GALOIS OBJECTS OVER GENERALIZED DRINFELD DOUBLES WITH AN APPLICATION TO $u_q(\mathfrak{sl}_2)$

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ABSTRACT. Ralf Günther has determined all the cleft extensions over the finite quotient Hopf algebra $u_q(\mathfrak{sl}_2)$ of the quantized universal enveloping algebra of $\mathfrak{sl}_2$ at a root of unity [5]. His techniques (applications of the Diamond Lemma) are similar to those used by Masuoka [7] for the two-generator Taft algebras. In the present paper we will give another proof of a special case of Günther's classification, namely the case of (cleft) Galois extensions of the base field. The idea is that $u_q(\mathfrak{sl}_2)$ is the quotient of the Drinfeld double of a Taft algebra by a normal Hopf subalgebra. We will use techniques that allow to calculate all Galois objects of such a composed Hopf algebra.

1. INTRODUCTION

Throughout the paper k is a fixed commutative base ring. All algebras, tensor products etc. are over k. We denote comultiplications of Hopf algebras by Δ, counits by ε, right (resp. left) comodule structures by ρ (resp. λ). We make use of the Sweedler notation without summation symbols: $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $\rho(x) = x_{(0)} \otimes x_{(1)}$, $\lambda(x) = x_{(-1)} \otimes x_{(0)}$.

Let H be a k-flat Hopf algebra with bijective antipode. A right H-comodule algebra A is said to be a right H-Galois extension of $B = A^{\mathrm{co}\,H} := \{a \in A | \rho(a) = a \otimes 1\}$ if the (right) Galois map $\kappa : A \underset{B}{\otimes} A \ni x \otimes y \mapsto xy_{(0)} \otimes y_{(1)} \in A \otimes H$ is a bijection. We will denote by $\mathrm{Gal}_B(H)$ the set of all isomorphism classes of right H-Galois extensions of B which are faithfully flat over B. In the case that $B = k$ we say that A is an H-Galois object and write $\mathrm{Gal}(H) = \mathrm{Gal}_B(H)$. An H-comodule algebra A is said to be cleft if there is an H-colinear convolution invertible map $H \to A$. Any cleft comodule algebra is a Galois extension, and if H is finite dimensional over a field, any Galois object is cleft.

If k is a field and contains a primitive n-th root of unity ζ then the two-generator Taft algebra is defined to be

$$T = k\langle G, X\rangle \big/ (G^n - 1, X^n, XG - \zeta GX)$$

with comultiplication determined by $\Delta(G) = G \otimes G$ (so G is grouplike) and $\Delta(X) =$

$1\otimes X+X\otimes G$ (so X is $(1,G)$-primitive). Masuoka [7] has classified all cleft extensions over the two-generator Taft algebras. Doi and Takeuchi [4] have also classified all their Galois extensions. Now assume that $q\in k$ is such that $\zeta=q^2$ is a primitive n-th root of unity. Then the finite quotient $u_q(\mathfrak{sl}_2)$ of the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is

$$u_q(\mathfrak{sl}_2)\cong k\langle G,X,Y\rangle\Big/(G^n-1,X^n,Y^n,XG-q^2GX,YG-q^{-2}GY,YX-XY-\frac{G-G^{-1}}{q-q^{-1}})$$

with grouplike G, $(1,G)$-primitive X and $(G^{-1},1)$-primitive Y. Günther [5] has classified all cleft extensions of $u_q(\mathfrak{sl}_2)$. The Taft algebras are related to $u_q(\mathfrak{sl}_2)$ in the following way: The Borel-like subalgebra $u_q(\mathfrak{b}_+)=\langle G,X\rangle\subset u_q(\mathfrak{sl}_2)$ is isomorphic to the Taft algebra. The Drinfeld double of $u_q(\mathfrak{b}_+)$ is, roughly speaking, $u_q(\mathfrak{sl}_2)$ with two copies of the grouplike generators. More precisely $u_q(\mathfrak{sl}_2)$ is the quotient of the Drinfeld double of the two generator Taft algebra by a normal subalgebra. In the language of [3] we can write

$$u_q(\mathfrak{sl}_2)=(T\bowtie_\tau T')\Big/I$$

where $T'\cong T^{\mathrm{op}\,*}$, τ is the canonical skew pairing between T' and T, and I is the Hopf ideal generated by $G(G^*)^{-1}$, where G^* is a certain grouplike in the dual of the Taft algebra.

Now if $u_q(\mathfrak{sl}_2)$ can be built up in this way from the Taft algebra, there should be a way to build up $\mathrm{Gal}(u_q(\mathfrak{sl}_2))$, which has been determined by Günther, from $\mathrm{Gal}(T)$, which has been determined by Masuoka. This would be expected to be done in two steps: The first is to relate Galois objects over a twisted product $H_1\bowtie H_2$ to Galois objects over the factors, and the second to relate Galois objects over a Hopf algebra to Galois objects over a normal quotient Hopf algebra.

