Hopf bimodules, coquasibialgebras, and an exact sequence of Kac

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1 Introduction

Besides being interesting objects in themselves, Hopf algebras (in some situations also known as quantum groups [11]) have found various applications in other areas of mathematics, and in mathematical physics. These include the construction of invariants of knots and three-manifolds [28, 40]; generalizations of the classical Galois theory of fields and rings [6, 14]; generalized symmetries of conformal field theory [13]; subfactors of von Neumann algebras [38].

In all applications it is a key property of Hopf algebras that their representations admit tensor products: Given two modules V and W over a Hopf algebra

H, one can form their tensor product over the base field k, and obtains another representation $V \otimes W$. One should think of this as a generalization of behavior familiar for groups and Lie algebras: If V and W above were representations of a group G, then G would act on $V \otimes W$ by $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$. If V and W were representations of a Lie algebra $\mathfrak{g}$, then this would act on $V \otimes W$ by $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$. The group G gives rise to a Hopf algebra $k[G]$, the group algebra; the Lie algebra $\mathfrak{g}$ gives rise to another Hopf algebra, the universal enveloping algebra $U(\mathfrak{g})$. In the case of modules V and W over a Hopf algebra H, the Hopf algebra acts on $V \otimes W$ via its comultiplication, a map $\Delta\colon H \to H \otimes H$. In the case of a group algebra, we have $\Delta(g) = g \otimes g$ for $g \in G$, in the case of a universal enveloping algebra we have $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. In the general case we follow Sweedler and write formally $\Delta(x) =: x_{(1)} \otimes x_{(2)}$, so that $x \cdot (v \otimes w) = x_{(1)} \cdot v \otimes x_{(2)} \cdot w$. In more abstract terminology the nice properties of the tensor product of modules over a Hopf algebra are summarized in the statement that the module category ${}_H\mathcal{M}$ is a monoidal category. Similarly, the comodule category ${}^H\mathcal{M}$ is a monoidal category, with tensor product induced via the multiplication of H; a first example here is ${}^{k[G]}\mathcal{M}$, which is canonically equivalent to the category of G-graded vector spaces. This is also the category of modules over the dual Hopf algebra k^G if G is finite. Note that $U(\mathfrak{g})$ as well as $k[G]$ are cocommutative (the comultiplication is insensitive to a switch of tensor factors) while the dual k^G is a commutative algebra.

How does one get nontrivial (in particular, noncommutative, non-cocommutative) examples of Hopf algebras? Besides deformations of classical groups or universal enveloping algebras of classical Lie algebras that lead to quantum groups, an important technique that is the source of many interesting examples is to build them from smaller, better understood pieces. An easy way of doing this is to form the tensor product $K \otimes Q$ of two Hopf algebras K and Q with 'component-wise' multiplication $(x \otimes p)(y \otimes q) = xy \otimes pq$ and 'component-wise' comultiplication $\Delta(x \otimes p) = x_{(1)} \otimes p_{(1)} \otimes x_{(2)} \otimes p_{(2)}$. One immediately gets noncommutative non-cocommutative examples this way — just take $k[G] \otimes k^G$ for a nonabelian group G— but these examples do not carry us very far.

The tensor product Hopf algebra of K and Q is a particularly simple example of a mapping system of the following type:

$$K \underset{\iota}{\overset{\pi}{\leftrightarrows}} H \underset{j}{\overset{\nu}{\rightleftarrows}} Q \text{ with } \pi\iota = \mathrm{id}_K, \nu j = \mathrm{id}_Q, \nu\iota = \eta\varepsilon, \pi j = \eta\varepsilon, \iota\pi * j\nu = \mathrm{id}_H, \qquad (*)$$

with a Hopf algebra H and linear maps ι, π, j, ν. In the case of a tensor product Hopf algebra, all four maps are fully structure preserving, that is, Hopf algebra maps. One should think of $(*)$ as an analog of a direct sum decomposition of modules with its associated inclusion and projection maps.

Now many interesting Hopf algebras arise as the middle term in a mapping system like $(*)$, in which, however, the four maps are not quite as structure preserving. The case where ι and j are Hopf algebra maps is called a double crossproduct [19]; this means that H is a generalized product of two Hopf subalgebras K and Q. The case where ι and ν are Hopf algebra maps, π is a left K-module coalgebra map, and j is a right Q-comodule algebra map is called a bismash product [39, 19]. The case where both ι and π are Hopf algebra maps is called a Radford biproduct [27]. In each of these cases H is isomorphic as a vector space to the tensor product $K \otimes Q$. One can analyze completely the general form that multiplication and comultiplication on $K \otimes Q$ take in terms of algebra and coalgebra structures on the two factors, and certain interactions between the factors that modify the simple form as found in an ordinary tensor product. A first example of such a modification is the smash product of an algebra A with a bialgebra B that acts on it: The smash product algebra $A\#B$ is $A \otimes B$ as a vector space, but with modified algebra structure $(a\#b)(a'\#b') = a(b_{(1)} \rightharpoonup a')\#b_{(2)}b'$, where $\rightharpoonup$ denotes the action. The smash product construction generalizes the construction of the skew group ring over a ring on which a group acts by automorphisms. A dual notion is the smash coproduct coalgebra, where one needs a bialgebra coacting on a coalgebra.

Each of the three types of mapping systems above can be constructed from a multiplication and comultiplication in each of K and Q by suitably modifying multiplication as well as comultiplication in $K \otimes Q$ in a similar manner using various actions and coactions of the two factors on each other. In each of the three cases, there are certain 'combinatorial' compatibility conditions that one has to impose on the actions, coactions, algebra and coalgebra structures in order that the resulting multiplication and comultiplication are associative and coassociative, and compatible to form a bialgebra.

From the combinatorial viewpoint double crossproducts and bismash products are closely connected by an obvious duality: If K is finite-dimensional, then the data necessary to build a bismash product $K\#Q$ are in bijection with the data necessary to build a double crossproduct $Q \bowtie K^*$: The action and coaction needed for the one give, by just dualizing K, the two actions necessary to build the other, and the combinatorics turn out to be equivalent. Thus, bismash products $K\#Q$ are in bijection with double crossproducts $Q \bowtie K^*$. But is there any intrinsic connection between the two resulting bialgebras, or between the mapping systems?

The combinatorics of a Radford biproduct are quite conceptual: In more recent terminology it turns out that a Radford biproduct $K \star Q$ is built from a bialgebra Q in the braided monoidal category of Yetter-Drinfeld K-modules. Since right Yetter-Drinfeld K-modules are canonically identified with left Yetter-Drinfeld K^*-modules for finite-dimensional K, we get a bijection between Radford biproducts $K \star Q$ and Radford biproducts $Q \star K^*$. But what is the intrinsic connection between a Hopf algebra with a projection onto K and the corresponding Hopf algebra with a projection onto K^*?

A far-reaching generalization of bismash products is given by the bicrossproduct construction [15, 21], going back to [17, 36] in special cases. This is characterized as a middle term of a mapping system $(*)$ in which ι and ν are bialgebra maps, π is a left K-module map and j is a right Q-comodule map. Stated another way, we are talking about the Hopf algebra analog [35] of a short exact sequence, or extension, of groups. Classifying the Hopf algebra extensions $K \to H \to Q$ in which K and Q are group algebras or their duals often constitutes an important step in the classification of semisimple Hopf algebras of low dimensions. Multiplication in a bicrossed product $K^\alpha\#_\sigma Q$ has the general form $(x\#p)(y\#q) = x(p_{(1)} \rightharpoonup y)\sigma(p_{(2)} \otimes q_{(1)})\#p_{(3)}q_{(2)}$ in terms of a weak action $\rightharpoonup\colon Q \otimes K \to K$ and a two-cocycle $\sigma\colon Q \otimes Q \to K$. Comultiplication is given by a dual formula involving a weak coaction $\rho\colon Q \to Q \otimes K$ and a dual two-cocycle $\alpha\colon Q \to K \otimes K$. The combinatorics that the data have to satisfy in order to yield a bialgebra are unwieldy. Even the equations that arise when we consider the algebra structure (a crossed product algebra) alone are in general hard to solve, and in addition to this, and the dual problem of finding a crossed coproduct coalgebra, there are equations coupling the action, coaction, cocycle and dual cocycle.

The situation is somewhat better when K is commutative and Q is cocommutative. In this case, the weak action and weak coaction above are really a module action and comodule coaction, determined uniquely by the extension, and forming a matched pair. Sweedler cohomology [37] of H with coefficients in K is an actual cohomology theory whose degree two term classifies crossed products (with fixed action), with σ as above as two-cocycle. α as above belongs to a dual homology theory. Moreover, one can interpret the pair (σ, α) as a cocycle in the homology of a certain double complex, hence the set of isomorphism classes of extensions inducing a fixed action and coaction as a homology group of that double complex [16]. Much like for group extensions with abelian kernel, there is a notion of Baer sum for Hopf algebra extensions corresponding to the operation in the homology group — the bismash product is the unit of this Baer sum. In the noncommutative non-cocommutative case, there is no group structure on the set of extensions, and no real homological interpretation of the restrictions on $(\rightharpoonup, \rho, \sigma, \alpha)$. One should note, though, that even in the abelian case (commutative K, cocommutative Q) the homology theory governing Hopf algebra extensions is not easy to compute due to the coupling between the algebra and coalgebra structures.

An important tool for dealing with groups of extensions is the exact sequence of Kac. Let K and Q be finite-dimensional cocommutative Hopf algebras and $(K^*, Q, \rightharpoonup, \rho)$ an abelian matched pair. Denote by $K^*\#Q$ the associated bismash product, by $Q \bowtie K$ the corresponding double crossproduct, by $\mathrm{Opext}(Q, K^*)$ the group of extensions of Q by K^* inducing the operations in the matched pair.

Then the Kac sequence

$$0 \to \mathcal{H}^1(Q \bowtie K) \overset{\text{res}}{\to} \mathcal{H}^1(Q) \oplus \mathcal{H}^1(K) \to \operatorname{Aut}(K^*\#Q) \to \mathcal{H}^2(Q \bowtie K) \overset{\text{res}}{\to}$$
$$\overset{\text{res}}{\to} \mathcal{H}^2(Q) \oplus \mathcal{H}^2(K) \to \operatorname{Opext}(Q, K^*) \to \mathcal{H}^3(Q \bowtie K) \overset{\text{res}}{\to} \mathcal{H}^3(Q) \oplus \mathcal{H}^3(K)$$

is the long exact sequence arising from a certain short exact sequence of double complexes. In Kac' original work [17] the base field is the complex numbers, and K and Q are group algebras (this is no restriction for cocommutative finite dimensional Hopf algebras over the complex numbers), so that the cohomology groups are really those of group cohomology. The Hopf algebra extensions in [17] are really semisimple $*$-Hopf algebras, now also known as Kac algebras. That Kac' work applies to ordinary Hopf algebras in place of Kac algebras was clarified in [23]. The generalization to cocommutative Hopf algebras in place of group rings does not seem to be in the literature.

The Kac sequence is a striking and powerful result. It connects the (homological, but intricate) description of the set $\operatorname{Opext}(Q, K^*)$ with the ordinary cohomology of K, Q and another Hopf algebra (or group), the double crossproduct. In particular it is a vast extension of the connection between a double crossproduct and the corresponding bismash product in the abelian case, and it is extremely useful for the computation and classification of examples of low-dimensional Hopf algebras [22, 23]. But in tune with the questions above the Kac sequence raises more questions. How is it that certain Hopf algebras (the extensions in the Opext group) give rise to three-cocycles on a totally different Hopf algebra (the double crossproduct)? The homological techniques behind the Kac sequence are, of course, purely combinatorial in the sense in which we used this term above: One breaks multiplication and comultiplication down to cocycles and then interprets these data in another fashion. But is there an intrinsic connection?

In this thesis we will construct a functor $\mathfrak{F}$ taking as its input, by and large, certain mapping systems like $(*)$, with a finite-dimensional Hopf algebra K, and having another such mapping system as its output, with K^* instead of K (and a switching of sides which we will ignore in this introduction). $\mathfrak{F}(H)$ will be defined 'intrinsically' through rather simple principles, without reference to the conditions necessary to build up H from the pieces K and Q. The functor $\mathfrak{F}$ takes double crossproducts to the corresponding bismash products, and vice versa; it takes Hopf algebras with a projection onto K to Hopf algebras with a projection onto K^*. If we fix an abelian matched pair $(K^*, Q, \rightharpoonup, \rho)$, the functor $\mathfrak{F}$ will apply to arbitrary extensions in $\operatorname{Opext}(Q, K^*)$ to yield the double crossproduct $Q \bowtie K$ and in addition a three-cocycle on it. While the object map of $\mathfrak{F}$ is in this way 'responsible' for the map $\operatorname{Opext}(Q, K^*) \to \mathcal{H}^3(Q \bowtie K)$ in the Kac sequence, the morphism map will be 'responsible' for the map $\operatorname{Aut}(K^*\#Q) \to \mathcal{H}^2(Q \bowtie K)$. It will also be possible to apply $\mathfrak{F}$ to certain representatives of an element of the kernel of the restriction $\mathcal{H}^3(Q \bowtie K) \to \mathcal{H}^3(Q) \bowtie \mathcal{H}^3(K)$; these representatives will be mapped by $\mathfrak{F}$ to an inverse image under the map $\operatorname{Opext}(Q, K^*) \to \mathcal{H}^3(Q \bowtie$

K). Finally, the functor $\mathfrak{F}$ will also apply to extensions of Q by K in the case where Q and K are noncommutative and non-cocommutative, allowing us to prove partial generalizations of the Kac seqence for general extensions.

The idea that gives rise to the functor $\mathfrak{F}$ is another basic principle that one can use to obtain examples of Hopf algebras. Not only do Hopf algebras produce monoidal categories — under favorable conditions this process can be reversed. Suppose we have a monoidal category $\mathcal{C}$ and a functor $\mathcal{C} \to {}_k\mathcal{M}$ that preserves tensor products. Then (under additional hypotheses) the category $\mathcal{C}$ is equivalent, as a monoidal category, to the category of comodules over a suitable Hopf algebra. The process thus assigning a Hopf algebra to a monoidal category is sometimes referred to as Tannaka-Krein reconstruction in quantum group theory; it goes back to [29]. It is almost a tautology that every Hopf algebra arises by this process: it can be reconstructed from the category of its comodules, generalizing Tannaka-Krein duality. More importantly, Hopf algebras may be construced (or explained) in this fashion through abstractly defined monoidal categories; an example is [26].

Now given an inclusion $K \subset H$ of Hopf algebras, one can form a natural monoidal category: ${}^H_K\mathcal{M}_K$, the category of Hopf bimodules; objects in it are bimodules over K as well as comodules over H, with two natural compatibility conditions. The abstract tensor product is the tensor product over K. Thus, the obvious functor to the category of vector spaces — taking a Hopf bimodule to its underlying vector space — will not be suitable for our purposes. Instead, we will use the functor $M \mapsto M/K^+M$ dividing out the left K-action. This can be made a tensor product preserving functor, provided the inclusion $\iota\colon K \to H$ has a convolution invertible left module section $\pi\colon H \to K$ (a cleaving map). In a first approximation, the functor $\mathfrak{F}$ takes as its input the mapping system of type $(*)$ built from ι and π, and assigns to it the Hopf algebra $\tilde{H}$ reconstructed from ${}^H_K\mathcal{M}_K$ and the functor dividing out the left actions of K. This works well for the mapping systems of double crossproducts, bismash products, and Radford biproducts. However, it fails for extensions. The reason is that the reconstruction principles are not quite as outlined above: To make the category of H-modules monoidal with tensor product over k, one does not need a Hopf algebra H, but only a quasi-bialgebra, in which by definition [12] comultiplication is coassociative only up to conjugation by a certain element in the third tensor power of H. Since we are reconstructing $\tilde{H}$ via its comodule category, the dual notion of coquasibialgebra is relevant here. Thus, in a second approximation, the functor $\mathfrak{F}$ takes as its input a Hopf algebra inclusion with a cleaving map, and yields a coquasibialgebra. This fits in well with the Kac sequence, since a cocommutative coquasibialgebra is the same as a cocommutative ordinary bialgebra, equipped in addition with a fixed Sweedler three-cocycle. But now the construction is sadly asymetric: We have to put in a bialgebra, but what we get out is only a coquasibialgebra. So in a third and final approximation, the functor $\mathfrak{F}$ takes as its input a mapping system looking like $(*)$, but with H only a coquasibialgebra, and its output is again such a mapping system, with K replaced by K^*. To make the construction work, we have to impose certain conditions on the coquasibialgebra structure with respect to the inclusion and the cleaving. While these conditions may seem unpleasant at the outset, it turns out that they are stable under our construction; thus we get a functor $\mathfrak{F}$ that has the same type of mapping system as input and output (with K replaced by K^* and sides switched). Applied to an extension $K^* \to H \to Q$ of Hopf algebras, the functor $\mathfrak{F}$ yields what we will call a generalized product coquasibialgebra: a coquasibialgebra containing K and Q as subbialgebras and isomorphic to their tensor product. This, then, is the non-cocommutative analog of the key map from extensions to degree three cohomology in the Kac sequence, where a generalized product coquasibialgebra specializes to a double crossproduct with a three-cocycle.

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2 Preliminaries

Throughout the paper k denotes a fixed commutative base ring; tensor products, algebras, coalgebras etc. are over k if nothing else is indicated. We use $\nabla\colon A \otimes A \to A$ and $\eta\colon k \to A$ to denote multiplication and unit of an algebra A; we use $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ to denote comultiplication and counit of a coalgebra C. We use $S\colon H \to H$ to denote the antipode of a Hopf algebra H. We write A^{op} for the opposite algebra of an algebra, C^{cop} for the opposite coalgebra of a coalgebra, $H^{\mathrm{bop}} := (H^{\mathrm{op}})^{\mathrm{cop}} = (H^{\mathrm{cop}})^{\mathrm{op}}$ for the biopposite of a bialgebra. We denote the categories of left (right, bi-) modules over A by ${}_A\mathcal{M}$ ($\mathcal{M}_A$, ${}_A\mathcal{M}_A$). We denote the categories of left (right, bi-) comodules over C by ${}^C\mathcal{M}$ ($\mathcal{M}^C$, ${}^C\mathcal{M}^C$). We make constant use of variations of Sweedler's notation for coalgebra structures $\Delta(c) =: c_{(1)} \otimes c_{(2)}$, and for left and right comodule structures $\lambda(v) = v_{(-1)} \otimes v_{(0)}$ and $\rho(v) = v_{(0)} \otimes v_{(1)}$. The convolution product of $f, g \in \mathrm{Hom}(C, A)$ for an algebra A and coalgebra C will be denoted by $f * g$. In a coalgebra C we write $C^+ := \mathrm{Ker}(\varepsilon\colon C \to k)$. A Hopf algebra that is finitely generated projective as a k-module will be called a finite Hopf algebra for short. We refer to [24] for background material on Hopf algebra theory.

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2.1 Monoidal categories

A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \Phi, I, \lambda, \rho)$ consists of a category $\mathcal{C}$, a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a "unit" object I and coherent natural isomorphisms $\Phi = \Phi_{XYZ}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $\lambda\colon I \otimes X \to X$ and $\rho\colon X \otimes I \to X$. We refer to [18] (where monoidal categories are called tensor categories) for more details and background information. We note the famous "pentagon" axiom

$$(\mathrm{id}_W \otimes \Phi_{XYZ})\Phi_{W,X\otimes Y,Z}(\Phi_{WXY} \otimes Z) = \Phi_{W,X,Y\otimes Z}\Phi_{W\otimes X,Y,Z},$$

where both sides are morphisms $((W \otimes X) \otimes Y) \otimes Z \to W \otimes (X \otimes (Y \otimes Z))$. In our use of monoidal categories the structure morphisms λ and ρ will be somewhat trivial (typically the natural isomorphisms $R \underset{R}{\otimes} M \cong M \cong M \underset{R}{\otimes} R$ for an R-bimodule M over some ring R), whereas Φ is often nontrivial. We will often suppress all considerations involving the neutral object and the structure morphisms attached to it.

Obviously structure preserving functors are of interest in relation to monoidal categories. We define a tensor functor

$$(\mathcal{F}, \xi, \xi_0)\colon (\mathcal{C}, \otimes, \Phi, I, \lambda, \rho) \to (\mathcal{D}, \otimes, \Phi', I', \lambda', \rho')$$

to consist of a functor $\mathcal{F}$, a natural isomorphism $\xi\colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ and an isomorphism $\xi_0\colon I' \to \mathcal{F}(I)$. A tensor functor $(\mathcal{F}, \xi, \xi_0)$ is strict provided ξ and ξ_0 are identity morphisms. A tensor functor $(\mathcal{F}, \xi, \xi_0)$ is neutral if

$$\begin{tikzcd}
I' \otimes \mathcal{F}(X) \arrow[r, "\lambda'"] \arrow[d, "\xi_0 \otimes \mathrm{id}"'] & \mathcal{F}(X) \\
\mathcal{F}(I) \otimes \mathcal{F}(X) \arrow[r, "\xi"'] & \mathcal{F}(I \otimes X) \arrow[u, "\mathcal{F}(\lambda)"']
\end{tikzcd}$$

and an analogous diagram for ρ commute. We will only encounter neutral tensor functors, and we will suppress ξ_0 and all considerations of neutral objects. A neutral tensor functor is called a monoidal functor if the diagrams

$$\begin{tikzcd}
(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) \arrow[r, "\Phi'"] \arrow[d, "\xi \otimes \mathrm{id}"'] & \mathcal{F}(X) \otimes (\mathcal{F}(Y) \otimes \mathcal{F}(Z)) \arrow[d, "\mathrm{id} \otimes \xi"] \\
\mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) \arrow[d, "\xi"'] & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) \arrow[d, "\xi"] \\
\mathcal{F}((X \otimes Y) \otimes Z) \arrow[r, "\mathcal{F}(\Phi)"'] & \mathcal{F}(X \otimes (Y \otimes Z))
\end{tikzcd}$$

commute. (Warning: In [18] monoidal functors are called tensor functors.)

The composition of tensor functors $(\mathcal{F},\xi)\colon \mathcal{C} \to \mathcal{D}$ and $(\mathcal{G},\zeta)\colon \mathcal{B} \to \mathcal{C}$ is defined to be $(\mathcal{F}\mathcal{G},\mathcal{F}(\zeta)\xi)\colon \mathcal{B} \to \mathcal{D}$, where

$$\mathcal{F}(\zeta)\xi = \left(\mathcal{F}\mathcal{G}(X)\otimes\mathcal{F}\mathcal{G}(Y) \xrightarrow{\xi} \mathcal{F}(\mathcal{G}(X)\otimes\mathcal{F}(X)) \xrightarrow{\mathcal{F}(\zeta)} \mathcal{F}\mathcal{G}(X\otimes Y)\right).$$

2.2 Coquasibialgebras and coquasimorphisms

The notion of a coquasibialgebra [20] is formally dual to the notion of a quasibialgebra due to Drinfeld [12]. A coquasibialgebra $H = (H,\nabla,\phi)$ consists of a coalgebra H, equipped with a coalgebra map $\nabla\colon H\otimes H \to H$, a grouplike element $1 \in H$ which is a unit for ∇, and a convolution invertible map $\phi\colon H\otimes H\otimes H \to k$ such that $\phi(g\otimes 1\otimes h) = \varepsilon(g)\varepsilon(h)$,

$$(f_{(1)}g_{(1)})h_{(1)}\phi(f_{(2)}\otimes g_{(2)}\otimes h_{(2)}) = \phi(f_{(1)}\otimes g_{(1)}\otimes h_{(1)})f_{(2)}(g_{(2)}h_{(2)}),$$

and

$$\begin{aligned}\phi(d_{(1)}f_{(1)}\otimes g_{(1)}\otimes h_{(1)})\phi(d_{(2)}\otimes f_{(2)}\otimes g_{(2)}h_{(2)})&\\ = \phi(d_{(1)}\otimes f_{(1)}\otimes g_{(1)})\phi(d_{(2)}\otimes f_{(2)}g_{(2)}\otimes h_{(1)})&\phi(f_{(3)}\otimes g_{(3)}\otimes h_{(2)})\end{aligned}$$

for $d,f,g,h \in H$. As a consequence of the definition we note that also $\phi(1\otimes g\otimes h) = \phi(g\otimes h\otimes 1) = \varepsilon(gh)$ for all $g,h \in H$. If ϕ is trivial, a coquasibialgebra is of course just a bialgebra. A cocommutative coquasibialgebra is the same as a cocommutative bialgebra endowed with a normalized Sweedler three-cocycle [37] on H with values in k.

It is the 'purpose' of coquasibialgebras that the category of, say, left comodules over a coquasibialgebra H is a monoidal category: The tensor product of $V,W \in {}^H\mathcal{M}$ is their tensor product over k, with the comodule structure induced via multiplication, and the associator map is

$$(U\otimes V)\otimes W \ni u\otimes v\otimes w \mapsto \phi(u_{(-1)}\otimes v_{(-1)}\otimes w_{(-1)})u_{(0)}\otimes v_{(0)}\otimes w_{(0)} \in U\otimes(V\otimes W)$$

for $U,V,W \in {}^H\mathcal{M}$. Note that the underlying functor ${}^H\mathcal{M} \to {}_k\mathcal{M}$ is a strict neutral tensor functor.

If H is a coquasibialgebra with coassociator ϕ, then H^{op} is a coquasibialgebra with coassociator ϕ^{op} defined by $\phi^{\mathrm{op}}(f\otimes g\otimes h) = \phi^{-1}(h\otimes g\otimes f)$, H^{cop} is a coquasibialgebra with coassociator $\phi^{\mathrm{cop}} = \phi^{-1}$, and H^{bop} is a coquasibialgebra with $\phi^{\mathrm{bop}}(f\otimes g\otimes h) = \phi(h\otimes g\otimes f)$.

There is an obvious notion of coquasibialgebra map $F\colon H \to H'$: This is a multiplicative coalgebra map with $\phi'(F\otimes F\otimes F) = \phi$.

We will need the following more general notion:

Let (H,ϕ) and (H',ϕ') be two coquasibialgebras. A coquasimorphism $F = (F,\theta)\colon (H,\phi) \to (H',\phi')$ consists of a unital coalgebra map $F\colon H \to H'$ and a convolution invertible $\theta\colon H \otimes H \to k$ satisfying $\theta(1 \otimes h) = \theta(h \otimes 1) = \varepsilon(h)$,

$$\theta(g_{(1)} \otimes h_{(1)})F(g_{(2)}h_{(2)}) = F(g_{(1)})F(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})$$

and

$$\begin{aligned}\theta(f_{(1)} \otimes g_{(1)})\theta(f_{(2)}g_{(2)} \otimes h_{(1)})\phi(f_{(3)} \otimes g_{(3)} \otimes h_{(2)})\\ = \phi'(F(f_{(1)}) \otimes F(g_{(1)}) \otimes F(h_{(1)}))\theta(g_{(2)} \otimes h_{(2)})\theta(f_{(2)} \otimes g_{(3)}h_{(3)})\end{aligned}$$

for all $f,g,h \in H$. Coquasibialgebra maps are precisely those coalgebra maps F such that (F,ε) is a coquasimorphism. It is the 'purpose' of coquasimorphisms that they induce monoidal functors: If (F,θ) is a coquasimorphism as above, then ${}^{(F,\theta)}\mathcal{M} := {}^{F}\mathcal{M}\colon {}^{H}\mathcal{M} \to {}^{H'}\mathcal{M}$ is a monoidal functor when equipped with the monoidal functor structure

$$V \otimes W \ni v \otimes w \mapsto \theta(v_{(-1)} \otimes w_{(-1)})v_{(0)} \otimes w_{(0)} \in V \otimes W$$

for $V,W \in {}^{H}\mathcal{M}$. (The domain of the morphism is ${}^{F}\mathcal{M}(V) \otimes {}^{F}\mathcal{M}(W)$, the codomain is ${}^{F}\mathcal{M}(V \otimes W)$.)

The composition of coquasimorphisms $(F,\theta)\colon H \to H'$ and $(F',\theta')\colon H' \to H''$ is given by $(F',\theta')(F,\theta) = (F'F, \theta'(F \otimes F) * \theta)$. With this definition we have $({}^{(F',\theta')}\mathcal{M})({}^{(F,\theta)}\mathcal{M}) = {}^{(F',\theta')(F,\theta)}\mathcal{M}$ as monoidal functors.

If $(F,\theta)\colon H \to H'$ is a coquasimorphism, then so are $(F,\theta)^{\mathrm{op}} = (F,\theta^{\mathrm{op}})\colon H^{\mathrm{op}} \to H'^{\mathrm{op}}$, $(F,\theta)^{\mathrm{cop}} = (F,\theta^{\mathrm{cop}})\colon H^{\mathrm{cop}} \to H'^{\mathrm{cop}}$, and $(F,\theta)^{\mathrm{bop}} = (F,\theta^{\mathrm{bop}})\colon H^{\mathrm{bop}} \to H'^{\mathrm{bop}}$, where $\theta^{\mathrm{op}}(g \otimes h) = \theta(h \otimes g)$, $\theta^{\mathrm{cop}} = \theta^{-1}$, and $\theta^{\mathrm{bop}}(g \otimes h) = \theta^{-1}(h \otimes g)$.

A special case of coquasimorphisms are cotwists [9] dual to the twists defined in [12]: These can be considered as coquasimorphisms in which the underlying coalgebra map is the identity. Let (H,∇,ϕ) be a coquasibialgebra, and $\theta\colon H \otimes H \to k$ a convolution invertible map satisfying $\theta(1 \otimes h) = \theta(h \otimes 1) = \varepsilon(h)$ for all $h \in H$. Then there is a unique coquasibialgebra structure $(H,\nabla,\phi)^\theta$ on the coalgebra $H^\theta := H$ such that $(\mathrm{id}_H,\theta)\colon H \to H^\theta$ is a coquasimorphism: One can easily solve the equations defining a coquasimorphism for ∇^θ and ϕ^θ. Note that every coquasimorphism $(F,\theta)\colon H \to H'$ factors into a cotwist and a coquasibialgebra map:

$$(F,\theta) = \left(H \xrightarrow{(\mathrm{id}_H,\theta)} H^\theta \xrightarrow{(F,\varepsilon)} H'\right).$$

If H is a bialgebra and $\theta\colon H \otimes H \to k$ is a cotwist, then H^θ is a bialgebra (with trivial coassociator) if and only if $\theta(f_{(1)} \otimes g_{(1)})\theta(f_{(2)}g_{(2)} \otimes h) = \theta(g_{(1)} \otimes h_{(1)})\theta(f \otimes g_{(2)}h_{(2)})$ holds for all $f,g,h \in H$. We say that θ is a (normalized) two-cocycle on H. Note that for cocommutative H a two-cocycle in this sense is precisely a normalized Sweedler two-cocycle [37] on H with coefficients in k. If H is a Hopf algebra, then any two-cocycle (co)twist H^θ is also a Hopf algebra.

The dual (and older, cf. [12]) notion to a cotwist is the twist of a quasibialgebra H by an invertible element $t \in H \otimes H$. We will only need the case where H is an ordinary bialgebra, and t is a dual two-cocycle, that is, satisfies $t_{23}(\mathrm{id} \otimes \Delta)(t) = t_{12}(\Delta \otimes \mathrm{id})(t) \in H \otimes H \otimes H$, where $t_{23} = 1 \otimes t$ and $t_{12} = t \otimes 1$, and $(\varepsilon \otimes \mathrm{id})(t) = (\mathrm{id} \otimes \varepsilon)(t) = 1$. Then H_t is a bialgebra, with underlying algebra the same as H, and comultiplication $\Delta_t(h) = t\Delta(h)t^{-1}$. If H is a Hopf algebra, then so is H_t. Note that if H is finite, then t is a dual two-cocycle if and only if t, considered as a map $H^* \otimes H^* \to k$, is a two-cocycle on H^*.

2.3 Tannaka-Krein reconstruction principles

Not only is it an important property of coquasibialgebras that they give rise to monoidal categories, it even characterizes them: If H is a coalgebra and we are given a monoidal category structure $(\otimes, \Phi)$ on ${}^H\mathcal{M}$ such that the underlying functor ${}^H\mathcal{M} \to {}_k\mathcal{M}$ is a strict neutral tensor functor, then there is a unique structure (∇, ϕ) of coquasibialgebra on H such that the monoidal category structure induced by it on ${}^H\mathcal{M}$ coincides with the given one. Explicitly, the multiplication and coassociator on H can be expressed by the formulas

$$
\begin{aligned}
gh &= (g \otimes h)_{(-1)}\varepsilon^{\otimes 2}((g \otimes h)_{(0)}) \\
\phi(f \otimes g \otimes h) &= \varepsilon^{\otimes 3}\Phi(f \otimes g \otimes h)
\end{aligned}
$$

where $f, g, h \in H$, and H is considered a left H-comodule in the usual way.

