

$$
\begin{aligned}
{}_{A^B}\left(\mathcal{YD}_H^H\right) &\to {}_A\mathcal{M}_A^H \\
T &\mapsto A \underset{A^B}{\otimes} T
\end{aligned}
$$

which is monoidal and left adjoint to

$$
\begin{aligned}
{}_A\mathcal{M}_A^H &\to {}_{A^B}\left(\mathcal{YD}_H^H\right) \\
M &\mapsto M^B
\end{aligned}
$$

PROOF: We identify ${}_{A^B}\left(\mathcal{YD}_H^H\right)$ with the category of those A^B-bimodules in $\mathcal{YD}_H^H$ on which A^B acts right symmetrically.

Let $T \in {}_{A^B}\left(\mathcal{YD}_H^H\right)$. First note that (as for any Yetter-Drinfel'd module T) we have $A \otimes T \in {}_A\mathcal{M}_A^H$ with the codiagonal comodule structure, the obvious left module structure and $(a \otimes t)x := ax_{(0)} \otimes t \leftharpoondown x_{(1)}$ for $a, x \in A$ and $t \in T$. Compatibility of the right module structure and the comodule structure uses the fact that T is Yetter-Drinfel'd, see the calculation following [11, (4.2)]. We have to check that $A \underset{A^B}{\otimes} T$ is a quotient Hopf module. This is obvious for the comodule structure and the left module structure. For the right module structure we check that, for $r \in A^B$, we have

$$\begin{aligned}
ax_{(0)} \underset{A^B}{\otimes} (rt) \leftharpoondown x_{(1)} &= ax_{(0)} \underset{A^B}{\otimes} (r \leftharpoondown x_{(1)})(t \leftharpoondown x_{(2)}) \\
&= ax_{(0)}(r \leftharpoondown x_{(1)}) \underset{A^B}{\otimes} t \leftharpoondown x_{(2)} \\
&= arx_{(0)} \underset{A^B}{\otimes} t \leftharpoondown x_{(1)}.
\end{aligned}$$

In this way we have defined a functor ${}_{A^B}\left(\mathcal{YD}_H^H\right) \to {}_A\mathcal{M}_A^H$ mapping T to $A \underset{A^B}{\otimes} T$. This functor is obviously monoidal.

We have to check that the claimed right adjoint is well defined. For $M \in {}_A\mathcal{M}_A^H$, that is, M an A-bimodule in $\mathcal{M}^H$, we conclude that M^B is an A^B-bimodule in $\mathcal{M}^H$, since A^B is a sub-comodule algebra of A, and M^B is a subcomodule and A^B-A^B-sub-bimodule of M. Formula (3.10) shows M^B to be an A^B-bimodule in $\mathcal{M}_H$, and thus in $\mathcal{YD}_H^H$. Finally, A^B acts right symmetrically on M^B by (3.11).

For $M \in {}_A\mathcal{M}_A^H$, we let $\mu : A \underset{A^B}{\otimes} M^B \to M$ be the map induced by the left module structure of M. We claim that μ is a map in ${}_A\mathcal{M}_A^H$: left linearity and colinearity are obvious, while right linearity is a consequence of (3.11):

$$\mu\left((a \otimes m)x\right) = \mu(ax_{(0)} \otimes m \leftharpoondown x_{(1)}) = ax_{(0)}(m \leftharpoondown x_{(1)}) = amx = \mu(a \otimes m)x$$

holds for $a, x \in A$ and $m \in M^B$.

For $T \in {}_{A^B}\left(\mathcal{YD}_H^H\right)$, we define $j : T \to \left(A \underset{A^B}{\otimes} T\right)^B$ to be the map sending $t \in T$ to $1 \otimes t \in A \underset{A^B}{\otimes} T$. This element is in $\left(A \underset{A^B}{\otimes} T\right)^B$ by definition of the right module structure on $A \underset{A^B}{\otimes} T$. The map j is obviously left A^B-linear and H-colinear. It is also right H-linear

$$\begin{aligned}
j(t) \leftharpoondown h &= \sum \ell_i(h)(1 \otimes t)r_i(h) \\
&= \sum \ell_i(h)r_i(h)_{(0)} \otimes t \leftharpoondown r_i(h)_{(1)} \\
&= 1 \otimes t \leftharpoondown h
\end{aligned}$$

and right A^B-linear:

$$j(t)r = r_{(0)} \otimes t \leftharpoonup r_{(1)} = 1 \otimes r_{(0)}(t \leftharpoonup r_{(1)}) = 1 \otimes tr = j(tr).$$

It is straightforward to check that j and μ satisfy the triangle equations for the unit and counit of adjunction. $\square$

4 The center of the category of relative Hopf bimodules

Let A be an H-comodule algebra over the Hopf algebra H. In [11], we have constructed a monoidal functor $A \underset{H}{\square} (-) : {}_H^H\mathcal{M}_H^H \to \mathcal{W}_\ell({}_A\mathcal{M}_A^H)$. Using the category equivalence ${}_H^H\mathcal{M}_H^H \cong \mathcal{YD}_H^H$ from [10] and the isomorphism $A \underset{H}{\square} \Gamma \cong A \otimes {}^{\mathrm{co}\,H}\Gamma$ in ${}_A\mathcal{M}_A^H$, we deduce the following result for the case that H is a Hopf algebra. We will omit the proof that the antipode is in fact not needed.