For each $A\in\mathrm{Gal}(H)$ there is, by [11], a unique up to isomorphism Hopf algebra $L:=L(A,H)$ and left L-comodule structure on A such that A is a left L-Galois object and an L-H-bicomodule algebra; we shall say that A is an L-H-biGalois object. Kreimer [6] has found a formula for the *group* of Galois objects of a tensor product of cocommutative Hopf algebras, namely

$$\mathrm{Gal}(H_1\otimes H_2)\cong\mathrm{Gal}(H_1)\oplus\mathrm{Gal}(H_2)\oplus\mathrm{Hopf}(H_1,H_2^*).$$

The isomorphism assigns to Galois objects A_i over H_i and a Hopf algebra map $H_1\to H_2^*$ a twisted tensor product of the A_i. A partial analog in the noncocommutative case is [9, Prop.12], which says that any $A\in\mathrm{Gal}(H_1\otimes H_2)$ can be obtained in a unique way as a tensor product of $A_i\in\mathrm{Gal}(H_i)$ twisted by a skew pairing between $L(A_2,H_2)$ and $L(A_1,H_1)$. It is easy to generalize this result to twisted products $H_1\bowtie_\tau H_2$, where τ is a skew pairing between H_2 and H_1; we will do this in section 2. For our application to $u_q(\mathfrak{sl}_2)$ it then remains to see how $\mathrm{Gal}(H)$ is related to $\mathrm{Gal}(\overline{H})$, where $\overline{H}$ is a normal quotient Hopf algebra of H; we shall deal with this problem in section 3. Every $\overline{A} \in \operatorname{Gal}(\overline{H})$ yields $\overline{A} \mathop{\Box}_{\overline{H}} H \in \operatorname{Gal}(H)$ by cotensor product. One has to check which $A \in \operatorname{Gal}(H)$ arise in this way, and how many different $\overline{A}$ may yield isomorphic $\overline{A} \mathop{\Box}_{\overline{H}} H$. The first question is answered by conditions on $A^{\operatorname{co}\overline{H}}$: This subcomodule algebra has to be invariant under the Miyashita-Ulbrich action of H on A, and isomorphic to $H^{\operatorname{co}\overline{H}}$ as a Yetter-Drinfeld-module algebra over H. The second question can be answered by studying the relation between grouplikes in the duals of $L(A,H)$ and $H^{\operatorname{co}\overline{H}}$.

In the last section we then use the general results to obtain a complete description of $\operatorname{Gal}(u_q(\mathfrak{sl}_2))$. This is a special case of results in Günther's thesis, which both motivated the general theory presented in this paper, and served as an example to test it on.

2. Galois objects over twisted tensor products

Recall the following facts from [11]: Let H and H' be two Hopf algebras, and A an H-H'-biGalois object. Then we have equivalences of monoidal categories

$$\mathcal{M}^H \ni V \mapsto V \mathop{\Box}_H A \in \mathcal{M}^{H'}$$
$$^{H'}\mathcal{M} \ni V \mapsto A \mathop{\Box}_{H'} V \in {}^H\mathcal{M}$$

In particular, for any bialgebra B these induce bijections between isomorphism classes of B-H-bicomodule algebras and of B-H'-bicomodule algebras, and also between isomorphism classes of H'-B-bicomodule algebras and of H-B-bicomodule algebras. These bijections induce bijections between the sets of H- and H'-Galois objects (on the left or right) and between B-H-biGalois objects and B-H'-biGalois objects, and also between H-B-biGalois objects and H'-B-biGalois objects.

We shall be needing a special case of these bijections.

A 2-cocycle on a bialgebra H is a map $\sigma : H \otimes H \to k$ such that for all $f, g, h \in H$ we have

$$\sigma(f_{(1)}, g_{(1)})\sigma(f_{(2)}g_{(2)}, h) = \sigma(g_{(1)}, h_{(1)})\sigma(f, g_{(2)}h_{(2)})$$

and $\sigma(h, 1) = \sigma(1, h) = 1$. If σ is a (convolution) invertible two-cocycle, then Doi [1] constructs a double twisted Hopf algebra H^σ which is H as a coalgebra with multiplication $g \cdot h = \sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}\sigma^{-1}(g_{(3)}, h_{(3)})$.

An older construction based on a convolution invertible 2-cocycle is the the crossed product right H-comodule algebra $k_\sigma[H] := k\#_\sigma H$, which is defined to be H with multiplication $g \cdot h = \sigma(g_{(1)}, h_{(1)})g_{(2)}h_{(2)}$. It is a right H-Galois object, and by [11] in fact an H^σ-H-biGalois object via the comultiplication of H.

More generally, we can define, for any Hopf algebras L and H, invertible two-cocycles $\alpha : L \otimes L \to k$ and $\sigma : H \otimes H \to k$, and any L-H-bicomodule algebra A the twisted algebra $k^\sigma_\alpha[A]$ to be A with the multiplication

$$x\cdot y := \alpha(x_{(-1)},y_{(-1)})x_{(0)}y_{(0)}\sigma^{-1}(x_{(1)},y_{(1)}).$$

Note that $k_\sigma[H]=k^\varepsilon_\sigma[H]$ and $H^\sigma=k^\sigma_\sigma[H]$ are special cases. One checks that $k^\sigma_\alpha[A]$ is an L^α-H^σ-bicomodule algebra, and the canonical isomorphisms $A\cong A\underset{H}{\square}H$ and $A\cong L\underset{L}{\square}A$ induce bicomodule algebra isomorphisms

$$(k_\alpha[L])\underset{L}{\square}A\cong k_\alpha[A]\cong(k^\sigma_\alpha[A])\underset{H^\sigma}{\square}k_\sigma[H].$$

We deduce that

$$\mathrm{BiGal}(L,H)\ni A\mapsto k^\sigma_\alpha[A]\in\mathrm{BiGal}(L^\alpha,H^\sigma) \tag{1}$$

is a bijection.