The principles of Tannaka-Krein duality or reconstruction theory for quantum groups [29, 26, 31, 30] assert somewhat more. Not only can the structures of a coquasibialgebra on a given coalgebra be reconstructed from a monoidal category structure on its comodule category, but under favorable circumstances a coquasibialgebra can be reconstructed from any monoidal category $\mathcal{C}$. The key ingredient is a functor $\mathcal{V}\colon \mathcal{C} \to {}_k\mathcal{M}$ to the category of k-modules. A coendomorphism coalgebra of $\mathcal{V}$ is by definition a representing object $\mathrm{coend}(\mathcal{V})$ for the functor assigning to $M \in {}_k\mathcal{M}$ the set of natural transformations $\mathrm{Nat}(\mathcal{V}, M \otimes \mathcal{V})$. If it exists, it is a coalgebra, and the universal coalgebra for which the functor $\mathcal{V}$ factors over a functor $\hat{\mathcal{V}}$ making the diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\hat{\mathcal{V}}"] \arrow[dr, "\mathcal{V}"'] & & {}^{\mathrm{coend}(\mathcal{V})}\mathcal{M} \arrow[dl, "\mathcal{U}"] \\
& {}_k\mathcal{M} &
\end{tikzcd}
$$

commute, where $\mathcal{U}$ denotes the underlying functor. Even if $\mathrm{coend}(\mathcal{V})$ exists, $\hat{\mathcal{V}}$ need not be an equivalence. If k is a field, $\mathcal{C}$ is k-linear abelian, $\mathcal{V}$ is faithful and exact, and every object of $\mathcal{C}$ is a colimit of objects whose image under $\mathcal{V}$ is finite dimensional, then $\operatorname{coend}(\mathcal{V})$ exists, and the functor $\hat{\mathcal{V}}$ is an equivalence. Now if we assume further that $\mathcal{C}$ is a monoidal category and $\mathcal{V}$ is endowed with a structure ξ of a neutral tensor functor, then we can proceed to reconstruct a coquasibialgebra structure on $\operatorname{coend}(\mathcal{V})$: There is a unique such structure such that we can complete the above triangle to be a commutative triangle

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "(\hat{\mathcal{V}},\hat{\xi})"] \arrow[dr, "(\mathcal{V},\xi)"'] & & {}^{\operatorname{coend}(\mathcal{V})}\mathcal{M} \arrow[dl, "(\mathcal{U},\mathrm{id})"] \\ & {}_k\mathcal{M} & \end{tikzcd}$$

of tensor functors.

The reconstruction principles extend easily to morphisms: If H, H' are two coquasibialgebras and $(\mathcal{F},\xi)\colon {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$ is a monoidal functor that commutes as a functor with the underlying functors to the category ${}_k\mathcal{M}$, then there is a unique coquasimorphism $(F,\theta)\colon H \to H'$ such that $(\mathcal{F},\xi) = {}^{(F,\theta)}\mathcal{M}$. Explicitly, we have $F(h) = (\mathrm{id}_{H'} \otimes \varepsilon)\lambda'(h)$, where $\lambda'\colon H \to H' \otimes H$ is the H'-comodule structure of the image under $\mathcal{F}$ of the left H-comodule H, and we have $\theta(g \otimes h) = \varepsilon^{\otimes 2}\xi(g \otimes h)$.

2.4 Hopf modules

Stated most generally, a Hopf module M is simultaneously a module over an algebra A and a comodule over a coalgebra C, with the two structures subject to some kind of compatibility condition; recent very general formulations are in [8, 4].

We will be needing the following special types of Hopf modules: If H is a bialgebra, then a left H-comodule algebra is by definition an algebra in the monoidal category of left H-comodules, that is, an algebra and a left H-comodule satisfying $(ab)_{(-1)} \otimes (ab)_{(0)} = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}$ for all $a, b \in A$, and $1_{(-1)} \otimes 1_{(0)} = 1 \otimes 1$. If A is a left H-comodule algebra, then a Hopf module $M \in {}^H_A\mathcal{M}$ is by definition a left A-module in the monoidal category of H-comodules. Explicitly, M is a left H-comodule and a left A-module such that the module structure is a comodule map, that is, $(am)_{(-1)} \otimes (am)_{(0)} = a_{(-1)}m_{(-1)} \otimes a_{(0)}m_{(0)}$ holds for all $a \in A$ and $m \in M$. Similarly, the objects of ${}^H\mathcal{M}_A$ and ${}^H_A\mathcal{M}_A$ are by definition right A-modules and A-A-bimodules within the monoidal category of left H-comodules, respectively.

Dually, a left H-module coalgebra is by definition a coalgebra in the monoidal category of left H-modules, that is, a left H-module and coalgebra C such that $(h\cdot c)_{(1)} \otimes (h\cdot c)_{(2)} = h_{(1)}\cdot c_{(1)} \otimes h_{(2)}\cdot c_{(2)}$ and $\varepsilon(h\cdot c) = \varepsilon(h)\varepsilon(c)$ hold for all $h \in H$ and $c \in C$. Now for an H-module coalgebra C, a Hopf module in ${}^C_H\mathcal{M}$ is by definition a left C-comodule in the monoidal category of left H-modules, that is, a left C-comodule and H-module M satisfying $(h{\cdot}m)_{(-1)}\otimes(h{\cdot}m)_{(0)} = h_{(1)}{\cdot}m_{(-1)}\otimes h_{(2)}{\cdot}m_{(0)}$ for all $h \in H$ and $m \in M$.

2.5 Galois coextensions and cleft coextensions

Hopf-Galois coextensions were introduced in [34] (without being given a name). The key fact for our purposes is a structure theorem for their Hopf modules reviewed below.

The theory connecting Hopf-Galois coextensions, cleft coextensions, and crossed coproducts is formally dual to that of Hopf-Galois extensions, cleft extensions, and crossed products [10, 3]; it was developed in detail in [7] (for the case of a base field). Most importantly for our purposes, a cleft K-coextensions $H \to Q$ is a Galois coextension, and isomorphic to a crossed coproduct $H \cong K \ltimes Q$. Let us review some further details to fix notations:

Let K be a Hopf algebra and H a left K-module coalgebra. Let $Q := H/K^+H$, a quotient coalgebra of H. We say that H is a faithfully coflat K-Galois coextension of Q if H is a faithfully coflat right Q-comodule, and the Galois map

$$K \otimes H \ni x \otimes h \mapsto xh_{(1)} \otimes h_{(2)} \in H \underset{Q}{\Box} H$$

is a bijection. Here for $V \in \mathcal{M}^Q$ and $W \in {}^Q\mathcal{M}$ the cotensor product $V \underset{Q}{\Box} W$ is defined by $V \underset{Q}{\Box} W = \{\sum v_i \otimes w_i | \sum v_{i(0)} \otimes v_{i(1)} \otimes w_i = \sum v_i \otimes w_{i(-1)} \otimes w_{i(0)}\}$. If H is a faithfully coflat K-Galois coextension of Q, then by [34] one has a category equivalence

$$\begin{aligned} {}^H_K\mathcal{M} &\cong {}^Q\mathcal{M} \\ M &\mapsto \overline{M} := M/K^+M \\ H \underset{Q}{\Box} V &\leftarrow\!\shortmid V \end{aligned}$$

where, for $M \in {}^H_K\mathcal{M}$, the left Q-comodule structure of $\overline{M}$ is given by $(\overline{m})_{(-1)} \otimes (\overline{m})_{(0)} = \overline{m_{(-1)}} \otimes \overline{m_{(0)}}$, and, for $V \in {}^Q\mathcal{M}$, the structure of left H-comodule on $H \underset{Q}{\Box} V$ is induced by the left tensor factor. In particular, one has the isomorphisms

$$M \ni m \mapsto m_{(-1)} \otimes \overline{m_{(0)}} \in H \underset{Q}{\Box} \overline{M} \quad \text{and} \quad \overline{H \underset{Q}{\Box} V} \ni \overline{\sum h_i \otimes v_i} \mapsto \sum \varepsilon(h_i) v_i \in V$$

for all M, V as above.

H is said to be a cleft K-coextension of Q if there exists a cleaving map for the K-module coalgebra H, that is, a convolution invertible K-module map $\pi \colon H \to K$.

If H is cleft, a cleaving map π with $\varepsilon\pi = \varepsilon$ can be chosen: If π_0 is a cleaving map, then so is π defined by $\pi(h) = \pi(h_{(1)})\varepsilon\pi^{-1}(h_{(2)})$.

Assume H is cleft and fix a cleaving map π with $\varepsilon\pi = \varepsilon$. We have $\pi^{-1}(xh) = \pi^{-1}(h)S(x)$ for $x \in K$ and $h \in H$. For every Hopf module $M \in {}^H_K\mathcal{M}$ put $\overline{M} := M/K^+M \in {}^Q\mathcal{M}$. A natural section $j = j_M \colon \overline{M} \to M$ for the canonical epi $\nu = \nu_M \colon M \to \overline{M}$ is given by $j(\overline{m}) = \pi^{-1}(m_{(-1)})m_{(0)}$. One checks that $j_M(v)_{(-1)} \otimes \overline{j_M(v)_{(0)}} = j_H(v_{(-1)}) \otimes v_{(0)} \in H \otimes \overline{M}$ holds for all $v \in \overline{M}$. In particular, $j_H \colon Q \to H$ is right Q-colinear.

One has an isomorphism

$$\begin{aligned} M &\cong K \otimes \overline{M} \\ m &\mapsto \pi(m_{(-1)}) \otimes \overline{m_{(0)}} \\ xj(v) &\leftarrow\!\shortmid x \otimes v. \end{aligned}$$

In particular, H itself is isomorphic to $K \otimes Q$ as a left K-module and right Q-comodule.

A (left) crossed coproduct is a K-module coalgebra $H = K \ltimes_\alpha Q$ of the following form: As a left K-module, we have $H = K \otimes Q$; the comultiplication is given in terms of a coalgebra structure on Q, a 'weak coaction' $\rho \colon Q \ni q \mapsto q_{[0]} \otimes q_{[1]} \in Q \otimes K$ and a 'dual cocycle' $\alpha \colon Q \ni q \mapsto \alpha^{(1)}(q) \otimes \alpha^{(2)}(q) \in K \otimes K$ by the formula

$$\Delta(x \ltimes q) = x_{(1)}\alpha^{(1)}(q_{(1)}) \ltimes q_{(2)[0]} \otimes x_{(2)}\alpha^{(2)}(q_{(1)})q_{(2)[1]} \ltimes q_{(3)},$$

and $\varepsilon(x \ltimes q) = \varepsilon(x)\varepsilon(q)$ for $x \in K$ and $q \in Q$, where we write $x \ltimes q := x \otimes q \in K \otimes Q =: K \ltimes_\alpha Q$. We will not be needing to know explicitly the conditions that one has to impose on ρ and α to ensure that the above formula really defines a coalgebra. If α is convolution invertible, then a crossed coproduct is a cleft K-module coalgebra with cleaving map $\pi = \mathrm{id}_K \otimes \varepsilon \colon K \ltimes Q \to K$; the quotient $(K \ltimes_\alpha Q)/K^+(K \ltimes_\alpha Q)$ can be identified with Q via $\nu = (\varepsilon \otimes \mathrm{id}_Q) \colon K \ltimes_\alpha Q \to Q$.

If H is a cleft K-coextension of $Q = H/K^+H$, then H is isomorphic to a crossed coproduct with convolution invertible dual cocycle: The isomorphism $K \otimes Q \ni x \otimes q \mapsto xj(q) \in H$ becomes a coalgebra isomorphism $H \cong K \ltimes_\alpha Q$ if we define the weak coaction and dual cocycle by the formulas $\rho := (\nu \otimes \pi)\Delta j$ and $\alpha := (\pi \otimes \pi)\Delta j$.

3 Coquasibialgebras constructed via Hopf bimodules

The subject of this section is a general construction for coquasibialgebras of the following type: Given a suitable coquasibialgebra containing a certain finite Hopf algebra K as a subbialgebra, one can form another coquasibialgebra containing K^* as a subbialgebra. The construction arises naturally from rather simple principles: The basic idea of reconstruction theory is that (coquasi)bialgebras come naturally with any monoidal category that we can find; we apply this to a monidal category that one can form in the most obvious way from any Hopf algebra inclusion.

The simplicity of the basic idea may easily be forgotten in the thick of the technical details involved — these are due in particular to a generalization to Hopf algebra inclusions $K \subset H$ in which H is not a bialgebra but only a coquasibialgebra. This generalization is natural since the output of the construction is not an ordinary bialgebra anyway; the generalized construction is a functor between analogous categories associated to K and K^*, respectively.

To help keep the general strategy in mind we will start by a short outline of things ahead.

3.1 An outline of the programme

Consider a bialgebra H and a subbialgebra $K \subset H$ which is a Hopf algebra. Then K is an algebra in the monoidal category ${}^H\mathcal{M}$ of left H-comodules. Hence, one can consider the category ${}^H_K\mathcal{M}_K$ of K-bimodules in the monoidal category ${}^H\mathcal{M}$. These are just ordinary K-bimodules which are also H-comodules and satisfy the two compatibility conditions for a Hopf module with respect to the K-module coalgebra H. Of course, ${}^H_K\mathcal{M}_K$ is a monoidal category with the tensor product over K (it does not matter if we take this within the category ${}^H\mathcal{M}$, or in the ordinary sense).

Assume now that H is cleft as a left K-module coalgebra. This means in particular that a Hopf module $M \in {}^H_K\mathcal{M}$ is isomorphic to $H \underset{Q}{\square} \overline{M}$ where $\overline{M} := M/K^+M$, and $Q = \overline{H} = H/K^+H$. Moreover, $H \cong K \otimes Q$ as left K-module and right Q-comodule, so that $M \cong K \otimes \overline{M}$ as left K-module. Now considering $M, N \in {}^H_K\mathcal{M}_K$, we find $\overline{M \underset{K}{\otimes} N} \cong \overline{M \underset{K}{\otimes} K \otimes \overline{N}} \cong \overline{M \otimes \overline{N}} \cong \overline{M} \otimes \overline{N}$; in short, since every Hopf bimodule in ${}^H_K\mathcal{M}_K$ is a free left K-module, the functor $\mathcal{V}\colon {}^H_K\mathcal{M}_K \ni M \mapsto \overline{M} \in {}_k\mathcal{M}$ preserves tensor products: it maps the tensor product over K to the tensor product over k.

Now the principles of reconstruction theory almost tell us that the tensor product preserving functor $\mathcal{V}\colon {}^H_K\mathcal{M}_K \to {}_k\mathcal{M}$ factors through a monoidal category equivalence ${}^H_K\mathcal{M}_K \to {}^{\tilde{H}}\mathcal{M}$, where $\tilde{H}$ is a suitable coquasibialgebra, the coendomorphism coalgebra of the functor $\mathcal{V}$. Surely coendomorphism coalgebras do not always exist, so that one has to take care of this by explicit construction rather than reference to general principles (see however Remark 3.4.1 below). In the case that K is finite, we find that $\mathcal{V}$ does factor over an equivalence with the category ${}^{\tilde{H}}\mathcal{M}$ for a suitably constructed coalgebra $\tilde{H}$. From here, general principles take over to guarantee a coquasibialgebra structure on $\tilde{H}$ coming from the monoidal category structures.

Now the output of the construction outlined so far is of a different type than the input: We put in a bialgebra and get out a coquasibialgebra. Below, the construction will be done more generally for an inclusion $K \subset H$ in which H is a coquasibialgebra subject to certain restrictions. The output of the construction will be again a coquasibialgebra which will turn out to satisfy the same restrictions we had to impose on the input.

3.2 Hopf bimodules associated to an inclusion

Let H be a coquasibialgebra with coassociator ϕ. If K is a subcoquasibialgebra (a subcoalgebra closed under multiplication and containing the unit; this is naturally a coquasibialgebra with coassociator the restriction of ϕ) and the restriction of ϕ to $K^{\otimes 3}$ happens to be trivial, then K is an ordinary bialgebra. It is natural to call K a subbialgebra of H in this situation. However, we still do not have associativity for products that involve elements of K as well as of H. Thus, H is not a K-module in the ordinary sense. Of course K, which is naturally an H-comodule, say, on the left, can also be considered as an associative algebra within the monoidal category of H-comodules (since the coassociator is trivial on $K \otimes K \otimes K$). However, H is not a left or right K-module in this category either; this would mean

$$(xg)h = \phi(x_{(1)} \otimes g_{(1)} \otimes h_{(1)})x_{(2)}(g_{(2)}h_{(2)})$$

holds for all $x \in K$ and $g, h \in H$, which differs from the associativity axiom in H. We can ensure that H is an ordinary left K-module as well as a left K-module in the category of left H-comodules by imposing an additional restriction on ϕ.

Lemma 3.2.1. *Let $H = (H, \Delta, \nabla, \phi)$ be a k-flat coquasibialgebra, and $K \subset H$ a subcoquasibialgebra such that $\phi|_{K \otimes H \otimes H} = \varepsilon$. Then*

(1) $\phi(xf \otimes g \otimes h) = \varepsilon(x)\phi(f \otimes g \otimes h) \quad \forall x \in K, f, g, h \in H.$

(2) $\phi^{-1}(xf \otimes g \otimes h) = \varepsilon(x)\phi^{-1}(f \otimes g \otimes h) \quad \forall x \in K, f, g, h \in H.$

(3) K is an ordinary bialgebra. More generally, $(xg)h = x(gh)$ holds for all $x \in K$ and $g, h \in H$; in particular, H is a left K-module coalgebra.

(4) K is an algebra in the monoidal category ${}^H\mathcal{M}$, and H is a left K-module in the monoidal category ${}^H\mathcal{M}$.

Proof. For the first claim we calculate

$$\begin{aligned}\phi(xf \otimes g \otimes h) &= \phi(x_{(1)}f_{(1)} \otimes g_{(1)} \otimes h_{(1)})\phi(x_{(2)} \otimes f_{(2)} \otimes g_{(2)}h_{(2)}) \\ &= \phi(x_{(1)} \otimes f_{(1)} \otimes g_{(1)})\phi(x_{(2)} \otimes f_{(2)}g_{(2)} \otimes h_{(1)})\phi(f_{(3)} \otimes g_{(3)} \otimes h_{(2)}) \\ &= \varepsilon(x)\phi(f \otimes g \otimes h).\end{aligned}$$

The second claim follows easily from the first. The third claim as well as the fourth are direct consequences of the assumption on ϕ and the definitions. $\square$

Note that (1) implies that ϕ induces a well-defined map $H/K^+H \otimes H \otimes H \ni \overline{f} \otimes g \otimes h \mapsto \phi(f \otimes g \otimes h) \in k$, which we will denote again by ϕ.

Definition 3.2.2. Let H be a k-flat coquasibialgebra with associator ϕ and $K \subset H$ a subcoquasibialgebra with $\phi|_{K \otimes H \otimes H} = \varepsilon$. We denote the categories of left K-modules (resp. right K-modules, K-bimodules) in the monoidal category ${}^H\mathcal{M}$ by ${}^H_K\mathcal{M}$ (resp. ${}^H\mathcal{M}_K$, ${}^H_K\mathcal{M}_K$). The category ${}^H_K\mathcal{M}_K$ has a natural monoidal category structure. We will denote the tensor product of $M, N \in {}^H_K\mathcal{M}_K$ by $M \underset{K}{\otimes} N$.

Remark 3.2.3. Assume the conditions of the preceding definition.

A left K-module M in ${}^H\mathcal{M}$ is just an ordinary left K-module and left H-comodule satisfying $(xm)_{(-1)} \otimes (xm)_{(0)} = x_{(1)}m_{(-1)} \otimes x_{(2)}m_{(0)}$ for all $x \in K$ and $m \in M$; that is, M is a relative Hopf module in the ordinary sense with respect to the K-module coalgebra H. In particular, the notation ${}^H_K\mathcal{M}$ is in accordance with the familiar notations for Hopf modules, whereas the other two notations are somewhat abusive. In detail, an object of ${}^H\mathcal{M}_K$ is a left H-comodule M, equipped with a colinear map $\mu_r\colon M \otimes K \ni m \otimes x \mapsto mx \in M$ satisfying $(mx)y = \phi(m_{(-1)} \otimes x_{(1)} \otimes y_{(1)})m_{(0)}(x_{(2)}y_{(2)})$ and $m1_K = m$ for all $m \in M$, $x, y \in K$ (so M is not in general a right K-module). An object $M \in {}^H_K\mathcal{M}_K$ is an object of ${}^H_K\mathcal{M}$ as well as ${}^H\mathcal{M}_K$ satisfying in addition $(xm)y = x(my)$ for all $x, y \in K$ and $m \in M$.

The notation $M \underset{K}{\otimes} N$ for the tensor product of $M, N \in {}^H_K\mathcal{M}_K$ has to be read with caution as well. As a left K-module and H-comodule, $M \underset{K}{\otimes} N$ is the quotient of $M \otimes N$ (equipped with the left K-module structure of the left tensor factor and the codiagonal comodule structure) modulo the relation $mx \otimes n = \phi(m_{(-1)} \otimes x_{(1)} \otimes n_{(-1)})m_{(0)} \otimes x_{(2)}n_{(0)}$ for $m \in M$, $n \in N$ and $x \in K$. The right K-structure is given by $(m \otimes n)x = \phi(m_{(-1)} \otimes n_{(-1)} \otimes x_{(1)})m_{(0)} \otimes n_{(0)}x_{(2)}$.

Definition 3.2.4. Let (H, ϕ) be a k-flat coquasibialgebra and $K \subset H$ a subcoquasibialgebra such that $\phi|_{K \otimes H \otimes H} = \varepsilon$. Put $Q := H/K^+H$.

Define the category ${}^Q\mathcal{M}_{\phi,K}$ as follows: An object of ${}^Q\mathcal{M}_{\phi,K}$ is a left Q-comodule V equipped with a map $\mu\colon V \otimes K \ni v \otimes x \mapsto vx \in V$ satisfying

$$\begin{aligned}(vx)_{(-1)} \otimes (vx)_{(0)} &= v_{(-1)}x_{(1)} \otimes v_{(0)}x_{(2)}\\ (vx)y &= \phi(v_{(-1)} \otimes x_{(1)} \otimes y_{(1)})v_{(0)}(x_{(2)}y_{(2)})\\ v1_K &= v\end{aligned}$$

for all $v \in V$, $x, y \in K$.

Proposition 3.2.5. *Let (H, ϕ) be a k-flat coquasibialgebra and $K \subset H$ a subcoquasibialgebra such that $\phi|_{K \otimes H \otimes H} = \varepsilon$. Assume that H is a faithfully coflat K-Galois coextension of $Q := H/K^+H$ (and K is a Hopf algebra).*

Then we have a category equivalence

$$\begin{aligned}\hat{\mathcal{V}}\colon {}^H_K\mathcal{M}_K &\cong {}^Q\mathcal{M}_{\phi,K}\\ M &\mapsto \overline{M} := M/K^+M\\ H\underset{Q}{\square}V &\leftarrow\!\shortmid V\end{aligned}$$

described as follows: For $M \in {}^H_K\mathcal{M}_K$, we have $\overline{M} \in {}^Q\mathcal{M}_{\phi,K}$ by the Q-comodule structure induced by the H-comodule structure of M, and $\overline{m}x = \overline{mx}$ For $V \in {}^Q\mathcal{M}_{\phi,K}$, we have $H\underset{Q}{\square}V \in {}^H_K\mathcal{M}_K$ with the left H-comodule and K-module structures induced by the left tensor factor, and $(\sum h_i \otimes v_i)x = \sum h_i x_{(1)} \otimes v_i x_{(2)}$ for $\sum h_i \otimes v_i \in H\underset{Q}{\square}V$ and $x \in K$.

Proof. The equivalence is based on Schneider's category equivalence ${}^H_K\mathcal{M} \cong {}^Q\mathcal{M}$ reviewed in Section 2.5.

If $M \in {}^H_K\mathcal{M}_K$, then the map $\overline{M} \otimes K \ni \overline{m} \otimes x \mapsto \overline{mx} \in \overline{M}$ is well-defined, since $y(mx) = (ym)x$ for $x, y \in K$ and $m \in M$, since $\phi|_{K\otimes H\otimes K} = \varepsilon$. We have

$$\begin{aligned}(\overline{m}x)y = \overline{mx}y = \overline{(mx)y} &= \phi(m_{(-1)} \otimes x_{(1)} \otimes y_{(1)})\overline{m_{(0)}(x_{(2)}y_{(2)})}\\ &= \phi(m_{(-1)} \otimes x_{(1)} \otimes y_{(1)})\overline{m_{(0)}}(x_{(2)}y_{(2)}) = \phi(\overline{m}_{(-1)} \otimes x_{(1)} \otimes y_{(1)})\overline{m}_{(0)}(x_{(2)}y_{(2)})\end{aligned}$$

and $\overline{m}1 = \overline{m}$, showing that $\overline{M} \in {}^Q\mathcal{M}_{\phi,K}$.

If $V \in {}^Q\mathcal{M}_{\phi,K}$, then

$$(H\underset{Q}{\square}V) \otimes K \ni \sum h_i \otimes v_i \otimes x \mapsto \sum h_i x_{(1)} \otimes v_i x_{(2)} \in H\underset{Q}{\square}V$$

is well-defined by the first condition in the definition of ${}^Q\mathcal{M}_{\phi,K}$. We have

$$\begin{aligned}((\sum h_i \otimes v_i)x)y = (\sum h_i x_{(1)} \otimes v_i x_{(2)})y &= \sum (h_i x_{(1)})y_{(1)} \otimes (v_i x_{(2)})y_{(2)}\\ &= \sum (h_i x_{(1)})y_{(1)} \otimes \phi(v_{i(-1)} \otimes x_{(2)} \otimes y_{(2)})v_{i(0)}(x_{(3)}y_{(3)})\\ &= \sum (h_{i(1)}x_{(1)})y_{(1)}\phi(h_{i(2)} \otimes x_{(2)} \otimes y_{(2)}) \otimes v_i(x_{(3)}y_{(3)})\\ &= \sum \phi(h_{i(1)} \otimes x_{(1)} \otimes y_{(1)})h_{i(2)}(x_{(2)}y_{(2)}) \otimes v_i(x_{(3)}y_{(3)})\\ &= \sum \phi(h_{i(1)} \otimes x_{(1)} \otimes y_{(2)})(h_{i(2)} \otimes v_i)(x_{(2)}y_{(2)})\end{aligned}$$

We omit checking that the isomorphisms $M \ni m \mapsto m_{(-1)} \otimes \overline{m_{(0)}} \in H\underset{Q}{\square}\overline{M}$ and $\overline{H\underset{Q}{\square}V} \ni \overline{\sum h_i \otimes v_i} \mapsto \sum \varepsilon(h_i)v_i \in V$ are in ${}^H_K\mathcal{M}_K$, resp. ${}^Q\mathcal{M}_{\phi,K}$ when M and V are. $\square$

3.3 Tensorial underlying functors

The next step in our programme is to make the functor $\mathcal{V}\colon {}^H_K\mathcal{M}_K \to {}_k\mathcal{M}$ a tensor functor under the assumption that H is a cocleft K-module coalgebra. This will be somewhat more complicated than indicated in the outline, since we want to retain the possibility that H is a coquasibialgebra rather than an ordinary bialgebra. In this case it is still true that any $M \in {}^H_K\mathcal{M}_K$ is free as a left K-module, but since the tensor product in ${}^H_K\mathcal{M}_K$ is not the ordinary tensor product of K-bimodules, the desired consequences won't follow until we assume additional technical hypotheses on the cleaving map.

Lemma 3.3.1. *Let H be a k-flat coquasibialgebra and $K \subset H$ a subcoquasibialgebra such that $\phi|_{K\otimes H\otimes H} = \varepsilon$ and K is a Hopf algebra. Let $\pi\colon H \to K$ be a cleaving map satisfying $\varepsilon\pi = \varepsilon$ and $\pi(1) = 1$. Put $Q = H/K^+H$ and define j as in Section 2.5.*

The following are equivalent:

(1) $\phi(g \otimes x \otimes j(q)_{(1)})j(q)_{(2)} = \varepsilon(g)\varepsilon(x)j(q)\,\forall g \in H, x \in K, q \in Q.$

(2) $\phi(g \otimes x \otimes yj(v)_{(-1)})j(v)_{(0)} = \phi(g \otimes x \otimes y)j(v)\,\forall M \in {}^H_K\mathcal{M}, v \in \overline{M}, g \in H, x, y \in K.$

(3) $\phi(g \otimes x \otimes h_{(1)})\pi(h_{(2)}) = \phi(g \otimes x \otimes \pi(h)_{(1)})\pi(h)_{(2)}\,\forall g, h \in H, x \in K.$

(4) Any one of (1) through (3) with ϕ replaced by ϕ^{-1}

If K is finite, then the above conditions are further equivalent to

(5) *(a)* $\phi(\mathrm{id}_H \otimes \iota \otimes \pi) = \phi(\mathrm{id}_H \otimes \iota \otimes \mathrm{id}_H)$

(b) $\tilde{\alpha}^{(1)}(p)\langle\tilde{\alpha}^{(2)}(p)|\alpha^{(1)}(q)\rangle \otimes \alpha^{(2)}(q) = \varepsilon(p)\varepsilon(q)\,\forall p, q \in Q$

where α is the cocycle in the cleft K-module coalgebra H, and $\tilde{\alpha}\colon Q \ni q \mapsto \tilde{\alpha}^{(1)}(q) \otimes \tilde{\alpha}^{(2)}(q) \in K^* \otimes K^*$ is defined by $\langle\tilde{\alpha}(q)|x \otimes y\rangle = \phi(q \otimes x \otimes y)$ for $q \in Q$ and $x, y \in K$.

Proof. Clearly (2) is a generalization of (1). Assuming (1), the calculation

$$\begin{aligned}\phi(g\otimes x\otimes yj(q)_{(1)})j(q)_{(1)} &= \phi(g_{(1)}x_{(1)}\otimes y_{(1)}\otimes j(q)_{(1)})\phi(g_{(2)}\otimes x_{(2)}\otimes y_{(2)}j(q)_{(2)})j(q)_{(3)}\\ &= \phi(g_{(1)} \otimes x_{(1)} \otimes y_{(1)})\phi(g_{(2)} \otimes x_{(2)}y_{(2)} \otimes j(q)_{(1)})\phi(x_{(3)} \otimes y_{(3)} \otimes j(q)_{(2)})j(q)_{(3)}\\ &= \phi(g \otimes x \otimes y)j(q)\end{aligned}$$

shows (2) for the special case $M = H$. Assuming this, the calculation

$$\begin{aligned}\phi(g \otimes x \otimes yj(v)_{(-1)})j(v)_{(0)} &= \phi(g \otimes x \otimes yj(v)_{(-2)})\pi(j(v)_{(-1)})j(\overline{j(v)_{(0)}})\\ &= \phi(g \otimes x \otimes yj(v_{(-1)})_{(1)})\pi(j(v_{(-1)})_{(2)})j(v_{(0)})\\ &= \phi(g \otimes x \otimes y)\pi(j(v_{(-1)}))j(v_{(0)}) = \phi(g \otimes x \otimes y)j(v)\end{aligned}$$

proves (2) in the general case.

To see that (1) follows from (3) we calculate

$$\begin{aligned}\phi(g\otimes x\otimes j(q)_{(1)})j(q)_{(2)} &= \phi(g\otimes x\otimes j(q)_{(1)})\pi(j(q)_{(2)})j(\overline{j(q)_{(3)}})\\ &= \phi(g\otimes x\otimes j(q_{(1)})_{(1)})\pi(j(q_{(1)})_{(2)})j(q_{(2)})\\ &= \phi(g\otimes x\otimes \pi(j(q_{(1)}))_{(1)})\pi(j(q_{(1)}))_{(2)}j(q_{(2)}) = j(q).\end{aligned}$$

Assume (2). To show (3) it is enough to consider $h = yj(q)$ for $y\in K$ and $q\in Q$; here

$$\begin{aligned}\phi(g\otimes x\otimes (yj(q))_{(1)})\pi((yj(q))_{(2)}) &= \phi(g\otimes x\otimes y_{(1)}j(q)_{(1)})\pi(y_{(2)}j(q)_{(2)})\\ = \phi(g\otimes x\otimes y_{(1)})\pi(y_{(2)}j(q)) &= \phi(g\otimes x\otimes y_{(1)})y_{(2)}\varepsilon(q)\\ &= \phi(g\otimes x\otimes \pi(yj(q))_{(1)})\pi(yj(q))_{(2)}\end{aligned}$$

The equivalence with (4) follows from the following more general observation:

Lemma 3.3.2. *Let H be a cleft K-module coalgebra with cleaving map π, and j as in Section 2.5. Let A be an algebra and $T\colon H\to A$ a convolution invertible map.*

(1) For a Hopf module $M\in {}^H_K\mathcal{M}$ the condition

$$T(xj(v)_{(-1)})\otimes j(v)_{(0)} = T(x)\otimes j(v)\in A\otimes M\quad \forall x\in K, v\in M/K^+M$$

implies the same for T^{-1} in place of T.

(2) The condition

$$T(h_{(1)})\otimes \pi(h_{(2)}) = T(\pi(h)_{(1)})\otimes \pi(h)_{(2)}\in A\otimes K\quad \forall h\in H$$

implies the same for T^{-1} in place of T.