4.1. ***Proposition:*** *Let H be a bialgebra and A an H-comodule algebra. Then there is a monoidal functor*

$$\begin{array}{rcl} \mathcal{YD}_H^H & \to & \mathcal{W}_\ell({}_A\mathcal{M}_A^H) \\ V & \mapsto & (A \otimes V, \sigma_{A\otimes V,-}) \end{array}$$

where $A \otimes V$ is equipped with the structures

$$\begin{array}{rcl} \delta(a \otimes v) & = & a_{(0)} \otimes v_{(0)} \otimes a_{(1)}v_{(1)} \\ x(a \otimes v) & = & xa \otimes v \\ (a \otimes v)x & = & ax_{(0)} \otimes v \leftharpoonup x_{(1)} \end{array}$$

($a, x \in A$, $v \in V$) and, for $\ell \in L \in {}_A\mathcal{M}_A^H$,

$$\sigma_{A\otimes V,L}(a \otimes v \underset{A}{\otimes} \ell) = a\ell_{(0)} \underset{A}{\otimes} 1 \otimes v \leftharpoonup \ell_{(1)}$$

The image of the functor is in $\mathcal{Z}({}_A\mathcal{M}_A^H)$ if H has a skew antipode $\overline{S}$, in which case the inverse of σ is given by

$$\sigma^{-1}_{A\otimes V,L}(\ell \underset{A}{\otimes} a \otimes v) = 1 \otimes v \leftharpoonup \overline{S}(\ell_{(1)}a_{(1)}) \underset{A}{\otimes} \ell_{(0)}a_{(0)}.$$

4.2. ***Example:*** We use the notations and results of [6], only switching sides from left to right comodule algebras and modifying the position of some indices. We consider quadratic algebras

$$A = k\langle x\rangle/(x \circ x - Ax \circ x) = k\langle x^1, \dots, x^N\rangle/(x^i x^j - A^{ij}_{k\ell}x^k x^\ell)$$

$$B = k\langle\xi\rangle/(\xi\circ\xi + B\xi\circ\xi) = k\langle\xi^1,\dots,\xi^N\rangle/(\xi^i\xi^j + B^{ij}_{k\ell}\xi^k\xi^\ell)$$

where N is an integer and $A = (A^{ij}_{k\ell})$ and $B = (B^{ij}_{k\ell})$ are $N^2\times N^2$ matrices (we have employed the summation convention for upper and lower indices). Let $H = M(A,B)$ denote the universal coacting bialgebra; H is generated by elements Z^i_j with $i,j = 1,\dots,N$ and coacts on A and B via $\delta(x^i) = x^j\otimes Z^i_j$ and $\delta(\xi^i) = \xi^j\otimes Z^i_j$, the comultiplication is $\Delta(Z^i_j) = Z^k_j\otimes Z^i_k$. Assume that there is a de Rham complex $\Omega(A,B)$ on A in the sense of [6, Sec.3.1], i.e. in particular $\Omega(A,B)$ is a right $M(A,B)$-comodule algebra with A and B as subcomodule algebras, $\Omega(A,B)$ is bigraded and $A\otimes B\cong\Omega(A,B)\cong B\otimes A$ as bigraded spaces (the two gradings on $A\otimes B$ being induced by those of A and B), there is a differential $d:\Omega(A,B)\to\Omega(A,B)$ of bidegree $(-1,1)$ satisfying $d^2=0$ and the Leibniz rule $d(fg) = d(f)g + (-1)^i f d(g)$ when i is the second component of the bidegree of f. By [6, Sec. 3.3, Thm.1 and Thm.2] there is an $N^2\times N^2$-matrix R satisfying the quantum Yang-Baxter equation $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$ such that, first, defining relations for $M(A,B)$ take the form $R(Z\circ Z) = (Z\circ Z)R$, and, second,

$$\Omega(A,B)\cong k\langle x,\xi\rangle/(x\circ x - Ax\circ x, \xi\circ\xi + B\xi\circ\xi, \xi\circ x - Rx\circ\xi)$$

We can define an action of H on B as follows. First we can define the symbols $\omega\leftharpoonup Z^j_k$ for $\omega\in B$ by the equations

$$\omega x^j = (x^j)_{(0)}(\omega\leftharpoonup (x^j)_{(1)}) = x^k(\omega\leftharpoonup Z^j_k) \tag{4.1}$$

in $\Omega(A,B)$, using the bigraded isomorphism $\Omega(A,B)\cong A\otimes B$. Note that we have

$$\xi^i\leftharpoonup Z^j_k = R^{ij}_{k\ell}x^\ell.$$

That this action of the generators Z^j_k extends to a well defined module algebra structure follows from the quantum Yang-Baxter equation for R. It makes B a Yetter-Drinfel'd module algebra because

$$\begin{aligned}
(\xi^i)_{(0)}\leftharpoonup (Z^j_k)_{(1)}\otimes(\xi^i)_{(1)}(Z^j_k)_{(2)} &= \xi^\ell\leftharpoonup Z^m_k\otimes Z^i_\ell Z^j_m\\
&= R^{\ell m}_{kn}\xi^n\otimes Z^i_\ell Z^j_m\\
&= \xi^n\otimes Z^\ell_k Z^m_n R^{ij}_{\ell m}\\
&= (\xi^m)_{(0)}\otimes Z^\ell_k(\xi^m)_{(1)}R^{ij}_{\ell m}\\
&= (\xi^i\leftharpoonup Z^j_\ell)_{(0)}\otimes Z^\ell_k(\xi^i\leftharpoonup Z^j_\ell)_{(1)}\\
&= (\xi^i\leftharpoonup (Z^j_k)_{(2)})_{(0)}\otimes (Z^j_k)_{(1)}(\xi^i\leftharpoonup (Z^j_k)_{(2)})_{(1)}
\end{aligned}$$