Let L,H be two bialgebras. A skew pairing of L and H is a map $\tau:L\otimes H\to k$ satisfying

$$\tau(\ell\ell',h)=\tau(\ell,h_{(2)})\tau(\ell',h_{(1)})$$
$$\tau(\ell,hh')=\tau(\ell_{(1)},h)\tau(\ell_{(2)},h')$$

and $\tau(\ell,1)=\varepsilon(\ell)$, $\tau(1,h)=\varepsilon(h)$ for all $\ell,\ell'\in L$ and $h,h'\in H$. We will denote the set of skew pairings of L and H by $\mathrm{SPair}(L,H)$. A skew pairing is said to be invertible if it is as an element of $(L\otimes H)^*$. The trivial skew pairing is by definition $\varepsilon:=\varepsilon_{L\otimes H}$. Note that if H is a Hopf algebra then any skew pairing $\tau:L\otimes H\to k$ is invertible with $\tau^{-1}=\tau(\mathrm{id}_L\otimes S)$.

Let $\tau:L\otimes H\to k$ be an (invertible) skew pairing. Then $\hat\tau:=\varepsilon\otimes\tau\otimes\varepsilon:(H\otimes L)\otimes(H\otimes L)\to k$ is an (invertible) two-cocycle.

As in [3] we denote the twisted Hopf algebra $(H\otimes L)^{\hat\tau}$ by $H\bowtie_\tau L$. Its multiplication is given by $(h\bowtie 1)(1\bowtie\ell)=h\bowtie\ell$ and $(1\bowtie\ell)(h\bowtie 1)=\tau(\ell_{(1)},h_{(1)})(h_{(2)}\bowtie\ell_{(2)})\tau^{-1}(\ell_{(3)},h_{(3)})$

Let H_1,H_2,L_1,L_2 be Hopf algebras, $\tau\in\mathrm{SPair}(L_2,L_1)$ and $\sigma\in\mathrm{SPair}(H_2,H_1)$. Then for L_i-H_i-bicomodule algebras A_i we abbreviate

$$A_1\overset{\sigma}{\underset{\tau}{\#}}A_2:=k^{\hat\sigma}_{\hat\tau}[A_1\otimes A_2].$$

Thus, multiplication in $A_1\overset{\sigma}{\underset{\tau}{\#}}A_2$ is given by

$$(x\#y)(x'\#y')=x\tau(y_{(-1)},x'_{(-1)})x'_{(0)}\#y_{(0)}\sigma^{-1}(y_{(1)},x'_{(1)})y'$$

Theorem 2.1. *Let H_1 and H_2 be Hopf algebras, and $\tau\in\mathrm{SPair}(H_2,H_1)$.*

1. *For $A_1\in\mathrm{Gal}(H_1)$, $A_2\in\mathrm{Gal}(H_2)$ and $\sigma\in\mathrm{SPair}(L_2,L_1)$, where $L_i=L(A_i,H_i)$, we have $A_1\overset{\tau}{\underset{\sigma}{\#}}A_2\in\mathrm{Gal}(H_1\bowtie_\tau H_2)$ and $L(A_1\overset{\tau}{\underset{\sigma}{\#}}A_2,H_1\bowtie_\tau H_2)\cong L_1\bowtie_\sigma L_2$.*
2. *Any $A\in\mathrm{Gal}(H_1\bowtie_\tau H_2)$ has the form in (1) for unique A_i and σ.*

Proof. This is a corollary of the bijection (1) and [9, Prop. 12]. $\square$

3. Galois objects over normal quotient Hopf algebras

Throughout the section we assume the following situation: Let H be a k-flat Hopf algebra with bijective antipode over the commutative base ring k. Let H' be a normal k-flat Hopf subalgebra such that H is a faithfully flat H'-module, and let $\overline{H} := H/H'^{+}H$ be the corresponding quotient Hopf algebra. Then H is a faithfully coflat $\overline{H}$-comodule and ${}^{\mathrm{co}\,\overline{H}}H = H^{\mathrm{co}\,\overline{H}} = H$. We will study the relation between the classes of H-Galois objects and $\overline{H}$-objects. First, we consider more generally faithfully flat Hopf Galois extensions of an arbitrary algebra B. An easy way to get H-Galois extensions from F-Galois extensions and a Hopf algebra map $H \to F$ for any k-flat Hopf algebra F is by forming the cotensor product $A \mathbin{\underset{F}{\square}} H$, as proved in [12, Rem.3.11.(3)]; this yields a map $\mathrm{Gal}_B(F) \to \mathrm{Gal}_B(H)$. We will study this map in the special case $F = \overline{H}$. The question then will be what the image and the fibers of this map are.

Recall that for $A \in \mathrm{Gal}_B(H)$ the Miyashita-Ulbrich action of H on the centralizer A^B of B in A is defined as the unique mapping $A^B \otimes H \ni x \otimes h \mapsto x \leftharpoonup h$ such that $xy = y_{(0)}(x \leftharpoonup y_{(1)})$ holds for all $x \in A^B$ and $y \in A$ (see [13, 2]). In the special case that $A = H$, this is just the adjoint action of H on itself by $g \leftharpoonup h = S(h_{(1)})gh_{(2)}$. The Miyashita-Ulbrich action makes A^B a Yetter-Drinfeld-H-module algebra; that is, A^B is an H-module algebra, and the condition $x_{(0)} \leftharpoonup h_{(1)} \otimes x_{(1)}h_{(2)} = (x \leftharpoonup h_{(1)})_{(0)} \otimes (x \leftharpoonup h_{(1)})_{(1)}h_{(2)}$ holds for all $x \in A^B$ and $h \in H$.