For the first claim we compute

$$\begin{aligned}T^{-1}(xj(v)_{(-1)})\otimes j(v)_{(0)} &= T^{-1}(x_{(1)})T(x_{(2)})T^{-1}(x_{(3)}j(v)_{(-1)})\otimes j(v)_{(0)}\\ &= T^{-1}(x_{(1)})T(x_{(2)}j(v)_{(-2)})T^{-1}(x_{(3)}j(v)_{(-1)})\otimes j(v)_{(0)} = T^{-1}(x_{(1)})j(v),\end{aligned}$$

for the second

$$\begin{aligned}T^{-1}(\pi(h)_{(1)})\otimes\pi(h)_{(2)} &= T^{-1}(h_{(1)})T(h_{(2)})T^{-1}(\pi(h_{(3)})_{(1)})\otimes\pi(h_{(3)})_{(2)}\\ &= T^{-1}(h_{(1)})T(\pi(h_{(2)})_{(1)})T^{-1}(\pi(h_{(2)})_{(2)})\otimes\pi(h_{(2)})_{(3)} = T^{-1}(h_{(1)})\otimes\pi(h_{(2)}).\end{aligned}$$

Now assume K is finite (5)(a) follows from (3) by applying ε. Using (5)(a) and (1) we have, for all $x\in K$, $p,q\in Q$,

$$\begin{aligned}&\langle\tilde\alpha^{(1)}(p)|x\rangle\langle\tilde\alpha^{(2)}(p)|\alpha^{(1)}(q)\rangle\alpha^{(2)}(q)\\ &\quad= \phi(j(p)\otimes x\otimes\alpha^{(1)}(q))\alpha^{(2)}(q) = \phi(j(p)\otimes x\otimes\pi(j(q)_{(1)}))\pi(j(q)_{(2)})\\ &\qquad= \phi(j(p)\otimes x\otimes j(q)_{(1)})\pi(j(q)_{(2)}) = \varepsilon(p)\varepsilon(x)\pi j(q) = \varepsilon(p)\varepsilon(q)\varepsilon(x),\end{aligned}$$

showing (5)(b).

Finally assume (5). Then the calculation

$$
\begin{aligned}
&\phi(g \otimes x \otimes j(q)_{(1)})j(q)_{(2)} \\
&\qquad = \phi(g \otimes x \otimes \alpha^{(1)}(q_{(1)})j(q_{(2)[0]}))\alpha^{(2)}(q_{(1)})q_{(2)[1]}j(q_{(3)}) \\
&\qquad = \phi(g \otimes x \otimes \alpha^{(1)}(q_{(1)}))\alpha^{(2)}(q_{(1)})j(q_{(2)}) \\
&\qquad = \langle \tilde{\alpha}^{(1)}(\overline{g})|x\rangle\langle \tilde{\alpha}^{(2)}(\overline{g})|\alpha^{(1)}(q_{(1)})\rangle\alpha^{(1)}(q_{(1)})j(q_{(2)}) \\
&\qquad = \varepsilon(g)\varepsilon(x)\varepsilon(q_{(2)})j(q_{(2)}) = \varepsilon(g)\varepsilon(x)j(q)
\end{aligned}
$$

shows (1). □

Definition 3.3.3. Let K be a k-Hopf algebra. The category $\mathfrak{E}_\ell(K)$ is defined as follows:

(1) An object $H = (H, \nabla, \Delta, \phi, \iota, \pi) \in \mathfrak{E}_\ell(K)$ consists of a k-flat coquasibialgebra $(H, \nabla, \Delta, \phi)$, a multiplicative unital coalgebra map $\iota\colon K \to H$ such that

$$\phi(\iota \otimes \mathrm{id}_{H\otimes H}) = \varepsilon,$$

and a convolution invertible left K-module map $\pi\colon H \to K$ satisfying $\pi(1) = 1$, $\varepsilon\pi = \varepsilon$, and

$$\phi(g \otimes x \otimes h_{(1)})\pi(h_{(2)}) = \phi(g \otimes x \otimes \pi(h)_{(1)})\pi(h)_{(2)} \quad \forall g, h \in H, x \in K.$$

(2) A morphism $(H, \nabla, \Delta, \phi, \iota, \pi) \to (H', \nabla', \Delta', \phi', \iota', \pi')$ in $\mathfrak{E}_\ell(K)$ is a coquasimorphism $(F, \theta)\colon (H, \nabla, \Delta, \phi) \to (H', \nabla', \Delta', \phi')$ such that $F\iota = \iota'$ and $\theta(\iota \otimes \mathrm{id}_H) = \varepsilon$.

Besides the notion of isomorphism within the category $\mathfrak{E}_\ell(K)$, there is an obvious stronger notion of isomorphism between objects of $\mathfrak{E}_\ell(K)$, which we will refer to as a strong isomorphism: a coalgebra isomorphism compatible with all the structure maps that constitute the objects.

The category $\mathfrak{E}_r(K)$ has objects those $(H, \nabla, \Delta, \phi, \iota, \pi)$ such that $(H^{\mathrm{bop}}, \nabla^{\mathrm{op}}, \Delta^{\mathrm{cop}}, \phi^{\mathrm{bop}}, \iota, \pi) \in \mathfrak{E}_\ell(K^{\mathrm{bop}})$ and morphisms those coquasimorphisms (F, θ) such that $(F, \theta^{-1}\tau)$ is a morphism in $\mathfrak{E}_\ell(K^{\mathrm{bop}})$.

We will most of the time treat ι as above tacitly as an inclusion map.

Assume that π_0 is a cleaving map and $\pi = \pi_0 * \varepsilon\pi_0^{-1}$ is the corresponding cleaving map with $\varepsilon\pi = \varepsilon$. It is interesting to note that $\pi_0(1) = 1$ implies

$\pi(1)=1$, and if condition (3) in Lemma 3.3.1 is satisfied for π_0 in place of π, then

$$\begin{aligned}\phi(g\otimes x\otimes h_{(1)})\pi(h_{(2)}) &= \phi(g\otimes x\otimes h_{(1)})\pi_0(h_{(2)})\varepsilon\pi_0^{-1}(h_{(3)})\\ &= \phi(g\otimes x\otimes \pi_0(h_{(1)})_{(1)})\pi_0(h_{(1)})_{(2)}\varepsilon\pi_0^{-1}(h_{(2)})\\ &= \phi(g\otimes x\otimes (\pi_0(h_{(1)})\varepsilon\pi_0^{-1}(h_{(2)}))_{(1)})(\pi_0(h_{(1)})\varepsilon\pi_0^{-1}(h_{(2)}))_{(2)}\\ &= \phi(g\otimes x\otimes \pi(h)_{(1)})\pi(h)_{(2)}\end{aligned}$$

also holds for $g,h\in H$ and $x\in K$.

Proposition 3.3.4. *Let K be a k-flat Hopf algebra, and $(H,\nabla,\Delta,\phi,\iota,\pi)\in\mathfrak{E}_\ell(K)$.*

For $M\in{}^H_K\mathcal{M}_K$ write $\overline{M}:=M/K^+M$, and define j as in Section 2.5.

Then we have the isomorphism, natural in $M,N\in{}^H_K\mathcal{M}_K$,

$$\gamma\colon M\underset{K}{\otimes}N\ni m\otimes n\mapsto m\pi(n_{(-1)})\otimes\overline{n_{(0)}}\in M\otimes\overline{N}$$

with inverse given by $\gamma^{-1}(m\otimes v)=m\otimes j(v)$.

Proof. To show that γ is well-defined, consider $\gamma_0\colon M\otimes N\to M\otimes\overline{N}$ with $\gamma_0(m\otimes n)=m\pi(n_{(-1)})\otimes\overline{n_{(0)}}$. Since

$$\begin{aligned}\gamma_0(mx\otimes n) &= (mx)\pi(n_{(-1)})\otimes\overline{n_{(0)}}\\ &= \phi(m_{(-1)}\otimes x_{(1)}\otimes\pi(n_{(-1)})_{(1)})m_{(0)}(x_{(2)}\pi(n_{(-1)})_{(2)})\otimes\overline{n_{(0)}}\\ &= \phi(m_{(-1)}\otimes x_{(1)}\otimes n_{(-2)})m_{(0)}(x_{(2)}\pi(n_{(-1)})\otimes x_{(3)}n_{(0)}\\ &= \phi(m_{(-1)}\otimes x_{(1)}\otimes n_{(-2)})m_{(0)}\pi(x_{(2)}n_{(-1)})\otimes x_{(3)}n_{(0)}\\ &= \gamma_0(\phi(m_{(-1)}\otimes x_{(1)}\otimes n_{(-1)})m_{(0)}\otimes x_{(2)}n_{(0)})\end{aligned}$$

holds for $m\in M$, $n\in N$ and $x\in K$, γ_0 factors to give the desired map γ. To show that $\overline{\gamma}\colon M\otimes\overline{N}\ni m\otimes v\mapsto m\otimes j(v)\in M\underset{K}{\otimes}N$ is inverse to γ, we calculate

$$\begin{aligned}\overline{\gamma}\gamma(m\otimes n) &= \overline{\gamma}(m\pi(n_{(-1)})\otimes\overline{n_{(0)}}) = m\pi(n_{(-1)})\otimes j(\overline{n_{(0)}})\\ &= \phi(m_{(-1)}\otimes\pi(n_{(-1)})_{(1)}\otimes j(\overline{n_{(0)}})_{(-1)})m_{(0)}\otimes\pi(n_{(-1)})_{(2)}j(\overline{n_{(0)}})_{(0)}\\ &= m\otimes\pi(n_{(-1)})j(\overline{n_{(0)}}) = m\otimes n\end{aligned}$$

and

$$\gamma\overline{\gamma}(m\otimes v)=\gamma(m\otimes j(v))=m\pi(j(v)_{(-1)})\otimes\overline{j(v)_{(0)}}=m\pi(j(v_{(-1)}))\otimes v_{(0)}=m\otimes v$$

for $m\in M$, $n\in N$ and $v\in\overline{N}$. Naturality is obvious. □

Lemma 3.3.5. *Let K be a k-flat Hopf algebra, and $(H,\nabla,\Delta,\phi,\iota,\pi)\in\mathfrak{E}_\ell(K)$.*

Then for all $M,N\in{}^H_K\mathcal{M}_K$ we have an isomorphism

$$\begin{aligned}\xi=\xi_\pi=\xi_{\pi,M,N}\colon \overline{M}\otimes\overline{N}&\to\overline{M\underset{K}{\otimes}N}\\ \overline{m}\otimes\overline{n}&\mapsto\overline{m\otimes\pi^{-1}(n_{(-1)})n_{(0)}}\\ \overline{m\pi(n_{(-1)})}\otimes\overline{n_{(0)}}&\leftarrow\!\shortmid\overline{m\otimes n}\end{aligned}$$

making

$$(\mathcal{V},\xi_\pi)\colon({}^H_K\mathcal{M}_K,\underset{K}{\otimes})\to({}_k\mathcal{M},\otimes)$$

a neutral tensor functor.

Proof. To show the existence of the isomorphism ξ it suffices to note that the map γ from the preceding proposition is a left K-module map. The claim that $(\mathcal{V},\xi_\pi)$ is a neutral tensor functor means that $\xi_{\pi,K,N}\colon\overline{N}\cong\overline{K}\otimes\overline{N}\to\overline{K\underset{K}{\otimes}N}$ as well as $\xi_{\pi,M,K}\colon\overline{M}\cong\overline{M}\otimes\overline{K}\to\overline{M\underset{K}{\otimes}K}$ coincide with the obvious canonical isomorphisms. This is trivial to read off from the definition. □

Given a monoidal category $\mathcal{C}$ and another category $\mathcal{D}$ equivalent to it, it is trivial to conclude that $\mathcal{D}$ is also monoidal — in such a way that the given equivalence is a monoidal functor. In particular, we could have concluded above that ${}^Q\mathcal{M}_{\phi,K}$ is a monoidal category, just because it is equivalent to ${}^H_K\mathcal{M}_K$. Now the resulting abstract tensor product in ${}^Q\mathcal{M}_{\phi,K}$ resulting from these considerations is quite unpleasant: It is obtained by transporting objects back and forth between the categories by means of the equivalence in Proposition 3.2.5, that is, the tensor product of $V,W\in{}^Q\mathcal{M}_{\phi,K}$ is given by $\overline{(H\underset{Q}{\Box}V)\underset{K}{\otimes}(H\underset{Q}{\Box}W)}$. Given a suitable cleaving map π as above we can conclude that the latter is isomorphic to $V\otimes W$ as a k-module. In other words, given a suitable cleaving map we can give ${}^Q\mathcal{M}_{\phi,K}$ a more manageable structure of monoidal category, with the abstract tensor product modelled on the tensor product over k of the underlying k-modules. The technical details are laid out in the following

Theorem 3.3.6. *Let K be a k-flat Hopf algebra, $(H,\nabla,\Delta,\phi,\iota,\pi)\in\mathfrak{E}_\ell(K)$, and $Q:=H/K^+H$. Consider the functor $\mathcal{V}\colon{}^H_K\mathcal{M}_K\ni M\mapsto\overline{M}:=M/K^+M\in{}_k\mathcal{M}$.*

There is a unique monoidal category structure on ${}^Q\mathcal{M}_{\phi,K}$ and a unique natural isomorphism $\hat{\xi}_\pi$ such that the underlying functor $\mathcal{U}\colon{}^Q\mathcal{M}_{\phi,K}\to{}_k\mathcal{M}$ is a strict neutral tensor functor, $(\hat{\mathcal{V}},\hat{\xi}_\pi)$ is a monoidal functor, and

$$\begin{tikzcd}
{}^H_K\mathcal{M}_K \arrow[rr, "(\hat{\mathcal{V}},\hat{\xi}_\pi)"] \arrow[dr, "(\mathcal{V},\xi_\pi)"'] & & {}^Q\mathcal{M}_{\phi,K} \arrow[dl, "(\mathcal{U},\mathrm{id})"] \\
& \mathcal{M}_k &
\end{tikzcd}$$

is a commutative diagram of neutral tensor functors.

Explicitly, the tensor product of $V, W \in {}^Q\mathcal{M}_{\phi,K}$ *is their tensor product over* k*, equipped with the following structures:*

$$(v \otimes w)x = \phi(j(v_{(-1)}) \otimes j(w_{(-1)})_{(1)} \otimes x_{(1)})v_{(0)}\pi(j(w_{(-1)})_{(2)}x_{(2)}) \otimes w_{(0)}x_{(3)}$$
$$(v \otimes w)_{(-1)} \otimes (v \otimes w)_{(0)} = \overline{j(v_{(-1)})j(w_{(-1)})_{(1)}} \otimes v_{(0)}\pi(j(w_{(-1)})_{(2)}) \otimes w_{(0)}$$

(where $v \in V$*,* $w \in W$ *and* $x \in K$*). For* $U, V, W \in {}^Q\mathcal{M}_{\phi,K}$*, the associator map* $\Phi\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ *is given by*

$$\begin{aligned}\Phi(u \otimes v \otimes w) = \ & \phi(j(u_{(-1)}) \otimes j(v_{(-1)})_{(1)} \otimes j(w_{(-1)})_{(1)}) \\ & \quad u_{(0)}\pi(j(v_{(-1)})_{(2)}j(w_{(-1)})_{(2)}) \otimes v_{(0)}\pi(j(w_{(-1)})_{(3)}) \otimes w_{(0)}\end{aligned}.$$

Proof. Suppose that we have a monoidal category and monoidal functor structure as claimed. That $\mathcal{U}$ is a strict neutral tensor functor means that the tensor product in ${}^Q\mathcal{M}_{\phi,K}$ is modelled on the tensor product of k-modules over k. The claimed formulas for the Q-comodule structure and K-action on a tensor product only have to be verified in the case that the objects involved are in the image of $\hat{\mathcal{V}}$, thanks to the fact that $\hat{\mathcal{V}}$ is an equivalence. That $(\mathcal{U}, \mathrm{id})(\hat{\mathcal{V}}, \hat{\xi}_\pi) = (\mathcal{V}, \xi_\pi)$ as neutral tensor functors means that the k-module homomorphism underlying $\hat{\xi}_\pi$ is just ξ_π. Hence, the Q-comodule structure and right K-action on $\overline{M} \otimes \overline{N}$ are determined by the requirement that $\xi_\pi\colon \overline{M} \otimes \overline{N} \to \overline{M \underset{K}{\otimes} N}$ be a morphism in ${}^Q\mathcal{M}_{\phi,K}$. We compute

$$\begin{aligned}
&(\overline{m}\otimes v)_{(-1)} \otimes (\overline{m} \otimes v)_{(0)} \\
&\quad = \overline{\xi(\overline{m} \otimes v)_{(-1)}} \otimes \xi^{-1}(\xi(\overline{m} \otimes v)_{(0)}) \\
&\quad = \overline{m_{(-1)}j(v)_{(-1)}} \otimes \xi^{-1}(\overline{m_{(0)} \otimes j(v)_{(0)}}) \\
&\quad = \overline{m_{(-1)}j(v)_{(-2)}} \otimes \overline{m_{(0)}}\pi(j(v)_{(-1)}) \otimes \overline{j(v)_{(0)}} \\
&\quad = \overline{m_{(-1)}j(v_{(-1)})_{(1)}} \otimes \overline{m_{(0)}}\pi(j(v_{(-1)})_{(2)}) \otimes v_{(0)} \\
&\quad = \overline{j(\overline{m}_{(-1)})j(v_{(-1)})_{(1)}} \otimes \overline{m}_{(0)}\pi(j(v_{(-1)})_{(2)}) \otimes v_{(0)}
\end{aligned}$$

and

$$\begin{aligned}
(\overline{m}\otimes v)x &= \xi^{-1}(\xi(\overline{m} \otimes v)x) \\
&= \xi^{-1}(\overline{(m \otimes j(v))x}) \\
&= \xi^{-1}(\phi(m_{(-1)} \otimes j(v)_{(-1)} \otimes x_{(1)})\overline{m_{(0)} \otimes j(v)_{(0)}x_{(2)}}) \\
&= \phi(m_{(-1)} \otimes j(v)_{(-2)} \otimes x_{(1)})\overline{m_{(0)}}\pi(j(v)_{(-1)}x_2) \otimes \overline{j(v)_{(0)}x_3} \\
&= \phi(j(m_{(-1)}) \otimes j(v_{(-1)})_{(1)} \otimes x_{(1)})\overline{m}_{(0)}\pi(j(v_{(-1)})_{(2)}x_{(2)}) \otimes v_{(0)}x_{(3)}
\end{aligned}$$

for $M, N \in {}^H_K\mathcal{M}_K$, $m \in M$ and $v \in \overline{N}$. The form of the associator map also has to be verified only for three objects in the image of $\hat{\mathcal{V}}$. For $L, M, N \in {}^H_K\mathcal{M}_K$ the condition that $(\hat{\mathcal{V}},\hat{\xi})$ is a monoidal functor means that

$$\begin{tikzcd}
(\overline{L}\otimes\overline{M})\otimes\overline{N} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\xi\otimes\mathrm{id}"'] & \overline{L}\otimes(\overline{M}\otimes\overline{N}) \arrow[d, "\mathrm{id}\otimes\xi"] \\
\overline{L\underset{K}{\otimes}M}\otimes\overline{N} \arrow[d, "\xi"'] & \overline{L}\otimes\overline{M\underset{K}{\otimes}N} \arrow[d, "\xi"] \\
\overline{(L\underset{K}{\otimes}M)\underset{K}{\otimes}N} \arrow[r, "\overline{\Phi}"'] & \overline{L\underset{K}{\otimes}(M\underset{K}{\otimes}N)}
\end{tikzcd}$$

commutes. We compute

$$\xi(\xi\otimes\mathrm{id})(\overline{\ell}\otimes v\otimes w)=\xi(\overline{\ell\otimes j(v)}\otimes w)=\overline{(\ell\otimes j(v))\otimes j(w)}$$

and

$$\begin{aligned}(\mathrm{id}\otimes\xi)^{-1}\xi^{-1}(\overline{\ell\otimes(m\otimes n)})&=(\mathrm{id}\otimes\xi)^{-1}(\overline{\ell}\pi(m_{(-1)}n_{(-1)})\otimes\overline{m_{(0)}\otimes n_{(0)}})\\ &\qquad\overline{\ell}\pi(m_{(-1)}n_{(-2)})\otimes\overline{m_{(0)}}\pi(n_{(-1)})\otimes\overline{n_{(0)}}\end{aligned}$$

for $\ell\in L$, $m\in M$, $n\in N$, $v\in\overline{M}$, and $w\in\overline{N}$, whence

$$\begin{aligned}\Phi((\overline{\ell}\otimes v)\otimes w)&=(\mathrm{id}\otimes\xi)^{-1}\xi^{-1}\Phi\xi(\xi\otimes\mathrm{id})((\overline{\ell}\otimes v)\otimes w)\\
&=(\mathrm{id}\otimes\xi)^{-1}\xi^{-1}\Phi(\overline{(\ell\otimes j(v))\otimes j(w)})\\
&=(\mathrm{id}\otimes\xi)^{-1}\xi^{-1}(\phi(\ell_{(-1)}\otimes j(v)_{(-1)}\otimes j(w)_{(-1)})\overline{\ell_{(0)}\otimes(j(v)_{(0)}\otimes j(w)_{(0)})})\\
&=\ \phi(\ell_{(-1)}\otimes j(v)_{(-2)}\otimes j(w)_{(-3)})\\
&\qquad\overline{\ell_{(0)}}\pi(j(v)_{(-1)})j(w)_{(-2)})\otimes\overline{j(v)_{(0)}}\pi(j(w)_{(-1)})\otimes\overline{j(w)_{(0)}}\\
&=\ \phi(j(\overline{\ell}_{(-1)})\otimes j(v_{(-1)})_{(1)}\otimes j(w_{(-1)})_{(1)})\\
&\qquad\overline{\ell}_{(0)}\pi(j(v_{(-1)})_{(2)}j(w_{(-1)})_{(2)})\otimes w_{(0)}\pi(j(v_{(-1)})_{(3)})\otimes w_{(0)}\end{aligned}$$

follows.

Existence of the claimed structures is proved by running the proof of uniqueness backwards: For $M,N\in{}^H_K\mathcal{M}_K$, we make $\overline{M}\otimes\overline{N}$ an object of ${}^Q\mathcal{M}_{\phi,K}$ by transporting the structures of $\overline{M\underset{K}{\otimes}N}$ along $\hat{\xi}_\pi$. For general objects $V,W\in{}^Q\mathcal{M}_{\phi,K}$ we utilize the fact that they are isomorphic to $\overline{M}$ and $\overline{N}$ for suitable M,N as above. Similarly, $\tilde{\Phi}\colon(\overline{L}\otimes\overline{M})\otimes\overline{N}\to\overline{L}\otimes(\overline{M}\otimes\overline{N})$ for $L,M,N\in{}^H_K\mathcal{M}_K$ can be defined by the same diagram used in the proof of uniqueness for computing it, and $\tilde{\Phi}\colon(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ for $U,V,W\in{}^Q\mathcal{M}_{\phi,K}$ can be defined by utilizing isomorphisms $U\cong\overline{L}$, $V\cong\overline{M}$, $W\cong\overline{N}$ for suitable $L,M,N\in{}^H_K\mathcal{M}_K$. We omit further details. □

3.4 Reconstruction

We will now proceed to construct a coquasibialgebra $\tilde{H}$ associated to $H \in \mathfrak{E}_\ell(K)$ for finite K.

The purpose of the following remark is to show that, at least in the case where k is a field, the object $\tilde{H}$ we will construct exists by the general Tannaka-Krein reconstruction principles, applied to the tensor functor $(\mathcal{V}, \xi_\pi)$ — we could have defined it, for that matter, using only the definitions of $\mathfrak{E}_\ell(K)$, $\mathcal{V}$ and ξ_π.

Surely our results will be more general and, more importantly, more explicit.

Remark 3.4.1. Assume k is a field, K is a finite dimensional Hopf algebra and $H \in \mathfrak{E}_\ell(K)$. Denote by ${}^H_K\mathcal{M}^f_K$ the category of those objects of ${}^H_K\mathcal{M}_K$ that are finite dimensional over k (or, equivalently, finitely generated K-modules). Then $\mathcal{V}$ induces a functor $\mathcal{V}_0\colon {}^H_K\mathcal{M}^f_K \to {}_k\mathcal{M}$ that maps into the subcategory of finite dimensional vector spaces; by [41] there exists a coendomorphism object $\tilde{H} := \operatorname{coend}(\mathcal{V}_0)$ of $\mathcal{V}_0$, and $\mathcal{V}_0$ factors through an equivalence $\hat{\mathcal{V}}_0\colon {}^H_K\mathcal{M}^f_K \to {}^{\tilde{H}}\mathcal{M}^f$ with the category of finite dimensional $\tilde{H}$-comodules. By [20, 30] $\tilde{H}$ has a coquasibialgebra structure such that $\hat{\mathcal{V}}_0$ is a monoidal functor with monoidal functor structure induced by the isomorphism ξ_π from Lemma 3.3.5. It is easy to check that $\hat{\mathcal{V}}_0$ extends to an equivalence $\hat{\mathcal{V}}\colon {}^H_K\mathcal{M}_K \to {}^{\tilde{H}}\mathcal{M}$: both $\tilde{H}$-comodules and Hopf modules in ${}^H_K\mathcal{M}_K$ can be written as inductive limits of finite dimensional subobjects. $\tilde{H}$ comes with an intrinsically defined bialgebra map $\tilde{\imath}\colon K^* \to \tilde{H}$, because ${}^K_K\mathcal{M}_K$ is canonically a monoidal subcategory of ${}^H_K\mathcal{M}_K$, and ${}^K_K\mathcal{M}_K$ is monoidally equivalent to ${}^{K^*}\mathcal{M} \cong \mathcal{M}_K$ by the special case $H = K$ of Proposition 3.2.5 and Theorem 3.3.6 (or [32]). There is also an intrinsically defined map $\tilde{\pi}\colon \tilde{H} \to K^*$, since we have a natural transformation in $\mathcal{V} \otimes K \to \mathcal{V}$ whose components are the right K-actions, and $\tilde{H}$ is a representing object of $\operatorname{Nat}(\mathcal{V}, -\otimes\mathcal{V})$.

We now start to determine $\tilde{H}$ more explicitly and in a way that also works if k is not a field. The first step constructs only the coalgebra $\tilde{H}$ and does not need a tensorial structure on $\mathcal{V}$.

Definition and Lemma 3.4.2. *Let (H, ϕ) be a k-flat coquasibialgebra and $K \subset H$ a subcoquasibialgebra such that $\phi|_{K\otimes H\otimes H} = \varepsilon$. Put $Q := H/K^+H$.*

Assume that K is finite.

Define $\tilde{\rho}\colon Q \ni q \mapsto q_{[-\tilde{1}]} \otimes q_{[\tilde{0}]} \in K^* \otimes Q$ by $\langle q_{[-\tilde{1}]}|x\rangle q_{[\tilde{0}]} = qx$ for $x \in K$.

Define $\tilde{\alpha}\colon Q \ni q \mapsto \tilde{\alpha}^{(1)}(q) \otimes \tilde{\alpha}^{(2)}(q) \in K^* \otimes K^*$ by $\langle \tilde{\alpha}^{(1)}(q)|x\rangle\langle\tilde{\alpha}^{(2)}(q)|y\rangle := \phi(q \otimes x \otimes y)$ for $q \in Q$ and $x, y \in K$. Then

(1) $Q \rtimes_{\tilde{\alpha}} K^* := Q \otimes K^*$ is a (crossed coproduct) right K^*-module coalgebra with

$$\Delta(q \rtimes \varphi) = q_{(1)} \rtimes q_{(2)[-\tilde{1}]}\tilde{\alpha}^{(1)}(q_{(3)})\varphi_{(1)} \otimes q_{(2)[\tilde{0}]} \rtimes \tilde{\alpha}^{(2)}(q_{(3)})\varphi_{(2)}$$

and $\varepsilon(q \rtimes \varphi) = \varepsilon(q)\varepsilon(\varphi)$.

(2) *One has an isomorphism of categories* ${}^{Q}\mathcal{M}_{\phi,K} \cong {}^{Q\rtimes_{\tilde\alpha}K^*}\mathcal{M}$ *described as follows: Given* $V \in {}^{Q}\mathcal{M}_{\phi,K}$*, the map*

$$V \ni v \mapsto v_{(-1)} \rtimes x^i \otimes v_{(0)}x_i \in Q \rtimes_{\tilde\alpha} K^* \otimes V,$$

where $x_i \otimes x^i \in K \otimes K^*$ *are dual bases, makes* V *a left* $Q \rtimes_{\tilde\alpha} K^*$*-comodule. Given* $V \in {}^{Q\rtimes_{\tilde\alpha}K^*}\mathcal{M}$ *with comodule structure* $v \mapsto v_{(-\tilde 1)} \otimes v_{(\tilde 0)}$*, one has* $V \in {}^{Q}\mathcal{M}_{\phi,K}$ *by*

$$V \ni v \mapsto \tilde\nu(v_{(-\tilde 1)}) \otimes v_{(\tilde 0)} \in Q \otimes V$$

and $vx = \langle \tilde\pi(v_{(-\tilde 1)})|x\rangle v_{(\tilde 0)}$ *for* $v \in V$ *and* $x \in K$*, where* $\tilde\nu = \mathrm{id}\otimes\varepsilon\colon Q \rtimes_{\tilde\alpha} K^* \to Q$ *and* $\tilde\pi = \varepsilon \otimes \mathrm{id}\colon Q \rtimes_{\tilde\alpha} K^* \to K^*$.

Proof. For $x \in K$ define

$$t_x\colon Q \rtimes_{\tilde\alpha} K^* \ni q \rtimes \varphi \mapsto q\langle\varphi|x\rangle \in Q.$$

The claimed coalgebra structure in (1) is obviously right K^*-linear, so that to prove coassociativity, it is enough to check

$$(t_x \otimes t_y \otimes t_z)(\Delta \otimes \mathrm{id})\Delta(q \rtimes 1) = (t_x \otimes t_y \otimes t_z)(\mathrm{id}\otimes\Delta)\Delta(q \rtimes 1)$$

for all $x, y, z \in K$ and $q \in Q$. Using

$$\begin{aligned}
&(t_x \otimes t_y)\Delta(q \rtimes \varphi)\\
&\qquad = q_{(1)}\langle q_{(2)[-\tilde 1]}\tilde\alpha^{(1)}(q_{(3)})\varphi_{(1)}|x\rangle \otimes q_{(2)[\tilde 0]}\langle\tilde\alpha^{(2)}(q_{(3)})\varphi_{(2)}|y\rangle\\
&\qquad\qquad = q_{(1)} \otimes q_{(2)}x_{(1)}\phi(q_{(3)} \otimes x_{(2)} \otimes y_{(1)})\langle\varphi|x_{(3)}y_{(2)}\rangle
\end{aligned}$$

we find

$$\begin{aligned}
&(t_x\otimes t_y \otimes t_z)(\Delta \otimes \mathrm{id})\Delta(q \rtimes 1)\\
&\quad = (t_x \otimes t_y)\Delta(q_{(1)} \rtimes q_{(2)[-\tilde 1]}\tilde\alpha^{(1)}(q_{(3)})) \otimes q_{(2)[\tilde 0]}\langle\tilde\alpha^{(2)}(q_{(3)})|z\rangle\\
&\quad = q_{(1)} \otimes q_{(2)}x_{(1)}\phi(q_{(3)} \otimes x_{(2)} \otimes y_{(1)})\langle q_{(1)[-\tilde 1]}\tilde\alpha^{(1)}(q_{(5)})|x_{(3)}y_{(2)}\rangle\\
&\qquad\qquad \otimes q_{(4)[\tilde 0]}\langle\tilde\alpha^{(2)}(q_{(5)})|z\rangle\\
&\quad = q_{(1)} \otimes q_{(2)}x_{(1)}\phi(q_{(3)} \otimes x_{(2)} \otimes y_{(1)}) \otimes q_{(4)}(x_{(3)}y_{(2)})\phi(q_{(5)} \otimes x_{(4)}y_{(3)} \otimes z)\\
&\quad = q_{(1)} \otimes q_{(2)}x_{(1)} \otimes (q_{(3)}x_{(2)})y_{(1)}\phi(q_{(4)} \otimes x_{(3)} \otimes y_{(2)})\phi(q_{(5)} \otimes x_{(4)}y_{(3)} \otimes z)\\
&\quad = q_{(1)} \otimes q_{(2)}x_{(1)} \otimes (q_{(3)}x_{(2)})y_{(1)}\\
&\qquad\qquad \phi(q_{(4)} \otimes x_{(3)} \otimes y_{(2)})\phi(q_{(5)} \otimes x_{(4)}y_{(3)} \otimes z_{(1)})\phi(x_{(5)} \otimes y_{(4)} \otimes z_{(2)})\\
&\quad = q_{(1)} \otimes q_{(2)}x_{(1)} \otimes (q_{(3)}x_{(2)})y_{(1)}\phi(q_{(4)}x_{(3)} \otimes y_{(2)} \otimes z_{(1)})\phi(q_{(5)} \otimes x_{(4)} \otimes y_{(3)}z_{(2)})
\end{aligned}$$

and also

$$
\begin{aligned}
&(t_x\otimes t_y\otimes t_z)(\mathrm{id}\otimes\Delta)\Delta(q\rtimes 1)\\
&\qquad = q_{(1)}\langle q_{(2)[-\tilde{1}]}\tilde{\alpha}^{(1)}(q_{(3)})|x\rangle\otimes(t_y\otimes t_z)\Delta(q_{(2)[\tilde{0}]}\rtimes\tilde{\alpha}^{(2)}(q_{(3)}))\\
&\qquad = q_{(1)}\otimes(t_x\otimes t_y)\Delta(q_{(2)}x_{(1)}\rtimes\langle\tilde{\alpha}^{(1)}(q_{(3)})|x_{(2)}\rangle\tilde{\alpha}^{(2)}(q_{(3)}))\\
&\qquad = \ q_{(1)}\otimes q_{(2)}x_{(1)}\otimes(q_{(3)}x_{(2)})y_{(1)}\phi(q_{(4)}x_{(3)}\otimes y_{(2)}\otimes z_{(1)})\\
&\qquad\qquad\quad \langle\tilde{\alpha}^{(1)}(q_{(5)})|x_{(4)}\rangle\langle\tilde{\alpha}^{(2)}(q_{(5)})|y_{(3)}z_{(2)}\rangle\\
&\qquad = q_{(1)}\otimes q_{(2)}x_{(1)}\otimes(q_{(3)}x_{(2)})y_{(1)}\phi(q_{(4)}x_{(3)}\otimes y_{(2)}\otimes z_{(1)})\phi(q_{(5)}\otimes x_{(4)}\otimes y_{(3)}z_{(2)}).
\end{aligned}
$$

We now turn to the claimed category equivalence. Let V be a k-module. If V is a left $Q\rtimes_{\tilde{\alpha}}K^*$-comodule with structure map $\Lambda\colon V\ni v\mapsto v_{(-\tilde{1})}\otimes v_{(\tilde{0})}\in Q\otimes K^*\otimes V$, and if we define $\lambda\colon V\ni v\mapsto v_{(-1)}\otimes v_{(0)}\in Q\otimes V$ and $\mu\colon V\otimes K\ni v\otimes x\mapsto vx\in V$ by $vx=\langle\tilde{\pi}(v_{(-\tilde{1})})|x\rangle v_{(\tilde{0})}$ and $\lambda=(\tilde{\nu}\otimes\mathrm{id})\Lambda$, then $\Lambda(v)=\tilde{\nu}(v_{(-\tilde{2})})\otimes\tilde{\pi}(v_{(-\tilde{1})})\otimes v_{(\tilde{0})}=v_{(-1)}\otimes x^i\otimes v_{(0)}x_i$.