Let σ be the prebraiding in $\mathcal{YD}^H_H$. We have

$$\sigma_{BB}(\xi^i\otimes\xi^j) = (\xi^i)_{(0)}(\xi^j\leftharpoonup(\xi^i)_{(1)}) = \xi^k(\xi^i\leftharpoonup Z^j_k) = \xi^k R^{ij}_{k\ell}\xi^\ell = R^{ij}_{k\ell}\xi^k\xi^\ell.$$

Reasoning as in [6, Sec.3, Lem.2 c)], we find that B is an anticommutative graded algebra in $\mathcal{Y}D^H_H$, in the sense that $fg = (-1)^{ij}\nabla_B\sigma_{BB}(f\otimes g)$ holds for elements $f \in B$ of degree i and $g \in B$ of degree j; in fact we only have to check this on the generators, where

$$\xi\circ\xi + R\xi\circ\xi = d(-\xi\circ x + Rx\circ\xi) = 0.$$

By (4.1) we can consider $\Omega(A, B)$ as the image of $B \in \mathcal{Y}D^H_H$ under the functor in Proposition 4.1 for the H-comodule algebra A, and consequently it is an anticommutative graded algebra in the prebraided category $\mathcal{W}_\ell({}_A\mathcal{M}^H_A)$.

Examples of the situation just described are given by the multiparameter quantum planes, $A = k\langle x\rangle/(x^i x^j - q^{-1}_{ij}x^j x^i)$ and $B = k\langle\xi\rangle/(\xi^i\xi^j - p_{ij}\xi^j\xi^i)$ (no summation convention), if the parameters q_{ij}, p_{ij} satisfy $p_{ii} = q_{ii} = 1, q_{ij} = q^{-1}_{ji}, p_{ij} = p^{-1}_{ji}$ and $p_{ij}q_{ij} = c \neq -1$ (a constant), see [6, Sec.2.4]. In these examples, $H = M(A, B)$ is a multiparameter deformation of $\mathrm{GL}(N)$.

Our task is to show, under restrictive hypotheses on A, that the functor constructed in Proposition 4.1 is actually a category equivalence. For the first steps of the proof, we need only assume that A is any H-comodule algebra. We put $B := A^{\mathrm{co}\,H}$.

Let (M, σ) be any object in $\mathcal{W}_\ell({}_A\mathcal{M}^H_A)$. In the following we will abbreviate $A^2 := A\otimes A \in {}_A\mathcal{M}^H_A$ with the codiagonal comodule structure and the left and right module structures induced by the respective tensorands. The map $\sigma_{M,A^2} : M \underset{A}{\otimes} A^2 \to A^2 \underset{A}{\otimes} M$ of Hopf bimodules induces a left A-linear and H-colinear map $\theta : M \to A\otimes M$, for which we use the notation $\theta(m) =: \sum\lambda_j(m)\otimes\rho_j(m)$. It is characterized by

$$\sigma_{M,A^2}(m \underset{A}{\otimes} 1\otimes 1) = \sum\lambda_j(m)\otimes 1\underset{A}{\otimes}\rho_j(m)$$

and satisfies

$$\sigma_{M,A^2}(m \underset{A}{\otimes} x\otimes y) = \sum\lambda_j(mx)\otimes 1\underset{A}{\otimes}\rho_j(mx)y.$$

We define a morphism $f : A^2 \to A^2 \underset{A}{\otimes} A^2$ in ${}_A\mathcal{M}^H_A$ by $f(x\otimes y) = x\otimes 1\underset{A}{\otimes} 1\otimes y$. Applying naturality of $\sigma_{M,-}$ and the axiom (2.1) for $(M, \sigma_{M,-}) \in \mathcal{W}_\ell({}_A\mathcal{M}^H_A)$, we obtain a commutative diagram

$$\begin{tikzcd}
M \underset{A}{\otimes} A^2 \arrow[rr, "\sigma_{M,A^2}"] \arrow[d, "M \underset{A}{\otimes} f"'] & & A^2 \underset{A}{\otimes} M \arrow[d, "f \underset{A}{\otimes} M"] \\
M \underset{A}{\otimes} A^2 \underset{A}{\otimes} A^2 \arrow[r, "\sigma_{M,A^2} \underset{A}{\otimes} A^2"] & A^2 \underset{A}{\otimes} M \underset{A}{\otimes} A^2 \arrow[r, "A^2 \underset{A}{\otimes} \sigma_{M,A^2}"] & A^2 \underset{A}{\otimes} A^2 \underset{A}{\otimes} M
\end{tikzcd}$$

Chasing the element $m \underset{A}{\otimes} 1 \otimes 1$ around this pentagon yields the equation

$$\begin{aligned}
\sum \lambda_j(m) \otimes 1 \underset{A}{\otimes} 1 \otimes 1 \underset{A}{\otimes} \rho_j(m) = (f \underset{A}{\otimes} M)\sigma(m \underset{A}{\otimes} 1 \otimes 1) \\
= \quad (A^2 \underset{A}{\otimes} \sigma)(\sigma \underset{A}{\otimes} A^2)(M \underset{A}{\otimes} f)(m \underset{A}{\otimes} 1 \otimes 1) \\
= \quad (A^2 \underset{A}{\otimes} \sigma)(\sigma \underset{A}{\otimes} A^2)(m \underset{A}{\otimes} 1 \otimes 1 \underset{A}{\otimes} 1 \otimes 1) \\
= \quad (A^2 \underset{A}{\otimes} \sigma)(\sum \lambda_j(m) \otimes 1 \underset{A}{\otimes} \rho_j(m) \underset{A}{\otimes} 1 \otimes 1) \\
= \quad \sum \lambda_j(m) \otimes 1 \underset{A}{\otimes} \lambda_k\rho_j(m) \otimes 1 \underset{A}{\otimes} \rho_k\rho_j(m)
\end{aligned}$$

from which we read off

$$(A \otimes \theta)\theta = (\eta \otimes M)\theta, \tag{4.2}$$

where $\eta : k \to A$ is the unit map of A.