We will use the following simple fact:

Lemma 3.1. *Let H be a Hopf algebra and $M \in \mathcal{M}_H^H$ a Hopf module. Then the identity on M induces an isomorphism $M^{\mathrm{co}\,H} \cong M/MH^{+}$, whose inverse maps $\overline{m}$ to $m_{(0)}S(m_{(1)})$.*

Theorem 3.2. (1) *Let $\overline{A} \in \mathrm{Gal}_B(\overline{H})$. Then $\overline{A} \mathbin{\underset{\overline{H}}{\square}} H \in \mathrm{Gal}_B(H)$, and $f_{\overline{A}} : H' \ni h' \mapsto 1 \otimes h' \in (\overline{A} \mathbin{\underset{\overline{H}}{\square}} H)^B$ is a Yetter-Drinfeld-H-module algebra map.*

(2) *Let $A \in \mathrm{Gal}_B(H)$, and let $f : H' \to A^B$ be a homomorphism of Yetter-Drinfeld-H-module algebras. Then $Af(H'^{+})$ is an $\overline{H}$-subcomodule ideal of A, and $A/f := A/Af(H'^{+}) \in \mathrm{Gal}_B(\overline{H})$.*

(3) *In the situation of (1) we have $(A/f) \mathbin{\underset{\overline{H}}{\square}} H = A$ in $\mathrm{Gal}_B(H)$. In the situation of (2) we have $(\overline{A} \mathbin{\underset{\overline{H}}{\square}} H)\big/ f_{\overline{A}} = \overline{A}$ in $\mathrm{Gal}_B(\overline{H})$.*

(4) *Every $\overline{A} \in \mathrm{Gal}_B(\overline{H})$ arises as A/f for a unique $A \in \mathrm{Gal}_B(H)$ and a Yetter-Drinfeld-H-module algebra map $f : H' \to A^B$ which is determined uniquely up to an H-comodule algebra automorphism of A fixing B elementwise.*

Proof. (1) Let $\overline{A} \in \mathrm{Gal}(\overline{H})$. Then $\overline{A} \mathbin{\square}_{\overline{H}} H \in \mathrm{Gal}(H)$ by [12, Rem.3.11.(3)]. The Miyashita-Ulbrich action of H on $(\overline{A} \mathbin{\square}_{\overline{H}} H)^B = A^B \mathbin{\square}_{\overline{H}} H$ is given by $(\sum x_i \otimes h_i) \leftharpoonup g = \sum x_i \leftharpoonup g_{(2)} \otimes S(g_{(1)}) h_i g_{(3)}$. In fact, this action is well defined because A is a Yetter-Drinfeld-module, which entails $(x \leftharpoonup g)_{(0)} \otimes (x \leftharpoonup g)_{(1)} = x_{(0)} \leftharpoonup g_{(2)} \otimes S(g_{(1)}) x_{(1)} g_{(3)}$, and the formula $\xi\zeta = \zeta_{(0)}(\xi \leftharpoonup \zeta_{(1)})$ can be verified easily. It follows that $f_{\overline{A}}$, which is clearly an H-comodule algebra map to $(\overline{A} \mathbin{\square}_{\overline{H}} H)^B$, is also H-linear.

(2) Let $A \in \mathrm{Gal}_B(H)$ and $f : H' \to A^B$ an H-module, comodule and algebra map. Then $Af(H'^+)$ is an ideal because for $h' \in H'^+$ and $a \in A$ we have

$$f(h')a = a_{(0)}(f(h') \leftharpoonup a_{(1)}) = a_{(0)} f(S(a_{(1)}) h' a_{(2)}) \in Af(H'^+.)$$

Clearly $Af(H'^+)$ is an $\overline{H}$-subcomodule. In particular, A and $Af(H'^+)$ can be considered as relative Hopf modules in ${}_A\mathcal{M}^{\overline{H}}$. Since A is a faithfully flat $\overline{H}$-Galois extension of $A(H')$ we can use the structure theorem on Hopf modules [12, Thm. I]. Because $A(H')f(H'^+)$ is a left $A(H')$-submodule of $(Af(H'^+))^{\mathrm{co}\,\overline{H}}$ with $A(A(H')f(H'^+)) = Af(H'^+)$, this yields $A(H')f(H'^+) = (Af(H'^+))^{\mathrm{co}\,\overline{H}}$. Since $A(H')$ is an H'-Galois extension of B, and becomes, via f, an object of $\mathcal{M}^{H'}_{H'}$, we have, using [12, Thm.I] and Lemma 3.1,

$$\begin{aligned}\left(A/Af(H'^+)\right)^{\mathrm{co}\,\overline{H}} \cong A^{\mathrm{co}\,\overline{H}} \Big/ \left(Af(H'^+)\right)^{\mathrm{co}\,\overline{H}} &\cong A(H') \big/ A(H')f(H'^+) \\ &\cong A(H')^{\mathrm{co}\,H'} \cong B.\end{aligned}$$

As an $\overline{H}$-comodule algebra, A/f is a quotient of A, which in turn is a faithfully flat $\overline{H}$-Galois extension of $A(H')$ by [12, Cor.3.10]. It follows by [8, Prop.3.3] that A/f is a faithfully flat $\overline{H}$-Galois extension of B.