Now assume we are given maps $\lambda\colon V\ni v\mapsto v_{(-1)}\otimes v_{(0)}\in Q\otimes V$ and $\mu\colon V\otimes K\ni v\otimes x\mapsto vx\in V$ and define $\Lambda\colon V\ni v\mapsto v_{(-\tilde{1})}\otimes v_{(\tilde{0})}\in Q\otimes K^*\otimes V$ by $\Lambda(v)=v_{(-1)}\otimes x^i\otimes v_{(0)}x_i$. Then clearly $vx=\langle\tilde{\pi}(v_{(-\tilde{1})})|x\rangle v_{(\tilde{0})}$ and $\lambda=(\tilde{\nu}\otimes\mathrm{id})\Lambda$.We note the formula $(t_x\otimes\mathrm{id}_V)\Lambda(v)=v_{(-1)}\otimes v_{(0)}x$. It remains to show that Λ is a comodule structure if and only if λ and μ make V an object of ${}^{Q}\mathcal{M}_{\phi,K}$. First, if Λ is a comodule structure then so is λ, since $\tilde{\nu}$ is a coalgebra map. Under the assumption that λ is a comodule structure we compute

$$
\begin{aligned}
&(t_x\otimes t_y\otimes\mathrm{id}_V)(v_{(-\tilde{1})}\otimes v_{(\tilde{0})(-\tilde{1})}\otimes v_{(\tilde{0})(\tilde{0})})\\
&\qquad = v_{(-1)}\otimes t_y((v_{(0)}x)_{(-\tilde{1})})\otimes(v_{(0)}x)_{(\tilde{0})}=v_{(-1)}\otimes(v_{(0)}x)_{(-1)}\otimes(v_{(0)}x)_{(0)}y
\end{aligned}
$$

and

$$
\begin{aligned}
&(t_x\otimes t_y\otimes\mathrm{id}_V)(\Delta(v_{(-\tilde{1})})\otimes v_{(\tilde{0})})=(t_x\otimes t_y)\Delta(v_{(-1)}\otimes x^i)\otimes v_{(0)}x_i\\
&\qquad = v_{(-3)}\otimes v_{(-2)}x_{(1)}\phi(v_{(-1)}\otimes x_{(2)}\otimes y_{(1)})\langle x^i|x\rangle_{(3)}y_{(2)}\rangle\otimes v_{(0)}x_i\\
&\qquad\qquad = v_{(-3)}\otimes v_{(-2)}x_{(1)}\phi(v_{(-1)}\otimes x_{(2)}\otimes y_{(1)})\otimes v_{(0)}(x_{(3)}y_{(2)}).
\end{aligned}
$$

Thus, if Λ is a coalgebra structure, then for $x,y\in K$ and $v\in V$ we have

$$
\begin{aligned}
(vx)_{(-1)}\otimes(vx)_{(0)} &= (\varepsilon t_x\otimes t_1\otimes\mathrm{id}_V)(v_{(-\tilde{1})}\otimes v_{(\tilde{0})(-\tilde{1})}\otimes v_{(\tilde{0})(\tilde{0})})\\
&= (\varepsilon t_x\otimes t_1\otimes\mathrm{id}_V)(\Delta(v_{(-\tilde{1})})\otimes v_{(\tilde{0})})=v_{(-1)}x_{(1)}\otimes v_{(0)}x_{(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&(vx)y=(\varepsilon t_x\otimes\varepsilon t_y\otimes\mathrm{id}_V)(v_{(-\tilde{1})}\otimes v_{(\tilde{0})(-\tilde{1})}\otimes v_{(\tilde{0})(\tilde{0})})\\
&\quad = (\varepsilon t_x\otimes\varepsilon t_y\otimes\mathrm{id}_V)(\Delta(v_{(-\tilde{1})})\otimes v_{(\tilde{0})})=\phi(v_{(-1)}\otimes x_{(1)}\otimes y_{(1)})\otimes v_{(0)}(x_{(2)}y_{(2)}).
\end{aligned}
$$

Conversely, if $V \in {}^{Q}\mathcal{M}_{\phi,K}$, then

$$
\begin{aligned}
&(t_x \otimes t_y \otimes \mathrm{id}_V)(v_{(-\tilde{1})} \otimes v_{(\tilde{0})(-\tilde{1})} \otimes v_{(\tilde{0})(\tilde{0})}) \\
&\qquad = v_{(-1)} \otimes (v_{(0)}x)_{(-1)} \otimes (v_{(0)}x)_{(0)}y = v_{(-2)} \otimes v_{(-1)}x_{(1)} \otimes (v_{(0)}x_{(2)})y \\
&\qquad\quad = v_{(-3)} \otimes v_{(-2)}x_{(1)} \otimes \phi(v_{(-1)} \otimes x_{(2)} \otimes y_{(1)})v_{(0)}(x_{(3)}y_{(2)}) \\
&\qquad\qquad\qquad\qquad = (t_x \otimes t_y \otimes \mathrm{id}_V)(\Delta(v_{(-\tilde{1})}) \otimes v_{(\tilde{0})})
\end{aligned}
$$

for all $x, y \in K$ and $v \in V$, whence Λ is coassociative. $\square$

Remark 3.4.3. To simplify notations, we identify the categories ${}^{Q}\mathcal{M}_{\phi,K}$ and ${}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M}$ along the isomorphism described in Lemma 3.4.2.

Thus in the situation of Proposition 3.2.5 $\mathcal{V}\colon {}^{H}_{K}\mathcal{M}_{K} \ni M \mapsto \overline{M} \in {}_{k}\mathcal{M}$ factors over the category equivalence $\hat{\mathcal{V}}\colon {}^{H}_{K}\mathcal{M}_{K} \cong {}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M}$, giving a commutative triangle

$$
\begin{tikzcd}
{}^{H}_{K}\mathcal{M}_{K} \arrow[rr, "\hat{\mathcal{V}}"] \arrow[dr, "\mathcal{V}"'] & & {}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M} \arrow[dl, "\mathcal{U}"] \\
& {}_{k}\mathcal{M} &
\end{tikzcd}
$$

of functors.

Corollary 3.4.4. *Let K be a finite Hopf algebra, and $(H, \nabla, \phi, \iota, \pi) \in \mathfrak{E}_{\ell}(K)$. Define $Q, \tilde{\rho}$ and $\tilde{\alpha}$ as in Lemma 3.4.2. There exists a unique coquasibialgebra structure $(\tilde{\nabla}, \tilde{\phi})$ on $Q \rtimes_{\tilde{\alpha}} K^*$ and a unique natural isomorphism $\hat{\xi}_{\pi}$ such that the underlying functor $\mathcal{U}\colon {}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M} \to {}_{k}\mathcal{M}$ is a strict neutral tensor functor, $(\hat{\mathcal{V}}, \hat{\xi}_{\pi})$ is a monoidal functor, and*

$$
\begin{tikzcd}
{}^{H}_{K}\mathcal{M}_{K} \arrow[rr, "(\hat{\mathcal{V}}, \hat{\xi}_{\pi})"] \arrow[dr, "(\mathcal{V}, \xi_{\pi})"'] & & {}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M} \arrow[dl, "(\mathcal{U}, \mathrm{id})"] \\
& {}_{k}\mathcal{M} &
\end{tikzcd}
$$

is a commutative diagram of neutral tensor functors.

Theorem and Definition 3.4.5. *Let K be a finite Hopf algebra. A mapping $\mathfrak{F}\colon \mathfrak{E}_{\ell}(K) \to \mathfrak{E}_{r}(K^*)$ is defined as follows:*

For $(H, \nabla, \phi, \iota, \pi) \in \mathfrak{E}_{\ell}(K)$, define Q, $\tilde{\rho}$ and $\tilde{\alpha}$ as in Lemma 3.4.2.

Then

$$\mathfrak{F}(H, \nabla, \phi, \iota, \pi) := (Q \rtimes_{\tilde{\alpha}} K^*, \tilde{\nabla}, \tilde{\phi}, \tilde{\iota}, \tilde{\pi}),$$

where $(\tilde{\nabla}, \tilde{\phi})$ is the coquasibialgebra structure from Corollary 3.4.4, and

$$
\begin{aligned}
\tilde{\iota}\colon K^* \ni \varphi \mapsto 1 \rtimes \varphi \in Q \rtimes_{\tilde{\alpha}} K^* \\
\tilde{\pi}\colon Q \rtimes_{\tilde{\alpha}} K^* \ni q \rtimes \varphi \mapsto \varepsilon(q)\varphi \in K^*.
\end{aligned}
$$

Explicitly, we have

$$\tilde\nabla(p \rtimes \varphi \otimes q \rtimes \psi) = \begin{array}{l} \overline{j(p_{(1)})j(q_{(1)})_{(1)}} \rtimes x^i(\varphi_{(2)} \leftharpoonup q_{(2)})\psi \\ \qquad \phi(j(p_{(2)}) \otimes j(q_{(1)})_{(2)} \otimes x_i)\langle\varphi_{(1)}|\pi(j(q_{(1)})_{(3)})\rangle \end{array},$$

where $\langle\varphi \leftharpoonup q|x\rangle := \langle\varphi|q \rightharpoonup x\rangle$ *for* $x \in K$, *and* $x_i \otimes x^i \in K \otimes K^*$ *are dual bases. and*

$$\begin{aligned}&\tilde\phi(p \rtimes \varphi \otimes q \rtimes \psi \otimes r \rtimes \vartheta)\\ &\qquad = \phi(j(p) \otimes j(q)_{(1)} \otimes j(r)_{(1)})\langle\varphi|\pi(j(q)_{(2)}j(r)_{(2)})\rangle\langle\psi|\pi(j(r)_{(3)})\rangle\varepsilon(\vartheta)\end{aligned}$$

for $p, q, r \in Q$ *and* $\varphi, \psi, \vartheta \in K^*$.

Proof. Considering $\tilde H := Q \rtimes_{\tilde\alpha} K^*$ as an object in ${}^Q\mathcal{M}_{\phi,K}$ we have $\tilde h = \tilde h_{(\tilde 1)}\varepsilon(\tilde h_{(\tilde 2)}) = \tilde h_{(-1)} \rtimes x^i\varepsilon(\tilde h_{(0)}x_i)$. If $\tilde h = p \rtimes \varphi$, we have $\tilde h_{(-1)}\varepsilon(\tilde h_{(0)}x) = \tilde h_{(-1)}\varepsilon(\tilde h_{(0)}x_i)\langle x^i|x\rangle = p\langle\varphi|x\rangle$.

Let $p, q, r \in Q$ and $\varphi, \psi, \vartheta \in K^*$, put $\tilde f = p \rtimes \varphi$, $\tilde g = q \rtimes \psi$ and $\tilde h = r \rtimes \vartheta$. Then

$$\begin{aligned}\phi(\tilde f \otimes \tilde g \otimes \tilde h) &= \phi(\tilde f_{(\tilde 1)} \otimes \tilde g_{(\tilde 1)} \otimes \tilde h_{(\tilde 1)})\varepsilon(\tilde f_{(\tilde 2)})\varepsilon(\tilde g_{(\tilde 2)})\varepsilon(\tilde h_{(\tilde 2)})\\ &= \varepsilon^{\otimes 3}\Phi(\tilde f \otimes \tilde g \otimes \tilde h)\\ &= \phi(j(\tilde f_{(-1)}) \otimes j(\tilde g_{(-1)})_{(1)} \otimes j(\tilde h_{(-1)})_{(1)})\\ &\qquad \varepsilon(\tilde f\pi(j(\tilde g_{(-1)})_{(2)}j(\tilde h_{(-1)})_{(2)}))\varepsilon(\tilde g\pi(j(\tilde h_{(-1)})_{(3)}))\varepsilon(\tilde h_{(0)})\\ &= \phi(j(p) \otimes j(q)_{(1)} \otimes j(r)_{(1)})\langle\phi|\pi(j(q)_{(2)}j(r)_{(2)})\rangle\langle\psi|\pi(j(r)_{(3)}\rangle\langle\vartheta|1\rangle\end{aligned}$$

and

$$\begin{aligned}\tilde f\tilde g &= (\tilde f \otimes \tilde g)_{(-\tilde 1)}\varepsilon^{\otimes 2}((\tilde f \otimes \tilde g)_{(\tilde 0)})\\ &= (\tilde f \otimes \tilde g)_{(-1)} \rtimes x^i\varepsilon^{\otimes 2}((\tilde f \otimes \tilde g)_{(0)}x_i)\\ &= \overline{j(\tilde f_{(-1)})j(\tilde g_{(-1)})_{(1)}} \rtimes x^i\varepsilon^{\otimes 2}((\tilde f_{(0)}\pi(j(\tilde g_{(-1)})_{(2)}) \otimes \tilde g_{(0)})x_i)\\ &= \overline{j(\tilde f_{(-2)})j(\tilde g_{(-2)})_{(1)}} \rtimes x^i\\ &\qquad \phi(j(\tilde f_{(-1)}\pi(j(\tilde g_{(-2)})_{(2)})_{(1)}) \otimes j(\tilde g_{(-1)})_{(1)} \otimes x_{i(1)})\\ &\qquad \varepsilon((\tilde f_{(0)}\pi(j(\tilde g_{(-2)})_{(2)})_{(2)})\pi(j(\tilde g_{(-1)})_{(2)}x_{i(2)}))\varepsilon(\tilde g_{(0)}x_{i(3)})\\ &= \overline{j(\tilde f_{(-2)})j(q_{(1)})_{(1)}} \rtimes x^i\\ &\qquad \phi(j(\tilde f_{(-1)}\pi(j(q_{(1)})_{(2)})_{(1)}) \otimes j(q_{(2)})_{(1)} \otimes x_{i(1)})\\ &\qquad \varepsilon((\tilde f_{(0)}\pi(j(q_{(1)})_{(2)})_{(2)})\pi(j(q_{(2)})_{(2)}x_{i(2)}))\langle\psi|x_{i(3)}\rangle\\ &= \overline{j(\tilde f_{(-3)})j(q_{(1)})_{(1)}} \rtimes x^i\psi\\ &\qquad \phi(j(\tilde f_{(-2)}\pi(j(q_{(1)})_{(2)})_{(1)}) \otimes j(q_{(2)})_{(1)} \otimes x_{i(1)})\\ &\qquad \phi(j(\tilde f_{(-1)}) \otimes \pi(j(q_{(1)})_{(2)})_{(2)} \otimes \pi(j(q_{(2)})_{(2)}x_{i(2)})_{(1)})\\ &\qquad \varepsilon(\tilde f_{(-0)}(\pi(j(q_{(1)})_{(2)})_{(3)}\pi(j(q_{(2)})_{(2)}x_{i(2)})_{(2)}))\end{aligned}$$

$$
\begin{aligned}
&= \overline{j(\tilde{f}_{(-2)})j(q_{(1)})_{(1)}} \rtimes x^i\psi \\
&\qquad \phi(j(\tilde{f}_{(-1)})_{(1)}\pi(j(q_{(1)})_{(2)})_{(1)} \otimes j(q_{(2)})_{(1)} \otimes x_{i(1)}) \\
&\qquad \phi(j(\tilde{f}_{(-1)})_{(2)} \otimes \pi(j(q_{(1)})_{(2)})_{(2)} \otimes j(q_{(2)})_{(2)}x_{i(2)}) \\
&\qquad \varepsilon(\tilde{f}_{(-0)}(\pi(j(q_{(1)})_{(2)})_{(3)}\pi(j(q_{(2)})_{(3)}x_{i(3)}))) \\
&= \overline{j(\tilde{f}_{(-2)})j(q_{(1)})_{(1)}} \rtimes x^i\psi \\
&\qquad \phi(j(\tilde{f}_{(-1)})_{(1)} \otimes \pi(j(q_{(1)})_{(2)})_{(1)} \otimes j(q_{(2)})_{(1)}) \\
&\qquad \phi(j(\tilde{f}_{(-1)})_{(2)} \otimes \pi(j(q_{(1)})_{(2)})_{(2)}j(q_{(2)})_{(2)} \otimes x_{i(1)}) \\
&\qquad \phi(\pi(j(q_{(1)})_{(2)})_{(3)} \otimes j(q_{(2)})_{(3)} \otimes x_{i(2)}) \\
&\qquad \varepsilon(\tilde{f}_{(0)}(\pi(j(q_{(1)})_{(2)})_{(4)}\pi(j(q_{(2)})_{(4)}x_{i(3)}))) \\
&= \overline{j(\tilde{f}_{(-2)})j(q)_{(1)}} \rtimes x^i\psi \\
&\qquad \phi(j(\tilde{f}_{(-1)}) \otimes j(q)_{(2)} \otimes x_{i(1)})\varepsilon(\tilde{f}_{(0)}\pi(j(q)_{(3)}x_{i(2)})) \\
&= \overline{j(p_{(1)})j(q)_{(1)}} \rtimes x^i\psi\phi(j(p_{(2)}) \otimes j(q)_{(2)} \otimes x_{i(1)})\langle\varphi|\pi(j(q)_{(3)}x_{i(2)}\rangle \\
&= \overline{j(p_{(1)})j(q_{(1)})_{(1)}} \rtimes x^i\psi \\
&\qquad \phi(j(p_{(2)}) \otimes j(q_{(1)})_{(2)} \otimes x_{i(1)})\langle\varphi|\pi(j(q_{(1)})_{(3)})\pi(j(q_{(2)})x_{i(2)})\rangle \\
&= \overline{j(p_{(1)})j(q_{(1)})_{(1)}} \rtimes x^ix^j\psi \\
&\qquad \phi(j(p_{(2)}) \otimes j(q_{(1)})_{(2)} \otimes x_i)\langle\varphi_{(1)}|\pi(j(q_{(1)})_{(3)})\rangle\langle\varphi_{(2)}|q_{(2)} \rightharpoonup x_j\rangle \\
&= \overline{j(p_{(1)})j(q_{(1)})_{(1)}} \rtimes x^i(\varphi_{(2)} \leftharpoonup q_{(2)})\psi \\
&\qquad \phi(j(p_{(2)}) \otimes j(q_{(1)})_{(2)} \otimes x_i)\langle\varphi_{(1)}|\pi(j(q_{(1)})_{(3)})\rangle
\end{aligned}
$$

It remains to show that $\tilde{H} = Q \rtimes_{\tilde{\alpha}} K^* \in \mathfrak{E}_r(K^*)$. Note first that the map $\tilde{\pi}$ is convolution invertible since so is $\tilde{\alpha}$, since so is ϕ. The property $\tilde{\phi}(\mathrm{id}_{\tilde{H}\otimes\tilde{H}} \otimes \tilde{\iota}) = \varepsilon$ is immediate from the general form of $\tilde{\phi}$, as is $\tilde{\phi}(\tilde{\pi} \otimes \tilde{\iota} \otimes \mathrm{id}_{\tilde{H}}) = \tilde{\phi}(\mathrm{id}_{\tilde{H}} \otimes \tilde{\iota} \otimes \mathrm{id}_{\tilde{H}})$. Moreover, we read off $\tilde{\phi}(\tilde{\iota}(\varphi) \otimes \tilde{\iota}(\psi) \otimes \tilde{j}(r)) = \langle\varphi|\pi(j(r)_{(1)})\rangle\langle\psi|\pi(j(r)_{(2)}\rangle = \langle\varphi|\alpha^{(1)}(r)\rangle\langle\psi|\alpha^{(2)}(r)\rangle$, so that the condition (5)(b) in Lemma 3.3.1 is invariant under replacing H with $\tilde{H}^{\mathrm{bop}}$. □

Definition 3.4.6. We define a map $\mathfrak{F}^{\mathrm{bop}} \colon \mathfrak{E}_r(K) \to \mathfrak{E}_\ell(K^*)$ by $\mathfrak{F}^{\mathrm{bop}}(H) = \mathfrak{F}(H^{\mathrm{bop}})^{\mathrm{bop}}$ and a map $\mathfrak{G} \colon \mathfrak{E}_\ell(K) \to \mathfrak{E}_\ell(K^*)$ by $\mathfrak{G}(H) = \mathfrak{F}(H)^{\mathrm{bop}}$.

Notations 3.4.7. To cut short the listings of data constituting the objects of $\mathfrak{E}_\ell(K)$ and related categories, we will start to adopt a somewhat sloppy notational strategy for the rest of the paper: We will frequently write $H \in \mathfrak{E}_\ell(K)$ instead of repeating the whole list. ϕ will be understood to be the coassociator of H, Q will be understood to be the quotient coalgebra H/K^+H, α the dual two-cocycle in the cleft coextension. Moreover, if H' is another object in $\mathfrak{E}_\ell(K)$, it will be understood to come with structure maps π', ι', ϕ' the map $j' \colon Q' \to H'$ will be analogous to $j \colon Q \to H$ as in Section 2.5, $\tilde{\phi}$ will be understood to denote the coassociator on $\tilde{H}$, and so on.

3.5 Functoriality

Let K be a finite Hopf algebra.

So far we have defined a map $\mathfrak{F}\colon \mathfrak{E}_\ell(K) \to \mathfrak{E}_r(K^*)$ on the level of objects. Our aim now is to make $\mathfrak{F}$ a functor between the categories $\mathfrak{E}_\ell(K)$ and $\mathfrak{E}_r(K^*)$ as in Definition 3.3.3.

Let us first note that if $(F,\theta)\colon H \to H'$ is a morphism in $\mathfrak{E}_\ell(K)$, then $\theta(xg \otimes h) = \varepsilon(x)\theta(g \otimes h)$ and $\theta^{-1}(xg \otimes h) = \varepsilon(x)\theta^{-1}(g \otimes h)$ for all $x \in K$, $g, h \in H$. The first claim follows by the calculation

$$\begin{aligned}&\theta(xg \otimes h) = \theta(x_{(1)} \otimes g_{(1)})\theta(x_{(2)}g_{(2)} \otimes h_{(1)})\phi(x_{(3)} \otimes g_{(3)} \otimes h_{(2)})\\ &= \phi'(F(x_{(1)}) \otimes F(g_{(1)}) \otimes F(h_{(1)}))\theta(g_{(2)} \otimes h_{(2)})\theta(x_{(2)} \otimes g_{(3)}h_{(3)}) = \varepsilon(x)\theta(g \otimes h),\end{aligned}$$

and the second from the first. It follows further that θ induces a well-defined map $\theta\colon Q \otimes H \to k$. Also, since $F(x) = x$ for all $x \in K$, the map $F_1\colon Q \ni \overline{h} \mapsto \overline{F(h)} \in Q'$ is well-defined.

Thus, let $H, H' \in \mathfrak{E}_\ell(K)$ for some Hopf algebra K, and let $(F,\theta)\colon H \to H'$ be a morphism in $\mathfrak{E}_\ell(K)$. Then the monoidal functor ${}^{(F,\theta)}\mathcal{M}\colon {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$ maps the algebra K in ${}^H\mathcal{M}$ to the algebra K in ${}^{H'}\mathcal{M}$: the multiplication in K does not change in view of the condition imposed on θ. In particular, ${}^{(F,\theta)}\mathcal{M}$ induces a monoidal functor ${}^{(F,\theta)}_K\mathcal{M}_K\colon {}^H_K\mathcal{M}_K \to {}^{H'}_K\mathcal{M}_K$. This functor has the following description: The underlying left K-module of ${}^{(F,\theta)}_K\mathcal{M}_K(M)$ is the same as that of M. The left H'-comodule structure is induced via F. The right K-module structure is given by

$$m \cdot x = \theta(m_{(-1)} \otimes x_{(1)})m_{(0)}x_{(2)}$$

and the monoidal functor structure is induced by that of the functor ${}^{(F,\theta)}\mathcal{M}$.

Now there are a unique monoidal functor $(\mathcal{F},\tilde{\zeta})$ and a unique coquasimorphism $(\tilde{F},\tilde{\theta})$ making the following

$$\begin{tikzcd}
{}^H_K\mathcal{M}_K \arrow[r, "\hat{\mathcal{V}}"] \arrow[d, "{}^{(F,\theta)}_K\mathcal{M}_K"'] & {}^Q\mathcal{M}_{\phi,K} \arrow[r, "\cong"] \arrow[d, "(\mathcal{F},\tilde{\zeta})"] & {}^{Q\rtimes_{\tilde{\alpha}}K^*}\mathcal{M} \arrow[d, "{}^{(\tilde{F},\tilde{\theta})}\mathcal{M}"] \\
{}^{H'}_K\mathcal{M}_K \arrow[r, "\hat{\mathcal{V}}"'] & {}^{Q'}\mathcal{M}_{\phi',K'} \arrow[r, "\cong"'] & {}^{Q'\rtimes_{\tilde{\alpha}'}K^*}\mathcal{M}
\end{tikzcd}$$

a commutative diagram of monoidal functors.

Definition 3.5.1. The functor $\mathfrak{F}\colon \mathfrak{E}_\ell(K) \to \mathfrak{E}_r(K^*)$ takes the morphism (F,θ) to the morphism $(\tilde{F},\tilde{\theta})$ described above.

The maps $\mathfrak{F}^{\mathrm{bop}}\colon \mathfrak{E}_r(K) \to \mathfrak{E}_\ell(K^*)$ and $\mathfrak{G}\colon \mathfrak{E}_\ell(K) \to \mathfrak{E}_\ell(K^*)$ are extended to functors by $\mathfrak{F}^{\mathrm{bop}}(F,\theta) = \mathfrak{F}((F,\theta)^{\mathrm{bop}})^{\mathrm{bop}}$ and $\mathfrak{G}(F,\theta) = \mathfrak{F}(F,\theta)^{\mathrm{bop}}$.

Explicitly, $\mathcal{F}(V)$ equals V as a k-module, with left Q-comodule structure induced via $F_1\colon Q \to Q'$, and right K-action given by $v \cdot x = \theta(v_{(-1)} \otimes x_{(1)})v_{(0)}x_{(2)}$ for $v \in V$ and $x \in K$. the map $\tilde{\zeta}$ is determined by commutativity of

$$\begin{array}{ccc} \overline{M} \otimes \overline{N} & \xrightarrow{\tilde{\zeta}} & \overline{M} \otimes \overline{N} \\ \Big\downarrow \xi_{\pi'} & & \Big\downarrow \xi_{\pi} \\ \overline{M \underset{K}{\otimes} N} & \xrightarrow[\overline{\zeta}]{} & \overline{M \underset{K}{\otimes} N} \end{array}$$

for all $M, N \in {}^H_K\mathcal{M}_K$. We compute

$$\begin{aligned} \tilde{\zeta}^{-1}(\overline{m} \otimes \overline{n}) = \xi_{\pi'}^{-1}\overline{\zeta}^{\,-1}\xi_\pi(\overline{m} \otimes \overline{n}) &= \xi_{\pi'}^{-1}(\overline{\zeta}^{\,-1}(\overline{m \otimes j(n)})) \\ &= \xi_{\pi'}(\theta^{-1}(m_{(-1)} \otimes j(n_{(0)})_{(-1)})\overline{m_{(0)} \otimes j(n_{(0)})_{(0)}} \\ = \theta^{-1}(m_{(-1)} \otimes j(n_{(0)})_{(-2)})\overline{m_{(0)}}\pi'(F(j(n_{(0)})_{(-1)})) &\otimes \overline{j(n_{(0)})_{(0)}} \\ = \theta^{-1}(m_{(-1)} \otimes j(\overline{n}_{(-1)})_{(1)})\overline{m}_{(0)}&\pi'(F(j(\overline{n}_{(-1)})_{(2)})) \otimes \overline{n}_{(0)} \end{aligned}$$

from which we deduce that

$$\tilde{\zeta}^{-1}(v \otimes w) = \theta^{-1}(v_{(-1)} \otimes j(w_{(-1)})_{(1)})v_{(0)}\pi'(F(j(w_{(-1)})_{(2)})) \otimes w_{(0)}$$

holds for all $V, W \in {}^Q\mathcal{M}_{\phi,K}$ and $v \in V$, $w \in W$.

Now for $\tilde{g} = p \rtimes \varphi \in Q \rtimes_{\tilde{\alpha}} K^*$ and $\tilde{h} = q \rtimes \psi \in Q \rtimes_{\tilde{\alpha}} K^*$ we have

$$\begin{aligned} \tilde{F}(p \rtimes \varphi) &= \tilde{F}(\tilde{g}_{(1)})\varepsilon(\tilde{g}_{(2)}) = F_1(g_{(-1)}) \rtimes x^i\varepsilon(\tilde{g}_{(0)} \cdot x_i) \\ &= F_1(g_{(-2)}) \rtimes x^i\theta(\tilde{g}_{(-1)} \otimes x_{i(1)})\varepsilon(\tilde{g}_{(0)}x_{i(2)}) = F_1(p_{(1)}) \rtimes x^i\theta(p_{(2)} \otimes x_{i(1)})\langle\varphi|x_{i(2)}\rangle \\ &= F_1(p_{(1)}) \rtimes \theta(p_{(2)} \otimes x_i)x^i\varphi \end{aligned}$$

and

$$\begin{aligned} \tilde{\theta}^{-1}(p \rtimes \varphi \otimes q \rtimes \psi) &= \varepsilon^{\otimes 2}\tilde{\zeta}^{-1}(\tilde{g} \otimes \tilde{h}) \\ &= \theta^{-1}(\tilde{g}_{(-1)} \otimes j(\tilde{h}_{(-1)})_{(1)})\varepsilon(\tilde{g}_{(0)}\pi'(F(j(\tilde{h}_{(-1)})_{(2)})))\varepsilon(\tilde{h}_{(0)}) \\ &= \theta^{-1}(p \otimes j(q)_{(1)})\langle\varphi|\pi'(F(j(q)_{(2)}))\rangle\varepsilon(\psi) \end{aligned}$$

Note in particular that $F(1 \rtimes \varphi) = 1 \rtimes \varphi$ and $\tilde{\theta}^{-1}(p \rtimes \varphi \otimes 1 \rtimes \psi) = \varepsilon(p)\varepsilon(\varphi\psi)$, which shows that $(\tilde{F}, \tilde{\theta})$ is really a morphism in $\mathfrak{E}_r(K^*)$.

4 Skeleta and a category equivalence

The construction $\mathfrak{F}$ of the preceding section yields as its output an object of almost the same type as the input. The immediate question comes to mind wether the construction is involutive. Stated more reasonably, given a finite Hopf algebra K, are the following compositions (which, of course, coincide) isomorphic to identity functors?

$$\mathfrak{E}_\ell(K) \xrightarrow{\mathfrak{F}} \mathfrak{E}_r(K^*) \xrightarrow{\mathfrak{F}^{\mathrm{bop}}} \mathfrak{E}_\ell(K)$$

$$\mathfrak{E}_\ell(K) \xrightarrow{\mathfrak{G}} \mathfrak{E}_\ell(K^{*\mathrm{bop}}) \xrightarrow{\mathfrak{G}} \mathfrak{E}_\ell(K)$$

To me, the answer is at the moment out of reach.