Next, we consider the morphism $\nabla : A^2 \to A$ in ${}_A\mathcal{M}_A^H$, and get a commutative square

$$\begin{tikzcd}
M \underset{A}{\otimes} A^2 \arrow[r, "\sigma_{M,A^2}"] \arrow[d, "M \underset{A}{\otimes} \nabla"'] & A^2 \underset{A}{\otimes} M \arrow[d, "\nabla \underset{A}{\otimes} M"] \\
M \underset{A}{\otimes} A \arrow[r, "\sigma_{MA}"'] & A \underset{A}{\otimes} M
\end{tikzcd}$$

By the second axiom for $(M, \sigma_{M,-}) \in \mathcal{W}_\ell({}_A\mathcal{M}_A^H)$, the lower horizontal arrow is the trivial one, so that we can conclude

$$\begin{aligned}
1 \underset{A}{\otimes} m \quad &= \quad \sigma_{MA}(M \underset{A}{\otimes} \nabla)(m \underset{A}{\otimes} 1 \otimes 1) = (\nabla \underset{A}{\otimes} M)\sigma_{M,A^2}(m \underset{A}{\otimes} 1 \otimes 1) \\
&= \quad \lambda_j(m) \underset{A}{\otimes} \rho_j(m)
\end{aligned}$$

and thus $\mu\theta = id_M$. By (4.2) and this equation, θ is a descent data on M, and for ${}^\theta M := \{m \in M | \theta(m) = 1 \otimes m\}$ we conclude that

$$\begin{aligned}
A \otimes {}^\theta M \quad &\cong \quad M \\
a \otimes m \quad &\mapsto \quad am \\
\theta(m) \quad &\leftarrow\!\shortmid \quad m
\end{aligned} \tag{4.3}$$

Note that since σ_{M,A^2} and thus θ is an H-colinear map, ${}^\theta M$ is a subcomodule of M, and the isomorphism $A \otimes {}^\theta M \cong M$ is colinear if we equip the source with the codiagonal comodule structure.

We have found that every object of $\mathcal{W}_\ell({}_A\mathcal{M}_A^H)$ is a free left A-object over ${}^\theta M$ in $\mathcal{M}^H$. This result will obviously hold more generally (with essentially the same proof) for the left weak center of the category of bimodules over any algebra in a monoidal category, provided the tensor product in that category is exact.

We can deduce some properties of θ from the naturality of σ_{M,A^2} with respect to endomorphisms of A^2 in ${}_A\mathcal{M}_A^H$. Note first that these endomorphisms $t : A^2 \to A^2$ are precisely the morphisms of the form $t = \psi_\xi$ for some $\xi \in (A^2)^{\operatorname{co} H}$, where $\psi_\xi(x \otimes y) = x\xi y$. Indeed this fact is just a reformulation of the bijections

$$\operatorname{Hom}^H_{A-A}(A \otimes A, X) \cong \operatorname{Hom}^H_{A-}(A, X) \cong \operatorname{Hom}^H(k, X) \cong X^{\operatorname{co} H}$$

for every $X \in {}_A\mathcal{M}_A^H$. Now let $\xi = \sum x_i \otimes y_i \in (A^2)^{\operatorname{co} H}$. Then we exploit the commutative square

$$\begin{tikzcd}
M \underset{A}{\otimes} A^2 \arrow[r, "\sigma_{M,A^2}"] \arrow[d, "M \underset{A}{\otimes} \psi_\xi"'] & A^2 \underset{A}{\otimes} M \arrow[d, "\psi_\xi \underset{A}{\otimes} M"] \\
M \underset{A}{\otimes} A^2 \arrow[r, "\sigma_{M,A^2}"'] & A^2 \underset{A}{\otimes} M
\end{tikzcd}$$

(which is naturality of σ with respect to ψ_ξ), to get

$$\begin{aligned}
\sum \lambda_j(mx_i) \otimes 1 \underset{A}{\otimes} \rho_j(mx_i)y_i &= \sigma(m \underset{A}{\otimes} \sum x_i \otimes y_i) \\
&= \sigma(M \underset{A}{\otimes} \psi_\xi)(m \underset{A}{\otimes} 1 \otimes 1) \\
&= (\psi_\xi \underset{A}{\otimes} M)\sigma(m \underset{A}{\otimes} 1 \otimes 1) \\
&= (\psi_\xi \underset{A}{\otimes} M)(\sum \lambda_j(m) \otimes 1 \underset{A}{\otimes} \rho_j(m)) \\
&= \sum \lambda_j(m)x_i \otimes y_i \underset{A}{\otimes} \rho_j(m)
\end{aligned}$$

and conclude

$$\sum \lambda_j(mx_i) \otimes \rho_j(mx_i)y_i = \sum \lambda_j(m)x_i \otimes y_i\rho_j(m) \tag{4.4}$$

as an equation in $A \otimes M$.