(3) Since any comodule algebra map between faithfully flat Hopf-Galois extensions of B that fixes B elementwise is an isomorphism by [12, Rem.3.11(1)], it is enough to point out the obvious maps

$$\begin{aligned}(\overline{A} \mathbin{\square}_{\overline{H}} H)\big/ f_{\overline{A}} \ni \overline{\textstyle\sum x_i \otimes h_i} &\mapsto \textstyle\sum x_i \varepsilon(h_i) \in \overline{A} \\ A \ni a &\mapsto \overline{a_{(0)}} \otimes a_{(1)} \in (A/f) \mathbin{\square}_{\overline{H}} H\end{aligned}$$

(4) By (3) it is clear that every element of $\mathrm{Gal}(\overline{H})$ arises as A/f for a unique $\mathrm{Gal}(H)$ and some Yetter-Drinfeld-H-module algebra map $f : H' \to A^B$. If t is an H-colinear algebra automorphism of A fixing the elements of B, then it is clear that t induces an isomorphism $\overline{t} : A/f \to A/tf$. Conversely, assume given two Yetter-Drinfeld-H-module algebra homomorphisms $f, g : H' \to A^B$ and an isomorphism $\overline{t} : A/f \to A/g$ of $\overline{H}$-Galois extensions of B. Then an

automorphism t of the H-Galois extension A of B can be defined by commutativity of the outer square of the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "\pi_f"] \arrow[dd, "t"'] & & A/f \underset{\overline{H}}{\square} H \arrow[dd, "\overline{t} \,\square\, H"] \\
& H' \arrow[ul, "f"] \arrow[ur, "f_{A/f}"'] \arrow[dl, "g"] \arrow[dr, "f_{A/g}"'] & \\
A \arrow[rr, "\pi_g"'] & & A/g \underset{\overline{H}}{\square} H'
\end{tikzcd}$$

It remains to show that the left triangle commutes. But the right triangle commutes trivially, and the top triangle commutes because $f(h'_{(1)}) \otimes h'_{(1)} \equiv \overline{1} \otimes h'$ modulo $f(H'^+)$ for $h' \in H'$; the the bottom triangle is analogous to the top one. Since π_g is an isomorphism, we can conclude that the left triangle commutes also. $\square$

A more concise way of stating the content of the preceding theorem is the following:

Corollary 3.3. *Define*

$$\mathcal{I} : \mathrm{Gal}_B(\overline{H}) \ni \overline{A} \mapsto \overline{A} \underset{\overline{H}}{\square} H \in \mathrm{Gal}_B(H).$$

Then $A \in \mathrm{Im}(\mathcal{I})$ iff $\mathrm{Alg}^H_H(H', A^B) \neq \emptyset$. The group $\mathrm{Alg}^H_B(A, A)$ acts on the left on $\mathrm{Alg}^H_H(H', A^B)$ by composition, and the quotient of the action is in bijection with $\mathcal{I}^{-1}(A)$ via

$$\mathrm{Alg}^H_B(A, A)\Big\backslash \mathrm{Alg}^H_H(H', A^B) \ni [f] \mapsto A/f \in \mathcal{I}^{-1}(A).$$

Let us turn to the special case of Hopf-Galois objects, that is $B = k$. Let $A \in \mathrm{Gal}(H)$. Put $L := L(A, H)$ and $L' := L(A(H'), H')$.

Let us recall some results from [10]: We can consider L' as a Hopf subalgebra of L in a unique way such that $A(H') \subset A$ is an L-H-bicomodule subalgebra of A. Moreover, we then have $A(H') = {}^{\mathrm{co}\,A}\overline{L}$ and $L' = {}^{\mathrm{co}\,L}\overline{L}$ for $\overline{L} := L/LL'^+$. The Hopf subalgebra $L' \subset L$ is normal if and only if $A(H') \subset A$ is an H-submodule with respect to the Miyashita-Ulbrich action.

Now assume A is in the image of $\mathcal{I} : \mathrm{Gal}(\overline{H}) \to \mathrm{Gal}(H)$. Then there exists $f \in \mathrm{Alg}^H_H(H', A)$. Obviously, the image of f is contained in $A(H')$, and since H' and $A(H')$ are H'-Galois objects, $H' \cong A(H')$ via f. In particular, $A(H')$ is an

H-submodule of the Miyashita-Ulbrich action of H on A, and isomorphic to H' as a Yetter-Drinfeld-H-module algebra. It follows that we have an isomorphism of Hopf algebras $H' \cong L(H',H') \cong L(A(H'),H') = L'$, which is by definition the unique vector space map $\phi_f : H' \to L'$ such that $\phi_f(h'_{(1)}) \otimes f(h'_{(2)}) = f(h')_{(-1)} \otimes f(h')_{(0)}$ holds for all $h' \in H'$.