This section introduces a tool to deal with $\mathfrak{F}$ more explicitly, and gives a partial answer to the above question. The tool is a variation on a well-known technique: One can study "constituents" of the structure maps of an object $H \in \mathfrak{E}_\ell(K)$, maps that involve only few copies of the components K and Q that compose H. For example, the comultiplication of H, a map $K \otimes Q \to K \otimes Q \otimes K \otimes Q$ is studied in terms of maps $K \to K \otimes K$, $Q \to Q \otimes Q$, $Q \to Q \otimes K$ and $Q \to K \otimes K$ (two comultiplications, a coaction, and a dual two-cocycle). We call a certain collection of maps of this type the skeleton of $H \in \mathfrak{E}_\ell(K)$ and show that $\mathfrak{F}$ acts on the level of skeleta by dualizing every occurrence of K in a suitable way. In particular, the skeleton of $\mathfrak{G}(\mathfrak{G}(H))$ is the same as the skeleton of H. We prove the same for skeleta of morphisms.

This falls short of a positive answer to the above question because the skeleton does not seem to contain the complete information needed to reconstruct all of an object (or all of a morphism), mainly due to lack of associativity in the general situation. However, we show that the skeleta of some special types of objects and morphisms of $\mathfrak{E}_\ell(K)$ do contain all the necessary information, and obtain a category equivalence induced by $\mathfrak{F}$ between two subcategories.

4.1 Skeleta of objects and a bijection

Definition 4.1.1. Let K be a Hopf algebra and $H = (H, \nabla, \phi, \iota, \pi) \in \mathfrak{E}_\ell(K)$. Put $Q := H/K^+H$ and define j as in Section 2.5.

The skeleton $\mathfrak{S}(H) = (\Delta, \rho, \alpha, \nabla_Q, \rightharpoonup, \leftharpoonup, \sigma, \phi_{QQ}, \phi_{QK}, \phi_{KK})$ of H consists of the following maps:

$$\begin{aligned}
\Delta &= (\nu\otimes\nu)\Delta_H j, \text{ the comultiplication of } Q\\
\rho &: Q \ni q \mapsto q_{[0]}\otimes q_{[1]} := (\nu\otimes\pi)\Delta j(q) \in Q\otimes K\\
\alpha &: Q \ni q \mapsto \alpha^{(1)}(q)\otimes\alpha^{(2)}(q) := (\pi\otimes\pi)\Delta j(q) \in K\otimes K\\
\nabla_Q &: Q\otimes Q \ni p\otimes q \mapsto p\bullet q := \nu(j(p)j(q)) \in Q\\
\rightharpoonup &: Q\otimes K \ni q\otimes x \mapsto q\rightharpoonup x := \pi(j(q)x) \in K\\
\leftharpoonup &: Q\otimes K \ni q\otimes x \mapsto q\leftharpoonup x := \nu(j(q)x) \in Q\\
\sigma &: Q\otimes Q \ni p\otimes q \mapsto \sigma(p\otimes q) := \pi(j(p)j(q)) \in K\\
\phi_{QQ} &= \phi(j\otimes j\otimes j)\colon Q\otimes Q\otimes Q \to k\\
\phi_{QK} &= \phi(j\otimes j\otimes\iota)\colon Q\otimes Q\otimes K \to k\\
\phi_{KK} &= \phi(j\otimes\iota\otimes\iota)\colon Q\otimes K\otimes K \to k
\end{aligned}$$

The skeleton $\mathfrak{S}(\tilde{H})$ of $\tilde{H}\in\mathfrak{E}_r(K)$ is defined similarly.

Note that the first three components of the skeleton already appeared before: the comultiplication of the quotient Q, the weak coaction ρ and dual two-cocycle α in the cleft module coalgebra $H\cong K\ltimes_\alpha Q$.

As we already did with the map ϕ, we will consider, for example, ϕ_{QQ} as a map from $H\otimes Q\otimes Q$ instead of from $Q\otimes Q\otimes Q$, to k whenever it is notationally convenient.

We will in the rest of the paper extend the notational strategy set forth in 3.4.7 to skeleta. Thus, whenever considering some object $H'\in\mathfrak{E}_\ell(K)$, we will freely refer to ϕ'_{KK} or $\rightharpoonup'\colon Q'\otimes K\to K$ with the obvious meanings.

Proposition 4.1.2. *Let K be a finite Hopf algebra and $H\in\mathfrak{E}_\ell(K)$; put $\tilde{H}=\mathfrak{F}(H)$. Let*

$$\begin{aligned}
\mathfrak{S}(H) &= (\Delta,\rho,\alpha,\nabla_Q,\rightharpoonup,\leftharpoonup,\sigma,\phi_{QQ},\phi_{QK},\phi_{KK})\\
\mathfrak{S}(\tilde{H}) &= (\tilde{\Delta},\tilde{\rho},\tilde{\alpha},\tilde{\nabla}_Q,\tilde{\leftharpoonup},\tilde{\rightharpoonup},\tilde{\sigma},\tilde{\phi}_{QQ},\tilde{\phi}_{K^*Q},\tilde{\phi}_{K^*K^*})
\end{aligned}$$

be the skeleta of H and $\tilde{H}$. Then $\tilde{\Delta}=\Delta$, $\tilde{\nabla}_Q=\tilde{\nabla}$, $\tilde{\phi}_{QQ}=\phi_{QQ}$,

$$\begin{aligned}
\langle q_{[-\tilde{1}]}|x\rangle q_{[\tilde{0}]} &= q\leftharpoonup x\\
\langle\tilde{\alpha}^{(1)}(q)|x\rangle\langle\tilde{\alpha}^{(2)}(q)|y\rangle &= \phi_{KK}(q\otimes x\otimes y)\\
\langle\varphi\tilde{\leftharpoonup}q|x\rangle &= \langle\varphi|q\rightharpoonup x\rangle\\
\varphi\tilde{\rightharpoonup}q &= q_{[0]}\langle\varphi|q_{[1]}\rangle\\
\langle\tilde{\sigma}(p\otimes q)|x\rangle &= \phi_{QK}(p\otimes q\otimes x)\\
\tilde{\phi}_{K^*Q}(\varphi\otimes p\otimes q) &= \langle\varphi|\sigma(p\otimes q)\rangle\\
\tilde{\phi}_{K^*K^*}(\varphi\otimes\psi\otimes q) &= \langle\varphi|\alpha^{(1)}(q)\rangle\langle\psi|\alpha^{(2)}(q)\rangle
\end{aligned}$$

where $\tilde{\rho}(q)=:q_{[-\tilde{1}]}\otimes q_{[\tilde{0}]}\in K^*\otimes Q$ and $\tilde{\alpha}(q)=:\tilde{\alpha}^{(1)}(q)\otimes\tilde{\alpha}^{(2)}(q)\in K^*\otimes K^*$.

Proof. By construction of $\tilde{H}$, it is clear that $\tilde{\Delta} = \Delta$: We have $\tilde{H}/\tilde{H}K^{*+} = Q$ as coalgebras. As to the equalities concerning $\tilde{\alpha}$, $\tilde{\rho}$, and $\tilde{\leftharpoonup}$, we first have to acknowledge a clash of notations: These three maps were introduced in Lemma 3.4.2 and Theorem 3.4.5, well before the definition of the skeleton, and satisfied by definition the equalities that we now claim are true. However, by inspecting the formulas for comultiplication and multiplication in $\tilde{H}$, we find that these three previously defined maps are also the components of the skeleton of $\tilde{H}$ that we have denoted above by the same symbols: For example

$$\begin{aligned}\tilde{\pi}(\tilde{\nabla}(1 \rtimes \varphi \otimes q \rtimes 1)) \\ &= \tilde{\pi}(\overline{j(q_{(1)})_{(1)}} \rtimes x^i(\varphi_{(2)} \tilde{\leftharpoonup} q_{(2)})\phi(1 \otimes j(q_{(1)})_{(2)} \otimes x_i)\langle \varphi_{(1)} | \pi(j(q_{(1)})_{(3)})\rangle) \\ &= x^i(\varphi_{(2)} \tilde{\leftharpoonup} q_{(2)})\varepsilon(x_i)\langle \varphi_{(1)} | \pi(j(q_{(1)}))\rangle = \varphi \tilde{\leftharpoonup} q\end{aligned}$$

holds for $\varphi \in K^*$ and $q \in Q$ by Theorem 3.4.5. The remaining equations are proved by specializing the formulas in Theorem 3.4.5 in a similar way. $\square$

Corollary 4.1.3. *Let K be a finite Hopf algebra and $H \in \mathfrak{E}_\ell(K)$. Then*

$$\mathfrak{S}(\mathfrak{F}^{\mathrm{bop}}(\mathfrak{F}(H))) = \mathfrak{S}(H), \quad \textit{that is,} \quad \mathfrak{S}(\mathfrak{G}^2(H)) = \mathfrak{S}(H).$$

Now if the skeleton of an object $H \in \mathfrak{E}_\ell(K)$ contained enough information to reconstruct the whole of H from it, the above result would mean that $\mathfrak{G}$ was involutive, at least on the level of objects. However, we cannot prove this in complete generality.

Of course, the coalgebra structure of H is contained in the skeleton: H is isomorphic to a crossed coproduct coalgebra. A certain part of the multiplication can also always be reconstructed from the skeleton by the following calculation for $x \in K$ and $p \in Q$:

$$\begin{aligned}j(p)x &= \pi(j(p)_{(1)}x_{(1)}) \ltimes \nu(j(p)_{(2)}x_{(2)}) \\ &= \pi(\alpha^{(1)}(p_{(1)})j(p_{(2)[0]})x_{(1)}) \ltimes \nu(\alpha^{(2)}(p_{(1)})p_{(2)[1]}j(p_{(3)})x_{(2)}) \\ &= \pi(j(p_{(1)})x_{(1)}) \ltimes \nu(j(p_{(2)})x_{(2)}) = p_{(1)} \rightharpoonup x_{(1)} \ltimes p_{(2)} \leftharpoonup x_{(2)}.\end{aligned}$$

In trying to reconstruct the whole of the multiplication, we will run into problems caused by the lack of associativity. We can deal with these problems in two special cases.

Proposition 4.1.4. *Let $H \in \mathfrak{E}_\ell(K)$. If α is trivial, then H is determined by its skeleton.*

Proof. The assumption $\alpha = \varepsilon$ means that $\Delta j(q) = j(q_{(1)[0]}) \otimes q_{(1)[1]}j(q_{(2)})$ holds for all $q \in Q$. We identify $H = K \ltimes_\alpha Q$. The claim means that ϕ as well as the multiplication on H can be expressed in terms of the skeleton of H. Then, for $x, y \in K$ and $p, q \in Q$ we have

$$\begin{aligned}
(x \ltimes p)(y \ltimes q) &= (xj(p))(yj(q)) \\
&= x\phi^{-1}(j(p)_{(1)} \otimes y_{(1)} \otimes j(q)_{(1)})(j(p)_{(2)}y_{(2)})j(q)_{(2)}\phi(j(p)_{(3)} \otimes y_{(3)} \otimes j(q)_{(3)}) \\
&= x(j(p)_{(1)}y_{(1)})j(q)_{(1)}\phi(j(p)_{(2)} \otimes y_{(2)} \otimes j(q)_{(2)}) \\
&= x(j(p_{(1)[0]})y_{(1)})j(q_{(1)[0]})\phi(p_{(1)[1]}j(p_{(2)}) \otimes y_{(2)} \otimes p_{(1)[1]}j(q_{(2)})) \\
&= x(j(p_{(1)})y_{(1)})j(q_{[0]})\phi(j(p_{(2)}) \otimes y_{(2)} \otimes q_{[1]}) \\
&= x(p_{(1)} \rightharpoonup y_{(1)})j(p_{(2)} \leftharpoonup y_{(2)})j(q_{[0]})\phi_{KK}(p_{(2)} \otimes y_{(3)} \otimes q_{[1]})
\end{aligned}$$

In particular, the map $\nu\nabla \colon H \otimes H \to Q$ can be expressed in terms of the skeleton by

$$\nu\nabla(x \ltimes p \otimes y \ltimes q) = \varepsilon(x)(p_{(1)} \leftharpoonup y_{(2)}) \bullet q_{[0]}\phi_{KK}(p_{(2)} \otimes y_{(2)} \otimes q_{[1]}),$$

and to express multiplication in terms of the skeleton, it remains to find an expression for $j(p)j(q)$. Now

$$\begin{aligned}
j(p)j(q) &= \pi(j(p)_{(1)}j(q)_{(1)}) \ltimes \nu(j(p)_{(2)}j(q)_{(2)}) \\
&= \pi(j(p_{(1)[0]})j(q_{(1)[0]})) \ltimes \nu((p_{(1)[1]}j(p_{(2)}))(q_{(1)[1]}j(q_{(2)}))) \\
&= \pi(j(p_{(1)})j(q_{(1)[0]}) \ltimes \nu(j(p_{(2)})(q_{(1)[1]}j(q_{(2)}))) \\
&= \sigma(p_{(1)} \otimes q_{(1)[0]}) \ltimes (p_{(2)} \leftharpoonup q_{(1)[1]}) \bullet q_{(2)}\phi_{KK}(p_{(3)} \otimes q_{(1)[2]} \otimes q_{(3)}).
\end{aligned}$$

It remains to express $\phi \colon H \otimes H \otimes H \to k$ in terms of the skeleton. For $g, h \in H$ we have

$$\begin{aligned}
\phi(g \otimes xj(p) \otimes h) &= \phi(g_{(1)} \otimes x_{(1)} \otimes j(p)_{(1)})\phi(g_{(2)} \otimes x_{(2)}j(p)_{(2)} \otimes h_{(1)})\phi(x_{(3)} \otimes j(p)_{(3)} \otimes h_{(2)}) \\
&= \phi(g_{(1)}x_{(1)} \otimes j(p)_{(1)} \otimes h_{(1)})\phi(g_{(2)} \otimes x_{(2)} \otimes j(p)_{(2)}h_{(2)}) \\
&= \phi(g_{(1)}x_{(1)} \otimes j(p_{(1)[0]}) \otimes h_{(1)})\phi_{KK}(\overline{g_{(2)}} \otimes x_{(2)} \otimes \pi(p_{(1)[1]}j(p_{(2)})h_{(2)})) \\
&= \phi(g_{(1)}x_{(1)} \otimes j(p_{(1)[0]}) \otimes h_{(1)})\phi_{KK}(\overline{g_{(2)}} \otimes x_{(2)} \otimes p_{(1)[1]}\pi(j(p_{(2)})h_{(2)}))
\end{aligned}$$

In particular, the maps $\phi_{HK} = \phi(\mathrm{id} \otimes \mathrm{id} \otimes \iota) \colon Q \otimes H \otimes K \to k$ and $\phi_{HQ} = \phi(\mathrm{id} \otimes \mathrm{id} \otimes j) \colon Q \otimes H \otimes Q \to k$ can be expressed in terms of the skeleton:

$$\phi(g \otimes xj(p) \otimes y) = \phi_{QK}(g_{(1)}x_{(1)} \otimes p_{(1)[0]} \otimes y_{(1)})\phi_{KK}(g_{(2)} \otimes x_{(2)} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup y_{(2)})),$$

in particular

$$\begin{aligned}
&\phi(j(q) \otimes xj(p) \otimes y) \\
&\quad = \phi_{QK}(q_{(1)} \leftharpoonup x_{(1)} \otimes p_{(1)[0]} \otimes y_{(1)})\phi_{KK}(q_{(2)} \otimes x_{(2)} \otimes p_{(1)[1]}(p_{(2)} \rightharpoonup y_{(2)})),
\end{aligned}$$

$$\begin{aligned}
&\phi(g \otimes xj(p) \otimes j(q)) \\
&= \phi_{QQ}(g_{(1)}x_{(1)} \otimes p_{(1)[0]} \otimes q_{(1)[0]})\phi_{KK}(g_{(2)} \otimes x_{(2)} \otimes p_{(1)[1]}\pi(j(p_{(2)})q_{(1)[1]}j(q_{(2)}))),
\end{aligned}$$

and finally we compute

$$\begin{aligned}
\phi(g\otimes h\otimes yj(q)) &= \phi(g_{(1)}h_{(1)}\otimes y_{(1)}\otimes j(q)_{(1)})\phi(g_{(2)}\otimes h_{(2)}\otimes y_{(2)}j(q)_{(2)})\\
&= \phi(g_{(1)}\otimes h_{(1)}\otimes y_{(1)})\phi(g_{(2)}\otimes h_{(2)}y_{(2)}\otimes j(q)_{(1)})\phi(h_{(3)}\otimes y_{(3)}\otimes j(q)_{(2)})\\
&= \phi_{HK}(g_{(1)}\otimes h_{(1)}\otimes y_{(1)})\phi_{HQ}(g_{(2)}\otimes h_{(2)}y_{(2)}\otimes q_{[0]})\phi_{KK}(h_{(3)}\otimes y_{(3)}\otimes q_{[1]})
\end{aligned}$$

and have expressed all of ϕ in terms of its restrictions in the skeleton. □

Proposition 4.1.5. *Let $H\in\mathfrak{E}_\ell(K)$. If $\phi_{KK}=\varepsilon$, then H is determined by its skeleton.*

Proof. As a result of the hypothesis, we have associativity $(gx)h=g(xh)$ in H whenever $x\in K$ and $g,h\in H$. In particular, we find

$$\begin{aligned}
(1\ltimes p)(1\ltimes q) = {}& \pi((\alpha^{(1)}(p_{(1)})\ltimes p_{(2)[0]})(\alpha^{(1)}(q_{(1)})\ltimes q_{(2)[0]}))\\
&\otimes\nu((\alpha^{(2)}(p_{(1)})p_{(2)[1]}\ltimes p_{(3)})(\alpha^{(2)}(q_{(1)})q_{(2)[1]}\ltimes q_{(3)}))\\
= {}& \pi((1\ltimes p_{(1)})(\alpha^{(1)}(q_{(1)})\ltimes q_{(2)[0]}))\\
&\otimes\nu((1\ltimes p_{(2)})(\alpha^{(2)}(q_{(1)})q_{(2)[1]}\ltimes q_{(3)}))\\
= {}& (p_{(1)}\rightharpoonup\alpha^{(1)}(q_{(1)})_{(1)})\pi((1\ltimes p_{(2)}\leftharpoonup\alpha^{(1)}(q_{(1)})_{(2)})(1\ltimes q_{(2)[0]}))\\
&\otimes\nu((1\ltimes p_{(3)}\leftharpoonup\alpha^{(2)}(q_{(1)})q_{(2)[1]})(1\ltimes q_{(3)}))\\
= {}& (p_{(1)}\rightharpoonup\alpha^{(1)}(q_{(1)})_{(1)})\sigma(p_{(2)}\leftharpoonup\alpha^{(1)}(q_{(1)})_{(2)}\otimes q_{(2)[0]})\\
&\otimes(p_{(3)}\leftharpoonup\alpha^{(2)}(q_{(1)})q_{(2)[1]})\bullet q_{(3)}
\end{aligned}$$

and

$$(x\ltimes p)(y\ltimes q)=(xj(p))(yj(q))=x(j(p)y)j(q)=x(p_{(1)}\rightharpoonup y_{(1)})j(p_{(2)}\leftharpoonup y_{(2)})j(q)$$

shows that we can express multiplication in H in terms of the skeleton. As for the associator ϕ, we first compute

$$\begin{aligned}
&\phi(g\otimes xj(q)\otimes h)\\
&\quad=\phi(g_{(1)}\otimes x_{(1)}\otimes j(q)_{(1)})\phi(g_{(2)}\otimes x_{(2)}j(q)_{(2)}\otimes h_{(1)})\phi(x_{(3)}\otimes j(q)_{(3)}\otimes h_{(2)})\\
&\quad=\phi(g_{(1)}x_{(1)}\otimes j(q)_{(1)}\otimes h_{(1)})\phi(g_{(2)}\otimes x_{(2)}\otimes j(q)_{(2)}h_{(2)})=\phi(gx\otimes j(q)\otimes h)
\end{aligned}$$

for $g,h\in H$, $x\in K$ and $q\in Q$, from which we read off that $\phi_{HK}=\phi(\mathrm{id}\otimes\mathrm{id}\otimes\iota)$ and $\phi_{HQ}=\phi(\mathrm{id}\otimes\mathrm{id}\otimes j)$ can be expressed in terms of ϕ_{QK} and ϕ_{QQ}, respectively. Now

$$\begin{aligned}
\phi(g\otimes h\otimes yj(q)) &= \phi(g_{(1)}h_{(1)}\otimes x_{(1)}\otimes j(q)_{(1)})\phi(g_{(2)}\otimes h_{(2)}\otimes x_{(2)}j(q)_{(2)})\\
&= \phi(g_{(1)}\otimes h_{(1)}\otimes x_{(1)})\phi(g_{(2)}\otimes h_{(2)}x_{(2)}\otimes j(q)_{(1)})\phi(h_{(3)}\otimes x_{(3)}\otimes j(q)_{(2)})\\
&= \phi_{HK}(g_{(1)}\otimes h_{(1)}\otimes x_{(1)})\phi_{HQ}(g_{(2)}\otimes h_{(2)}x_{(2)}\otimes q).
\end{aligned}$$

for $g,h\in H$, $x\in K$ and $q\in Q$. □

Corollary 4.1.6. *Let K be a finite Hopf algebra. Then the object map of the functor $\mathfrak{F}$ restricts to a bijection between (strong isomorphism classes of) objects $H\in\mathfrak{E}_\ell(K)$ with trivial α, and objects $\tilde{H}\in\mathfrak{E}_r(K^*)$ with trivial $\tilde{\phi}_{K^*K^*}$.*

4.2 Skeleta of morphisms and a category equivalence

Definition 4.2.1. Let K be a Hopf algebra and

$$(F,\theta)\colon (H,\phi,\iota,\pi) \to (H',\phi',\iota',\pi')$$

a morphism in $\mathfrak{E}_\ell(K)$. The skeleton $\mathfrak{S}(F,\theta) := (F_0, F_1, \theta_K, \theta_Q)$ of (F,θ) consists of the following maps:

$$\begin{aligned} F_0 &= \pi' F j\colon Q \to K \\ F_1 &:= \nu' F j\colon Q \ni \overline{h} \mapsto \overline{F(h)} \in Q' \\ \theta_K &= \theta(j \otimes \iota) \\ \theta_Q &= \theta(j \otimes j) \end{aligned}$$

The skeleton of a morphism (F,θ) in $\mathfrak{E}_r(K)$ is $\mathfrak{S}(F,\theta) := (F_0, F_1, \theta_{K^*}^{-1}, \theta_Q^{-1})$, where

$$\begin{aligned} F_0 &= \pi' F j\colon Q \to K^* \\ F_1 &= \nu' F j\colon Q \to Q' \\ \theta_{K^*}^{-1} &= \theta^{-1}(\iota \otimes j) \\ \theta_Q^{-1} &= \theta^{-1}(j \otimes j) \end{aligned}$$

Proposition 4.2.2. *Let K be a finite Hopf algebra, and $(F,\theta)\colon H \to H'$ a morphism in $\mathfrak{E}_\ell(K)$. The skeleton of $\mathfrak{F}(F,\theta) = (\tilde{F},\tilde{\theta})$ is determined by*

$$\begin{aligned} \langle \tilde{F}_0(q)|x\rangle &= \theta_K(q \otimes x) \\ \tilde{F}_1 &= F_1 \\ \tilde{\theta}_{K^*}^{-1}(\varphi \otimes q) &= \langle \varphi | F_0(q)\rangle \\ \tilde{\theta}_Q^{-1}(p \otimes q) &= \theta^{-1}(j(p) \otimes j(q)) \end{aligned}$$

The proof consists in specializing the results of Section 3.5.

Corollary 4.2.3. *Let K be a finite Hopf algebra and (F,θ) a morphism in $\mathfrak{E}_\ell(K)$. Then*

$$\mathfrak{S}(\mathfrak{F}^{\mathrm{bop}}(\mathfrak{F}(F,\theta))) = \mathfrak{S}(F,\theta), \quad \textit{that is,} \quad \mathfrak{S}(\mathfrak{G}^2(F,\theta)) = \mathfrak{S}(F,\theta)$$

Again, if we knew that the skeleton of a morphism contained "enough" information (and if $\mathfrak{G}$ were involutive on objects), we could conclude that $\mathfrak{G}$ is involutive on morphisms, in particular a category equivalence.

Remark 4.2.4. In a morphism $(F,\theta)\colon H \to H'$ in $\mathfrak{E}_\ell(K)$, the map F can be expressed in terms of the skeleton by

$$\begin{aligned} &F(xj(q)) = xF(j(q)) = x\pi(F(j(q))_{(1)})j(\overline{F(j(q))_{(2)}}) \\ &= x\pi(F(j(q)_{(1)}))j(\overline{F(j(q)_{(2)})}) = x\pi(F(j(q_{(1)}))j(\overline{F}(j(q_{(2)})) = xF_0(q_{(1)})j(F_1(q_{(2)})) \end{aligned}$$

for $x \in K$ and $q \in Q$.

Proposition 4.2.5. *Let K be a Hopf algebra and $(F,\theta)\colon H \to H'$ a morphism in $\mathfrak{E}_\ell(K)$. Assume $H, H' \in \mathfrak{E}_\ell(K)$ satisfy $\phi_{KK} = \varepsilon$ and $\phi'_{KK} = \varepsilon$.*

Then (F,θ) is determined by its skeleton.

Proof. It remains to express θ in terms of the skeleton. Now we have

$$\begin{aligned}
&\theta(p \otimes xj(q))\\
&\quad = \phi'(F(j(p)_{(1)}) \otimes x_{(1)} \otimes F(j(q)_{(1)})\theta(x_{(2)} \otimes j(q)_{(2)})\theta(j(p)_{(2)} \otimes x_{(3)}j(q)_{(3)})\\
&\quad\quad = \theta(j(p)_{(1)} \otimes x_{(1)})\theta(j(p)_{(2)}x_{(2)} \otimes j(q)_{(1)})\phi(j(p)_{(3)} \otimes x_{(3)} \otimes j(q)_{(2)})\\
&\quad\quad\quad = \theta(j(p)_{(1)} \otimes x_{(1)})\theta(j(p)_{(2)}x_{(2)} \otimes j(q))\\
&\quad\quad\quad\quad = \theta_K(p_{(1)[0]} \otimes x_{(1)})\theta_Q(p_{(2)} \leftharpoonup x_{(2)} \otimes q)
\end{aligned}$$

for $p, q \in Q$ and $x \in K$. $\square$

Proposition 4.2.6. *Let K be a Hopf algebra and $(F,\theta)\colon H \to H'$ a morphism in $\mathfrak{E}_\ell(K)$. If $\alpha = \varepsilon$ and $F_0 = \eta\varepsilon$, then (F,θ) is determined by its skeleton.*

Proof. Again, it remains to express θ in terms of the skeleton. Using that $\phi'(F(g) \otimes x \otimes F(j(q)_{(1)}))j(q)_{(2)} = \phi'(F(g) \otimes x \otimes \pi F(j(q_{(1)[0]}))q_{(1)[1]}j(q_{(2)}) = j(q)$ for $g \in H$, $x \in K$ and $q \in Q$, we find

$$\begin{aligned}
&\theta(p \otimes xj(q))\\
&\quad = \phi'(F(j(p)_{(1)}) \otimes x_{(1)} \otimes F(j(q)_{(1)}))\theta(x_{(2)} \otimes j(q)_{(2)})\theta(j(p)_{(2)} \otimes x_{(3)}j(q)_{(3)})\\
&\quad\quad = \theta(j(p)_{(1)} \otimes x_{(1)})\theta(j(p)_{(2)}x_{(2)} \otimes j(q)_{(1)})\phi(j(p)_{(3)} \otimes x_{(3)} \otimes j(q)_{(2)})\\
&\quad\quad\quad = \theta_K(p_{(1)} \otimes x_{(1)})\theta_Q(p_{(2)} \leftharpoonup x_{(2)} \otimes q_{[0]})\phi_{KQ}(p_{(3)} \otimes x_{(3)} \otimes q_{[1]})
\end{aligned}$$

for $p, q \in Q$ and $x \in K$. $\square$

Corollary 4.2.7. *Let K be a finite Hopf algebra. The functor $\mathfrak{F}$ induces a category equivalence between*

- *the subcategory of $\mathfrak{E}_\ell(K)$ whose objects are those $H \in \mathfrak{E}_\ell(K)$ with trivial α, and whose morphisms are those (F,θ) with trivial F_0, and*
- *the subcategory of $\mathfrak{E}_r(K^*)$ whose objects are those $\tilde{H} \in \mathfrak{E}_r(K^*)$ with $\tilde{\phi}_{K^*K^*} = \varepsilon$, and whose morphisms are those $(\tilde{F},\tilde{\theta})$ with $\tilde{\theta}^{-1}_{K^*} = \varepsilon$.*

Proposition 4.2.8. *Let K be a Hopf algebra and $(F,\theta)\colon H \to H'$ a morphism in $\mathfrak{E}_\ell(K)$. If $j\colon Q \to H$ is a coalgebra map, then (F,θ) is determined by its skeleton.*

Proof. We have

$$\begin{aligned}
\theta(p \otimes xj(q)) &= \theta(x_{(1)} \otimes j(q)_{(1)})\theta(p \otimes x_{(2)}j(q)_{(2)})\\
&= (\phi')^{-1}(F(j(p)_{(1)}) \otimes x_{(2)} \otimes F(j(q)_{(2)})\\
&\qquad \theta(j(p)_{(2)} \otimes x_{(3)})\theta(j(p)_{(3)}x_{(4)} \otimes j(q)_{(3)})\phi(j(p)_{(4)} \otimes x_{(5)} \otimes j(q)_{(4)})\\
&= (\phi')^{-1}(Fj(p_{(1)}) \otimes x_{(1)} \otimes F_0(q_{(2)}))\theta(p_{(2)} \otimes x_{(2)})\theta(p_{(3)} \leftharpoonup x_{(3)} \otimes q_{(3)})
\end{aligned}$$

for $x \in K$ and $p, q \in Q$. $\square$

5 Inspection of some special cases

5.1 Hopf algebra inclusions

An important special class of objects in $\mathfrak{E}_\ell(K)$ are those where $K \subset H$ is a bialgebra inclusion, with trivial coassociator on H. With the ordinary bialgebra maps as morphisms, these form the category $\mathfrak{E}_\ell^{\mathrm{bialg}}(K)$ below, which is equivalent via $\mathfrak{F}$ to a suitable subcategory $\mathfrak{E}_r^{\mathrm{wprod}}(K^*) \subset \mathfrak{E}_r(K^*)$.

Definition 5.1.1. Let K be a Hopf algebra. The subcategory $\mathfrak{E}_\ell^{\mathrm{bialg}}(K) \subset \mathfrak{E}_\ell(K)$ has objects those $H \in \mathfrak{E}_\ell(K)$ with $\phi = \varepsilon$, and morphisms those (F, θ) with $\theta = \varepsilon$.

The subcategory $\mathfrak{E}_\ell^{\mathrm{wprod}}(K) \subset \mathfrak{E}_\ell(K)$ has objects those $H \in \mathfrak{E}_\ell(K)$ in which $\alpha = (\pi \otimes \pi)\Delta j$ and $\sigma = \pi\nabla(j \otimes j)$ are trivial and $\phi(\mathrm{id}_H \otimes j \otimes j) = \varepsilon$. Morphisms in $\mathfrak{E}_\ell^{\mathrm{wprod}}(K)$ are those $(F, \theta)\colon H \to H'$ in $\mathfrak{E}_\ell(K)$ in which $F_0 = \pi' F j$ is trivial and $\theta(\mathrm{id}_H \otimes j) = \varepsilon$.

The categories $\mathfrak{E}_r^{\mathrm{bialg}}(K)$ and $\mathfrak{E}_r^{\mathrm{wprod}}(K^*)$ are defined similarly.

One can view an object of $\mathfrak{E}_\ell^{\mathrm{wprod}}(K)$ as some kind of generalized product of K with the nonassociative bialgebra Q, yielding a coquasibialgebra. Indeed, let $H \in \mathfrak{E}_\ell(K)$. Then the image of $j\colon Q \to H$ is a right H-subcomodule of H if and only if α is trivial. If α is trivial then σ is trivial if and only if the image of $j\colon Q \to H$ is multiplicatively closed in $\mathfrak{E}_\ell^{\mathrm{wprod}}(K)$. In this situation H is, via multiplication, a tensor product of a subbialgebra K and a nonassociative subalgebra Q such that the restriction of the coassociator to $H \otimes Q \otimes Q$ is trivial.