First, we apply (4.4) to $\xi := 1 \otimes b$ for $b \in B$ to get

$$\sum \lambda_j(m) \otimes \rho_j(m)b = \sum \lambda_j(m) \otimes b\rho_j(m)$$

or, denoting by $\ell_b, r_b : M \to M$ left and right multiplication with b, respectively,

$$\theta(m) \in \operatorname{Ker}(A \otimes (r_b - \ell_b)) = A \otimes \operatorname{Ker}(r_b - \ell_b).$$

This means $\theta(M) \subset A \otimes M^B$, or, equivalently, ${}^\theta M \subset M^B$.

Next, we apply (4.4) to $\xi = b \otimes 1$ for $b \in B$ to get

$$\theta(mb) = \sum \lambda_j(mb) \otimes \rho_j(mb) = \sum \lambda_j(m)b \otimes \rho_j(m)$$

Assuming in addition that $m \in M^B$, this yields

$$\sum \lambda_j(m)b \otimes \rho_j(m) = \theta(bm) = b\theta(m),$$

hence

$$\theta(m) \in \operatorname{Ker}((r_b - \ell_b) \otimes M) = \operatorname{Ker}(r_b - \ell_b) \otimes M$$

and we conclude that

$$\theta(M^B) \subset A^B \otimes M^B.$$

Because of this and (4.3) we can in the following assume that for all $m \in M$ the elements $\lambda_j(m)$ and $\rho_j(m)$ are so chosen that $\rho_j(m) \in {}^\theta M \subset M^B$ for all $m \in M$, and $\lambda_j(m) \in A^B$ if $m \in M^B$.

For the rest of this section we assume that A is a right H-Galois extension of B with ${}_BA$ flat.

Let $a \in A$. Then we claim that the element

$$t(a) := \sum a_{(0)} \otimes \ell_i(a_{(1)}) \underset{B}{\otimes} r_i(a_{(1)}) \in A \otimes A \underset{B}{\otimes} A$$

is in fact contained in $(A \otimes A)^{\mathrm{co}\, H} \underset{B}{\otimes} A$. Since ${}_BA$ is flat, we only have to verify

$$\begin{aligned}
&\sum (a_{(0)} \otimes \ell_i(a_{(1)}))_{(0)} \underset{B}{\otimes} r_i(a_{(1)}) \otimes (a_{(0)} \otimes \ell_i(a_{(1)}))_{(1)} \\
&= \sum a_{(0)} \otimes \ell_i(a_{(2)})_{(0)} \underset{B}{\otimes} r_i(a_{(2)}) \otimes a_{(1)}\ell_i(a_{(2)})_{(1)} \\
&= \sum a_{(0)} \otimes \ell_i(a_{(1)}) \underset{B}{\otimes} r_i(a_{(1)}) \otimes 1,
\end{aligned}$$

using (3.8). Now we can, roughly speaking, apply (4.4) to the $(A \otimes A)^{\mathrm{co}\, H}$-component of the tensor $t(a)$. To see that our calculations are correct, note first that the expression

$$\sum \lambda_j(m)a_k \otimes a'_k\rho_j(m)a''_k \in A \otimes M$$

for $m \in M$ really depends only of the tensor $\sum a_k \otimes a'_k \underset{B}{\otimes} a''_k \in A \otimes A \underset{B}{\otimes} A$, since we have chosen all $\rho_j(m) \in M^B$. Now we can write

$$\begin{aligned}
\theta(ma) &= \sum \lambda_j(ma) \otimes \rho_j(ma) \\
&= \sum \lambda_j(ma_{(0)}) \otimes \rho_j(ma_{(0)})\ell_i(a_{(1)})r_i(a_{(1)}) \\
&\overset{*}{=} \sum \lambda_j(m)a_{(0)} \otimes \ell_i(a_{(1)})\rho_j(m)r_i(a_{(1)}) \\
&= \sum \lambda_j(m)a_{(0)} \otimes \rho_j(m) \leftharpoonup a_{(1)},
\end{aligned}$$

where the equality marked $*$ follows from (4.4) and the fact that $\sum a_{(0)} \otimes \ell_i(a_{(1)}) \underset{B}{\otimes} r_i(a_{(1)}) \in (A \otimes A)^{\mathrm{co}\, H} \underset{B}{\otimes} A$.

For an element $m \in M^B$ we can conclude

$$\begin{aligned}
\theta(m \leftharpoonup h) &= \sum \theta(\ell_i(h) m r_i(h)) \\
&= \sum \ell_i(h) \theta(m r_i(h)) \\
&= \sum \ell_i(h) \lambda_j(m) r_i(h)_{(0)} \otimes \rho_j(m) \leftharpoonup r_i(h)_{(1)} \\
&= \sum \ell_i(h_{(1)}) \lambda_j(m) r_i(h_{(1)}) \otimes \rho_j(m) \leftharpoonup h_{(2)} \\
&= \sum \lambda_j(m) \leftharpoonup h_{(1)} \otimes \rho_j(m) \leftharpoonup h_{(2)}
\end{aligned}$$

If we equip $A^B \otimes M^B$ with the diagonal action composed from the Miyashita-Ulbrich actions on A^B and M^B, we have shown that the map $\theta' : M^B \to A^B \otimes M^B$ induced by θ is a map in $\mathcal{YD}_H^H$. As an immediate consequence we see that ${}^\theta M$ is a subobject of $M^B \in \mathcal{YD}_H^H$.