Since any other $g \in \mathrm{Alg}_H^H(H',A)$ also induces an isomorphism $H' \cong A(H')$, it has the form $g = f\theta$ for a unique H-linear and -colinear algebra automorphism θ of H'. Being an H'-colinear map, θ has the form $\theta(h') = u(h'_{(1)})h'_{(2)}$ with $u = \varepsilon\theta : H' \to k$. Since θ is an H-linear algebra map, so is u, and conversely, any H-linear algebra map $u : H' \to k$ defines an H-colinear and H-linear algebra automorphism θ of H' by $\theta(h') = u(h'_{(1)})h'_{(2)}$. This shows that the group $\mathrm{Alg}_H(H',k)$ acts simply and transitively on the left on $\mathrm{Alg}_H^H(H',A)$ by the formula $(u * f)(h') = u(h'_{(1)})f(h'_{(2)})$. By the universal properties of $L(A,H)$, we have a bijection

$$\mathrm{Alg}(L,k) \ni v \mapsto t_v \in \mathrm{Alg}^H(A,A),$$

given by $t_v(a) = v(a_{(-1)})a_{(0)}$. This bijection is easily checked to be a group anti-isomorphism. Now for $v \in \mathrm{Alg}(L,k)$, $f \in \mathrm{Alg}_H^H(H',A)$ and $u \in \mathrm{Alg}_H(H',k)$ we have

$$\begin{aligned} t_v(u * f)(h') = t_v(u(h'_{(1)})f(h'_{(2)})) &= u(h'_{(1)})v(f(h'_{(2)})_{(-1)})f(h'_{(2)})_{(0)} \\ &= u(h'_{(1)})v\phi_f(h'_{(2)})f(h'_{(3)}) = ((u * v\phi_f) * f)(h'). \end{aligned}$$

Summarizing, we have proved:

Theorem 3.4. *Let $A \in \mathrm{Gal}(H)$ be in the image of $\mathcal{I} : \mathrm{Gal}(\overline{H}) \to \mathrm{Gal}(H)$, and put $L := L(A,H)$. Fix $f \in \mathrm{Alg}_H^H(H',A)$. Then there is a unique Hopf algebra map $\phi_f : H' \to L$ for which f is left L-colinear. The image L' of ϕ_f is normal in L and independent of the choice of f. We have $L(A/f,\overline{H}) \cong L/LL'^+$. There is a group homomorphism*

$$\mathrm{Alg}(L,k) \ni v \mapsto \phi_f^* := v\phi_f \in \mathrm{Alg}_H(H',k)$$

and the space of left cosets of the image of ϕ_f^* is in bijection with $\mathcal{I}^{-1}(A)$ via

$$\mathrm{Alg}_H(H',k)\big/\phi_f^*(\mathrm{Alg}(L,k)) \ni [u] \mapsto A/(u * f) \in \mathcal{I}^{-1}(A).$$

Note that ϕ_f can depend on the choice of f; in fact, one can check that $\phi_{u*f} = \phi_f \ \mathrm{coinn}(u^{-1})$.

4. Galois objects over $u_q(\mathfrak{sl}_2)$

Fix a natural number $n \geq 2$, and assume that the base field k contains an element q such that q^2 is a primitive n-th root of unity.

Consider the following two Hopf algebras:

$$T = k\langle G, X\rangle\big/(G^n - 1, X^n, XG - q^2GX)$$

with grouplike G and $(1,G)$-primitive X, and

$$T' = k\langle J, Y\rangle \big/ (J^n - 1, Y^n, YJ - q^{-2}JY)$$

with grouplike J and $(J^{-1},1)$-primitive Y. T and T' are copies of two-generator Taft algebras.

The Hopf algebra T is known to be selfdual; there is a Hopf algebra isomorphism $D : T \to T^*$ with $D(G)(G) = q^2$, $D(G)(X) = 0$, $D(X)(G) = 0$ and $D(X)(X) = 1$.

The Hopf algebra T' is isomorphic to T^{op} via the unique Hopf algebra map $g : T' \to T^{\mathrm{op}}$ with $g(J) = G$ and $g(Y) = G^{-1}X$.

In particular, skew pairings $\tau : T' \otimes T \to k$ are in bijection with Hopf algebra maps $f : T \to T$ by the formula $\tau(\zeta, \xi) = D(fg(\zeta), \xi)$. All such f are given by $f(G) = G^\ell$ and $f(X) = 0$ for any $\ell \in \mathbb{Z}/(N)$, or by $f(G) = G$ and $f(X) = rX$ for any $r \in k$. We define

$$\mathcal{M} := \{(\ell, r) \in \mathbb{Z}/(N) \times k | \ell = 1 \text{ or } k = 0\} \tag{2}$$

and conclude that there is a bijection

$$\mathcal{M} \ni (\ell, r) \mapsto [\ell, r] \in \mathrm{SPair}(T', T)$$

with $[\ell, r]$ determined by $[\ell, r](J, G) = q^{2\ell}$, $[\ell, r](J, X) = 0 = [\ell, r](Y, G)$ and $[\ell, r](Y, X) = r$. One checks that $[\ell, r]^{-1}(J, G) = q^{-2\ell}$, $[\ell, r]^{-1}(J, X) = 0 = [\ell, r]^{-1}(Y, G)$ and $[\ell, r]^{-1}(Y, X) = -r$.