Corollary 5.1.2. *Let K be a finite Hopf algebra The functor $\mathfrak{F}\colon \mathfrak{E}_\ell(K) \to \mathfrak{E}_r(K^*)$ restricts to a category equivalence $\mathfrak{E}_\ell^{\mathrm{bialg}}(K) \to \mathfrak{E}_r^{\mathrm{wprod}}(K^*)$, whose inverse is the restriction of $\mathfrak{F}^{\mathrm{bop}}$. More precisely, $\mathfrak{F}(\mathfrak{F}^{\mathrm{bop}}(H))$ is strongly isomorphic to H whenever $H \in \mathfrak{E}_r^{\mathrm{wprod}}(K^*)$.*

In [33] we have analyzed what axioms a collection of maps has to satisfy to be the skeleton of some object of $\mathfrak{E}_\ell(K)$ which is a bialgebra and has trivial α. One axiom in the list in [33] remained particularly mysterious: Q is a nonassociative algebra (and a coalgebra), with the non-associativity "controlled" by a formula containing the other parts of the skeleton:

$$(p \bullet q) \bullet r = (p \leftharpoonup \sigma(q_{(1)} \otimes r_{(1)[0]})) \bullet ((q_{(2)} \leftharpoonup r_{(1)[1]}) \bullet r_{(2)})$$

Of course, nonassociative bialgebras are in general far from mysterious or exotic. Coquasibialgebras are a class of nonassociative bialgebras whose nonassociativity is controlled in a specified way; they are well understood both technically and conceptually. Unfortunately, no coquasibialgebras were around in [33] to account for the nonassociativity encountered there. Using the connection between bialgebra inclusions and coquasibialgebras furnished by $\mathfrak{F}$ we can do better now.

Lemma 5.1.3. *Let $H \in \mathfrak{E}_\ell(K)$ be a bialgebra. Then the associativity axiom for the coquasibialgebra $\mathfrak{F}(H)$ specializes to the following "associativity" axiom for multiplication in Q:*

$$\begin{aligned}(p \bullet q) \bullet r = \ & (p \leftharpoonup (q_{(1)} \rightharpoonup \alpha^{(1)}(r_{(1)})_{(1)})\sigma(q_{(2)} \leftharpoonup \alpha^{(1)}(r_{(1)})_{(2)} \otimes r_{(2)[0]})) \\ & \bullet((q_{(3)} \leftharpoonup \alpha^{(2)}(r_{(1)})r_{(2)[1]}) \bullet r_{(3)})\end{aligned}$$

for $p, q, r \in Q$.

Proof. Let us first examine associativity in Q for an object $H \in \mathfrak{E}_\ell(K)$ with trivial α and $\phi_{QQ} = \varepsilon$. It follows immediately that $\phi(g \otimes j(q)_{(1)} \otimes j(r)_{(1)})j(q)_{(2)} \otimes j(r)_{(2)} = \varepsilon(g)j(q) \otimes j(r)$ for $q, r \in Q$. Then

$$\begin{aligned}j(p)(j(q)j(r)) &= \phi(j(p)_{(1)} \otimes j(q)_{(1)} \otimes j(r)_{(1)})j(p)_{(2)}(j(q)_{(2)}j(r)_{(2)}) \\ &= (j(p)_{(1)}j(q)_{(1)})j(r)_{(1)}\phi(j(p)_{(2)} \otimes j(q)_{(2)} \otimes j(r)_{(2)}) \\ &= (j(p_{(1)})j(q_{(1)[0]}))j(r_{(1)[0]})\phi(j(p_{(2)}) \otimes q_{(1)[1]} \otimes r_{(2)[1]}j(r_{(2)})) \\ &= (j(p_{(1)})j(q_{(1)[0]}))j(r_{[0]})\phi(j(p_{(2)}) \otimes q_{(1)[1]}j(q_{(2)}) \otimes r_{[1]}) \\ &= \ (j(p_{(1)})j(q_{(1)[0]}))j(r_{[0]})\phi_{QK}(p_{(2)} \leftharpoonup q_{(1)[1]} \otimes q_{(2)[0]} \otimes r_{[1]}) \\ &\qquad \phi_{KK}(p_{(2)} \otimes q_{(1)[2]} \otimes q_{(2)[1]}(q_{(3)} \rightharpoonup r_{[2]}))\end{aligned}$$

Now we apply the opposite of this formula to $\tilde{H} = \mathfrak{F}(H)$, where H is as in the statement of the lemma. Thus

$$\begin{aligned}(p \bullet q) \bullet r = \ & \tilde{\phi}_{K^*K^*}((p_{[-\tilde{2}]} \tilde{\leftharpoonup} q_{(1)})q_{(2)[-\tilde{1}]} \otimes q_{(3)[-\tilde{2}]} \otimes r_{(2)}) \\ & \quad \tilde{\phi}_{K^*Q}(p_{[-\tilde{1}]} \otimes q_{(2)[\tilde{0}]} \otimes q_{(3)[-\tilde{1}]} \tilde{\rightharpoonup} r_{(2)})p_{[\tilde{0}]} \bullet (q_{(3)[\tilde{0}]} \bullet r_{(3)}) \\ = \ & \langle (p_{[-\tilde{2}]} \tilde{\leftharpoonup} q_{(1)})q_{(2)[-\tilde{1}]} | \alpha^{(1)}(r_{(1)}) \rangle \langle q_{(3)[-\tilde{2}]} | \alpha^{(2)}(r_{(1)}) \rangle \\ & \quad \langle p_{[-\tilde{1}]} | \sigma(q_{(2)[\tilde{0}]} \otimes q_{(3)[-\tilde{1}]} \tilde{\rightharpoonup} r_{(2)}) \rangle p_{[\tilde{0}]} \bullet (q_{(3)[\tilde{0}]} \bullet r_{(3)}) \\ = \ & \langle p_{[-\tilde{2}]} | q_{(1)} \rightharpoonup \alpha^{(1)}(r_{(1)})_{(1)} \rangle \langle q_{(2)[-\tilde{1}]} | \alpha^{(1)}(r_{(1)})_{(2)} \rangle \\ & \quad \langle q_{(3)[-\tilde{2}]} | \alpha^{(2)}(r_{(1)}) \rangle \langle p_{[-\tilde{1}]} | \sigma(q_{(2)[\tilde{0}]} \otimes r_{(2)[0]}) \rangle \langle q_{(3)[-\tilde{1}]} | r_{(2)[1]} \rangle \\ & \quad p_{[\tilde{0}]} \bullet (q_{(3)[\tilde{0}]} \bullet r_{(3)}) \\ = \ & p \leftharpoonup (q_{(1)} \rightharpoonup \alpha^{(1)}(r_{(1)})_{(1)})\sigma(q_{(2)} \leftharpoonup \alpha^{(1)}(r_{(1)})_{(2)} \otimes r_{(2)[0]})) \\ & \quad \bullet((q_{(3)} \leftharpoonup \alpha^{(2)}(r_{(1)})r_{(2)[1]}) \bullet r_{(3)})\end{aligned}$$

holds for $p, q, r \in Q$. ☐

Recall the construction of $\mathfrak{F}(H)$ for $H \in \mathfrak{E}_\ell(K)$. In a first step, a coalgebra $Q \rtimes_{\tilde{\alpha}} K^*$ and a category equivalence ${}^H_K\mathcal{M}_K \to {}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M}$ are constructed which do not depend on the choice of a cleaving map $\pi\colon H \to K$. The coquasibialgebra structure on $Q \rtimes_{\tilde{\alpha}} K^*$ is then reconstructed from a monoidal category structure on ${}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M}$, which does depend on the choice of π. However, by construction of the monoidal category structures on ${}^{Q \rtimes_{\tilde{\alpha}} K^*}\mathcal{M}$ associated to different cleaving maps, the identity functor is a monoidal equivalence between them; thus the resulting coquasibialgebra varies only by a cotwist when we vary π. This proves:

Corollary 5.1.4. *Let K be a finite Hopf algebra. The functor $\mathfrak{F}$ induces a well-defined map from the set of bialgebras H containing K as a subbialgebra such that H is a cleft K-module coalgebra, to the set of cotwist classes of coquasibialgebras containing K^* as a subbialgebra.*

5.2 Trivalent products

Let K be a Hopf algebra and Q an algebra and coalgebra. A trivalent product of K and Q is a bialgebra H that is the middle term of a mapping system

$$K \underset{\iota}{\overset{\pi}{\leftrightarrows}} H \underset{j}{\overset{\nu}{\rightleftarrows}} Q \text{ with } \pi\iota = \mathrm{id}_K, \nu j = \mathrm{id}_Q, \nu\iota = \eta\varepsilon, \pi j = \eta\varepsilon, \iota\pi * j\nu = \mathrm{id}_H,$$

subject to the following conditions: ι is a bialgebra map, ν is a coalgebra map, π is a left K-module coalgebra map, and j is a Q-colinear algebra map. Trivalent products can be constructed as follows: As a coalgebra, $H = K \ltimes Q$ is the semidirect coproduct coalgebra with respect to a coaction $\rho\colon Q \ni q \mapsto q_{[0]} \otimes q_{[1]} \in Q \otimes K$; that is, comultiplication is given by

$$\Delta(x \otimes q) = (x_{(1)} \otimes q_{(1)[0]}) \otimes (x_{(2)} q_{(1)[1]} \otimes q_{(2)}).$$

Multiplication in H is defined by the same formula as in a double crossproduct:

$$(x \otimes q)(y \otimes q) = x(p_{(1)} \rightharpoonup y_{(1)}) \otimes (p_{(2)} \leftharpoonup y_{(2)})q,$$

using two maps $\rightharpoonup\colon Q \otimes K \to K$ and $\leftharpoonup\colon Q \otimes K \to Q$. Of course there are certain axioms that $\rightharpoonup$, $\leftharpoonup$ and ρ have to satisfy so that the result of this construction is a bialgebra. Trivalent products (so called since there are three operations between the two factors involved) appeared in [2]. They also appear in [33] as a special case of Hopf algebras with a module coalgebra projection onto a Hopf subalgebra, and in [5]. They are designed to be simultaneous generalizations of Radford biproducts [27], double crossproducts, and bismash products [15, 16, 19]. Radford biproducts are recovered as the special case that $\rightharpoonup$ is trivial, that is, π is a bialgebra map. Double crossproducts are the special case where ρ is trivial, that is, j is an algebra map. Bismash products are the special case where $\leftharpoonup$ is trivial, that is, ν is a bialgebra map. To fix terminology we call a quadruple $(K, Q, \rightharpoonup, \rho)$ (or just the maps $\rightharpoonup$, ρ) satisfying the axioms necessary to form a bismash product a matched pair of Hopf algebras (as in [15, 16]); we call a quadruple $(K, Q, \rightharpoonup, \leftharpoonup)$ satisfying the axioms necessary to build a double crossproduct a crossed pair (warning: crossed pairs are called matched pairs in [21]).

Now the above says that the functor $\mathfrak{F}$ from Section 3 maps left (i. e. defined as above) trivalent products of K with Q to right (i. e. defined as above with switched sides) trivalent products of K^* with Q, in particular Radford biproducts to Radford biproducts, bismash products to double crossproducts, and double crossproducts to bismash products.

Moreover, trivalent products are precisely those $H \in \mathfrak{E}_\ell(K)$ such that both H and $\mathfrak{F}(H)$ have trivial coassociators (so that they are in particular usual bialgebras).

Note that a correspondence between left trivalent products over K and right trivalent products over K^* is easily deduced from the list of axioms that $\rho, \rightharpoonup, \leftharpoonup$ have to satisfy to build a trivalent product. The novelty here is that we have a direct connection between the Hopf algebras built from those data; in fact, notice that we never needed to know the whole list of axioms to transform one kind of product into the other. Without repeating the exact technical details, we close the section by listing the characterization of the correspondence in terms of monoidal category equivalences for the double crossproducts, bismash products, and biproducts:

$$\begin{aligned}
({}^{K\#Q}_{K}\mathcal{M}_K, \underset{K}{\otimes}) &\cong ({}^{Q\bowtie K^*}\mathcal{M}, \otimes) & \qquad ({}_{K^*}\mathcal{M}^{Q\bowtie K^*}_{K^*}, \underset{K^*}{\otimes}) &\cong (\mathcal{M}^{K\#Q}, \otimes)\\
({}^{K\bowtie Q}_{K}\mathcal{M}_K, \underset{K}{\otimes}) &\cong ({}^{Q\#K^*}\mathcal{M}, \otimes) & \qquad ({}_{K^*}\mathcal{M}^{Q\#K^*}_{K^*}, \underset{K^*}{\otimes}) &\cong (\mathcal{M}^{K\bowtie Q}, \otimes)\\
({}^{K\star Q}_{K}\mathcal{M}_K, \underset{K}{\otimes}) &\cong ({}^{Q\star K^*}\mathcal{M}, \otimes) & \qquad ({}_{K^*}\mathcal{M}^{Q\star K^*}_{K^*}, \underset{K^*}{\otimes}) &\cong (\mathcal{M}^{K\star Q}, \otimes)
\end{aligned}$$

6 Hopf algebra extensions and the Kac sequence

In this section, we will apply the functor $\mathfrak{F}$ from Section 3 to cleft Hopf algebra extensions $K \to H \to Q$. In a first step, this means that we just specialize the previous more general results to the case of those $H \in \mathfrak{E}_\ell(K)$ where H is a bialgebra and K is a normal subbialgebra, hence Q a quotient bialgebra. We find that these $H \in \mathfrak{E}_\ell(K)$ correspond via $\mathfrak{F}$ to certain product coquasibialgebras of K and Q. Specializing Corollary 5.1.4 we find that changing the cleaving map that we need to consider a Hopf algebra extension as an object of $\mathfrak{E}_\ell(K)$ only results in a cotwist of the resulting product coquasibialgebra, so that we have a well defined map from isomorphism classes of extensions to cotwisting classes of coquasibialgebras.

After this is done in Section 6.1, we start a closer analysis of the situation, with the aim of arriving at a generalized version of Kac' exact sequence connecting group cohomology with the description of extensions of group algebras by dual group algebras. We collect some machinery of homological nature (though, of course, not strictly connected to any homology theory in the noncommutative non-cocommutative situation) in Section 6.2, and we proceed to construct the maps and prove exactness of our generalized version of Kac' sequence in Section 6.3 (the higher degree part) and Section 6.5 (the lower degree part).

6.1 Just a special case

As a special case of the general construction in Section 3, we obtain a category equivalence between certain subcategories of $\mathfrak{E}_\ell(K)$, resp. $\mathfrak{E}_r(K^*)$, whose objects are Hopf algebra extensions and coquasibialgebra products, respectively.

Definition 6.1.1. The category $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ has objects those $H \in \mathfrak{E}_\ell(K)$ in which ϕ and $\leftharpoonup : Q \otimes K \to Q$ are trivial. Morphisms in $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ are those morphisms in $\mathfrak{E}_\ell(K)$ that are bialgebra homomorhisms rather than coquasimorphisms.

The category $\mathfrak{E}_\ell^{\mathrm{gpc}}(K)$ has objects those $(H, \phi, \iota, \pi) \in \mathfrak{E}_\ell(K)$ for which α, σ, $\rho : Q \to Q \otimes K$, and ϕ_{QQ} are trivial. A morphism in $\mathfrak{E}_\ell^{\mathrm{gpc}}(K)$ is a morphism $(F, \theta): H \to H'$ such that $\theta(j \otimes j) = \varepsilon$ and $\pi' F j = \varepsilon$.

One can define analogous subcategories $\mathfrak{E}_r^{\mathrm{ext}}(K)$ and $\mathfrak{E}_r^{\mathrm{gpc}}(K)$ of $\mathfrak{E}_r(K)$.

Corollary 6.1.2. *The functor $\mathfrak{F}: \mathfrak{E}_\ell(K) \to \mathfrak{E}_r(K^*)$ induces a category equivalence $\mathfrak{E}_\ell^{\mathrm{ext}}(K) \cong \mathfrak{E}_r^{\mathrm{gpc}}(K^*)$, whose inverse is induced by $\mathfrak{F}^{\mathrm{bop}}$. More precisely, $\mathfrak{F}(\mathfrak{F}^{\mathrm{bop}}(H)) \cong H$ for all $H \in \mathfrak{E}_r^{\mathrm{gpc}}(K^*)$.*

Proof. Both objects and morphisms of $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ and $\mathfrak{E}_r^{\mathrm{gpc}}(K^*)$ are determined by their skeleta. Moreover, by Proposition 4.1.2 $\mathfrak{F}(H)$ is in $\mathfrak{E}_r^{\mathrm{gpc}}(K^*)$ whenever H is in $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$, and vice versa. Similarly by Proposition 4.2.2, Proposition 4.2.5, and Proposition 4.2.8 $\mathfrak{F}$ induces a bijection between morphisms in $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ and $\mathfrak{E}_r^{\mathrm{gpc}}(K^*)$. ◻

Remark 6.1.3. If $H \in \mathfrak{E}_\ell^{\mathrm{ext}}(K)$, then $\nu : H \to Q$ is multiplicative, hence Q is a bialgebra. Thus, $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$ is a bialgebra extension in the sense of the following definition, which varies [15, 35]

Definition 6.1.4. Let K be a Hopf algebra and Q a bialgebra. An extension $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$ of Q by K is a triple (H, ι, ν) where H is a bialgebra, $\iota : K \to H$ is an injective bialgebra map, and $\nu : H \to Q$ is a surjective bialgebra map such that H is a cleft left K-module coalgebra and ν induces an isomorphism $H/K^+H \to Q$.

If Q is a Hopf algebra, then it follows by [33, Thm. A.2, Rem. A.3] that H is a Hopf algebra and H is a cleft right Q-comodule algebra. In this case we have an extension in the sense of [15], whereas [35] does not assume cleftness.

Two extensions $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$ and $K \xrightarrow{\iota'} H \xrightarrow{\nu'} Q$ are isomorphic if there exists a bialgebra isomorphism $F: H \to H'$ with $F\iota = \iota'$ and $\nu F = \nu'$. We denote the set of isomorphism classes of extensions of Q by K by $\mathrm{Ext}(Q, K)$.

As in [15], every extension H of Q by K is isomorphic to a bicrossproduct with a cocycle and a dual cocycle: For every choice of cleaving map π for the extension (H, ι, ν) there is an isomorphism $F: H \to K^\alpha \#_\sigma Q$ of extensions, where $K^\alpha \#_\sigma Q = K \ltimes_\alpha Q$ is a crossed coproduct coalgebra with multiplication

$$(x\#p)(y\#q) = x(p_{(1)} \rightharpoonup y)\sigma(p_{(2)} \otimes q_{(2)})\#p_{(3)}q_{(2)}$$

for $x, y \in K$ and $p, q \in Q$, and becomes an extension $(K^\alpha \#_\sigma Q, \iota_\#, \nu_\#)$ by $\nu_\#(x\#q) = q\varepsilon(x)$ and $\iota_\#(x) = x\#1$. The cleaving map π is recovered by $\pi(h) = \pi_\# F(h)$ where $\pi_\#(x\#q) = x\varepsilon(q)$. Note that every extension of Q by K can in this way be considered an object of $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$; morphisms of extensions are precisely those morphisms $F\colon H \to H'$ in $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ in which $Q = Q'$ and $F_1 = \mathrm{id}_Q$.

Remark 6.1.5. If $H \in \mathfrak{E}_\ell^{\mathrm{gpc}}(K)$, then $j\colon Q \to H$ is a coalgebra map, and $\nabla(\iota \otimes j)\colon K \otimes Q \to H$ is a coalgebra isomorphism. Moreover, $\phi(\mathrm{id}_{H\otimes H} \otimes j)$ implies that Q is a bialgebra, and since σ is trivial, j is a (coquasi)bialgebra map. This shows that an object of $\mathfrak{E}_\ell^{\mathrm{gpc}}(K)$ can be considered as a product coquasibialgebra of K and Q in the sense of the following definition.

Definition 6.1.6. Let K, Q be coquasibialgebras. A (generalized) product coquasibialgebra of K and Q is a triple (H, ι, j) in which H is a coquasibialgebra and $\iota\colon K \to H$ and $j\colon Q \to H$ are coquasibialgebra maps such that $\nabla(\iota \otimes j)\colon K \otimes H \to H$ is bijective. A generalized product bialgebra is a generalized product coquasibialgebra that is a bialgebra with trivial coassociator.

Two generalized product coquasibialgebras (H, ι, j) and (H', ι', j') are isomorphic if there is a coquasibialgebra isomorphism $F\colon H \to H'$ such that $F\iota = \iota'$ and $Fj = j'$.

If K and Q are bialgebras, then we denote by $\mathrm{Gpb}(K, Q)$ the set of isomorphism classes of generalized product bialgebras of K and Q.

By its very definition, any product coquasibialgebra of K and Q is isomorphic to one that equals $K \otimes Q$ as a coalgebra, with $\iota(x) = x \otimes 1$ and $j(q) = 1 \otimes q$. If, moreover, $\phi(\iota \otimes \mathrm{id}_{H\otimes H})$ and $\phi(\mathrm{id}_{H\otimes H} \otimes j)$ are trivial, and K is a Hopf algebra, then we can consider H as an object of $\mathfrak{E}_\ell^{\mathrm{gpc}}(K)$ by choosing $\pi(x \otimes q) = x\varepsilon(q)$. Then the multiplication in $K \otimes Q$ is given by $(x \otimes p)(y \otimes q) = x(p_{(1)} \rightharpoonup y_{(1)}) \otimes (p_{(2)} \leftharpoonup y_{(2)})q$.

The restriction of $\mathfrak{F}$ to a functor $\mathfrak{E}_\ell^{\mathrm{ext}}(K) \to \mathfrak{E}_r^{\mathrm{gpc}}(K^*)$ can be written out quite explicitly, using either the description of $\mathfrak{F}$ in Theorem 3.4.5 directly, or the description on skeleta. If $H \in \mathfrak{E}_\ell^{\mathrm{ext}}(K)$ is given as a bicrossproduct $H = K^\alpha \#_\sigma Q$, then $\mathfrak{F}(H) = Q \otimes K^*$ as a coalgebra, with multiplication

$$(p \otimes \varphi)(q \otimes \psi) = p(\varphi_{(1)} \tilde{\leftharpoonup} q_{(1)}) \otimes (\varphi_{(2)} \tilde{\rightharpoonup} q_{(2)})\psi$$

with $\langle \varphi \tilde{\leftharpoonup} q | x \rangle = \langle \varphi | q \rightharpoonup x \rangle$ and $\varphi \tilde{\rightharpoonup} q = q_{[0]} \langle \varphi | q_{[1]} \rangle$, where $\rightharpoonup$ and $q \mapsto q_{[0]} \otimes q_{[1]}$ are the weak action and weak coaction in the bicrossproduct. The coassociator on $\mathfrak{F}(H)$ is given in terms of the cocycle σ and dual cocycle α by

$$\begin{aligned}\tilde{\phi}&(p \otimes \varphi \otimes q \otimes \psi \otimes r \otimes \vartheta)\\ &= \varepsilon(p)\langle \varphi | (q \rightharpoonup \alpha^{(1)}(r_{(1)}))\sigma(q_{(2)} \otimes r_{(2)[0]})\rangle \langle \psi | \alpha^{(2)}(r_{(1)}) r_{(2)[1]} \rangle \varepsilon(\vartheta).\end{aligned}$$

Let H and H' be two extensions of Q by K, and put $\tilde{H} = \mathfrak{F}(H)$, $\tilde{H}' = \mathfrak{F}(H')$. Then H and H' are isomorphic as extensions if and only if the associated product coquasibialgebras satisfy $\tilde{H}' = \tilde{H}^\theta$ for some cotwist θ satisfying $\theta(\tilde{j} \otimes \mathrm{id}_{\tilde{H}}) = \varepsilon$ and $\theta(\mathrm{id}_{\tilde{H}} \otimes \tilde{\iota}) = \varepsilon$. In fact, considering H and H' as objects in $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ (after choosing cleaving maps!), isomorphisms of extensions between them are precisely isomorphisms $F\colon H \to H'$ in $\mathfrak{E}_\ell^{\mathrm{ext}}(K)$ satisfying $F_1 = \mathrm{id}_Q$. By the description of the morphism map of $\mathfrak{F}$ in Section 3.5 (or, more easily, by the description of the morphism map in terms of skeleta in Proposition 4.2.2) such isomorphism correspond bijectively to cotwists of the form described above. Note that by the same reasoning as in Section 3.5 we have $\theta(\tilde{h} \otimes \tilde{j}(q)\tilde{\iota}(\varphi)) = \varepsilon(\varphi)\theta(\tilde{h} \otimes \tilde{j}(q))$ and $\theta(\tilde{j}(q)\tilde{\iota}(\varphi) \otimes \tilde{h}) = \varepsilon(q)\theta(\tilde{\iota}(\varphi) \otimes \tilde{h})$ for $\tilde{h} \in \tilde{H}$, $q \in Q$ and $\varphi \in K^*$.

6.2 "Homological" technicalities

We have seen that the functor $\mathfrak{F}$ maps extensions to generalized products, when we consider the latter up to cotwist. In the cocommutative case, coquasibialgebra structures are Sweedler three-cocycles and changing them by a cotwist is changing them by a Sweedler coboundary. In short, we really started to consider noncommutative cohomology classes of some sort. This section collects some technical results of a cohomological nature in the sense that under sufficient cocommutativity assumptions they have a cohomological meaning.

Let us start with a very simple observation:

Remark 6.2.1. Let K and Q be two k-coquasibialgebras, and (H, ι, j) a generalized product coquasibialgebra of K and Q. Let $\theta\colon H \otimes H \to k$ be convolution invertible, and denote the restrictions of θ to $K \otimes K$ and $Q \otimes Q$ again by θ. Then (H^θ, ι, j) is a generalized product coquasibialgebra of K^θ and Q^θ.

In fact, if we treat ι and j as inclusions, then we have $\nabla^\theta(\iota \otimes j)(x \otimes q) = \theta(x_{(1)} \otimes q_{(1)})x_{(2)}q_{(2)}\theta^{-1}(x_{(3)} \otimes q_{(3)})$ for $x \in K$ and $q \in Q$, so that $\nabla^\theta(\iota \otimes j)$ is a bijection.

If a generalized product coquasibialgebra is obtained by another one by a cotwist, we would like to call them homologous. The following definition hides a small technical problem: If $K \otimes Q$ is a generalized product coquasibialgebra, with $(x \otimes 1)(1 \otimes q) = x \otimes q$ for $x \in K$ and $q \in Q$, and if θ is a cotwist on $K \otimes Q$, then we need not have $(x \otimes 1) \cdot (1 \otimes q) = x \otimes q$ with respect to the twisted multiplication on $K \otimes Q$. So although any generalized product coquasibialgebra is canonically isomorphic to one of the form $K \otimes Q$ with $(x \otimes 1)(1 \otimes q) = x \otimes q$, it is sometimes better not to use this.

Definition 6.2.2. Let K, Q be two k-coquasibialgebras. Two generalized product coquasibialgebras (H, ι, j) and (H', ι', j') of K and Q are homologous if there exists a coquasiisomorphism $(F, \theta)\colon H \to H'$ such that $F\iota = \iota'$ and $Fj = j'$ (as k-module maps).

We denote by $\mathrm{Gpc}(K, Q)$ the set of homology classes of generalized product coquasibialgebras of K and Q.

A special case of coquasibialgebra structures is provided by Sweedler three-cocycles on cocommutative Hopf algebras. The following Proposition provides the exact comparison between homology classes of generalized products as defined above, and degree three Sweedler cohomology $\mathcal{H}^3(B)$ of a bialgebra B with values in the base ring k.

Proposition 6.2.3. *Let K and Q be cocommutative bialgebras. Then a bijection*

$$\mathrm{Gpc}(K,Q) \cong \bigcup_{H\in\mathrm{Gpb}(K,Q)} \mathrm{Ker}\left(\mathcal{H}^3(H) \xrightarrow{\mathrm{res}} \mathcal{H}^3(K)\oplus\mathcal{H}^3(Q)\right)$$

is given by assigning $\phi\in\mathcal{H}^3(H)$ to $((H,\phi),\iota,j)\in\mathrm{Gpc}(K,Q)$.

Proof. By cocommutativity, every generalized product coquasibialgebra is actually an ordinary bialgebra H, equipped with a possibly nontrivial coassociator ϕ, which is just a three-cocycle on H whose restrictions to both K and Q are trivial by definition of a generalized product coquasibialgebra. Moreover, if two generalized product coquasibialgebras are homologous, then their underlying generalized product bialgebras are isomorphic. If two generalized product coquasibialgebras have the same underlying generalized product bialgebra, then they are homologous if and only if their coassociators are cohomologous three-cocylces. This shows that the claimed map exists and is injective. Now let $(H,\iota,j)\in\mathrm{Gpb}(K,Q)$ and $\phi\in\mathcal{H}^3(H)$ such that the restrictions $\phi|_{K\otimes K\otimes K}$ and $\phi|_{Q\otimes Q\otimes Q}$ are cohomologous to ε. We can assume that $H=K\otimes Q$, $\iota(x)=x\otimes 1$ and $j(p)=1\otimes p$. Choose invertible $\theta'\in(K\otimes K)^*$ and $\theta''\in(Q\otimes Q)^*$ with $(\phi|_{K\otimes K\otimes K})^{\theta'}=\varepsilon$ and $(\phi|_{Q\otimes Q\otimes Q})^{\theta'}=\varepsilon$. Define $\theta\in(H\otimes H)^*$ by $\theta(x\otimes p\otimes y\otimes q)=\theta'(x\otimes y)\theta''(p\otimes q)$. Then $\phi^\theta|_{K\otimes K\otimes K}=\varepsilon$ and $\phi^\theta|_{Q\otimes Q\otimes Q}$, that is, $((H,\phi),\iota,j)\in\mathrm{Gpc}(K,Q)$, showing surjectivity of our map. □

The output of the functor $\mathfrak{F}$ applied to an extension is not the most general type of product coquasibialgebra allowed by Definition 6.1.6. However, the next Proposition shows that this is true up to homology.

Proposition 6.2.4. *Let K and Q be k-bialgebras. Every homology class in $\mathrm{Gpc}(K,Q)$ contains a representative $((H,\phi),\iota,j)$ with $\phi(\iota\otimes\mathrm{id}_{H\otimes H})=\varepsilon$ and $\phi(\mathrm{id}_{H\otimes H}\otimes j)=\varepsilon$.*

Proof. Choose a representative $((H,\phi),\iota,j)$. We treat both ι and j as multiplicative inclusions of subbialgebras K and Q of H.

Define $t\in(H\otimes Q)^*$ and $t'\in(K\otimes H)^*$ by

$$\begin{aligned} t(xp\otimes q) &= \phi^{-1}(x\otimes p\otimes q)\\ t'(x\otimes yq) &= \phi(x\otimes y\otimes q)\end{aligned}$$

for $x, y \in K$ and $p, q \in Q$. Evidently t and t' are invertible. Define $\theta \in (H \otimes H)^*$ by

$$\begin{aligned}&\theta(xp \otimes yq)\\ &= t(x_{(1)}(p_{(1)}y_{(1)})\otimes q_{(1)})t'(x_{(2)}\otimes p_{(2)}y_{(2)})\phi^{-1}(x_{(3)}\otimes p_{(3)}\otimes y_{(3)})\phi(x_{(4)}p_{(4)}\otimes y_{(4)}\otimes q_{(2)})\end{aligned}$$

for $x, y \in K$ and $p, q \in Q$. One checks that $\theta|_{K^{\otimes 2}} = \varepsilon$ and $\theta|_{Q^{\otimes 2}} = \varepsilon$.

We claim that $\phi^\theta|_{K\otimes H\otimes Q} = \varepsilon$. Note first that

$$\begin{aligned}\theta(xp \otimes q) &= \phi^{-1}(x \otimes p \otimes q)\\ \theta(x \otimes yq) &= \phi(x \otimes y \otimes q)\end{aligned}$$

holds for $x, y \in K$ and $p, q \in Q$. Hence

$$\begin{aligned}&\phi^\theta(x_{(1)} \otimes y_{(1)}p_{(1)} \otimes q_{(1)})\theta(y_{(2)}p_{(2)} \otimes q_{(2)})\theta(x_{(2)} \otimes (y_{(3)}p_{(3)})q_{(3)})\\
&\quad= \theta(x_{(1)} \otimes y_{(1)}p_{(1)})\theta(x_{(2)}(y_{(2)}p_{(2)}) \otimes q_{(1)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes q_{(3)})\\
&\quad= \phi(x_{(1)} \otimes y_{(1)} \otimes p_{(1)})\theta(x_{(2)}(y_{(2)}p_{(2)}) \otimes q_{(1)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes q_{(2)})\\
&\quad= \theta((x_{(1)}y_{(1)})p_{(1)} \otimes y_{(1)})\phi(x_{(2)} \otimes y_{(2)} \otimes p_{(2)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes q_{(2)})\\
&\quad= \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes q_{(1)})\phi(x_{(2)} \otimes y_{(2)} \otimes p_{(2)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes q_{(2)})\\
&\quad= \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes q_{(1)})\phi(x_{(2)} \otimes y_{(2)} \otimes p_{(2)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes q_{(2)})\\
&\qquad\quad \phi(y_{(4)} \otimes p_{(4)} \otimes q_{(3)})\phi^{-1}(y_{(5)} \otimes p_{(5)} \otimes q_{(4)})\\
&\quad= \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes q_{(1)})\phi(x_{(2)}y_{(2)} \otimes p_{(2)} \otimes q_{(2)})\phi(x_{(3)} \otimes y_{(3)} \otimes p_{(3)}q_{(3)})\\
&\qquad\quad \phi^{-1}(y_{(4)} \otimes p_{(4)} \otimes q_{(4)})\\
&\quad= \phi(x_{(1)} \otimes y_{(1)} \otimes p_{(1)}q_{(1)})\phi^{-1}(y_{(2)} \otimes p_{(2)} \otimes q_{(2)})\\
&\quad= \theta(x \otimes y_{(1)}(p_{(1)}q_{(1)}))\phi^{-1}(y_{(2)} \otimes p_{(2)} \otimes q_{(2)})\\
&\quad= \phi^{-1}(y_{(1)} \otimes p_{(1)} \otimes q_{(1)})\theta(x \otimes (y_{(2)}p_{(2)})q_{(2)})\\
&\quad= \theta(y_{(1)}p_{(1)} \otimes q_{(1)})\theta(x \otimes (y_{(2)}p_{(2)})q_{(2)})\end{aligned}$$

and $\phi^\theta(x \otimes yp \otimes q) = \varepsilon(xy)\varepsilon(pq)$ follows by cancelling.