We are ready to prove our main result.

4.3. Theorem: *Let H be a bialgebra and A a right H-Galois extension of B with ${}_BA$ flat. Then the functor $\mathcal{YD}_H^H \to \mathcal{W}_\ell({}_A\mathcal{M}_A^H)$ from 4.1 is a category equivalence. In particular $\mathcal{YD}_H^H \cong \mathcal{W}_\ell({}_A\mathcal{M}_A^H) \cong \mathcal{Z}({}_A\mathcal{M}_A^H)$ if H has a skew antipode.*

PROOF: We claim that $\mathcal{W}_\ell({}_A\mathcal{M}_A^H) \ni (M, \sigma) \mapsto {}^\theta M \in \mathcal{YD}_H^H$ is a quasiinverse for the functor from 4.1. First, let $V \in \mathcal{YD}_H^H$. Then we have to show that ${}^\theta(A \otimes V) \cong V$ as Yetter-Drinfel'd modules. Now as $\sigma(a \otimes v \underset{A}{\otimes} 1 \otimes 1) = a \otimes 1 \underset{A}{\otimes} 1 \otimes v$ holds for $a \in A$ and $v \in V$, we obtain $\theta(a \otimes v) = a \otimes 1 \otimes v$.

Thus $j : V \ni v \mapsto 1 \otimes v \in A \otimes V$ obviously maps to ${}^\theta(A \otimes V)$. Conversely, let $t \in {}^\theta(A \otimes V)$. Choosing a k-basis (a_i) of A with $a_1 = 1$, and writing $t = \sum a_i \otimes v_i$ with $v_i \in V$, we find $1 \otimes t = \theta(t) = \sum a_i \otimes 1 \otimes v_i$, and, comparing coefficients, $t = 1 \otimes v_1 \in j(V)$. It remains to show that $j : V \cong {}^\theta(A \otimes V)$ is an isomorphism of Yetter-Drinfel'd modules. We check

$$\begin{aligned}
j(v) \leftharpoonup h &= \sum \ell_i(h)(1 \otimes v) r_i(h) = \sum \ell_i(h) r_i(h)_{(0)} \otimes v \leftharpoonup r_i(h)_{(1)} \\
&= 1 \otimes v \leftharpoonup h = j(v \leftharpoonup h)
\end{aligned}$$

while colinearity of j is trivial.

Second, let $(M, \sigma) \in \mathcal{W}_\ell({}_A\mathcal{M}_A^H)$. We have to verify that the isomorphism (4.3) is an isomorphism in the left weak center of the category of Hopf bimodules, where we equip $A \otimes {}^\theta M$ with the structures from 4.1. Left linearity and colinearity are trivial, and the calculation

$$\begin{aligned}(av)x &= \mu\theta(avx) = \sum \lambda_j(avx)\rho_j(avx) = a\sum \lambda_j(vx)\rho_j(vx)\\ &= a\sum \lambda_j(v)x_{(0)}(\rho_j(v) \leftharpoonup x_{(1)}) = ax_{(0)}(v \leftharpoonup x_{(1)})\\ &= \mu(ax_{(0)} \otimes v \leftharpoonup x_{(1)})\end{aligned}$$

for $v \in {}^\theta M$ and $a, x \in A$ proves right linearity. Denote $\sigma_{A\otimes^\theta M,-}$ the structure of $A\otimes{}^\theta M$ of an object of the left weak center. We have to prove commutativity of

$$\begin{tikzcd} A \otimes {}^\theta M \underset{A}{\otimes} X \arrow[r, "\sigma_{A\otimes^\theta M,X}"] \arrow[d, "\mu \otimes X"'] & X \underset{A}{\otimes} A \otimes {}^\theta M \arrow[d, "X \otimes \mu"] \\ M \underset{A}{\otimes} X \arrow[r, "\sigma_{MX}"'] & X \underset{A}{\otimes} M \end{tikzcd}$$

for every $X \in {}_A\mathcal{M}_A^H$. First, let $X = A^2$. Then, for $a \in A$ and $m \in {}^\theta M$, we have

$$\begin{aligned}\sigma_{M,A^2}(am \underset{A}{\otimes} 1 \otimes 1) &= a\lambda_j(m) \otimes 1 \underset{A}{\otimes} \rho_j(m)\\ &= a \otimes 1 \underset{A}{\otimes} m\\ &= (A^2 \underset{A}{\otimes} \mu)\sigma_{A\otimes^\theta M,A^2}(a \otimes m \underset{A}{\otimes} 1 \otimes 1),\end{aligned}$$

and conclude that the diagram commutes. The proof is complete when we show that A^2 is a generator in ${}_A\mathcal{M}_A^H$. Let $X \in {}_A\mathcal{M}_A^H$ be arbitrary. We know that $X \cong X^{\mathrm{co}\,H} \underset{B}{\otimes} A$ in $\mathcal{M}_A^H$ by [2, (2.11)] (Note that the proof of that result does not need H to have an antipode). As a consequence, X is a quotient of $A \underset{B}{\otimes} X^{\mathrm{co}\,H} \underset{B}{\otimes} A$ in ${}_A\mathcal{M}_A^H$. Since $X^{\mathrm{co}\,H} \in {}_B\mathcal{M}_B$ is a quotient of a direct sum of copies of $B \otimes B$, and $A \underset{B}{\otimes} B \otimes B \underset{B}{\otimes} A \cong A^2$, we conclude that X is a quotient of a direct sum of copies of A^2. □