Let $\mathcal{A}$ be a set of representatives of $\dot{k}/(\dot{k})^n$. Masuoka [7] has classified all cleft extensions over T, and in particular we know that

$$\mathcal{A} \times k \ni (\alpha, \beta) \mapsto A_{\alpha,\beta} \in \mathrm{Gal}(T)$$

is a bijection, where

$$A_{\alpha,\beta} = k\langle g, x\rangle \big/ (g^n - \alpha, x^n - \beta, xg - q^2 gx)$$

with right comodule algebra structure determined by $\rho(g) = g \otimes G$ and $\rho(x) = 1 \otimes X + x \otimes G$. We have checked in [9] that $A_{\alpha,\beta}$ is a T-T-biGalois extension with left comodule algebra structure determined by $\lambda(g) = G \otimes g$ and $\lambda(x) = 1 \otimes x + X \otimes g$.

One concludes that

$$\mathcal{A} \times k \ni (\alpha, \beta) \mapsto A'_{\alpha,\beta} \in \mathrm{Gal}(T')$$

is also a bijection, where

$$A'_{\alpha,\beta} = k\langle j, y\rangle \big/ (j^n - \alpha, y^n - \beta, yj - q^{-2} jy)$$

with right comodule algebra structure $\rho(j) = j \otimes J$ and $\rho(y) = j^{-1} \otimes Y + y \otimes 1$. Moreover, $A'_{\alpha,\beta}$ is a T'-biGalois object with $\lambda(j) = J \otimes j$ and $\lambda(y) = J^{-1} \otimes y + Y \otimes 1$.

We can apply Theorem 2.1 to conclude that for every $(m,s) \in \mathcal{M}$ we have a bijection

$$\begin{aligned} \mathcal{A} \times k \times \mathcal{A} \times k \times \mathcal{M} &\to \operatorname{Gal}(T \bowtie_{[m,s]} T') \\ (\alpha,\beta,\alpha',\beta',\ell,r) &\mapsto A_{\alpha,\beta} \mathop{\#}_{[\ell,r]}^{[m,s]} A'_{\alpha',\beta'} \end{aligned}$$

We shall do some computations in $A_{\alpha,\beta} \mathop{\#}_{[\ell,r]}^{[m,s]} A'_{\alpha',\beta'}$. We have

$$\begin{aligned} (1\#j)(g\#1) &= [\ell,r](J,G)g\#j[m,s]^{-1}(J,G) = q^{2\ell-2m}g\#j \\ (1\#j)(x\#1) &= [\ell,r](J,x_{(-1)})x_{(0)}\#j[m,s]^{-1}(J,x_{(1)}) \\ &= [\ell,r](J,1)x\#j[m,s]^{-1}(J,G) \\ &= q^{-2m}x\#j \\ (1\#y)(g\#1) &= [\ell,r](y_{(-1)},G)g\#y_{(0)}[m,s]^{-1}[y_{(1)},G) \\ &= [\ell,r](J^{-1},G)g\#y[m,s]^{-1}(1,G) \\ &= q^{-2\ell}g\#y \\ (1\#y)(x\#1) &= [\ell,r](J^{-1},1)x\#y[m,s]^{-1}(1,G) \\ &\quad + [\ell,r](Y,X)g\#1[m,s]^{-1}(1,G) \\ &\quad + [\ell,r](J^{-1},1)1\#j^{-1}[m,s]^{-1}(Y,X) \\ &= x\#y + r\,g\#1 - s\,1\#j^{-1} \end{aligned}$$

In the sequel we shall write $g = g\#1$, $x = x\#1$, $y = 1\#y$ and $j = 1\#j$ in $A_{\alpha,\beta} \mathop{\#}_{[\ell,r]}^{[m,s]} A'_{\alpha',\beta'}$, and $G = G \bowtie 1$, $J = 1 \bowtie J$ etc in $T \bowtie_{[\ell,r]} T'$.

The right coaction of $T \bowtie_{[m,s]} T'$ on $A_{\alpha,\beta} \mathop{\#}_{[\ell,r]}^{[m,s]} A'_{\alpha',\beta'}$ satisfies $\rho(x) = 1 \otimes X + x \otimes G$ and $\rho(y) = j^{-1} \otimes Y + y \otimes 1$. One checks easily that the Galois map κ satisfies

$$\begin{aligned} \kappa(1 \otimes x - xg^{-1} \otimes g) &= 1 \otimes X \\ \kappa(j \otimes y - jy \otimes 1) &= 1 \otimes Y \\ \kappa(g^{-1} \otimes g) &= 1 \otimes G \\ \kappa(j^{-1} \otimes j) &= 1 \otimes J \end{aligned}$$

whence the Miyashita-Ulbrich action of $T \bowtie_{[m,s]} T'$ on $A_{\alpha,\beta} \mathop{\#}\limits_{[\ell,r]}^{[m,s]} A'_{\alpha',\beta'}$ satisfies

$$\begin{aligned}
j^{-1}g \leftharpoonup G &= g^{-1}j^{-1}g^2 = q^{2m-2\ell}j^{-1}g \\
j^{-1}g \leftharpoonup J &= j^{-2}gj = q^{2m-2\ell}j^{-1}g \\
j^{-1}g \leftharpoonup X &= j^{-1}gx - xg^{-1}j^{-1}g^2 = (1-q^{2-2\ell})xj^{-1}g \\
j^{-1}g \leftharpoonup Y &= jj^{-1}gy - jyj^{-1}g = (1-q^{2-2\ell})gy.
\end{aligned}$$

In particular, the subalgebra $\mathcal{A}'$ in $A_{\alpha,\beta} \mathop{\#}\limits_{[\ell,r]}^{[m,s]} A'_{\alpha',\beta'}$ generated by $j^{-1}g$ is stable under the Miyashita-Ulbrich action if and only if $\ell = 1$.