We have shown that the homology class under consideration contains a representative (H^θ, ι, j) with $\phi^\theta|_{K\otimes H\otimes Q} = \varepsilon$. We start over and assume that (H, ι, j) already satisfies $\phi|_{K\otimes H\otimes Q} = \varepsilon$. Then for $x, y, z \in K$ and $r \in Q$ we have

$$\begin{aligned}&\phi(x \otimes y \otimes zr) = \phi(x_{(1)}y_{(1)} \otimes z_{(1)} \otimes z_{(1)})\phi(x_{(2)} \otimes y_{(2)} \otimes z_{(2)}r_{(2)})\\ &= \phi(x_{(1)} \otimes y_{(1)} \otimes z_{(1)})\phi(x_{(2)} \otimes y_{(2)}z_{(2)} \otimes r_{(1)})\phi(y_{(3)} \otimes z_{(3)} \otimes r_{(2)}) = \varepsilon(xyz)\varepsilon(r),\end{aligned}$$

that is $\phi|_{K\otimes K\otimes H} = \varepsilon$. Similarly $\phi|_{H\otimes Q\otimes Q} = \varepsilon$.

Define $\theta \in (H \otimes H)^*$ by

$$\theta(xp \otimes yq) = \phi^{-1}(x_{(1)} \otimes p_{(1)} \otimes y_{(1)})\phi(x_{(2)}p_{(2)} \otimes y_{(2)} \otimes q).$$

Note that $\theta|_{K\otimes H} = \varepsilon$ and $\theta|_{H\otimes Q} = \varepsilon$. Now

$$\begin{aligned}
&\phi^\theta(x_{(1)} \otimes y_{(1)}p_{(1)} \otimes z_{(1)}q_{(1)})\theta(y_{(2)}p_{(2)} \otimes z_{(2)}q_{(2)})\theta(x_{(2)} \otimes (y_{(3)}p_{(3)})(z_{(3)}q_{(3)})) \\
&\quad = \theta(x_{(1)} \otimes y_{(1)}p_{(1)})\theta(x_{(2)}(y_{(2)}p_{(2)}) \otimes z_{(1)}q_{(1)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes z_{(2)}q_{(2)}) \\
&\quad = \theta((x_{(1)}y_{(1)})p_{(1)} \otimes z_{(1)}q_{(1)})\phi(x_{(2)} \otimes y_{(2)}p_{(2)} \otimes z_{(2)}q_{(2)}) \\
&\quad = \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes z_{(1)})\phi(x_{(2)}y_{(2)}p_{(2)} \otimes z_{(2)} \otimes q_{(1)}) \\
&\qquad\quad \phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes z_{(3)}q_{(2)}) \\
&\quad = \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes z_{(1)})\phi(x_{(2)} \otimes y_{(2)}p_{(2)} \otimes z_{(2)}) \\
&\qquad\quad \phi(x_{(2)} \otimes (y_{(2)}p_{(2)})z_{(2)} \otimes q_{(1)})\phi(y_{(3)}p_{(3)} \otimes z_{(3)} \otimes q_{(2)}) \\
&\quad = \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes z_{(1)})\phi(x_{(2)} \otimes y_{(2)}p_{(2)} \otimes z_{(2)})\phi(y_{(3)}p_{(3)} \otimes z_{(3)} \otimes q) \\
&\quad = \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes z_{(1)})\phi(x_{(2)} \otimes y_{(2)} \otimes p_{(2)})\phi(x_{(3)} \otimes y_{(3)}p_{(3)} \otimes z_{(2)}) \\
&\qquad\quad \phi(y_{(4)} \otimes p_{(4)} \otimes z_{(3)})\phi^{-1}(y_{(5)} \otimes p_{(5)} \otimes z_{(4)})\phi(y_{(6)}p_{(6)} \otimes z_{(5)} \otimes q) \\
&\quad = \phi^{-1}(x_{(1)}y_{(1)} \otimes p_{(1)} \otimes z_{(1)})\phi(x_{(2)}y_{(2)} \otimes p_{(2)} \otimes z_{(2)})\phi(x_{(3)} \otimes y_{(3)} \otimes p_{(3)}z_{(3)}) \\
&\qquad\quad \phi^{-1}(y_{(4)} \otimes p_{(4)} \otimes z_{(4)})\phi(y_{(5)}p_{(5)} \otimes z_{(5)} \otimes q) \\
&\quad = \varepsilon(x)\phi^{-1}(y_{(1)} \otimes p_{(1)} \otimes z_{(1)})\phi(y_{(2)}p_{(2)} \otimes z_{(2)} \otimes q) \\
&\quad = \theta(yp \otimes zq)\varepsilon(x) \\
&\quad = \theta(y_{(1)}p_{(1)} \otimes z_{(1)}q_{(1)})\theta(x \otimes (y_{(2)}p_{(2)})(z_{(2)}q_{(2)}))
\end{aligned}$$

so that $\phi^\theta|_{K\otimes H\otimes H} = \varepsilon$. Further

$$\begin{aligned}
&\phi^\theta(x_{(1)}p_{(1)} \otimes y_{(1)}q_{(1)} \otimes r_{(1)})\theta(y_{(2)}q_{(2)} \otimes r_{(2)})\theta(x_{(2)}p_{(2)} \otimes (y_{(3)}q_{(3)})r_{(3)}) \\
&\quad = \theta(x_{(1)}p_{(1)} \otimes y_{(1)}q_{(1)})\theta((x_{(2)}p_{(2)})(y_{(2)}q_{(2)}) \otimes r_{(1)}) \\
&\qquad\quad \phi(x_{(3)}p_{(3)} \otimes y_{(3)}q_{(3)} \otimes r_{(2)}) \\
&\quad = \phi^{-1}(x_{(1)} \otimes p_{(1)} \otimes y_{(1)})\phi(x_{(2)}p_{(2)} \otimes y_{(2)} \otimes q_{(1)})\phi(x_{(3)}p_{(3)} \otimes y_{(3)}q_{(2)} \otimes r) \\
&\quad = \phi^{-1}(x_{(1)} \otimes p_{(1)} \otimes y_{(1)})\phi(x_{(2)}p_{(2)} \otimes y_{(2)} \otimes q_{(1)})\phi(x_{(3)}p_{(3)} \otimes y_{(3)}q_{(2)} \otimes r_{(1)}) \\
&\qquad\quad \phi(y_{(4)} \otimes q_{(3)} \otimes r_{(2)}) \\
&\quad = \phi^{-1}(x_{(1)} \otimes p_{(1)} \otimes y_{(1)})\phi((x_{(2)}p_{(2)})y_{(2)} \otimes q_{(1)} \otimes r_{(1)}) \\
&\qquad\quad \phi(x_{(3)}p_{(3)} \otimes y_{(3)} \otimes q_{(2)}r_{(2)}) \\
&\quad = \phi^{-1}(x_{(1)} \otimes p_{(1)} \otimes y_{(1)})\phi(x_{(2)}p_{(2)} \otimes y_{(2)} \otimes qr) \\
&\quad = \theta(xp \otimes yqr) \\
&\quad = \theta(y_{(1)}q_{(1)} \otimes r_{(1)})\theta(xp \otimes y_{(2)}q_{(2)}r_{(2)})
\end{aligned}$$

so that $\phi^\theta|_{H\otimes H\otimes Q} = \varepsilon$. □

A special case of coquasimorphisms are Sweedler two-cocycles. The following definition generalizes that of cohomology of two such two-cocycles.

Definition 6.2.5. Let H and H' be coquasibialgebras. Two coquasihomomorphisms $(F, \theta), (F', \theta')\colon H \to H'$ are homologous if there is a convolution invertible $t\colon H \to k$ with

$$\theta'(g_{(1)} \otimes h_{(1)})t(g_{(2)}h_{(2)}) = t(g_{(1)})t(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})$$

and

$$t(g_{(1)})F(g_{(2)}) = F'(g_{(1)})t(g_{(2)})$$

for all $g, h \in H$.

Lemma 6.2.6. *Let H and H' be coquasibialgebras, and $(F,\theta)\colon H \to H'$ a coquasimorphism. Let $t\colon H \to k$ be convolution invertible with $t(1) = 1$. Then $(F,\theta)^t := (F^t, \theta^t)\colon H \to H'$, defined by*

$$\theta^t(g \otimes h) = t(g_{(1)})t(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})t^{-1}(g_{(3)}h_{(3)})$$

and

$$F^t(h) = t(h_{(1)})F(h_{(2)})t^{-1}(h_{(3)}),$$

is also a coquasimorphism (and by definition homologous to (F,θ)).

Proof. We calculate

$$\begin{aligned}
&\theta^t(f_{(1)} \otimes g_{(1)})\theta^t(f_{(2)}g_{(2)} \otimes h_{(2)})\phi(f_{(3)} \otimes g_{(3)} \otimes h_{(3)})\\
&\quad= t(f_{(1)})t(g_{(1)})\theta(f_{(2)} \otimes g_{(2)})t^{-1}(f_{(3)}g_{(3)})t(f_{(4)}g_{(4)})t(h_{(1)})\theta(f_{(5)}g_{(5)} \otimes h_{(2)})\\
&\qquad\quad t^{-1}((f_{(6)}g_{(6)})h_{(3)})\phi(f_{(7)} \otimes g_{(7)} \otimes h_{(4)})\\
&\quad= t(f_{(1)})t(g_{(1)})t(h_{(1)})\theta(f_{(2)} \otimes g_{(2)})\theta(f_{(3)}g_{(3)} \otimes h_{(2)})\phi(f_{(4)} \otimes g_{(4)} \otimes h_{(3)})\\
&\qquad\quad t^{-1}(f_{(5)}(g_{(5)}h_{(4)}))\\
&\quad= t(f_{(1)})t(g_{(1)})t(h_{(1)})\phi'(F(f_{(2)}) \otimes F(g_{(2)}) \otimes F(h_{(2)}))\theta(g_{(3)} \otimes h_{(3)})\\
&\qquad\quad \theta(f_{(3)} \otimes g_{(4)}h_{(4)})t^{-1}(f_{(4)}(g_{(5)}h_{(5)}))\\
&\quad= \phi'(F^t(f_{(1)}) \otimes F^t(g_{(1)}) \otimes F^t(h_{(1)}))t(f_{(2)})t(g_{(2)})t(h_{(2)})\theta(g_{(3)} \otimes h_{(3)})\\
&\qquad\quad \theta(f_{(3)} \otimes g_{(4)}h_{(4)})t^{-1}(f_{(4)}(g_{(5)}h_{(5)}))\\
&\quad= \phi'(F^t(f_{(1)}) \otimes F^t(g_{(1)}) \otimes F^t(h_{(1)}))t(g_{(2)})t(h_{(2)})\theta(g_{(3)} \otimes h_{(3)})t^{-1}(g_{(4)}h_{(4)})\\
&\qquad\quad t(f_{(2)})t(g_{(5)}h_{(5)})\theta(f_{(3)} \otimes g_{(6)}h_{(6)})t^{-1}(f_{(4)}(g_{(7)}h_{(7)}))\\
&\quad= \phi'(F^t(f_{(1)}) \otimes F^t(g_{(1)}) \otimes F^t(h_{(1)}))\theta^t(g_{(2)} \otimes h_{(2)})\theta^t(f_{(2)} \otimes g_{(3)}h_{(3)})
\end{aligned}$$

and

$$\begin{aligned}
&\theta^t(g_{(1)} \otimes h_{(1)})F^t(g_{(2)}h_{(2)})\\
&\quad= t(g_{(1)})t(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})t^{-1}(g_{(3)}h_{(3)})F^t(g_{(4)}h_{(4)})\\
&\quad= t(g_{(1)})t(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})F(g_{(3)}h_{(3)})t^{-1}(g_{(4)}h_{(4)})\\
&\quad= t(g_{(1)})g(h_{(1)})F(g_{(2)})F(h_{(2)})\theta(g_{(3)} \otimes h_{(3)})t^{-1}(g_{(4)}h_{(4)})\\
&\quad= F^t(g_{(1)})F^t(h_{(1)})t(g_{(2)})t(h_{(2)})\theta(g_{(3)} \otimes h_{(3)})t^{-1}(g_{(4)}h_{(4)})\\
&\quad= F^t(g_{(1)})F^t(h_{(1)})\theta^t(g_{(2)} \otimes h_{(2)})
\end{aligned}$$

for $f, g, h \in H$. □

Lemma 6.2.7. *Let K and Q be bialgebras, $((H,\phi),\iota,j)$ and $((H',\phi'),\iota',j')$ two generalized product coquasibialgebras of K and Q, and $(F,\theta)\colon H\to H'$ a coquasi-homomorphism. Then there is a homologous coquasimorphism (F',θ') such that $\theta'(\iota\otimes j)=\varepsilon$, $\theta'(\iota\otimes\iota)=\theta(\iota\otimes\iota)$ and $\theta'(j\otimes j)=\theta(j\otimes j)$.*

If moreover $\phi(\iota\otimes\mathrm{id}_{H\otimes H})$, $\phi(\mathrm{id}_{H\otimes H}\otimes j)$, $\phi'(\iota'\otimes\mathrm{id}_{H'\otimes H'})$ and $\phi'(\mathrm{id}_{H'\otimes H'}\otimes j')$ are trivial, then $\theta'(\iota(x)\otimes\iota(y)j(q))=\theta(\iota(x)\otimes\iota(y))\varepsilon(q)$ and $\theta'(\iota(x)j(p)\otimes j(q))=\varepsilon(x)\theta(j(p)\otimes j(q))$ hold for all $x,y\in K$ and $p,q\in Q$.

Proof. We treat all the maps ι,ι',j, and j' as inclusions. Define $t\colon H\to k$ by $t(x\otimes q)=\theta(xq)$. Then $F^t(xq)=xq$ for $x\in K$ and $q\in Q$ by the calculation $t(x_{(1)}q_{(1)})F(x_{(2)}q_{(2)})=\theta(x_{(1)}\otimes q_{(1)})F(x_{(2)}q_{(2)})=F(x_{(1)})F(q_{(1)})\theta(x_{(2)}\otimes q_{(2)})=x_{(1)}q_{(1)}t(x_{(2)}q_{(2)})=$. By construction

$$\theta^t(x\otimes q)=t(x_{(1)})t(q_{(1)})\theta(x_{(2)}\otimes q_{(2)})t^{-1}(x_{(3)}q_{(3)})=\varepsilon(xq),$$

$\theta^t(x\otimes y)=\theta(x\otimes y)$ and $\theta^t(p\otimes q)=\theta(p\otimes q)$ for $x,y\in K$ and $p,q\in Q$.

Now assume the additional hypotheses on ϕ and ϕ'. Then

$$\begin{aligned}&\theta^t(x\otimes yq)=\phi(x_{(1)}\otimes y_{(1)}\otimes q_{(1)})\theta^t(y_{(2)}\otimes q_{(2)})\theta^t(x_{(2)}\otimes y_{(3)}q_{(3)})\\&=\theta^t(x_{(1)}\otimes y_{(1)})\theta^t(x_{(2)}y_{(2)}\otimes q_{(1)})\phi(x_{(3)}\otimes y_{(3)}\otimes q_{(2)})=\theta^t(x\otimes y)\varepsilon(q)=\theta(x\otimes y)\varepsilon(q)\end{aligned}$$

and similarly $\theta^t(xp\otimes q)=\varepsilon(x)\theta(p\otimes q)$ for $x,y\in K$ and $p,q\in Q$. □

We close the section by a list of (well-known or obvious) definitions of various 'cohomology' sets or 'cohomology' groups that generalize Sweedler cohomology of low degrees in various ways to the non-cocommutative case.

Definition 6.2.8. Let H be a bialgebra. We denote by $\mathcal{H}^2(H)$ the set of homology classes of normalized two-cocycles on H, where two cocycles $\tau,\sigma\colon H\otimes H\to k$ are homologous iff there is a convolution invertible $t\colon H\to k$ with $\sigma(g_{(1)}\otimes h_{(1)})t(g_{(2)}h_{(2)})=t(g_{(1)})t(h_{(1)})\tau(g_{(2)}\otimes h_{(2)})$. If this is the case, then $H^\sigma\ni h\mapsto t(h_{(1)})h_{(2)}t^{-1}(h_{(3)})\in H^\tau$ is a bialgebra isomorphism. If H is cocommutative, then $\mathcal{H}^2(H)$ coincides with degree two Sweedler cohomology of H with values in k.

If H is a coquasibialgebra, then we denote by $\mathcal{H}^2_c(H)$ the set of homology classes (in the sense of Definition 6.2.5) of self-twists of H, where a self-twist is a map $\theta\colon H\otimes H\to k$ such that $(\mathrm{id},\theta)\colon H\to H$ is a coquasi(iso)morphism. Clearly $\mathcal{H}^2_c(H)$ has a natural group structure. If H is a bialgebra, then a self-twist is a two-cocycle τ which is central in the sense that $\tau(g_{(1)}\otimes h_{(1)})g_{(2)}h_{(2)}=g_{(1)}h_{(1)}\tau(g_{(2)}\otimes h_{(2)})$ holds for all $g,h\in H$. Two central two-cocycles that are homologous as two-cocycles need not be homologous as self-twists: In fact if θ,θ' are homologous self-twists, then the form t as in Definition 6.2.5 is central in the sense that $t(h_{(1)})h_{(2)}=h_{(1)}t(h_{(2)})$ holds for all $h\in H$. Note that $\mathcal{H}^2_c(H)$ is the quotient of the group of all self-twists by the central subgroup consisting of all self-twists θ of the form $\theta(g\otimes h)=t^{-1}(g_{(1)}h_{(1)})t(g_{(2)})t(h_{(2)})$ for some invertible $t\colon H\to$

k that is central in the above sense. If the coquasibialgebra H is cocommutative, then $\mathcal{H}^2_c(H)$ coincides with degree two Sweedler cohomology of the underlying ordinary bialgebra H with values in k.

For any bialgebra H we let $\mathcal{H}^1(H)$ denote the group of all convolution invertible algebra maps from H to k. We let $\mathcal{H}^1_c(H)$ denote the (abelian) subgroup of all convolution invertible algebra maps $t\colon H \to k$ that are central in the sense that $t(h_{(1)})h_{(2)} = h_{(1)}t(h_{(2)})$; in the case where H is finite, these are the groups of (central) invertible grouplike elements in H^*. The notation is meant to remind of the fact that if H is cocommutative, then both $\mathcal{H}^1(H)$ and $\mathcal{H}^1_c(H)$ coincide with degree one Sweedler cohomology of H with values in k.

6.3 Part two of a generalized Kac sequence

This section will be devoted to a Hopf algebraic version of the portion

$$\mathcal{H}^2(Q \bowtie K) \overset{\text{res}}{\to} \mathcal{H}^2(Q) \oplus \mathcal{H}^2(K) \to \mathrm{Opext}(Q, K^*) \to \mathcal{H}^3(Q \bowtie K) \overset{\text{res}}{\to} \mathcal{H}^3(Q) \oplus \mathcal{H}^3(K)$$

of the Kac sequence; in the original Kac sequence the Q as well as K are group algebras of finite groups over the complex numbers, and Hopf algebras are what are now called Kac algebras. Masuoka [22] treats the case of a general base field, and ordinary Hopf algebras.

To make things work in the non-cocommutative case, we will have to replace Opext by the set of all isomorphism classes of extensions (since the operations are no longer fixed by the isomorphism class of the extension) and the kernel of the rightmost map by $\mathrm{Gpc}(Q, K)$, which is a reasonable generalization by Proposition 6.2.3, since generalized product bialgebras are given (up to isomorphism) by double crossproducts, see [21]. Finally there are not as many group structures in cohomology any more in the non-cocommutative case, so the exactness statements will have to be suitably interpreted to still make sense.

Lemma 6.3.1. *Let $K \overset{\iota}{\to} H \overset{\nu}{\to} Q$ be an exact sequence of Hopf algebras.*

(1) Let $t \in K \otimes K$ be a dual two-cocycle. Then $K_t \overset{\iota}{\to} H_t \overset{\nu}{\to} Q$ is an exact sequence.

(2) Let $\tau \in (Q \otimes Q)^*$ be a two-cocycle. Then $K \overset{\iota}{\to} H^\tau \overset{\nu}{\to} Q^\tau$ is an exact sequence.

Proof. Cotwisting by the cocycle $\tau\colon H \otimes H \to Q \otimes Q \to k$ affects neither the algebra structure of K nor the K-module structure of H. Hence, ν still induces an isomorphism $H^\tau/K^+H^\tau \cong Q^\tau$, and H^τ is still a cleft K-module coalgebra. This proves (2), and the proof of (1) is dual. $\square$

Lemma 6.3.2. *Let K and Q be Hopf algebras with K finite. An action of $\mathcal{H}^2_c(K) \times \mathcal{H}^2_c(Q)$ on $\mathrm{Ext}(Q, K^*)$ is given by*

$$\mathcal{H}^2_c(Q) \times \mathcal{H}^2_c(K) \times \mathrm{Ext}(Q, K^*) \ni (\tau, t, H) \mapsto H^\tau_t \in \mathrm{Ext}(Q, K^*).$$

Proof. Let $K^* \to H \to Q$ be an extension.

Let $\tau\colon Q \otimes Q \to k$ be a central two-cocycle. By the preceding Lemma, $K^* \to H^\tau \to Q$ is in fact an extension (since $Q^\tau = Q$ by the centrality condition). Now assume that we have a convolution invertible map $a\colon Q \to k$ that is central in the sense that $a(q_{(1)})q_{(2)} = q_{(1)}a(q_{(2)})$ holds for $q \in Q$, and that satisfies $\tau(p_{(1)} \otimes q_{(1)})a(p_{(2)}q_{(2)}) = a(p)a(q)$ for all $p, q \in Q$. Then $H^\tau \ni h \mapsto a\nu(h_{(1)})h_{(2)}a^{-1}\nu(h_{(2)}) \in H$ is an isomorphism of extensions (the restriction to K is trivial, and it induces the identity on Q by the centrality condition).

This shows that cotwisting gives a well-defined action of $\mathcal{H}^2_c(Q)$ on $\operatorname{Ext}(Q, K^*)$. The proof that we have a well-defined action of $\mathcal{H}^2_c(K)$ by twisting is dual. Since it is trivial to check that the actions commute, we are done. $\square$

Remark 6.3.3. The action of $\mathcal{H}^2_c(Q)$ on $H \in \operatorname{Ext}(Q, K^*)$ can easily be sent through the functor $\mathfrak{F}$, since the two-cocycle on H obtained by pulling back a cocycle τ on Q is a map in $\mathfrak{E}_\ell(K^*)$. By a look at Proposition 4.2.2 we find that there exists a cotwist $\hat{\tau}$ on $\mathfrak{F}(H)$ such that $\hat{\tau}|_{\tilde{H}\otimes K^*} = \varepsilon$, $\hat{\tau}|_{Q\otimes Q} = \tau$, and $\mathfrak{F}(H^\tau) = \mathfrak{F}(H)^{\hat{\tau}}$.

Lemma 6.3.4. *Let K be a finite Hopf algebra, $H \in \mathfrak{E}^{\mathrm{ext}}_\ell(K)$, and $t \in K \otimes K$ a dual two-cocycle. Identify t with a two-cocycle on K^*. Then there is a cotwist $\hat{t}\colon \mathfrak{F}(H) \otimes \mathfrak{F}(H) \to k$ with $\hat{t}|_{K^*\otimes K^*} = t$, $\hat{t}|_{Q\otimes\tilde{H}} = \varepsilon$, and*

$$\mathfrak{F}(H_t) = \mathfrak{F}(H)^{\hat{t}} \in \mathfrak{E}_r(K^{*t}).$$

Proof. A monoidal equivalence $\mathcal{T}\colon {}^H_K\mathcal{M}_K \to {}^{H_t}_{K_t}\mathcal{M}_{K_t}$ is described as follows: $\mathcal{T}(M) = M$ as a K-bimodule, with H_t-coaction $m \mapsto t\lambda(m)t^{-1}$, where λ is the H-coaction on M. The structure of $\mathcal{T}$ as a monoidal functor is trivial. $\mathcal{T}$ induces a monoidal category equivalence $(\tilde{\mathcal{T}}, \zeta)\colon {}^Q\mathcal{M}_K \to {}^Q\mathcal{M}_{K_t}$, which is the identical functor, with the monoidal functor structure determined by commutativity of the diagrams

$$\begin{tikzcd}
\overline{M} \otimes \overline{N} \arrow[rr, "\zeta"] \arrow[dr, "\xi_t"'] & & \overline{M} \otimes \overline{N} \arrow[dl, "\xi"] \\
& \overline{M \underset{K}{\otimes} N} &
\end{tikzcd}$$

where ξ_t is the monoidal functor structure on $\hat{\mathcal{V}}\colon {}^{H_t}_{K_t}\mathcal{M}_{K_t} \to {}^Q\mathcal{M}_{K_t}$. Writing $t = t^{(1)} \otimes t^{(2)}$ and $t^{-1} = \overline{t}^{(1)} \otimes \overline{t}^{(2)}$ we compute

$$\begin{aligned}
\zeta^{-1}(\overline{m} \otimes \overline{n}) = \xi_t^{-1}\xi(\overline{m} \otimes v) = \xi_t^{-1}\overline{(m \otimes j(v))}& \\
= \overline{m}\pi(t^{(1)}j(v)_{(-1)}\overline{t}^{(1)}) \otimes \overline{t^{(2)}j(v)_{(0)}\overline{t}^{(2)}} = \overline{m}\pi(v_{(-1)}\overline{t}^{(1)}) \otimes v_{(0)}\overline{t}^{(2)}& \\
= \overline{m}(v_{(-1)} \rightharpoonup \overline{t}^{(1)}) \otimes v_{(0)}\overline{t}^{(2)}&
\end{aligned}$$

for $m \in M$ and $v \in \overline{N}$, $M, N \in {}^H_K\mathcal{M}_K$. Identifying ${}^Q\mathcal{M}_K = {}^{\mathfrak{F}(H)}\mathcal{M}$ and ${}^Q\mathcal{M}_{K_t} = {}^{\mathfrak{F}(H_t)}\mathcal{M}$, we find that there is a coquasiisomorphism $(G, \hat{t})\colon \mathfrak{F}(H) \to \mathfrak{F}(H_t)$ in which G is the identity, and $\hat{t}^{-1}(p \otimes \varphi \otimes q \otimes \psi) = \varepsilon(p)\langle\varphi|q \rightharpoonup \overline{t}^{(1)}\rangle\langle\psi|\overline{t}^{(2)}\rangle$, proving the claim. □

Theorem 6.3.5. *Let K be a finite Hopf algebra, Q a Hopf algebra, and $K^* \to H \to Q$ a cleft extension. The stabilizer of the class of H under the coaction of $\mathcal{H}^2_c(Q) \times \mathcal{H}^2_c(K)$ on $\operatorname{Ext}(Q, K^*)$ is the image of the map*

$$\mathcal{H}^2_c(\mathfrak{F}(H)) \to \mathcal{H}^2_c(Q) \times \mathcal{H}^2_c(K)$$

induced by the restrictions.

Proof. Put $\mathfrak{F}(H) =: \tilde{H}$ and consider $\tau \in \mathcal{H}_c(Q)$ and $t \in \mathcal{H}_c(K)$. Combining Remark 6.3.3 and Lemma 6.3.4 we know that there is a cotwist $(\mathrm{id}, \kappa)\colon \tilde{H} \to \mathfrak{F}(H^\tau_t)$ such that $\kappa|_{K\otimes K} = t$, $\kappa|_{Q\otimes Q} = \tau$, and $\kappa|_{Q\otimes K} = \varepsilon$.

The class of H is stable under the action of (τ, t) if and only if there is an isomorphism $(F, \theta)\colon H^\tau_t \to H$ of extensions, if and only if there is a cotwist $(\mathrm{id}, \tilde{\theta})\colon \mathfrak{F}(H^\tau_t) \to \tilde{H}$ which is a morphism in $\mathfrak{E}_r(K)$ satisfying $\tilde{\theta}|_{Q\otimes Q} = \varepsilon$. Now such cotwists are, through the equation $\tilde{\theta} * \kappa = \chi$, in bijection with cotwists $(\mathrm{id}, \chi)\colon \tilde{H} \to \tilde{H}$ satisfying $\chi|_{K\otimes K} = t$, $\chi|_{Q\otimes Q} = \tau$, and $\chi|_{Q\otimes K} = \varepsilon$. Such a cotwist is an inverse image in $\mathcal{H}^2_c(\mathfrak{F}(H))$ of $(\tau, t) \in \mathcal{H}^2_c(Q) \times \mathcal{H}^2_c(K)$. Conversely, if there is any cotwist $(\mathrm{id}, \chi')\colon \tilde{H} \to \tilde{H}$ with $\chi'|_{K\otimes K} = t$ and $\chi'|_{Q\otimes Q} = \tau$, then a cotwist χ satisfying these equations and in addition $\chi|_{Q\otimes K} = \varepsilon$ exists by Lemma 6.2.7. □

Theorem 6.3.6. *Let K be a finite Hopf algebra and Q a Hopf algebra.*

The object map of the functor $\mathfrak{F}\colon \mathfrak{E}_\ell(K^*) \to \mathfrak{E}_r(K)$ induces a bijection

$$[\mathfrak{F}]\colon \operatorname{Ext}(K^*, Q)/\mathcal{H}^2_c(K) \times \mathcal{H}^2_c(Q) \to \operatorname{Gpc}(Q, K).$$

The inverse of $[\mathfrak{F}]$ is induced by $\mathfrak{F}^{\mathrm{bop}}$.

Proof. That the map $[\mathfrak{F}]$ under consideration is well-defined is clear from the descriptions of $\mathfrak{F}(H^\tau)$ and $\mathfrak{F}(H_t)$ obtained in Remark 6.3.3 and Lemma 6.3.4.

By Proposition 6.2.4 every element of $\operatorname{Gpc}(Q, K)$ has a representative (H, j, ι) such that $\phi(j \otimes \mathrm{id}_{H\otimes H})$ and $\phi(\mathrm{id}_{H\otimes H} \otimes \iota)$ are trivial. Then (H, j, ι) is an object of $\mathfrak{E}^{\mathrm{gpc}}_r(()K)$ and strongly isomorphic to $\mathfrak{F}(\mathfrak{F}^{\mathrm{bop}}(H, j, \iota))$, showing that $[\mathfrak{F}]$ is surjective, with a section induced by $\mathfrak{F}^{\mathrm{bop}}$.

Now let $H, H' \in \mathfrak{E}^{\mathrm{ext}}_\ell(K^*)$ such that $\tilde{H} := \mathfrak{F}(H)$ and $\tilde{H}' := \mathfrak{F}(H')$ are homologous generalized products. Let $(F, \theta)\colon \tilde{H} \to \tilde{H}'$ be a coquasiisomorphism with $F\tilde{\iota} = \tilde{\iota}'$ and $F\tilde{j} = \tilde{j}'$. Then the restriction $t := \theta|_{K\otimes K}$ is a central two-cocycle, which we consider as a dual central two-cocycle $t \in K^* \otimes K^*$. The restriction $\tau := \theta|_{Q\otimes Q}$ is also a central two-cocycle. Combining Lemma 6.3.4 and Remark 6.3.3 we can factor

$$(F, \theta) = \left(\tilde{H} \xrightarrow{(\mathrm{id},\chi)} \mathfrak{F}(H^\tau_t) \xrightarrow{(F',\theta')} \tilde{H}'\right)$$

where the cotwist χ satisfies $\chi|_{K\otimes K} = t = \theta|_{K\otimes K}$, and $\chi|_{Q\otimes Q} = \tau$, so that $\theta'|_{K\otimes K} = \varepsilon$ and $\theta'|_{Q\otimes Q} = \varepsilon$.