5 The centralizer of a subcategory

In 3.2 we have constructed a class of objects in ${}_A\mathcal{M}_A^H$ by induction from objects in ${}_{A^B}\left(\mathcal{Y}D_H^H\right)$. Obviously, not all relative Hopf bimodules have the form $A \underset{A^B}{\otimes} T$ for some $T \in {}_{A^B}\left(\mathcal{Y}D_H^H\right)$. In this section we will show that, under favourable conditions, Hopf bimodules of this form can be characterized in terms of the existence of maps twisting tensor factors; more precisely, via $T \mapsto A \underset{A^B}{\otimes} T$ the category ${}_{A^B}\left(\mathcal{YD}_H^H\right)$ is equivalent to the centralizer of a certain subcategory in ${}_A\mathcal{M}_A^H$. As a corollary, we will exhibit the category of dyslectic A^B-modules in $\mathcal{YD}_H^H$, see [7], as the center of a suitable category.

5.1. Definition: *Denote $\mathcal{C}$ the subcategory of ${}_A\mathcal{M}_A^H$ of those Hopf bimodules for which the relation $rm = m_{(0)}(r \leftharpoondown m_{(1)})$ holds for all $m \in M$ and $r \in A^B$.*

5.2. Theorem: *Let H be a bialgebra with skew antipode and A a right H-Galois extension of B with ${}_BA$ flat. Then the adjunction from 3.2 induces an adjunction*

$$A \underset{A^B}{\otimes} (-) \dashv (-)^B : \mathcal{Z}\left(\mathcal{C}; {}_A\mathcal{M}_A^H\right) \to {}_{A^B}\left(\mathcal{YD}_H^H\right)$$

Proof: Let $T \in {}_{A^B}\left(\mathcal{YD}_H^H\right)$. From 4.1 we know that $A \otimes T$ is in the center of ${}_A\mathcal{M}_A^H$ and by Theorem 3.2 the quotient $A \underset{A^B}{\otimes} T$ is in ${}_A\mathcal{M}_A^H$. To see that $A \underset{A^B}{\otimes} T \in \mathcal{Z}\left(\mathcal{C}; {}_A\mathcal{M}_A^H\right)$ it remains to check that, for $M \in \mathcal{C}$, the map

$$\begin{aligned} \sigma_{A\otimes T, M} : (A \otimes T) \underset{A}{\otimes} M &\rightarrow M \underset{A}{\otimes} (A \otimes T) \\ a \otimes t \otimes m &\mapsto am_{(0)} \otimes 1 \otimes t \leftharpoondown m_{(1)} \end{aligned}$$

induces a map $\sigma_{A \underset{A^B}{\otimes} T, M} : (A \underset{A^B}{\otimes} T) \underset{A}{\otimes} M \to M \underset{A}{\otimes} (A \underset{A^B}{\otimes} T)$. Abusing notation, we compute, for $r \in A^B$:

$$\begin{aligned} \sigma(ar \otimes t \otimes m) &= arm_{(0)} \underset{A}{\otimes} 1 \underset{A^B}{\otimes} t \leftharpoondown m_{(1)} \\ &= am_{(0)}(r \leftharpoondown m_{(1)}) \underset{A}{\otimes} 1 \underset{A^B}{\otimes} t \leftharpoondown m_{(2)} \\ &= am_{(0)} \underset{A}{\otimes} 1 \underset{A^B}{\otimes} (r \leftharpoondown m_{(1)})(t \leftharpoondown m_{(2)}) \\ &= am_{(0)} \underset{A}{\otimes} 1 \underset{A^B}{\otimes} (rt) \leftharpoondown m_{(1)} \\ &= \sigma(a \otimes rt \otimes m). \end{aligned}$$

The conditions of flatness and existence of a skew antipode only enter in the proof that we have a pair of adjoint functors. To see this, it only remains to check that the adjunction morphism $\mu : A \underset{A^B}{\otimes} L^B \to L$ is a morphism in the centralizer if $(L, \sigma) \in \mathcal{Z}\left(\mathcal{C}; {}_A\mathcal{M}_A^H\right)$. This amounts to commutativity of the diagrams

$$\begin{array}{ccc}(A \underset{A^B}{\otimes} L^B) \underset{A}{\otimes} M & \xrightarrow{\sigma} & M \underset{A}{\otimes} (A \underset{A^B}{\otimes} L^B) \\ {\scriptstyle \mu \underset{A}{\otimes} M}\downarrow & & \downarrow{\scriptstyle M \underset{A}{\otimes} \mu} \\ L \underset{A}{\otimes} M & \xrightarrow[\sigma]{} & M \underset{A}{\otimes} L\end{array} \tag{5.1}$$

for all $M \in \mathcal{C}$. We show first that $A \underset{A^B}{\otimes} A$ is a generator of the category $\mathcal{C}$. Indeed it is straightforward to check, from (3.11), that $A \underset{A^B}{\otimes} A$ is really in $\mathcal{C}$. Let $M \in \mathcal{C}$, and let $\phi : A \otimes A \to M$ be a map in ${}_A\mathcal{M}_A^H$. Then for $x, y \in A$ and $r \in A^B$ we have, denoting the skew antipode of H by $\overline{S}$,