As a special case, the Hopf subalgebra H' of $T \bowtie_{[1,s]} T'$ generated by $J^{-1}G$ is normal. H' is isomorphic to the group algebra of a cyclic group of order n. Fix some $s \in k$ and consider $H = T \bowtie_{[1,s]} T'$. We wish to use the results of section 3 to determine the isomorphism classes of Galois objects over the quotient Hopf algebra $\overline{H} := H/HH'^+$.

Clearly

$$\left(A_{\alpha,\beta} \mathop{\#}\limits_{[\ell,r]}^{[1,s]} A'_{\alpha',\beta'}\right) \mathop{\square}\limits_{H} H' \cong \mathcal{A}'$$

for any $\alpha, \beta, \alpha', \beta', \ell, r$. We have already checked that $\mathcal{A}'$ is invariant under the Miyashita-Ulbrich action of H if and only if $\ell = 1$. Assuming this, we have $jg = gj$ and thus $(j^{-1}g)^n = \alpha/\alpha'$, so that, as a H'-comodule algebra, that is, C_n-graded algebra, $\mathcal{A}'$ is isomorphic to $k[t]/(t^n - \alpha/\alpha')$. This in turn is isomorphic to H' as a H'-comodule algebra iff α/α' is an n-th power in k, and thus $\alpha = \alpha'$. Conversely, if $\alpha = \alpha'$ and $\ell = 1$, then $f : H' \ni J^{-1}G \mapsto j^{-1}g \in \mathcal{A}'$ is an isomorphism of Yetter-Drinfeld-H-module algebras. The corresponding Hopf algebra imbedding

$$\phi = \phi_f : H' \to L := L\left(A_{\alpha,\beta} \mathop{\#}\limits_{[1,r]}^{[1,s]} A'_{\alpha,\beta}, H\right) = T \bowtie_{[1,r]} T'$$

is given by $J^{-1}G \mapsto J^{-1}G$. It is easy to determine all algebra maps $L \to k$: the images of G and J have to be n-th roots of unity, and the images of X and Y have to be zero. The relation $YX - XY = rG - rJ^{-1}$ entails that the image of G is inverse to the image of J unless $r = 0$. So the general form of an algebra map $v : L \to k$ is determined by $v(G) = q^{2e}$, $v(J) = q^{-2e'}$ with the condition $e = e'$ if $r \neq 0$. The restriction of v to H' along j is given by $J^{-1}G \mapsto q^{2e+2e'}$. Thus, if $r = 0$ or N is odd (in which case q^2 is of the form q^{4e} for suitable e) any algebra map $H' \to k$ arises as the restriction of some v, while in the case $r \neq 0$ and N even, there are two classes of algebra maps $H' \to k$ modulo restrictions of algebra maps $L \to k$. Representatives of these two classes are given by $u(J^{-1}G) = q^{2t}$ with

$t \in \{0,1\}$, and $(u * f)(J^{-1}G) = q^{2t}j^{-1}g$. We conclude that any Galois object over $\overline{H} = T \bowtie_{[1,s]} T'/(T \bowtie_{[1,s]} T')H'^{+}$ arises uniquely as one of the following:

$$\left(A_{\alpha,\beta} \underset{[1,r]}{\overset{[1,s]}{\#}} A'_{\alpha,\beta'} \right) \Big/ f_t$$

where $\alpha \in \dot{k}$, $\beta, \beta' \in k$, $r \in k$ and $t \in \{0,1\}$ with $t = 0$ if $r = 0$ or N is odd, and f_t is given by $f_t(J^{-1}G) = q^{2t}j^{-1}g$.

Finally, note that

$$\begin{aligned} \left(T \bowtie_{[1,r]} T'\right) \Big/ HH'^{+} &\cong u_q(\mathfrak{sl}_2) \\ G \bowtie 1 = 1 \bowtie J &\mapsto G \\ X \bowtie 1 &\mapsto X \\ 1 \bowtie Y &\mapsto r(q - q^{-1}) \end{aligned} \qquad \text{if } r \neq 0$$

while $(T \bowtie_{[1,0]} T')/HH'^{+}$ has two commuting nonproportional nontrivial skew primitives if $r = 0$, hence is not isomorphic to $u_q(\mathfrak{sl}_2)$. We summarize

Theorem 4.1. *Every Galois object Γ over $u_q(\mathfrak{sl}_2)$ arises uniquely as*

$$k\langle g, x, y\rangle/(g^n - \alpha, x^n - \beta, y^n - \beta', xg - q^2gx, yg - q^{-2}gy, xy - yx - rg - \frac{q^{2t}}{q - q^{-1}}g^{-1})$$

for $\alpha \in \dot{k}$ uniquely determined up to multiplication by an n-th power, unique $\beta, \beta', r \in k$, and $t \in \{0,1\}$ which is unique unless $r = 0$ or N is odd, and redundant in that case.

We have $L(\Gamma, u_q(\mathfrak{sl}_2)) \cong u_q(\mathfrak{sl}_2)$ if and only if $r \neq 0$, and

$$L(\Gamma, u_q(\mathfrak{sl}_2)) \cong k\langle G, X, Y\rangle \big/ (G^n - 1, X^n, Y^n, XG - q^2GX, YG - q^{-2}GY, XY - XY)$$

if $r = 0$.

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