Thus, replacing H by another element of its orbit under the action of $\mathcal{H}^2_c(Q)\times\mathcal{H}^2_c(K)$, we may assume that $(F,\theta)\colon H \to H'$ satisfies $\theta|_{K\otimes K} = \varepsilon$ and $\theta|_{Q\otimes Q} = \varepsilon$. By Lemma 6.2.7 we may further assume $\theta|_{\tilde{H}\otimes K} = \varepsilon$, so that (F,θ) is an isomorphism in the category $\mathfrak{C}^{\mathrm{gpc}}_r(K)$. Thus, $(F,\theta) = \mathfrak{F}(G)$ for some isomorphism $G\colon H \to H'$ of extensions. $\square$

6.4 Comparison with the original

Suppose that K is a commutative finite Hopf algebra, and that Q is a cocommutative k-flat Hopf algebra. It is not hard to specialize the results obtained above to find an exact sequence

$$\mathcal{H}^2(Q\bowtie K) \overset{\mathrm{res}}{\to} \mathcal{H}^2(Q)\oplus\mathcal{H}^2(K) \to \mathrm{Opext}(Q,K^*) \overset{[\mathfrak{F}]}{\to} \mathcal{H}^3(Q\bowtie K) \overset{\mathrm{res}}{\to} \mathcal{H}^3(Q)\oplus\mathcal{H}^3(K)$$

for each matched pair containing Q and K. (We should note, though, that we never considered the group structure on Opext or $\mathcal{H}^3$, so that we have not shown that we have an exact sequence of groups; thus, we let 'exact' stand for what we proved in detail above).

Surely the terms of the sequence and the restriction maps look the same as for the classical Kac sequence, but are the sequences the same in the special case considered by Kac?

We will only show how to check this for the map $\mathrm{Opext}(Q,K^*) \to \mathcal{H}^3(Q\bowtie K)$, which, in Kac' case, is defined using a nonstandard resolution to compute group cohomology of a double crossproduct of groups. Note that this technique is not available for Sweedler cohomology.

For the remainder of this section we put ourselves in the situation of the appendix to [22], without repeating all the notations and definitions there. We depart from the notations in [22] by writing $L := F\bowtie G$.

We let $K = kG$, so that $K^* = k^G$, and $Q = kF$.

The map $\kappa\colon \mathrm{Opext}(Q,K^*) \to \mathcal{H}^3(Q\bowtie K, k^\cdot)$ in Kac' sequence is given by commutativity of

$$\begin{array}{ccc}
\mathcal{H}^3(\hat{\mathbb{U}}) & \xrightarrow{I} & \mathcal{H}^3(\hat{\mathbb{T}}) \\
{\scriptstyle P}\downarrow & & \downarrow{\scriptstyle c} \\
\mathrm{Opext}(Q,K^*) & \xrightarrow[\kappa]{} & \mathcal{H}^3(G\bowtie F, k^\cdot)
\end{array}$$

The right hand vertical arrow just identifies group cohomology of $F\bowtie G$ with the cohomology of the special resolution $\mathbb{T}$ of the trivial $F\bowtie G$-module $\mathbb{Z}$. The left hand vertical arrow P takes the class of a pair $(\beta,\tau)\in C^{21}\oplus C^{12}$ to the class of the bicrossproduct $K^{*\alpha}\#_\sigma Q$, where $\langle \alpha(f)|g\otimes g'\rangle = \beta(g,g',f)$ and $\langle \sigma(f\otimes f')|g\rangle = \tau(g,f,f')$ for $f,f'\in F$ and $g,g'\in G$. The top arrow is dual to the projection from the complex $\mathbb{T}$ onto its factor complex $\mathbb{U}$. We identify $\mathrm{Hom}_{\mathbb{Z}L}(B_n\otimes B'_m, k^\cdot)$ with the set $\tilde{C}^{nm}$ of maps $G^{(n)}\times F^{(m)}\to k^\cdot$ taking the value 1 whenever one of the arguments is the neutral element, extending the definition of C^{nm}. Then I is just the inclusion of $\mathbb{C}$ into $\tilde{\mathbb{C}}$.

We need to compare the map κ with the map $[\mathfrak{F}]$. Let $\mathbb{B}''$ denote the normalized bar resolution of $\mathbb{Z}$ by right L-modules. Consider the diagram

$$\begin{tikzcd}
\mathcal{H}^3(\hat{\mathbb{U}}) \arrow[r, "J"] \arrow[d, "P"'] & \mathcal{H}^3(\hat{\mathbb{B}}'') \arrow[d, "{[\mathfrak{F}]}"] \\
\mathrm{Opext}(Q,K^*) \arrow[r, "d"'] & \mathcal{H}^3(Q\bowtie K)
\end{tikzcd}$$

in which the right vertical arrow identifies the homology of the normalized bar resolution with Sweedler cohomology. From our general formulas for $\mathfrak{F}$ we read off how J acts on a representative $(\beta,\tau)\in C^{21}\oplus C^{12}$, with α,σ defined as above.

$$\begin{aligned}
&J(\beta,\tau)([f\bowtie g|f'\bowtie g'|f''\bowtie g''])\\
&\qquad = \langle (f'\rightharpoonup \alpha^{(1)}(f''))\sigma(f'\otimes f''_{[0]})|g\rangle\langle \alpha^{(2)}(f'')f''_{[1]}|g'\rangle\\
&\qquad\quad = \langle \alpha(f'')|g\triangleleft f'\otimes g'\rangle\langle \sigma(f'\otimes g'\triangleright f'')|g\rangle\\
&\qquad\qquad = \beta(g\triangleleft f',g',f'')\tau(g,f',g'\triangleright f'').
\end{aligned}$$

Now to compare $[\mathfrak{F}]$ with κ (up to the identification of Sweedler cohomology of a group algebra with group cohomology), we need to see if

$$\begin{tikzcd}
 & \mathcal{H}^3(\hat{\mathbb{B}}'') \arrow[dd, "\mathcal{H}^3(\hat{\gamma})"] \\
\mathcal{H}^3(\hat{\mathbb{U}}) \arrow[ur, "J"] \arrow[dr, "I"'] & \\
 & \mathcal{H}^3(\hat{\mathbb{T}})
\end{tikzcd}$$

commutes for some (hence any) chain map $\gamma\colon \mathbb{T}\to\mathbb{B}''$ lifting the identity on $\mathbb{Z}$. In fact we only need to construct a suitable chain map up to degree three. It will turn out that the following chain map, defined in terms of its components

$\gamma_{nm}: \mathbb{B}_n \otimes \mathbb{B}'_m \to \mathbb{B}''_{n+m}$ is convenient:

$$\begin{aligned}
\gamma_{00}([] \otimes []) &= [] \\
\gamma_{10}([s_1] \otimes []) &= [1 \bowtie s_1] \\
\gamma_{01}([] \otimes [x_1]) &= -[x_1 \bowtie 1](x_1^{-1} \bowtie 1) \\
\gamma_{20}([s_1|s_2] \otimes []) &= [1 \bowtie s_1|1 \bowtie s_2] \\
\gamma_{11}([s_1] \otimes [x_1]) &= [1 \bowtie s_1|x_1 \bowtie 1](x_1^{-1} \bowtie 1) \\
&\quad - [s_1 \triangleright x_1 \bowtie 1|1 \bowtie s_1 \triangleleft x_1](x_1^{-1} \bowtie 1) \\
\gamma_{02}([] \otimes [x_1|x_2]) &= -[x_1 \bowtie 1|x_2 \bowtie 1](x_1x_2 \bowtie 1)^{-1} \\
\gamma_{30}([s_1|s_2|s_3] \otimes []) &= [1 \bowtie s_1|1 \bowtie s_2|1 \bowtie s_3] \\
\gamma_{21}([s_1|s_2] \otimes [x_1]) &= -[1 \bowtie s_1|1 \bowtie s_2|x_1 \bowtie 1](x_1 \bowtie 1)^{-1} \\
&\quad + [1 \bowtie s_1|s_2 \triangleright x_1 \bowtie 1|1 \bowtie s_2 \triangleleft x_1](x_1 \bowtie 1)^{-1} \\
&\quad - [s_1s_2 \triangleright x_1 \bowtie 1|1 \bowtie s_1 \triangleleft (s_2 \triangleright x_1)|1 \bowtie s_2 \triangleleft x_1](x_1 \bowtie 1)^{-1} \\
\gamma_{12}([s_1] \otimes [x_1|x_2]) &= -[1 \bowtie s_1|x_1 \bowtie 1|x_2 \bowtie 1](x_1x_2 \bowtie 1)^{-1} \\
&\quad + [s_1 \triangleright x_1 \bowtie 1|1 \bowtie s_1 \triangleleft x_1|x_2 \bowtie 1](x_1x_2 \bowtie 1)^{-1} \\
&\quad - [s_1 \triangleright x_1 \bowtie 1|(s_1 \triangleleft x_1) \triangleright x_2 \bowtie 1|1 \bowtie s_1 \triangleleft x_1x_2](x_1x_2 \bowtie 1)^{-1} \\
\gamma_{03}([] \otimes [x_1|x_2|x_3]) &= -[x_1 \bowtie 1|x_2 \bowtie 1|x_3 \bowtie 1](x_1x_2x_3 \bowtie 1)^{-1}
\end{aligned}$$

It is tedious to verify that these equations really define a chain map, and we will omit the calculations. Assuming that γ is a chain map, it turns out that κ and $[\mathfrak{F}]$ coincide up to a sign: We first observe that $J(\beta, \tau)$ is trivial on the images of γ_{30} and γ_{03}. Moreover, $J(\beta, \tau)$ is trivial on all but the first summands of the expressions for γ_{21} and γ_{12}, and thus

$$\begin{aligned}
\hat{\gamma}_{21}J(\beta, \tau)(s_1, s_2, x_1) &= J(\beta, \tau)\gamma_{21}([s_1|s_2] \otimes [x_1]) \\
&= J(\beta, \tau)(-[1 \bowtie s_1|1 \bowtie s_2|x_1 \bowtie 1](x_1 \bowtie 1)^{-1}) \\
&= J(\beta, \tau)([1 \bowtie s_1|1 \bowtie s_2|x_1 \bowtie 1])^{-1} = \beta(s_1, s_2, x_1)^{-1}
\end{aligned}$$

and similarly $\hat{\gamma}_{12}J(\beta, \tau)(s_1, x_1, x_2) = \tau(s_1, x_1, x_2)^{-1}$.

6.5 Part one of a generalized Kac sequence

In this section we will deal with a Hopf algebraic version of the portion

$$\mathcal{H}^1(Q \bowtie K) \overset{\text{res}}{\to} \mathcal{H}^1(Q) \oplus \mathcal{H}^1(K) \to \operatorname{Aut}(K^*\#Q) \to \mathcal{H}^2(Q \bowtie K) \overset{\text{res}}{\to} \mathcal{H}^2(Q) \oplus \mathcal{H}^2(K)$$

of the Kac sequence. This time there will be two versions of such a sequence. The first is an exact sequence of groups, with certain commutativity restrictions on the cocycles in the terms of the sequence. In the second version we will use larger cohomology sets, which, however, will carry less group structures, so that we will have to prove suitable interpretations of exactness, much like for the higher terms in the Kac sequence which we treated earlier.

Let H be a Hopf algebra. Then for a grouplike element $g \in H$ and an algebra map $u\colon H \to k$ we denote by $\mathrm{inn}(g)\colon H \ni h \mapsto ghg^{-1} \in H$ and $\mathrm{coinn}(u)\colon H \ni h \mapsto u^{-1}(h_{(1)})h_{(2)}u(h_{(3)})$ the inner (resp. coinner) automorphism of H induced by g (resp. u). We have $\mathrm{inn}(gg') = \mathrm{inn}(g)\,\mathrm{inn}(g')$ and $\mathrm{coinn}(u*u') = \mathrm{coinn}(u)\,\mathrm{coinn}(u')$ for grouplikes $g, g' \in H$ and algebra maps $u, u'\colon H \to k$.

Let K be a finite Hopf algebra and $K^* \xrightarrow{\iota} H \xrightarrow{\nu} Q$ an extension. It is trivial to see that for each central grouplike element $g \in K^*$ the automorphism $\mathrm{inn}(g) = \mathrm{inn}(\iota(g))$ of H is an automorphism of the extension H, and for any algebra map $u\colon Q \to k$ satisfying $u(q_{(1)})q_{(2)} = q_{(1)}u(q_{(2)})$ for all $q \in Q$, the automorphism $\mathrm{coinn}(u) := \mathrm{coinn}(u\nu)$ of H is an automorphism of the extension H. Since the automorphisms $\mathrm{coinn}(u)$ and $\mathrm{inn}(g)$ commute, we have defined a group homomorphism $\mathcal{H}^1_c(Q) \oplus \mathcal{H}^1_c(K) \to \mathrm{Aut}_{\mathrm{ext}}(H)$.

Theorem 6.5.1. *Let K be a finite Hopf algebra, Q a Hopf algebra, and $K^* \xrightarrow{\iota} H \xrightarrow{\nu} Q$ an extension. Choose a cleaving map π and consider $H \in \mathfrak{E}_\ell(K^*)$. Then we have an exact sequence*

$$\mathcal{H}^1_c(\mathfrak{F}(H)) \xrightarrow{\mathrm{res}} \mathcal{H}^1_c(Q) \oplus \mathcal{H}^1_c(K) \to \mathrm{Aut}_{\mathrm{ext}}(H) \xrightarrow{[\mathfrak{F}]} \mathcal{H}^2_c(\mathfrak{F}(H)) \xrightarrow{\mathrm{res}} \mathcal{H}^2_c(Q) \times \mathcal{H}^2_c(K)$$

of groups, in which $[\mathfrak{F}]$ is induced by the object map of the functor $\mathfrak{F}$.

Proof. Note first that $[\mathfrak{F}]$ is trivially a group homomorphism since $\mathfrak{F}$ is a functor.

We put $\tilde{H} = \mathfrak{F}(H)$ and consider the maps $\tilde{j}\colon Q \to \tilde{H}$ and $\tilde{\iota}\colon K \to \tilde{H}$ as inclusions.

Let θ be a self-twist of $\mathfrak{F}(H)$ such that $\theta|_{Q\otimes Q}$ and $\theta|_{K\otimes K}$ are homologous to the trivial self-twists. Let $u\colon Q \to k$ and $v\colon K \to k$ be convolution invertible and central with $(\theta|_{Q\otimes Q})^u = \varepsilon$ and $(\theta|_{K\otimes K})^v = \varepsilon$, and define $w\colon \mathfrak{F}(H) \to k$ by $w(qx) = u(q)v(x)$ for $q \in Q$ and $x \in K$. Then we find that $\theta^w|_{Q\otimes Q} = \varepsilon$ and $\theta^w|_{K\otimes K} = \varepsilon$. By Lemma 6.2.7 we can replace θ^w, hence θ, by a homologous self-twist θ of $\mathfrak{F}(H)$ such that $\theta|_{\mathfrak{F}(H)\otimes K}$ and $\theta|_{Q\otimes\mathfrak{F}(H)}$ are trivial. Such a self-twist is the image under $\mathfrak{F}$ of a (unique) automorphism $\mathfrak{F}$ of the extension H, showing that $[\mathfrak{F}]$ is surjective.

Now assume that $F \in \mathrm{Aut}_{\mathrm{ext}}(H)$ is in the kernel of $[\mathfrak{F}]$. Putting $\mathfrak{F}(F) = (\mathrm{id}, \theta)$, this means that there is a central convolution invertible $t\colon \mathfrak{F}(H) \to k$ with $\theta(\tilde{g} \otimes \tilde{h}) = t(\tilde{g}_{(1)})t(\tilde{h}_{(1)})t^{-1}(\tilde{g}_{(2)}\tilde{h}_{(2)})$ for all $\tilde{g}, \tilde{h} \in \mathfrak{F}(H)$. Since the restrictions of θ to $K \otimes K$ as well as $Q \otimes Q$ are trivial, the restrictions $u := t|_Q$ and $g := t|_K$ are algebra maps. Since, moreover, $\theta(px \otimes qy) = \varepsilon(y)\theta(px \otimes q) = \varepsilon(p)\varepsilon(y)\theta(x \otimes q)$ holds for all $p, q \in Q$ and $x, y \in K$, we have $t(py) = \theta(p_{(1)} \otimes y_{(1)})t(p_{(2)}y_{(2)}) = t(p)t(y) = u(p)g(y)$ for $p \in Q$ and $y \in K$, thus $\theta^{-1}(x \otimes q) = t(x_{(1)}q_{(1)})t^{-1}(x_{(2)})t^{-1}(q_{(2)}) = t(x_{(1)} \overset{\sim}{\rightharpoonup} q_{(1)})t(x_{(2)} \overset{\sim}{\leftharpoonup} q_{(2)})t^{-1}(x_{(3)})t^{-1}(q_{(3)}) = u(x_{(1)} \overset{\sim}{\rightharpoonup} q_{(1)})u^{-1}(x_{(2)})g(x_{(3)} \overset{\sim}{\leftharpoonup} q_{(2)})g^{-1}(q_{(3)})$ for all $x \in K$ and $q \in Q$. We claim that $(\mathrm{id}, \theta) = \mathfrak{F}(\mathrm{inn}(g^{-1})\,\mathrm{coinn}(u^{-1}))$, so that $F = \mathrm{inn}(g^{-1})\,\mathrm{coinn}(u^{-1})$. In fact, writing $\mathfrak{F}(\mathrm{inn}(g^{-1})) = (\mathrm{id}, \chi)$, we have $\chi^{-1}(x \otimes q) = \langle \pi(\mathrm{inn}(g^{-1})(j(q)))|x\rangle = \langle \pi(g^{-1}j(q)g)|x\rangle = \langle g^{-1}(q \rightharpoonup g)|x\rangle = g^{-1}(x_{(1)})g(x_{(2)} \mathbin{\tilde{\leftharpoonup}} q)$ and writing $\mathfrak{F}(\mathrm{coinn}(u^{-1})) = (\mathrm{id}, \kappa)$ we have $\kappa^{-1}(x \otimes q) = \langle \pi(\mathrm{coinn}(u^{-1})(j(q)))|x\rangle = \langle u(q_{(1)[0]})q_{(1)[1]}u^{-1}(q_{(2)})|x\rangle = u(x \mathbin{\tilde{\rightharpoonup}} q_{(1)})u^{-1}(q_{(2)})$.

Finally, let $g \in \mathcal{H}^1_c(K)$ and $u \in \mathcal{H}^1_c(Q)$. Then $\mathrm{inn}(g)\,\mathrm{coinn}(u)$ is the identity if and only if $\mathrm{inn}(g^{-1})\,\mathrm{coinn}(u^{-1})$ is, if and only if $\mathfrak{F}(\mathrm{inn}(g^{-1})\,\mathrm{coinn}(u^{-1}))$ is the trivial self-twist of $\mathfrak{F}(H)$. By the calculation above, this means that $t\colon \mathfrak{F}(H) \to k$, defined by $t(py) = u(p)g(y)$ for $p \in Q$ and $y \in K$, is an algebra map, that is, (u, g) is in the image of the homomorphism $\mathcal{H}^1_c(\mathfrak{F}(H)) \to \mathcal{H}^1_c(Q) \oplus \mathcal{H}^1_c(K)$ induced by the restrictions. $\square$

It turns out that to have a Kac-like sequence in the non-cocommutative case that involves the cohomology sets $\mathcal{H}^2$ instead of the restricted versions $\mathcal{H}^2_c$, one has to replace the automorphism group in the sequence by something different. In [15] it was shown that automorphisms of the bismash product associated with an abelian matched pair are in bijection with splittings of the bismash products in the sense we are about to recall. In the nonabelian case, however, this does not work, again because the operations in an extension of Hopf algebras are no longer determined by the extension.

Definition 6.5.2. Let $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$ be an extension of Hopf algebras. We call a pair $(\pi, j) \in \mathrm{Hom}(H, K) \times \mathrm{Hom}(Q, H)$ a splitting of H if $\pi\colon H \to K$ is a left K-module coalgebra map and $j\colon Q \to H$ is a right Q-comodule algebra map with $\pi(h_{(1)})j(\nu(h_{(2)}))$ for all $h \in H$.

We write $\mathrm{Split}(H)$ for the set of all splittings of H, and call the extension split if a splitting exists.

In other words, if (π, j) is a splitting, then π is a cleaving map which is also a coalgebra map, $j\colon Q \to H$ is the usual associated comodule algebra map, and we require in addition that j is an algebra map.

Read this way, there is an obvious redundancy in the definition, which, however, makes it self-dual.

The immediate example of a split extension is a bismash product $K\#Q$ associated to a matched pair $(\rightharpoonup, \rho, K, Q)$, where $\pi(x\#q) = x\varepsilon(q)$ and $j(q) = 1\#q$. On the other hand, if H is split, then it is canonically isomorphic to a bismash product. More precisely, the cleaving map induces an isomorphism $H \ni h \mapsto \pi(h_{(1)}) \otimes \nu(h_{(2)}) \in K \otimes Q$ with inverse $K \otimes Q \ni x \otimes q \mapsto xj(q) \in H$ as before, where this time the induced multiplication and comultiplication are given by $(x \otimes p)(y \otimes q) = x(p_{(1)} \rightharpoonup y) \otimes p_{(2)}q$ and $\Delta(x \otimes p) = x_{(1)} \otimes p_{(1)[0]} \otimes x_{(2)}p_{(1)[1]} \otimes p_{(2)}$, where $p \rightharpoonup x = \pi(j(p)x)$ and $p_{[0]} \otimes p_{[1]} = \nu(j(p)_{(1)}) \otimes \pi(j(p)_{(2)})$.

Although splittings are no longer in bijection with automorphisms in the nonabelian case, there is a close connection between splittings and isomorphisms to other split extensions. This will allow us to interpret a splitting as a morphism in $\mathfrak{E}_\ell(K^*)$ and apply $\mathfrak{F}$ to it.

Corollary 6.5.3. *Let K and Q be Hopf algebras and $K \to H \to Q$ an extension. Then we have a bijection*

$$\Gamma\colon \dot{\bigcup_{\substack{(\rightharpoonup,\rho,K,Q)\\ \textit{matched pair}}}} \operatorname{Hom}_{\mathrm{ext}}(H, K\#Q) \ni F \mapsto (\pi F, F^{-1}j) \in \operatorname{Split}(H)$$

In particular, we have a bijection

$$\operatorname{Aut}_{\mathrm{ext}}(K\#Q) \ni F \mapsto (\pi F, F^{-1}j) \in \operatorname{Split}(K\#Q)$$

whenever $(\rightharpoonup, \rho, K, Q)$ is an abelian matched pair, that is, a matched pair with K commutative and Q cocommutative.

Proof. To prove the first assertion, we first compute

$$\pi F(h_{(1)})F^{-1}j(\overline{h_{(2)}}) = F^{-1}(\pi(F(h_{(1)}))j(\overline{F(h_{(2)})}) = F^{-1}(\pi(F(h)_{(1)})j(\overline{F(h)_{(2)}}) = h$$

for $h \in H$. It remains to check that a homomorphism $F\colon H \to K\#Q$ of extensions is determined by πF. Now $F(h) = \pi(F(h)_{(1)})\#\overline{F(h)_{(2)}} = \pi(F(h_{(1)}))\#\overline{F(h_{(2)})} = \pi(F(h_{(1)}))\#\overline{h_{(2)}}$.

For the second assertion note that the matched pair $(\rightharpoonup, \rho, K, Q)$ is determined by the isomorphism class of the extension $K\#Q$ in the abelian case (see e. g. [16]). $\square$

Lemma 6.5.4. *Let K be a finite Hopf algebra, and $K^* \xrightarrow{\iota} H \xrightarrow{\nu} Q$ an extension of Hopf algebras. There are commuting actions*

$$\mathcal{H}^1(Q) \times \operatorname{Split}(H) \ni (u, \pi, j) \mapsto (\pi^u, j^u) \in \operatorname{Split}(H)$$
$$\mathcal{H}^1(K) \times \operatorname{Split}(H) \ni (g, \pi, j) \mapsto (\pi_g, j_g) \in \operatorname{Split}(H)$$

of $\mathcal{H}^1(Q)$ and $\mathcal{H}^1(K)$ on the set of all splittings of H satisfying $j_g = \operatorname{inn}(g)j$ and $\pi^u = \pi \operatorname{coinn}(u^{-1})$.

Proof. Note that each of π or j in a splitting (π, j) determines the other, so the actions, if they exist, are determined by the formulas given. To see that the action of $\mathcal{H}^1(K)$ is well-defined, note that $j_g := \operatorname{inn}(g)j$ is obviously a Q-comodule algebra map (since j is), and π_g, defined by $\pi_g(h) = \pi(hg)g^{-1}$ is a left module coalgebra map since π is, and g is grouplike. Now the calculation $\pi_g(h_{(1)})j_g(\overline{h_{(2)}}) = \pi_g(h_{(1)}g)g^{-1}gj(\overline{h_{(2)}})g^{-1} = \pi_g(h_{(1)})j(\overline{h_{(2)}g})g^{-1} = hgg^{-1} = h$ for $h \in H$ completes the proof that (π_g, j_g) is a splitting. The proof of existence for the action of $\mathcal{H}^1(Q)$ is dual. Since $(\pi^u)_g(h) = \pi^u(hg)g^{-1} = \pi(\operatorname{coinn}(u)(hg))g^{-1} = \pi(\operatorname{coinn}(u)(h)g)g^{-1} = \pi_g(\operatorname{coinn}(u)(h)) = (\pi_g)^u(h)$, the actions commute. $\square$

Lemma 6.5.5. *Let K and Q be k-flat Hopf algebras with K finite, and $K^* \xrightarrow{\iota} H \xrightarrow{\nu} Q$ a split extension. Fix a splitting (π_0, j_0) for H and consider $H = (H, \pi_0) \in \mathfrak{E}_\ell(K^*)$. Let $u \in \mathcal{H}^1(Q)$ and $g \in \mathcal{H}^1(K)$, and define $t\colon \mathfrak{F}(H) =: \tilde{H} \to k$ by $t(qx) = u(q)g(x)$ for $q \in Q$ and $x \in K$. Let (π, j) be a splitting of H and put $(\mathrm{id}, \theta) := \mathfrak{F}(\Gamma^{-1}(\pi, j))$. Then $\mathfrak{F}(\Gamma^{-1}(\pi^u_g, j^u_g)) = (\mathrm{id}, \theta^t)$.*

Proof. It suffices to treat the cases $u = \varepsilon$ and $g = \varepsilon$. We let $\rightharpoonup, \rho$ denote the matched pair associated to (π_0, j_0), and $\stackrel{\rightharpoonup}{-}, \stackrel{\leftharpoonup}{-}$ the corresponding crossed pair. Put $\mathfrak{F}(\Gamma^{-1}(\pi^u_g, j^u_g)) =: (\mathrm{id}, \chi)$. If $u = \varepsilon$, then

$$\begin{aligned}\chi^{-1}(x \otimes q) = \langle \pi_g(j_0(q))|x\rangle &= \langle \pi(j_0(q)g)g^{-1}|x\rangle = \langle \pi((q_{(1)} \rightharpoonup g)j_0(q_{(2)})g^{-1}|x\rangle \\ &= g(x \stackrel{\leftharpoonup}{-} q_{(1)})\langle \pi(j_0(q_{(2)}))|x_{(2)}\rangle g^{-1}(x_{(3)}) \\ &= t(x_{(1)}q_{(1)})\theta^{-1}(x_{(2)} \otimes q_{(2)})t^{-1}(x_{(3)})t^{-1}(q_{(3)}) = (\theta^t)^{-1}(x \otimes q),\end{aligned}$$

while if $g = \varepsilon$, then

$$\begin{aligned}\chi^{-1}(x \otimes q) &= \langle \pi^u(j_0(q))|x\rangle = \langle u\nu(j_0(q)_{(1)})\pi(j_0(q)_{(2)})u^{-1}\nu(j_0(q)_{(3)})|x\rangle \\ &= u(q_{(1)[0]})\langle q_{(1)[1]}\pi(j_0(q_{(2)}))|x\rangle u^{-1}(q_{(3)}) = u(x_{(1)} \stackrel{\rightharpoonup}{-} q_{(1)})\theta(x_{(2)} \otimes q_{(2)})u^{-1}(q_{(3)}) \\ &= t(x_{(1)}q_{(1)})\theta^{-1}(x_{(2)} \otimes q_{(2)})t^{-1}(x_{(3)})t^{-1}(q_{(3)}) = (\theta^t)^{-1}(x \otimes q)\end{aligned}$$

for $x \in K$ and $q \in Q$. □

Theorem 6.5.6. *Let K be a finite Hopf algebra, Q a Hopf algebra, and $K^* \xrightarrow{\iota} H \xrightarrow{\nu} Q$ a split extension. Then*

(1) The stabilizer of a splitting $(\pi, j) \in \mathrm{Split}(H)$ under the action of $\mathcal{H}^1(Q) \times \mathcal{H}^1(K)$ is the image of the restriction $\mathcal{H}^1(\mathfrak{F}(H, \pi)) \to \mathcal{H}^1(Q) \times \mathcal{H}^1(K)$.

(2) The morphism map of the functor $\mathfrak{F}$ induces a bijection

$$\mathrm{Split}(H)/\mathcal{H}^1(Q) \times \mathcal{H}^1(K) \xrightarrow{[\mathfrak{F}]} \mathrm{Ker}\left(\mathcal{H}^2(\mathfrak{F}(H)) \stackrel{\mathrm{res}}{\to} \mathcal{H}^2(Q) \times \mathcal{H}^2(K)\right),$$

where the kernel means the kernel of a map of pointed sets.

Proof. We prove (2) first. We fix a splitting (π_0, j_0) of H and treat the corresponding maps from K and Q to $\tilde{H} := \mathfrak{F}(H) := \mathfrak{F}(H, \pi_0)$ as inclusions. Let $\theta\colon \tilde{H} \otimes \tilde{H} \to k$ be a two-cocycle whose restrictions $\theta|_{K \otimes K}$ and $\theta|_{Q \otimes Q}$ are homologous to ε. Choose $u\colon Q \to k$ and $v\colon K \to k$ convolution invertible with $(\theta|_{K\otimes K})^v = \varepsilon$ and $(\theta|_{Q\otimes Q})^u = \varepsilon$. Defining $w\colon \tilde{H} \ni qx \mapsto u(q)g(x) \in k$, we find $\theta^w|_{K\otimes K} = \varepsilon$ and $\theta^w|_{Q\otimes Q} = \varepsilon$. Again replacing θ^w by a homologous cocycle, we can assume by Lemma 6.2.7 that $\theta(px \otimes qy) = \varepsilon(p)\theta(x \otimes q)\varepsilon(y)$ holds for all $p, q \in Q$ and $x, y \in K$. Now such cocycles are in bijection with isomorphisms of extensions $H \cong K^*\#Q$ with bismash products $K^*\#Q$ associated to some matched pair $(\rightharpoonup, \rho, K^*, Q)$, with the associated double crossproduct $Q \bowtie K = (\mathfrak{F}(H))^\theta$. We have shown that the map $[\mathfrak{F}]$ is onto.

Now assume that (π, j) and (π', j') are two splittings that have the same image under $[\mathfrak{F}]$. Put $(\mathrm{id}, \theta) := \mathfrak{F}(\Gamma^{-1}(\pi, j))$ and $(\mathrm{id}, \theta') := \mathfrak{F}(\Gamma^{-1}(\pi', j'))$. Then there is $t\colon \tilde{H} \to k$ convolution invertible with $t(1) = 1$ and $\theta' = \theta^t$. Since $\theta|_{K\otimes K} = \theta'|_{K\otimes K} = \varepsilon$, the restriction $g = t|_K$ is an algebra map. Similarly $u = t|_Q$ is an algebra map. Moreover, for $x \in K$ and $q \in Q$ we have $t(qx) = \theta(q_{(1)} \otimes x_{(1)})t(q_{(2)}x_{(2)}) = t(q_{(1)})t(x_{(1)})\theta(q_{(2)} \otimes x_{(2)}) = t(q)t(x)$. By Lemma 6.5.5 We have $\mathfrak{F}(\Gamma^{-1}(\pi', j')) = (\mathrm{id}, \theta^t) = \mathfrak{F}(\Gamma^{-1}(\pi^u_g, j^u_g))$, hence $(\pi', j') = (\pi^u_g, j^u_g)$. We have shown that $[\mathfrak{F}]$ is one-to-one.

Furthermore, if (π, j) is a splitting and $u \in \mathcal{H}^1(Q)$, $g \in \mathcal{H}^1(K)$ with $(\pi^u_g, j^u_g) = (\pi, j)$, then setting $t(qx) = u(q)g(x)$ for $q \in Q$ and $x \in K$, we find $\theta^t = \theta$ for $(\mathrm{id}, \theta) := \mathfrak{F}(\Gamma^{-1}(\pi, j))$. In other words $t(g)t(h) = \theta(g_{(1)} \otimes h_{(1)})t(g_{(2)}h_{(2)})\theta^{-1}(g_{(3)} \otimes h_{(3)})$, so that $t\colon \tilde{H}^\theta = \mathfrak{F}(H, \pi)\colon k$ is an algebra map, that is, (u, g) is the image of t under $\mathrm{res}\colon \mathcal{H}^1(\mathfrak{F}(H, \pi)) \to \mathcal{H}^1(Q) \times \mathcal{H}^1(K)$. $\square$

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