$$\begin{aligned}\phi(xr \otimes y) &= \phi\left(x_{(0)}(r \leftharpoondown \overline{S}(x_{(2)})x_{(1)}) \otimes y\right) \\ &= \phi\left((r \leftharpoondown \overline{S}(x_{(1)}))x_{(0)} \otimes y\right) \\ &= (r \leftharpoondown \overline{S}(x_{(1)}))\phi(x_{(0)} \otimes y) \\ &= \phi(x_{(0)} \otimes y_{(0)})(r \leftharpoondown \overline{S}(x_{(2)})x_{(1)}y_{(1)}) \\ &= \phi(x_{(0)} \otimes y_{(0)}(r \leftharpoondown y_{(1)})) \\ &= \phi(x \otimes ry)\end{aligned}$$

and thus ϕ factors through $A \underset{A^B}{\otimes} A$. Since $A \otimes A$ is a generator of ${}_A\mathcal{M}_A^H$, we conclude that $A \underset{A^B}{\otimes} A$ is a generator of $\mathcal{C}$.

Consequently, it is sufficient to consider the diagram (5.1) for $M = A \underset{A^B}{\otimes} A$ only. As in section 4 the map $\sigma_{L, A \underset{A^B}{\otimes} A}$ induces a left A-linear map $\theta : L \to A \underset{A^B}{\otimes} L$, which is sufficient to recover $\sigma_{L, A \underset{A^B}{\otimes} A}$. We then have to check that the diagram

$$\begin{array}{ccc}A \underset{A^B}{\otimes} L^B & \xrightarrow{A \otimes \eta \otimes L} & A \underset{A^B}{\otimes} A \underset{A^B}{\otimes} L^B \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle A \underset{A^B}{\otimes} \mu} \\ L & \xrightarrow[\theta]{} & A \underset{A^B}{\otimes} L\end{array}$$

commutes. Let $j : L^B \to L$ denote the inclusion. Then the way "first to the right, then down" around the square apparently equals $A \underset{A^B}{\otimes} j$. But the same is true for the way "first down, then to the right", since θ is A-linear on the left. $\square$

5.3. Theorem: *Let H be a bialgebra with skew antipode and A a right H-Galois extension of B. Assume that ${}_BA$ is flat and A_{A^B} is faithfully flat. Then the adjunction in 5.2 is an equivalence.*

PROOF: We prove first that the counit $\mu : A \underset{A^B}{\otimes} L^B \to L$ of the adjunction in 5.2 is a bijection for every $(L, \sigma) \in \mathcal{Z}\left(\mathcal{C}; {}_A\mathcal{M}_A^H\right)$. We consider the map $\theta : L \to A \underset{A^B}{\otimes} L$ from the proof of 5.2 and write $\theta(\ell) = \sum \lambda_j(\ell) \otimes \rho_j(\ell)$. As in the proof of 4.3, we can deduce $\sum \lambda_j(\ell) \otimes b\rho_j(\ell) = \sum \lambda_j(\ell) \otimes \rho_j(\ell)b$ for $b \in B$ from naturality of $\sigma_{L, A \underset{A^B}{\otimes} A}$ with respect to $A \underset{A^B}{\otimes} A \ni x \otimes y \mapsto x \otimes by \in A \underset{A^B}{\otimes} A$. Now since the map $L \ni \ell \mapsto b\ell - \ell b \in L$ is left A^B-linear, and A_{A^B} is faithfully flat, we can deduce that $\theta(L) \subset A \underset{A^B}{\otimes} L^B$. From naturality of $\sigma_{L,-}$ with respect to $\nabla : A \underset{A^B}{\otimes} A \to A$ we deduce that $\mu\theta = id_L$. Since we have shown at the end of the proof of 5.2 that $\theta\mu = A \underset{A^B}{\otimes} j$, and the latter is injective since A_{A^B} is flat, we know that μ is a bijection.

We are now in the situation of a left adjoint functor $A \underset{A^B}{\otimes} -$ which (by faithful flatness) reflects isomorphisms, and whose counit of adjunction is an isomorphism. It is straightforward to deduce from this that the unit of adjunction is also an isomorphism. $\square$

5.4. Corollary: *Let H be a bialgebra with skew antipode and A an H-Galois extension of B with ${}_BA$ flat. Then we have an adjunction*

$$A \underset{A^B}{\otimes} (-) \dashv (-)^B : \mathcal{Z}(\mathcal{C}) \to {}_{A^B}\left(\widetilde{\mathcal{Y}D_H^H}\right).$$

It is an equivalence if in addition A_{A^B} is faithfully flat.

PROOF: If M is in $\mathcal{C}$, then M^B is clearly dyslectic. If $T \in {}_{A^B}\left(\mathcal{Y}D_H^H\right)$ is dyslectic, then $A \underset{A^B}{\otimes} T$ is in $\mathcal{Z}\left(\mathcal{C}; {}_A\mathcal{M}_A^H\right)$, but also in $\mathcal{C}$, since

$$\begin{aligned} r(a \otimes t) &= ra \otimes t = a_{(0)}(r \leftharpoonup a_{(1)}) \otimes t = a_{(0)} \otimes (r \leftharpoonup a_{(1)})t \\ &= a_{(0)} \otimes t_{(0)}(r \leftharpoonup a_{(1)}t_{(1)}) = (a_{(0)} \otimes t_{(0)})(r \leftharpoonup a_{(1)}t_{(1)}) \end{aligned}$$

The last equality holds since by 3.2 the action of A^B on $A \underset{A^B}{\otimes} T$ is actually that induced by the right tensorand (this is implicit in right linearity of the unit of adjunction). $\square$

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